Integral expression of Dirichlet L-series

Marie José BERTIN

Mahler measure 3, MOBILE, January 5-8, 2006

Introduction

Berndt and Zaharescu proved the following Ramanujan type formula. Let $0 < q < 1$ then

$$q^{\frac{1}{9}} \prod_{n=1}^{\infty} (1-q^n)^{\chi_{-3}(n)n} = \exp(-C_3 - \frac{1}{9} \int_q^1 \frac{f^9(-t)}{f^3(-t^3)} \frac{dt}{t})$$

where in Ramanujan's notation,

$$f(-q) = \prod_{n=1}^{\infty} (1 - q^n)$$

and

$$C_3 = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) =: d_3.$$

The integrand is a weight 3 Eisenstein series for $\Gamma_0(3)$.

If

$$h(t) = \frac{f^9(-t)}{f^3(-t^3)}$$

and

$$-\frac{1}{9}\int_q^1 h(t)\frac{dt}{t} = -\frac{1}{9}\int_q^1 (h(t)-1)\frac{dt}{t} + \frac{1}{9}\log(q),$$

for q tending to 0 one gets the integral expression of d_3

$$d_3 = -\frac{1}{9}\int_0^1 (h(t)-1)\frac{dt}{t}.$$

From Villegas, we can derive

$$\boxed{\frac{1}{\pi}\int_0^1 (K-\frac{\pi}{2})\frac{dk}{k} = \log 2 - d_4,}$$

where

$$K = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}},$$

and

$$d_4 := \frac{4^{3/2}}{4\pi}L(\chi_{-4},2) = L'(\chi_{-4},-1) = \frac{2G}{\pi}$$

G being the Catalan's constant,

$$G = L(\chi_{-4},2) = 1 - \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$

Another formula

$$\boxed{\frac{1}{\pi}\int_0^1 K dk = d_4}$$

can be also deduced.

These formulae were known long ago, cf. Byrd Friedman's book.

Recently, Benferhat proved

$$L'(\chi_{-8},-1) = \frac{2\sqrt{2}}{\pi}\int_0^1 K(\sqrt{\frac{u^2+1}{2}})du$$

$$= \frac{4\sqrt{2}}{\pi}\int_{\frac{1}{\sqrt{2}}}^1 K\frac{kdk}{\sqrt{2k^2-1}}$$

$$L'(\chi_{-8},-1) = \ln(4\sqrt{2}+6) - \frac{4\sqrt{2}}{\pi}\int_0^1 \frac{K-\pi/2}{k\sqrt{2-k^2}}dk$$

where K denotes the real period of the Jacobi quartic

$$y^2 = (1-x^2)(1-k^2x^2).$$

Benferhat's proof uses the expression of the Mahler's measure of polynomials in family, namely

$$y^2(x+1)^2 + lxy + (x+1)^2$$

Let me explain the method for Villegas'example

Consider the family of elliptic curves

$$Q_l = (X + \frac{1}{X})(Y + \frac{1}{Y}) - 4l = 0.$$

Q_l defines the quartic E_l

$$Y^2(X^2+1) - 4lXY + (X^2+1) = 0$$

Completing the square, we get

$$(Y(X^2+1)-2lX)^2 = 4l^2X^2-(X^2+1)^2$$

Now for $l > 1$

$$m(Q_l) :=$$

$$\frac{1}{(2\pi i)^2}\int\int_{|X|=|Y|=1} \log \mid (X+\frac{1}{X})(Y+\frac{1}{Y})-4l \mid \frac{dX}{X}\frac{dY}{Y}$$

and

$$m'(l) = \frac{dm(Q_l)}{dl} =$$

$$\frac{1}{(2\pi i)^2}\int_{|X|=1}(\int_{|Y|=1} \frac{4}{4l-(X+\frac{1}{X})(Y+\frac{1}{Y})}\frac{dY}{Y})\frac{dX}{X}$$

By residue computation

$$m'(l) = \frac{1}{\pi i}\int_{|X|=1} \frac{dX}{\sqrt{4l^2X^2-(X+1)^2}} = \frac{1}{\pi i}\int_{|X|=1}\frac{dX}{Z}$$

a period of the elliptic curve

$$Z^2 = 4l^2X^2 - (X^2+1)^2$$

isomorphic to E_l hence satisfying the Picard-Fuchs equation of the family. Instead, applying Jensen's formula,

$$m'(l) = \frac{1}{\pi}\int_0^\pi \frac{2d\phi}{\sqrt{4l^2 - 4\cos^2\phi}}.$$

Now, if $\cos\phi = X$,

$$m'(l) = \frac{2}{\pi}\int_0^1 \frac{dX}{l\sqrt{(1-X^2)(1-\frac{X^2}{l^2})}} = \frac{2}{\pi}kK(k)$$

if $k^2 = \frac{1}{l^2}$. So

$$m'(l)dl = -\frac{2}{\pi}K\frac{dk}{k}$$

Then,

$$m(l) - m(1) = -\frac{2}{\pi}\int_1^k K\frac{dk}{k}$$

$$= -\frac{2}{\pi}\int_1^k (K - \frac{\pi}{2})\frac{dk}{k} - \int_1^k \frac{dk}{k}$$

Now

$$m(1) = 2d_4 = 2\log(2) - \frac{2}{\pi}\int_0^1 (K - \frac{\pi}{2})\frac{dk}{k}$$

If $0 < l < 1$, by applying Jensen's formula and a change of variables, we get

$$m'(l) = \frac{2}{\pi}K$$

Here $k = l$. Thus integrating separately

$$m(l) - m(0) = \frac{2}{\pi}\int_0^k K dk$$

and since $m(0) = 0$,

$$m(1) = 2L'(\chi_{-4}, -1) = \frac{2}{\pi}\int_0^1 K dk$$

Remark 1. • Since the previous integral expressions were known long ago, they can be used to guess $m(1)$.

• For the family Q_l, we get the Weierstass equation

$$y^2 = x^3 + (4l^2 - 2)x^2 - 4x - 4(4l^2 - 2)$$

hence

$$J(l) = 256\frac{(l^4 - l^2 + 1)^3}{l^4(l-1)^2(l+1)^2}$$

and the change variables used to obtain the integral expressions are among those leaving J invariant, namely the roots of

$$J(l) - J(T) =$$

$$(Tl+1)(l-T)(l+T)(Tl-1)(l^2T^2-T^2+1)$$

$$(l^2T^2 - l^2 - T^2)(l^2T^2 + 1 - l^2)(l^2 - 1 + T^2)$$

the group of anharmonic ratios.

- For the family Q_l we can use the modular method (Villegas)

- Modular elliptic surfaces are rare and we need methods for computing Mahler measure for the others. The philosophy is not a global approach but a local one.

QUESTIONS

Among the seven Boyd's elliptic surfaces, which are modular elliptic?

For the remaining ones can we apply the previous method?

Boyd's elliptic surfaces

- (1.3)
$$y^2 + y(x^2 + lx + 1) + x^2 = 0$$

- (2.3)
$$y^2(x+1) + y(x^2 + lx + 1) + x^2 + x = 0$$

- (3.1)
$$y^2(x^2 + x + 1) + ylx + x^2 + x + 1 = 0$$

- (3.3)
$$y^2(x^2+x+1)+y(x^2+(l+1)x+1)+x^2+x+1 = 0$$

- (3s.1)
$$y^2(x+1)^2 + lxy + (x+1)^2 = 0$$

- (3s.3)

  $$y^2(x+1)^2+y(x^2+(l+2)x+1)+(x+1)^2=0$$

- (3s.3s)

  $$y^2(x+1)^2+y(2x^2+(l+4)x+2)+(x+1)^2=0$$

Elliptic surfaces over $\mathbf{P}^1(\mathbb{C})$

Denote the elliptic surface by X. The elliptic fibration is given by

$$f : X \to \mathbb{P}^1$$

Stable families of elliptic curves on $\mathbb{P}^1$ are such that the corresponding elliptic surface has at most ordinary double points. They give semi-stable fibrations where the singular fibers are of type I_c (a polygonal line with c rational curves as edges).

Type of singular fibers

I_0		smooth elliptic curve
I_1		nodal rational curve
I_2		two smooth rational curves meeting transversally at 2 points
I_3		three smooth rational curves meeting in a cycle:a triangle
I_N	$N \geq 3$	smooth rational curves meeting in a cycle with dual graph $\tilde{A}_N$
I_N^*	$N + 5$	smooth rational curves meeting with dual graph $\tilde{D}_{N+4}$
II		a cuspidal rational curve
III		two smooth rational curves meeting at one point to order 2
IV		three smooth rational curves all meeting at one point
IV^*	7	smooth rational curves meeting with dual graph $\tilde{E}_6$
III^*	8	smooth rational curves meeting with dual graph $\tilde{E}_7$
II^*	9	smooth rational curves meeting with dual graph $\tilde{E}_8$

Beauville's results

The first Beauville's result is that a semi-stable elliptic surface has at least four singular fibers. The second is the following theorem.

Theorem 1. *(Beauville) Let* $f : X \to \mathbb{P}^1$ *a semi-stable elliptic surface with four singular fibers. Then* X *is isomorphic to a modular elliptic surface associated to one of the six modular groups* Γ.

Group Γ	*Equation of the surface*
$\Gamma(3)$	$X^3 + Y^3 + Z^3 + tXYZ$
$\Gamma_0^0(4) \cap \Gamma(2)$	$X(X^2 + Z^2 + 2ZY) + Z(X^2 - Y^2)$
$\Gamma_0^0(5)$	$X(X - Z)(Y - Z) + tZY(X - Y) =$
$\Gamma_0^0(6)$	$(X + Y)(Y + Z)(Z + X) + tXYZ =$
$\Gamma_0(8) \cap \Gamma_0^0(4)$	$(X + Y)(XY - Z^2) + tXYZ = 0$
$\Gamma_0(9) \cap \Gamma_0^0(3)$	$X^2Y + Y^2Z + Z^2X + tXYZ = 0$

Shioda and Nori's results

Let X be an elliptic surface over $\mathbb{P}^1(\mathbb{C})$. The Picard number N is given by

$$N = r + 2 + \sum_{\nu=1}^{n} (m_\nu - 1),$$

where r is the rank of the divisor class group of the associated generic curve of genus one, m_ν the number of irreducible components in the singular fiber C_ν, $\{C_\nu\}_{1\leq\nu\leq n}$ denoting the singular fibers of the elliptic surface.

In our case, the elliptic surface X has a section over $\mathbb{P}^1$ so r is in fact the rank of the Mordell-Weil group of the associated elliptic curve over the generic point of $\mathbb{P}^1$,

$$N := \dim_{\mathbb{Q}} H^1(X, \Omega^1_X) \cap H^2(X, \mathbb{Q})$$

$$h^{1,1} = \dim_{\mathbb{C}} H^1(X, \Omega^1_X).$$

So

$$N \leq h^{1,1}$$

By Shioda, we know that for elliptic modular surfaces, one gets $N = h^{1,1}$ and $r = 0$. Then Nori characterized elliptic surfaces with $N = h^{1,1}$ and $r = 0$. We are very much interested in them since elliptic modular surfaces are nice situations for testing the Mahler measure. Such surfaces characterized by Nori are called extremal by Miranda.

Theorem 2. *(Nori) An elliptic surface X over a base B with J non constant, has $N = h^{1,1}$ and $r = 0$ if and only if J is ramified only over 0, 1, ∞ with $e_p = 1, 2$ or 3 when $J(p) = 0$, $e_p = 1$ or 2 when $J(p) = 1$ and X has no singular fibers of type I_0^*, II, III and IV.*

Results and comments
Classification of Boyd's surfaces

Theorem 3. *The* 2*-torsion elliptic surfaces* $3s.1$ *and* $3s.3$ *are not elliptic modular.*

Proof. Since they both have I_0^* fibers, this derives directly from Nori's results. $\square$

Theorem 4. *The* 1.3 *and* 2.3 *surfaces are elliptic modular. So we have the following nice expression of the Mahler measure.*

1. *For* 1.3 *we can take the Hauptmodul as this*

$$t = 4 + \frac{32}{l-4} = \frac{\eta(4\tau)^{12}}{\eta(2\tau)^4\eta(8\tau)^8}$$

$$= \frac{1}{q} + 4q + 2q^3 - 8q^5 - q^7 + \cdots$$

2. *For* 2.3*, The Hauptmodul may be*

$$t = \frac{\eta(6\tau)^8\eta(\tau)^4}{\eta(3\tau)^4\eta(2\tau)^8}$$

$$t = \frac{1}{l+3}$$

and

$$m(l) = \Re\left(\frac{9\sqrt{3}y}{4\pi^2}\sum{}' \frac{\chi(n)}{(3m\tau+n)^2(3m\bar{\tau}+n)}\right)$$

$$+ 8\Re\left(\frac{9\sqrt{3}y}{4\pi^2}\sum{}' \frac{\chi(n)}{(6m\tau+n)^2(6m\bar{\tau}+n)}\right)$$

For the three remaining surfaces, it is not so obvious

Theorem 5. *1. The surface* (3.3) *is a tempered realization of the* 5*-tuple* $[4,4,2,1,1]$ *obtained either from the modular Beauville's realization* $[4,4,2,2]$ *or from* $[8,2,1,1]$ *by a monodromy extra permutation. It introduces an apparent singularity in the Picard-Fuchs equation of the surface and so* $r = 1$ *in the previous formula. Such a surface is thus not modular.*

2. *The surface* (3.1) *is a tempered realization of the* 6*-tuple* $[4, 4, 1, 1, 1, 1]$ *obtained either from the modular Beauville's realization* $[4, 4, 2, 2]$ *or from* $[8, 2, 1, 1]$ *by two monodromy extra permutations. It introduces two apparent singularities in the Picard-Fuchs equation of the surface and so* $r = 2$ *in the previous formula. Such a surface is thus not modular.*

3. *But locally, between two maximal unipotent singularities (singular fiber of type* I_n*) with no apparent singularity between them (no extra ramification point), one gets a polynomial relation between* k^2 *and* l *giving the change variables in the Mahler's measure integral expression.*

Proof. For the way the monodromy group can be altered we refer to Miranda.

In the first case, denote P the polynomial in the l and J variables:

$$P = 4096 l^4 (l+9)(l+1)(l-3)^2 J$$

$$-(1296 + 576 l - 416 l^2 + 64 l^3 + 16 l^4)^3.$$

With the "poldisc" order of PARI we get

$$poldisc(P, l) = (J - 1728)^6 J^8 (3J - 2048).$$

Thus J is ramified over 0, 1728, ∞ and $\frac{2048}{3}$. This latter ramification point corresponds to $l = -3$. Thus -3 will be an apparent singularity of the Picard-Fuchs equation.

In the second case, we get

$$P = 4096 l^4 (l^2 - 36)(l^2 - 4) J - (l^2 - 12)^6$$

$$poldisc(P, l) = (J - 1728)^6 J^{10}$$

In that case, there is ramification only over 0, 1 and ∞ but over 0, the index is 6 and not 1, 2 or 3 as expected. By Nori and Shioda's results such a surface is not modular. Moreover, $l = \pm 2\sqrt{3}$ are two apparent singularities of the Picard-Fuchs equation.

$\square$

Finally, it remains the surface $(3s.3s)$.

Theorem 6. *The surface* $(3s.3s)$ *has* $h^{1,1} = N$ *and* $r = 0$. *This is a modular elliptic surface for the congruence group* $\Gamma^0(4)$. *This is the extremal rational elliptic surface* X_{141} *in Miranda's notation. In that case the Hauptmodul*

$$k^2 = -\frac{16}{l}$$

is the Jacobi modulus.

Proof. We apply Nori's theorem. If

$$P = l(l+16)J - (l^2 + 16l + 16)^3$$

we get

$$poldisc(P,l) = (J-1728)^3 J^4$$

Since

$$P(1728,l) = (l+8)^2(l^2+16l-8)^2$$

the index of ramification over $J = 1728$ is 2, 2, 2. Since

$$P(0,l) = (l^2+16l+16)^3$$

the index of ramification over $J = 0$ is 3, 3. And over $J = \infty$ we get the fibers I_1, I_4 and I_1^*. So $h^{1,1} = N$ and $r = 0$. □

Corollary 7.

$$m(y^2(x+1)^2 + 2y(x^2-6x+1) + (x+1)^2)$$

$$= L'(\chi_{-4}, -1)$$

Remark 2. 1. The previous formula, guessed numerically by Boyd, was also conjectured by Vandervelde.

2. The two non modular elliptic surfaces $(3s.1)$ and $(3s.3)$ having both a fiber of type I_0^* are twists of the elliptic modular one $(3s.3s)$. By just a change $l \mapsto l+4$, we get the same J and the same projective Picard-Fuchs equation.