Mahler measure of some multivariate polynomials: old and new

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Introduction

We are interested in two types of polynomials:

1. two-variable polynomials with Mahler measure expressed as a Dirichlet L-series
2. three-variable polynomials defining singular $K3$ surfaces with Mahler measure expressed as L-series of the $K3$ surface

all polynomials define $K3$ surfaces over $\mathbb{Q}$
all the Mahler measures are the logarithmic ones

Boyd's Conjectures

Let us begin with some of Boyd's conjectures (1998)

$$m(y^2+(x^3-4x^2-4x+1)y+x^3) \stackrel{?}{=} d_7 \tag{1}$$

$$m((x^2+x+1)(y^2+1)+2xy \stackrel{?}{=} \frac{1}{3}d_8 \tag{2}$$

$$m((x^2+x+1)(y^2+x)+3x(x+1)y) \stackrel{?}{=} \frac{1}{6}d_{15} \tag{3}$$

$$m((x^2+x+1)(y^2+x^2)+(x^4-x^3-6x^2-x+1)y \stackrel{?}{=} \frac{1}{3}d_7+\frac{1}{6}d_{15} \tag{4}$$

$$m((x^2+x+1)(y^2+1)+6xy) \stackrel{?}{=} \frac{1}{6}d_{24} \tag{5}$$

$$m(x^2+x+1)(y^2+x)+(x^3-4x^2-4x+1)y) \stackrel{?}{=} \frac{1}{18}d_{39} \tag{6}$$

$$m((x^4+x^3+x^2+x+1)(y^2+1)+(x^4-3x^3-6x^2-3x+1)y) \stackrel{?}{=} \frac{1}{30}d_{55} \tag{7}$$

$$m((x+1)^2y+x^2+x+1) \stackrel{?}{=} \frac{1}{3}d_7 \tag{8}$$

$$m((x^2+x+1)y+x^2+1) \stackrel{?}{=} \frac{1}{12}d_{15} \tag{9}$$

$$m((x+1)^2(x^2+x+1)y+(x^2-x+1)^2) \stackrel{?}{=} \frac{2}{3}d_{11} \tag{10}$$

The Dirichlet L-series d_f is defined by

$$d_f := \frac{f^{3/2}}{4\pi} L(\chi_{-f}, 2) = L'(\chi_{-f}, 1)$$

The logarithmic Mahler measure m of a non-zero Laurent polynomial $A \in \mathbb{C}[x_1^{\pm 1}, ..., x_n^{\pm 1}]$ is defined as

$$m(A) := \int_0^1 \dots \int_0^1 \log \mid A(e^{2\pi i\theta_1}, ..., e^{2\pi i\theta_n}) \mid d\theta_1 ... d\theta_n$$

and its Mahler measure is the exponential of the latter.
If $A(x, y)$ is in two variables we can write

$$A(x, y) = a_0(y) \prod_{j=1}^{d} (x - x_j(y))$$

with $x_j(y)$ algebraic functions in y.

By Jensen's formula

$$m(A) = m(a_0) + \sum_{j=1}^{d} \frac{1}{2\pi i} \int_{|y|=1} \log^+ \mid x_j(y) \mid \frac{dy}{y}$$

where $\log^+ \mid z \mid = \log \mid z \mid$ if $\mid z \mid \geq 1$ and 0 otherwise.
Defining

$$\eta(x, y) := \log \mid x \mid d\arg y - \log \mid y \mid d\arg x$$

a real differential 1-form on $X \setminus S$ (X the variety defined by the polynomial A, smooth projective completion of Y zero locus of A, S points of X where x or y has a zero or a pole), we get

$$m(A) = m(a_0) + \frac{1}{2\pi} \int_{\gamma} \eta(x, y)$$

γ oriented path on X projecting to $Y \cap \{\mid y \mid = 1, \mid x \mid \geq 1\}$ and

$$\delta\gamma = \sum_k \epsilon_k [w_k], \;\; \epsilon_k = \pm 1, \;\; \mid x(w_k) \mid = \mid y(w_k) \mid = 1, \;\; w_k \in \bar{\mathbb{Q}}$$

Since $\dim X = 1$,

$$d\eta = \Im\left(\frac{dx}{x} \wedge \frac{dy}{y}\right)$$

vanishes and η is a closed differential.
If A is "tempered", η extends to all of X.
"Tempered" means the roots of all the face polynomials of the Newton polygon of A are roots of unity.

Let X be a smooth projective algebraic curve defined over $\mathbb{C}$ and let $\mathbb{C}(X)$ be its function field. Let x, $y \in \mathbb{C}(X)$ be two non-constant rational functions and let $S \subset X$ be the set of zeros and poles of x or y. The image of the rational map $(x, y) : X \setminus S \to \mathbb{C}^* \times \mathbb{C}^*$ is of dimension 1; let $A \in \mathbb{C}[x, y]$ be a defining equation.

$$\{x, y\} \in K_2(X) \otimes \mathbb{Q} \Leftrightarrow A \text{“tempered”}$$

If, in addition,

$$\{x, y\} = 0 \ \text{ in } \ K_2(X) \otimes \mathbb{Q}$$

or equivalently, for some $r_j \in \mathbb{Q}$ and $z_j \in \mathbb{C}(X)^*$ we have

$$x \wedge y = r_1\langle z_1\rangle + \ldots r_n\langle z_n\rangle \tag{T}$$

in $\Lambda^2(\mathbb{C}(X)^*) \otimes \mathbb{Q}$ with $\langle z\rangle := z \wedge (1 - z)$,
we say we have a triangulation of the wedge $x \wedge y$.
Denote D the Bloch-Wigner dilogarithm

$$D(t) = \Im(Li_2(t)) + \arg(1 - t)\log \mid t \mid$$

One can prove, under (T) that

$$\eta(x, y) = dV, \quad V = D(\xi), \ \ \xi = r_1[z_1] + \ldots + r_n[z_n]$$

Using Stokes, it follows

$$2\pi m(A) = 2\pi m(a_0) + D(\xi), \quad \xi = \sum_k \epsilon_k \xi_k, \quad \xi_k = \sum_j [z_j(w_k)].$$

To make short, the strategy is the following:
Assume

1. A is tempered
2. $x \wedge y$ can be triangulated

then $\pi m(A)$ equals an explicit rational linear combination of values of the BlochWigner dilogarithm at algebraic arguments.
Moreover these algebraic values are linked to "toric points" i.e. points (x, y) satisfying $\mid x \mid = \mid y \mid = 1$ and $A(x, y) = 0$. In the previous Boyd's conjectures the fields F_j associated to these toric points are imaginary quadratic.

Also we can prove:

$$\xi_j \in \mathcal{B}(F_j)$$

where $\mathcal{B}(F_j)$ is the Bloch group of the field.
Hence use Zagier's theorem

$$\mathcal{B}(F_j)/\mathrm{T}ors \simeq \mathbb{Z}$$

$$D(\xi_j)\frac{\pi^2}{\sqrt{\mid D_{F_j} \mid}} = r\zeta_{F_j}(2)$$

where r is an unknown rational number.
Hence we need to determine r to achieve the proofs since

$$\zeta_{\mathbb{Q}(\sqrt{-f})}(2) = \frac{\pi^2}{6}L(\chi_{-f}, 2) = \frac{4\pi^3}{4f\sqrt{f}}d_f.$$

For the determination of r we need the following ingredient.
One associate to F_j an hyperbolic manifold M^3

$$M^3 = \mathbb{H}^3/\Gamma_{F_j}$$

Γ_{F_j} being a discrete co-finite subgroup of $Sl_2(\mathbb{C}) = Aut(\mathbb{H}^3)$.
$\zeta_{F_j}(2)$ is related to Vol(M^3) (Humbert (1919) by

$$\zeta_{F_j}(2) = \frac{4\pi^2}{\mid D_{F_j} \mid^{3/2}} Vol(M^3).$$

The hyperbolic manifold M^3 can be triangulated by hyperbolic ideal tetrahedra (Milnor and Thurston) and its volume expressed in terms of a sum of $N \leq 24$ Bloch-Wigner dilogarithms on algebraic numbers.
Hence the theorem:

Theorem

If the number field F has only one complex embedding, $\zeta_F(2)$ is a sum of Bloch-Wigner dilogarithms on algebraic numbers.

For example, Zagier obtained

$$\zeta_{\mathbb{Q}(\sqrt{-7})}(2) = \frac{4\pi^2}{21\sqrt{7}}(2D(\frac{1+\sqrt{-7}}{2}) + D(\frac{-1+\sqrt{-7}}{4})).$$

Also, denoting $a = \frac{1+\sqrt{-15}}{2}$, Gangl found the formula

$$\zeta_{\mathbb{Q}(\sqrt{-15})}(2) = \frac{4\pi^2}{45\sqrt{15}}[3D(a) + 4D(\frac{a+1}{2}) + 2D(a+1) + 2D(a-1)].$$

Back to Boyd's Conjectures

Using the previous method when the curve defined by the polynomial is parametrizable , the triangulation of η is straightforward because of Tate's formula
if $\alpha \neq \beta$,

$$\eta(t-\alpha, t-\beta) = \eta(\frac{t-\alpha}{\beta-\alpha}, 1-\frac{t-\alpha}{\beta-\alpha}) + \eta(t-\alpha, \alpha-\beta) + \eta(\beta-\alpha, t-\beta)$$

I proved (8) and (9) thanks to Zagier's and Gangl's formulae but not (10) because of lack of triangulation of the hyperbolic variety corresponding to $\mathbb{Q}(\sqrt{-11})$.
However, if (10) is true I propose an analog to Zagier's and Gangl's formula

$$\zeta_{\mathbb{Q}(\sqrt{-11})}(2) \stackrel{?}{=} \frac{4\pi^2}{66\sqrt{11}}[2D(\frac{-49+3\sqrt{-11}}{50}) + D(\frac{-49+3\sqrt{-11}}{72})$$

$$+6D(\frac{-1+3\sqrt{-11}}{10}) + D(\frac{-2+6\sqrt{-11}}{25}) + 2D(\frac{-3+9\sqrt{-11}}{25})]$$

Hope it corresponds to a nice triangulation!

Back to Boyd's Conjectures: comments

Formula (3) was proved by Boyd and Villegas.
In fact the genus of the curve (3) is 1: so it is birational to an elliptic curve. Using this, B-V found a triangulation of the wedge and concluded using Gangl's formula.
The genus of curves defined by (1) to (7) is respectively 2, 0, 1,1,0,1,1.
In case of genus 0 the triangulation of the wedge is easy, but not the triangulation of the hyperbolic variety.
In case of genus 1, the elliptic curve to which it is birational is a non singular plane model of the curve.

Example: curve (4)

It is defined by the homogenous polynomial

$$(x^2 + xt + t^2)(y^2 + x^2)t + (x^4 - x^3 t - 6x^2 t^2 - xt^3 + t^4)y$$

Compute the genus; hence we take homogenous coordinates. We find 3 multiple points $(0:1:0), (1:1:1), (-1,1,1)$
$(0:1:0)$ is an ordinary triple point since we have

$$t(x^2 + xt + t^2) + ...$$

$(1:1:1)$ is an ordinary double point since taking $x = 1 + X$, $y = 1 + Y$, $t = 1$ we get

$$3Y^2 - 6XY + 8X^2 + ...$$

so the tangents at this double point are distinct of slope $\frac{3 \pm \sqrt{-15}}{3}$. Notice that this point is toric hence its desingularization in the elliptic model gives two points above $(1:1:1)$ with coordinates expressed with $\mathbb{Q}(\sqrt{-15})$. Thus the term in d_{15} in the Mahler measure.
Finally $(-1:1:1)$ is a toric double point associated to $\mathbb{Q}(\sqrt{-7})$.
I found an elliptic model of (4) but got only a partial triangulation of $x \wedge y$!

Motivations concerning Mahler measure of polynomials defining $K3$ surfaces

1. Deninger's guess (1996) proved in 2011 by Rogers and Zudilin, again in 2013 by Zudilin

$$m(x + \frac{1}{x} + y + \frac{1}{y} + 1) = \frac{15}{4\pi^2} L(E_{15}, 2) = L'(E, 0)$$

2. Maillot's hint (2003) using a result of Darboux (1875): the Mahler measure of P which is the integration of a differential form on a variety, when P is non reciprocal, is in fact an integration on a smaller variety and the expression of the Mahler measure is encoded in the cohomology of the smaller variety.

This phenomenom can be seen in (Villegas's Conjecture (2004))

$$m(x_0+x_1+x_2+x_3+x_4) \stackrel{?}{=} \frac{675\sqrt{15}}{16\pi^3} L(f,4)$$

f cusp form of weight 3 and conductor 15.
The smaller variety is defined by

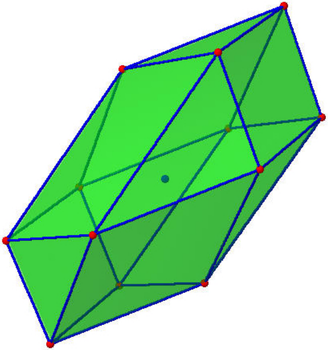
$$x_0+x_1+x_2+x_3+x_4=0$$
$$\frac{1}{x_0}+\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}=0.$$

It is the modular $K3$-surface studied by Peters, Top, van der Vlugt defined by a reciprocal polynomial. Its L-series is related to f.
Briefly, to guess the Mahler measure of a non reciprocal polynomial we need results on reciprocal ones.
In particuliar, it is very important to collect many examples of Mahler measures of $K3$-hypersurfaces.

What is a $K3$ surface?
The polynomials

$$x+\frac{1}{x}+y+\frac{1}{y}+z+\frac{1}{z}+xy+\frac{1}{xy}+xz+\frac{1}{xz}+yz+\frac{1}{yz}+\lambda=0$$

define a family of $K3$ surfaces.
But prior to the definition, let me show you the Newton polytope of that family.
More precisely, this Newton polytope is in the same class as the reflexive polytope of index 1529.



Reflexive polytope 1529

This Newton polytope is a reflexive polytope with 12 vertices and 14 facets. This polytope has the greatest number of facets of any three-dimensional reflexive polytope; furthermore, there is a unique three-dimensional reflexive polytope with this property, up to isomorphism. In the database of reflexive polytopes found in Sage, this polytope has index 1529.
Using the coordinates of the vertices we see that there are

1. 8 facets of Mahler measure 1
2. 6 facets of Mahler measure d_3

Another way of showing a surface is $K3$ is with the help of elliptic fibrations.

For example, in the previous family, for $\lambda = 0$, take $w := x + y + z$ (w is called an elliptic parameter), express the equation under a Weierstrass form and get

$$Y^2 + (w^2 + 3)XY + (w^2 - 1)^2 Y = X^3.$$

When you get such a Weierstrass equation

$$y^2 + a_1(t)\,yx + a_3(t)\,y = x^3 + a_2(t)\,x^2 + a_4(t)\,x + a_6(t)$$

with $a_i(t)$ polynomial in t of degree $\leq 2i$ and exactly $2i$ for one i, we have a $K3$ elliptic surface fibered in t.

Most of our results concern polynomials of the family

$$P_k = x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z} - k,$$

more precisely polynomials of the family defining $K3$ surfaces of Picard number 20 i.e. with a Néron-Severi lattice of rank 20, thanks to Schütt's result:

The last ingredient:Schütt's classification of CM-newforms of weight 3

Theorem

Consider the following classifications of singular $K3$ surfaces over $\mathbb{Q}$:

- *by the discriminant d of the transcendental lattice of the surface up to squares,*
- *by the associated newform up to twisting,*
- *by the level of the associated newform up to squares,*
- *by the CM-field $\mathbb{Q}(\sqrt{-d})$ of the associated newform.*

Then, all these classifications are equivalent. In particuliar, $\mathbb{Q}(\sqrt{-d})$ has exponent 1 or 2.

Theorem

$$m(P_0) = d_3 := \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) \quad (B.2005)$$

$$m(P_2) = \frac{|\det T(Y_2)|^{3/2}}{\pi^3} L(Y_2, 3) = \frac{8\sqrt{8}}{\pi^3} L(f_8, 3) \quad (B.\ 2005)$$

$$m(P_{10}) = \frac{|\det T(Y_{10})|^{3/2}}{9\pi^3} L(Y_{10}, 3) + 2d_3 = \frac{72\sqrt{72}}{9\pi^3} L(f_8, 3) + 2d_3 \quad (B.\ 2009)$$

$$m(P_3) = 2\frac{|\det T(Y_3)|^{3/2}}{4\pi^3} L(T(Y_3), 3) = \frac{15\sqrt{15}}{2\pi^3} L(f_{15}, 3) \quad (BFFLM\ 2013)$$

$$m(P_6) = \frac{|\det T(Y_6)|^{3/2}}{2\pi^3} L(Y_6, 3) = \frac{24\sqrt{24}}{2\pi^3} L(f_{24}, 3) \quad (BFFLM\ 2013)$$

$$m(P_{18}) = \frac{1}{5}\frac{|\det T(Y_{18})|^{3/2}}{4\pi^3} L(Y_{18}, 3) + \frac{14}{5} d_3 = \frac{120\sqrt{120}}{20\pi^3} L(f_{120}, 3) + \frac{14}{5} d_3$$

$$(BFFLM\ 2013)$$

Concerning that family, we have also to prove similar results for $k^2 \in \mathbb{Q}$ corresponding to singular $K3$, since these $K3$ are in fact defined over $\mathbb{Q}$ and we may apply Schütt's theorem.
It remains also one more Boyd's conjecture

$$m(P_{\sqrt{-45}}) \stackrel{?}{=} 16m(P_3) + d_{15}.$$

Question: Why d_{15} in the previous conjecture?
All the faces of the Newton polytope of family P_k have d_3 as the same Mahler measure.

I would like to end with some of Samart's results.
It concerns the family

$$(x+1/x)(y+1/y)(z+1/z)+s^{1/2}=0$$

τ	s	m
$\frac{\sqrt{-1}}{2}$	$64=8^2$	$8M_{16}$
$\frac{\sqrt{-3}}{2}$	$256=16^2$	$\frac{4}{3}(M_{12\otimes(-4)}+2d_4)$
$\frac{1+\sqrt{-3}}{4}$	$16=4^2$	$?8M_{12}$
$\frac{\sqrt{-7}}{2}$	$4096=64^2$	$?\frac{4}{7}(M_{7\otimes(-4)}+8d_4)$
$\frac{3+\sqrt{-7}}{8}$	1	$?8M_7$

where $M_N:=L'(g_N,0)$.

One can easily prove that the first Mahler measure can be expressed in terms of the L-series of the corresponding $K3$-surface.
We need an elliptic fibration of the surface

$$(x + \frac{1}{x})(y + \frac{1}{y})(z + \frac{1}{z}) + s^{1/2} = 0.$$

Take the elliptic parameter $z = w$. We get the elliptic fibration

$$\mathcal{E} : S \to \mathbb{P}^1_w$$

By standard birational transformations we get the following Weierstrass equation

$$Y^2 - 4(w+1)^2 XY = X(X - 8w(w^2+1))^2$$

With PARI, we find the singular fibers $I_4(w = 1, -1, 0, I, -I, \infty)$. Hence the Picard number of S is 20.

We see easily a cyclic torsion group of order 4. But by Shimada, a singular $K3$ surface with fibers $6I_4$ has necessary $\mathbb{Z}/(4) \times \mathbb{Z}/(4)$ as torsion group and transcendental lattice $[4, 0, 4]$.

Theorem

(Shimada) Let $f : X \to \mathbb{P}^1$ be an elliptic K3 surface. Then the following holds.

$$(MW)_{Tors} = \mathbb{Z}/(4) \times \mathbb{Z}/(4) \Longleftrightarrow f = 6A_3$$

Elliptic K3 surface with $(MW)_{Tors} = \mathbb{Z}/(4) \times \mathbb{Z}/(4)$ is constructed as elliptic modular surface corresponding to the congruence group $\Gamma(4) \subset Sl_2(\mathbb{Z})$.

We deduce, by Schütt 's theorem, the level 16 of the associated newform and counting points on S_w in $\mathbb{F}_p$ for $w \in \mathbb{P}^1(\mathbb{F}_p)$,

p	2	3	5	7	11	13	17	19
Newform $N = 16$	0	0	-6	0	0	10	-30	0
A_p	0	0	-6	0	0	10	-30	0

Hence

$$\frac{16\sqrt{16}}{4\pi^3} L(S,3) = \frac{16\sqrt{16}}{4\pi^3} L(g_{16},3)$$

Consider now the case $s = 1$. The corresponding Mahler measure is $M(g_7)$. We are tempted to think that the corresponding variety has determinant of its transcendental lattice equal to 7. Unfortunately it is not.
With the same elliptic parameter we get an elliptic fibration with fibers of type $4I_4 + 4I_2$ thus a Mordell-Weil with rank 2 and 4-torsion.
O. Lecacheux computed all elliptic fibrations of the $K3$ associated to $\Gamma_1(7)$. Among them, no fibration of type $4I_4 + 4I_2$ plus 4-torsion.
So the determinant of transcendendal lattice is of the form $7 \times a^2$.
So to compute the L-series of the $K3$, we have

- to know the two infinite generators of the Mordell-Weil (difficult)
- or to find an elliptic fibration with $r = 1$, using for example Elkies's method of 2-neighbours.

For similar reasons, the $K3$-surface associated to M_{12} has probably a discriminant equal to $12 \times b^2$, since we determined [BGHLMSW] all the fibrations of the singular $K3$-surface with transcendental lattice $\langle 6 \rangle \oplus \langle 2 \rangle$.