APÉRY-FERMI PENCIL OF $K3$-SURFACES AND THEIR 2-ISOGENIES

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1. Introduction

The Apéry-Fermi pencil $F$ is realised with the equations

$$X + \frac{1}{X} + Y + \frac{1}{Y} + Z + \frac{1}{Z} = k, \quad k \in \mathbb{Z},$$

and taking $k = s + \frac{1}{s}$, is seen as the Fermi threefold $Z$ with compactification denoted $\bar{Z}$ [PS]. The projection $\pi_s : \bar{Z} \to \mathbb{P}^1(s)$ is called the Fermi fibration. In their paper [PS], Peters and Stienstra proved that for $s \notin \{0, \infty, \pm 1, 3 \pm 2\sqrt{2}, -3 \pm 2\sqrt{2}\}$ the fibers of the Fermi fibration are $K3$ surfaces with the Néron-Severi lattice of the generic fiber isometric to $E$.

Thus, a question arises: are the corresponding 2-isogenies between $(i.e. \text{Picard number } 20)$ and observed that many of its elliptic fibrations are endowed with 2-torsion sections. Recently Bertin and Lecacheux [BL] found all the elliptic fibrations of a singular member $Y$ and this new $K3$-surface $X$ (with a section) with two II*-fibres, there exists a unique Kummer surface $S = Km(C_1 \times C_2)$ with two rational maps of degree 2, $X \to S$ and $S \to X$ where $C_1$ and $C_2$ are elliptic curves.

In van Geemen-Sarti [CG], Koike [Ko] and Schütt [Sc], sandwich Shioda-Inose structures are constructed via elliptic fibrations with 2-torsion sections.

Recently Bertin and Lecacheux [BL] found all the elliptic fibrations of a singular member $Y_2$ of $F$ (i.e. of Picard number 20) and observed that many of its elliptic fibrations are endowed with 2-torsion sections. Thus a question arises: are the corresponding 2-isogenies between $Y_2$ and this new $K3$-surface $S_2$ all Morrison-Nikulin meaning that $S_2$ is Kummer? Observing also that the Shioda’s Kummer sandwiching between a $K3$ surface $S$ and its Kummer $K$ is in fact a 2-isogeny between two elliptic fibrations of $S$ and $K$, we extended the above question to the generic member $Y_k$ of the family $F$ and obtained the following results.

**Theorem 1.1.** Suppose $Y_k$ is a generic $K3$ surface of the family with Picard number 19.

Let $\pi : Y_k \to \mathbb{P}^1$ be an elliptic fibration with a torsion section of order 2 which defines an involution $i$ of $Y_k$ (Van Geemen-Sarti involution) then the quotient $Y_k/i$ is either the Kummer surface $K_k$ associated to $Y_k$ given by its Shioda-Inose structure or a surface $S_k$ with transcendental lattice $T_{S_k} = (-2) \oplus (2) \oplus (6)$ and Néron-Severi lattice $NS(S_k) = U \oplus E_8(-1) \oplus E_8(-1) \oplus \langle -2 \rangle \oplus \langle -6 \rangle$, which is not a Kummer surface by a result of Morrison [M]. The $K3$ surface $S_k$ is the Hessian $K3$ surface of a general cubic surface with 3 nodes studied by Dardanelli and van Geemen [DG]. Thus,

\begin{itemize}
  \item \textbf{Date:} April 8, 2018.
  \item 1991 Mathematics Subject Classification. Primary 14J27,14J28,14J50,14H52; Secondary 11G05.
  \item \textbf{Key words and phrases.} Elliptic fibrations of $K3$ surfaces, Morrison-Nikulin involutions, Isogenies.
\end{itemize}
leads to an elliptic fibration either of $K_k$ or of $S_k$. Moreover there exist some genus 1 fibrations $\theta : K_k \to \mathbb{P}^1$ without section such that their Jacobian variety satisfies $J_{\theta}(K_k) = S_k$. More precisely, among the elliptic fibrations of $Y_k$ (up to automorphisms) 12 of them have a two-torsion section. And only 7 of them possess a Morrison-Nikulin involution $i$ such that $Y_k/i = K_k$.

**Theorem 1.2.** In the Apéry-Fermi pencil, the K3-surface $Y_2$ is singular, meaning that its Picard number is 20. Moreover $Y_2$ has many more 2-torsion sections than the generic K3 surface $Y_k$; hence among its 19 Van Geemen-Sarti involutions, 13 of them are Morrison-Nikulin involutions, 5 are symplectic automorphisms of order 2 (self-involutions) and one exchanges two elliptic fibrations of $Y_2$.

The specializations to $Y_2$ of the 7 Morrison-Nikulin involutions of a generic member $Y_k$ are verified among the 13 Morrison-Nikulin involutions of $Y_2$, as proved in a general setting by Schätt [Sc]. The specializations of the 5 involutions between $Y_k$ and the K3-surface $S_k$ are among the 6 Van Geemen-Sarti involutions of $Y_2$ which are not Morrison-Nikulin.

This theorem provides an example of a Kummer surface $K_2$ defined by the product of two isogenous elliptic curves (actually the same elliptic curve of $j$-invariant equal to 8000), having many fibrations of genus one whose Jacobian surface is not a Kummer surface. A similar result but concerning a Kummer surface defined by two non-isogenous elliptic curves has been exhibited by Keum [K]. Throughout the paper we use the following result [S]. If $E$ denotes an elliptic fibration with a 2-torsion point $(0,0)$:

$$E : y^2 = x^3 + Ax^2 + Bx,$$

the quotient curve $E/\langle(0,0)\rangle$ has a Weierstrass equation of the form

$$E/\langle(0,0)\rangle : y^2 = x^3 - 2Ax^2 + (A^2 - 4B)x.$$ 

The paper is organised as follows.

In section 2 we recall the Kneser-Nishiyama method and use it to find all the 27 elliptic fibrations of a generic K3 of the family $F$. In section 3, using Elkies’s method of ”2-neighbors” [El], we exhibit an elliptic parameter giving a Weierstrass equation of the elliptic fibration. The results are summarized in Table 2. Thus we obtain all the Weierstrass equations of the 12 elliptic fibrations with 2-torsion sections. Their 2-isogenous elliptic fibrations are computed in section 5 with their Mordell-Weil groups and discriminants. Section 4 recalls generalities about Nikulin involutions and Shioda-Inose structure. Section 5 is devoted to the proof of Theorem 1.1 while section 6 is concerned with the proof of Theorem 1.2. In the last section 7, using a theorem of Boissière, Sarti and Veniani [BSV], we explain why Theorem 1.2 cannot be generalised to the other singular K3 surfaces of the family. Computations were performed using partly the computer algebra system PARI [PA] and mostly the computer algebra system MAPLE and the Maple Library “Elliptic Surface Calculator” written by Kuwata [Ku].

2. **Elliptic fibrations of the family**

We refer to [BL], [Sc-Shio] for definitions concerning lattices, primitive embeddings, orthogonal complement of a sublattice into a lattice. We recall only what is essential for understanding this section and section 5.2.

2.1. **Discriminant forms.** Let $L$ be a non-degenerate lattice. The dual lattice $L^*$ of $L$ is defined by

$$L^* := \text{Hom}(L, \mathbb{Z}) = \{x \in L \otimes \mathbb{Q} / b(x, y) \in \mathbb{Z} \text{ for all } y \in L\}$$

and the discriminant group $G_L$ by

$$G_L := L^*/L.$$ 

This group is finite if and only if $L$ is non-degenerate. In the latter case, its order is equal to the absolute value of the lattice determinant $| \det(G(e)) |$ for any basis $e$ of $L$. A lattice $L$ is **unimodular** if $G_L$ is trivial.
Let $G_L$ be the discriminant group of a non-degenerate lattice $L$. The bilinear form on $L$ extends naturally to a $\mathbb{Q}$-valued symmetric bilinear form on $L^*$ and induces a symmetric bilinear form

$$b_L : G_L \times G_L \to \mathbb{Q}/\mathbb{Z}. $$

If $L$ is even, then $b_L$ is the symmetric bilinear form associated to the quadratic form defined by

$$q_L : G_L \to \mathbb{Q}/2\mathbb{Z},
q_L(x + L) \mapsto x^2 + 2\mathbb{Z}. $$

The latter means that $q_L(na) = n^2q_L(a)$ for all $n \in \mathbb{Z}$, $a \in G_L$ and $b_L(a, a') = \frac{1}{2}(q_L(a + a') - q_L(a) - q_L(a'))$, for all $a, a' \in G_L$, where $\frac{1}{2} : \mathbb{Q}/2\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ is the natural isomorphism. The pair $(G_L, b_L)$ (resp. $(G_L, q_L)$) is called the discriminant bilinear (resp. quadratic) form of $L$.

The lattices $A_n = \langle a_1, a_2, \ldots, a_n \rangle$ (n ≥ 1), $D_l = \langle d_1, d_2, \ldots, d_l \rangle$ (l ≥ 4), $E_p = \langle e_1, e_2, \ldots, e_p \rangle$ ($p = 6, 7, 8$) defined by the following Dynkin diagrams are called the root lattices. All the vertices $a_j$, $d_k$, $e_l$ are roots and two vertices $a_j$ and $a_j'$ are joined by a line if and only if $b(a_j, a_j') = 1$.

We use Bourbaki’s definitions [Bo2]. The discriminant groups of these root lattices are given below.

**A**n. $G_{A_n}$

Set

$[1]_n = \frac{1}{n+1} \sum_{j=1}^{n}(n - j + 1)a_j$

then $A'_n = \langle A_n, [1]_n \rangle$ and

$G_{A_n} = A'_n / A_n \simeq \mathbb{Z}/(n + 1)\mathbb{Z}.$

$q_{A_n}([1]_n) = -\frac{n}{n+1}.$

**D**l. $G_{D_l}$.

Set

$[1]_{D_l} = \frac{1}{2} \left( \sum_{i=1}^{l-2} i d_i + \frac{1}{2}(l - 2)d_{l-1} + \frac{1}{2}d_l \right)$

$[2]_{D_l} = \sum_{i=1}^{l-1} i d_i + \frac{1}{2}(d_{l-1} + d_l)$

$[3]_{D_l} = \frac{1}{2} \left( \sum_{i=1}^{l-1} i d_i + \frac{1}{2}(ld_{l-1} + \frac{1}{2}(l - 2)d_l) \right)$

then $D'_l = \langle [1]_{D_l}, [2]_{D_l} \rangle$.

$G_{D_l} = D'_l / D_l = \langle [1]_{D_l} \rangle \simeq \mathbb{Z}/4\mathbb{Z}$ if $l$ is odd,

$G_{D_l} = D'_l / D_l = \langle [1]_{D_l}, [2]_{D_l} \rangle \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ if $l$ is even.

$q_{D_l}([1]_{D_l}) = -\frac{1}{4}$, $q_{D_l}([2]_{D_l}) = -1$, $q_{D_l}([1], [2]) = -\frac{1}{2}.$

**E**p. $G_{E_p}$  $p = 6, 7, 8$.

Set

$[1]_{E_6} := \eta_6 = -\frac{1}{3}(2e_1 + 3e_2 + 4e_3 + 6e_4 + 5e_5 + 4e_6)$ and

$[1]_{E_7} := \eta_7 = -\frac{1}{2}(2e_1 + 3e_2 + 4e_3 + 6e_4 + 5e_5 + 4e_6 + 3e_7),$

then $E_6 = \langle E_6, \eta_6 \rangle$, $E_7 = \langle E_7, \eta_7 \rangle$ and $E_8 = E_8.$

$G_{E_6} = E_6 / E_6 \simeq \mathbb{Z}/3\mathbb{Z}$,  $G_{E_7} = E_7 / E_7 \simeq \mathbb{Z}/2\mathbb{Z}$,

$q_{E_6}(\eta_6) = -\frac{1}{3}$, $q_{E_7}(\eta_7) = -\frac{1}{2}.$

Let $L$ be a Niemeier lattice (i.e. an unimodular lattice of rank 24). Denote $L_{root}$ its root lattice. We often write $L = N_{L_{root}}.$ Elements of $L$ are defined by the glue code composed with glue vectors. Take for example $L = N_{E_6(A_{11}D_7E_6)}.$ Its glue code is generated by the glue vector $[1, 1, 1]$ where the first 1 means $[1]_{A_{11}},$ the second 1 means $[1]_{D_7}$ and the third 1 means $[1]_{E_6}.$ In the glue code $\langle [1], (0, 1, 2) \rangle$, the notation $(0, 1, 2)$ means any circular permutation of $(0, 1, 2)$: Niemeier lattices, their root lattices and glue codes used in the paper are given in Table 1 glue codes are taken from Conway and Sloane [Cd].

2.2. Knese-Nishiyama technique. We use the Knese-Nishiyama method to determine all the elliptic fibrations of $Y_k$. For further details we refer to [Nis], [Sc-Shio], [BL], [BGL]. In [Nis], [BL], [BGL] only singular K3 (i.e. of Picard number 20) are considered. In this paper we follow [Sc-Shio] we briefly recall.

Let $T(Y_k)$ be the transcendental lattice of $Y_k$, that is the orthogonal complement of $NS(Y_k)$ in $H^2(Y_k, \mathbb{Z})$ with respect to the cup-product. The lattice $T(Y_k)$ is an even lattice of rank $r = 22 - 19 =$
Table 1. Some Niemeier lattices and their glue codes \[\text{[Co]}\]

<table>
<thead>
<tr>
<th>$L_{\text{root}}$</th>
<th>$L/L_{\text{root}}$</th>
<th>glue vectors</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_8$</td>
<td>$\langle 0 \rangle$</td>
<td>0</td>
</tr>
<tr>
<td>$D_{16}E_8$</td>
<td>$\mathbb{Z}/2\mathbb{Z}$</td>
<td>$\langle 1, 0 \rangle$</td>
</tr>
<tr>
<td>$D_{10}E_7^+$</td>
<td>$(\mathbb{Z}/2\mathbb{Z})^2$</td>
<td>$\langle 1, 1, 0, [3, 0, 1] \rangle$</td>
</tr>
<tr>
<td>$A_{17}E_7$</td>
<td>$\mathbb{Z}/6\mathbb{Z}$</td>
<td>$\langle 3, 1 \rangle$</td>
</tr>
<tr>
<td>$D_{24}$</td>
<td>$\mathbb{Z}/2\mathbb{Z}$</td>
<td>$\langle 1 \rangle$</td>
</tr>
<tr>
<td>$D_{12}^2$</td>
<td>$(\mathbb{Z}/2\mathbb{Z})^2$</td>
<td>$\langle 1, 2, [2, 1] \rangle$</td>
</tr>
<tr>
<td>$D_8^4$</td>
<td>$(\mathbb{Z}/2\mathbb{Z})^4$</td>
<td>$\langle 1, 2, [1, 1, 1], [2, 2, 1] \rangle$</td>
</tr>
<tr>
<td>$A_{15}D_9$</td>
<td>$\mathbb{Z}/8\mathbb{Z}$</td>
<td>$\langle 2, 1 \rangle$</td>
</tr>
<tr>
<td>$E_6^4$</td>
<td>$(\mathbb{Z}/32\mathbb{Z})^2$</td>
<td>$\langle 1, (0, 1, 2) \rangle$</td>
</tr>
<tr>
<td>$A_{11}D_7E_6$</td>
<td>$\mathbb{Z}/12\mathbb{Z}$</td>
<td>$\langle 1, 1, 1 \rangle$</td>
</tr>
<tr>
<td>$D_6^4$</td>
<td>$(\mathbb{Z}/2\mathbb{Z})^4$</td>
<td>(even permutations of $[0, 1, 2, 3]$)</td>
</tr>
<tr>
<td>$E_8^4D_6$</td>
<td>$\mathbb{Z}/10\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$</td>
<td>$\langle 2, 4, 0, [5, 6, 1], [0, 5, 3] \rangle$</td>
</tr>
<tr>
<td>$E_8^4D_5^4$</td>
<td>$\mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$</td>
<td>$\langle 1, 1, 1, 2, [1, 7, 2, 1] \rangle$</td>
</tr>
</tbody>
</table>

3 and signature $(2, 1)$. Since $t = r - 2 = 1$, $T(Y_k)[-1]$ admits a primitive embedding into the following indefinite unimodular lattice:

$$T(Y_k)[-1] \hookrightarrow U \oplus E_8$$

where $U$ denotes the hyperbolic lattice and $E_8$ the unimodular lattice of rank 8. Define $M$ as the orthogonal complement of a primitive embedding of $T(Y_k)[-1]$ in $U \oplus E_8$. Since

$$T(Y_k)[-1] = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -12 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

it suffices to get a primitive embedding of $(-12)$ into $E_8$. From Nishiyama \[\text{[Nis]}\] we find the following primitive embedding:

$$v = (9e_2 + 6e_1 + 12e_3 + 18e_4 + 15e_5 + 12e_6 + 8e_7 + 4e_8) \hookrightarrow E_8,$$

giving $(v)^{\perp}_{E_8} = A_2 \oplus D_5$. Now the primitive embedding of $T(Y_k)[-1]$ in $U \oplus E_8$ is defined by $U \oplus v$; hence $M = (U \oplus v)^{\perp}_{U \oplus E_8} = A_2 \oplus D_5$. By construction, this lattice is negative definite of rank $t + 6 = 1 + 6 + 6 + 4 + 3 + 4 = 26 = \rho(Y_k) = 7$ with discriminant form $q_M = -q_{T(Y_k)}[-1] = q_{T(Y_k)} = -q_{NS(Y_k)}$. Hence $M$ takes exactly the shape required for Nishiyama’s technique.

All the elliptic fibrations come from all the primitive embeddings of $M = A_2 \oplus D_5$ into all the Niemeier lattices $L$. Since $M$ is a root lattice, a primitive embedding of $M$ into $L$ is in fact a primitive embedding into $L_{\text{root}}$. Whenever the primitive embedding is given by a primitive embedding of $A_2$ and $D_5$ in two different factors of $L_{\text{root}}$, or for the primitive embedding of $M$ into $E_8$, we use Nishiyama’s results \[\text{[Nis]}\]. Otherwise we have to determine the primitive embeddings of $M$ into $D_l$ for $l = 8, 9, 10, 12, 16, 24$. This is done in the following lemma.

**Lemma 2.1.** We obtain the following primitive embeddings.

1. $A_2 \oplus D_5 = \langle d_8, d_6, d_7, d_5, d_4, d_1, d_2 \rangle \hookrightarrow D_8$

$$\langle d_8, d_6, d_7, d_5, d_4, d_1, d_2 \rangle^{\perp}_{D_8} = \langle 2d_1 + 4d_2 + 6d_3 + 6d_4 + 6d_5 + 6d_6 + 3d_7 + 3d_8 \rangle = (-12)$$

2. $A_2 \oplus D_5 = \langle d_9, d_7, d_8, d_6, d_5, d_3, d_2 \rangle \hookrightarrow D_9$

$$\langle d_9, d_7, d_8, d_6, d_5, d_3, d_2 \rangle^{\perp}_{D_9} = \langle d_9 + 3d_8 + 2d_7 + 2d_6 + 2d_5 + 2d_4 + d_3 - 3d_1, d_3 + 2d_2 + 3d_4 \rangle$$

with Gram matrix \[\begin{pmatrix} -4 & 6 \\ 6 & -12 \end{pmatrix}\] of determinant 12.
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(3) \[ A_2 \oplus D_5 = \langle d_n, d_{n-2}, d_{n-1}, d_{n-3}, d_{n-4}, d_{n-7}, d_{n-6} \rangle \hookrightarrow D_n, n \geq 10 \]
\[ \langle d_n, d_{n-2}, d_{n-1}, d_{n-3}, d_{n-4}, d_{n-7}, d_{n-6} \rangle_{D_n} = \]
\[ \langle a = d_n + d_{n-1} + 2(d_{n-2} + \ldots + d_2) + d_1, d_{n-6} + 2d_{n-7} + 3d_{n-8}, d_{n-9}, \ldots, d_1 \rangle \]
\[ (A_2 \oplus D_5)_{D_n}^\perp = D_n - 8. \]

We have also the relation \[ 2, [2] D_n = a + d_1, a being the above root. \]

Theorem 2.1. There are 27 elliptic fibrations on the generic K3 surface of the Apéry-Fermi pencil (i.e. with Picard number 19). They are derived from all the non isomorphic primitive embeddings of \( A_2 \oplus D_5 \) into the various Niemeier lattices. Among them, 4 fibrations have rank 0, precisely with the type of singular fibers and torsion.

\[ A_{11}2A_22A_1 \quad 6 - \text{torsion} \]
\[ E_6D_{11} \quad 0 - \text{torsion} \]
\[ E_7A_3D_5 \quad 2 - \text{torsion} \]
\[ E_4E_6A_3 \quad 0 - \text{torsion}. \]

The list together with the rank and torsion is given in Table 2.

Proof. The torsion groups can be computed as explained in [BL] or [BGL]. Let us recall briefly the method.

Denote \( \phi \) a primitive embedding of \( M = A_2 \oplus D_5 \) into a Niemeier lattice \( L \). Define \( W = (\phi(M))^L \) and \( N = (\phi(M))_{L_{\text{root}}} \). We observe that \( W_{\text{root}} = N_{\text{root}} \). Thus computing \( N \) then \( N_{\text{root}} \) we know the type of singular fibers. Recall also that the torsion part of the Mordell-Weil group is \( W_{\text{root}}/W_{\text{root}}(\subset W/N) \)

and can be computed in the following way [BGL]: let \( l + L_{\text{root}} \) be a non trivial element of \( L/L_{\text{root}} \). If there exist \( k \neq 0 \) and \( u \in L_{\text{root}} \) such that \( k(l + u) \in N_{\text{root}} \), then \( l + u \in W \) and the class of \( l \) is a torsion element.

We use also several facts.

(1) If the rank of the Mordell-Weil group is 0, then the torsion group is equal to \( W/N \). Hence fibrations \#1(\( A_4E_6E_8 \)), \#3(\( D_{11}E_6 \)), \#7(\( D_5A_5E_7 \)), \#20(\( A_{11}2A_2A_1 \)) have respective torsion groups (0), (0), \( \mathbb{Z}/22\mathbb{Z}, \mathbb{Z}/6\mathbb{Z} \).

(2) If there is a singular fiber of type \( E_8 \), then the torsion group is (0). Hence the fibrations \#1, \#2 and \#6 have no torsion.

(3) Using lemma 2.2 below and the shape of glue vectors we prove that fibrations \#11, \#18, \#21, \#22, \#25, \#27 have no torsion.

Lemma 2.2. Suppose \( A_2 \) primitively embedded in \( A_n \), \( A_2 = \langle a_1, a_2 \rangle \hookrightarrow A_n \). Then for all \( k \neq 0 \), \( k[1]_{A_n} \notin \langle (A_2)_{A_n}^\perp \rangle_{\text{root}} \).

Proof. It follows from the fact that \( [1]_{A_n} \) is not orthogonal to \( a_1 \). \( \square \)

(4) Using lemma 2.3 below and the shape of glue vectors we can determine the torsion for elliptic fibrations \#5, \#10, \#13, \#15, \#23.

Lemma 2.3. Suppose \( A_2 \) primitively embedded in \( D_1 \), \( A_2 = \langle d_1, d_{1-2} \rangle \hookrightarrow D_1 \). Then \( 2, [2] D_i \in \langle (A_2)_{D_i}^\perp \rangle_{\text{root}} \) but there is no \( k \) satisfying \( k[i]_{D_i} \in \langle (A_2)_{D_i}^\perp \rangle_{\text{root}}, i = 1, 3 \).

Proof. It follows from Nishiyama [NIS]:

\[ (A_2)_{D_1}^\perp = \langle y, x_4, d_{1-4}, \ldots, d_1 \rangle \]
with $y = d_I + 2d_{I-1} + 2d_{I-2} + d_{I-3}$ and $x_4 = d_I + d_{I-1} + 2(d_{I-2} + d_{I-3} + ... + d_2) + d_1$ and

$$L_{l-3}^I = \begin{pmatrix} -4 & -1 & 0 & \ldots & 0 \\ -1 & 1 & D_{l-3} & \ldots \\ 0 & 0 & \ldots & \end{pmatrix}.$$  

Moreover $(A_2)^\dagger_{l-3,I}$ from there we compute easily the relation

$2,2|D_I = x_4 + d_I + 2d_{I-5} + ... + d_1$. The last assertion follows from the fact that $[i]_{D_I}$ is not orthogonal to $A_2$. 

We now give some examples showing the method in detail.
2.2.1. Fibration #17. It comes from a primitive embedding of $A_2 \oplus D_5$ into $D_5$ giving a primitive embedding of $A_2 \oplus D_5$ into $NI(A_15D_9)$ with glue code $\langle 2, 1 \rangle$. Since by lemma $2.1(2)$ $N_{\text{root}} = A_{15}$, among the elements $k[2, 1]$, only $4[2, 1] = [8, 4, 1] \in D_9$ satisfies $2[8, 0 + u] \in N_{\text{root}} = A_{15}$ with $u = 4.1$. Hence the torsion group is $\mathbb{Z}/2\mathbb{Z}$.

2.2.2. Fibration #19. It comes from a primitive embedding of $A_2 = \langle e_1, e_4 \rangle$ into $E_6^{(1)}$ and $D_5 = \langle e_2, e_3, e_4, e_5, e_6 \rangle$ into $E_6^{(2)}$ giving a primitive embedding of $A_2 \oplus D_5$ into $NI(E_6^{(1)})$. In that case $NI(E_6^{(1)})/E_6^{(1)} \cong (\mathbb{Z}/3\mathbb{Z})^2$ and the glue code is $\langle 1, (0, 1, 2) \rangle$. Moreover $(D_5)_{E_6} = 3e_2 + 4e_1 + 5e_3 + 6e_4 + 4e_5 + 2e_6 = a$, $(A_2)_{E_6} = \langle e_2, y \rangle \oplus \langle e_5, e_6 \rangle$ with $y = 2e_2 + e_1 + 2e_3 + 3e_4 + 4e_5 + e_6$. From the relation

$$[1]_{E_6} = -\frac{1}{3}(2e_1 + 3e_2 + 4e_3 + 6e_4 + 5e_5 + 4e_6)$$

we get

$$-3, [1]_{E_6} = a - 2e_1 - e_3 + e_5 + 2e_6 \in E_6$$

$$-3, [1]_{E_6} = 2y - e_2 + e_5 + 2e_6 \in (A_2)_{E_6}$$

deduce that only $[1, 0, 1, 2], [2, 0, 2, 1], [0, 0, 0, 0]$ contribute to the torsion thus the torsion group is $\mathbb{Z}/3\mathbb{Z}$.

2.2.3. Fibration #10. The embeddings of $A_2 = \langle d_{10}, d_8 \rangle$ into $D_{10}$ and $D_5 = \langle e_2, e_3, e_4, e_5, e_6 \rangle$ into $E_7^{(1)}$ lead to a primitive embedding of $A_2 \oplus D_5$ into $NI(D_{10}E_7^{(2)})$ satisfying $NI(D_{10}E_7^{(2)})/(D_{10}E_7^{(2)}) \cong (\mathbb{Z}/2\mathbb{Z})^2$ with glue code $\langle 1, 1, 0 \rangle$. (3, 0, 1). We deduce from lemma $2.3$ that no glue vector can contribute to the torsion which is therefore $0$.

2.2.4. Fibration #18. The embeddings of $A_2 = \langle a_1, a_2 \rangle$ into $A_{15}$ and $D_5 = \langle d_3, d_7, d_8, d_{10}, d_{15} \rangle$ into $D_5$ lead to a primitive embedding of $A_2 \oplus D_5$ into $NI(A_{15}D_9)$ satisfying $NI(A_{15}D_9)/(A_{15}D_9) \cong (\mathbb{Z}/8\mathbb{Z})$ with glue code $\langle 2, 1 \rangle$. We deduce from lemma $2.2$ that no glue vector can contribute to the torsion which is therefore $0$.

2.2.5. Fibration #8. The primitive embeddings of $A_2 = \langle e_1, e_2 \rangle$ into $E_7^{(1)}$ and $D_5 = \langle e_2, e_3, e_4, e_5, e_6 \rangle$ into $E_6^{(2)}$ lead to a primitive embedding of $A_2 \oplus D_5$ into $NI(D_{10}E_7^{(2)})$ satisfying $NI(D_{10}E_7^{(2)})/(D_{10}E_7^{(2)}) \cong (\mathbb{Z}/2\mathbb{Z})^2$ with glue code $\langle 1, 1, 0 \rangle$. (3, 0, 1). From Nishiyama [NiS] we get $(A_2)_{E_7} = \langle e_2, y, e_7, e_6, e_5 \rangle \cong A_7$ with $y = 2e_3 + e_1 + 2e_3 + 3e_4 + 2e_5 + e_6$ and $(D_5)_{E_7} = \langle (0, 0) D_{10} \cong (0, 0) \rangle$ and $W_{\text{root}} = N_{\text{root}} = D_{10} \oplus A_5 \oplus A_1$. Now

$$-2\eta = -\frac{1}{2}(1, e_7) = 2y - e_2 + e_5 + 2e_6 + 3e_7 \in ((A_2)_{E_7})_{\text{root}}$$

and for all $k \neq 0$, $[1, 0, 1, 0]$ can contribute to the torsion group which is therefore $\mathbb{Z}/2\mathbb{Z}$.

2.2.6. Fibration #24. The primitive embeddings $A_2 = \langle d_{10}, d_4 \rangle$ into $D_9^{(1)}$ and $D_5 = \langle d_6, d_5, d_4, d_3, d_2 \rangle$ into $D_9^{(2)}$ give a primitive embedding of $A_2 \oplus D_5$ into $L = NI(D_9^{(1)})$ with $L/I_{\text{root}} \cong (\mathbb{Z}/2\mathbb{Z})^4$ and glue code $\langle$ even permutations of $[0, 1, 2, 3] \rangle$. From Nishiyama [NiS] we get $(A_2)_{D_9} = \langle y = 2d_5 + d_6 + 2d_4 + d_3, d_3 = d_5 + d_6 + 2d_4 + d_3 + d_2, d_4, d_1 \rangle$, $((A_2)_{D_9})_{\text{root}} = \langle x_4, d_2, d_1 \rangle \cong A_3$ and $(D_5)_{D_9} = \langle x_6^7 \rangle = \langle d_6 + 2d_4 + d_3 + d_2 + d_1 \rangle = \langle -4 \rangle).$ We deduce $N_{\text{root}} = A_3 \oplus D_9 \oplus D_9$. From the relations $2[2]_{D_9} = x_4 + d_2 + 2d_1$ and $2[3]_{D_9} = y + x_4 + d_2 + d_1$ we deduce that the glue vectors having 1, 2, 3 or 0 in the first position may belong to $W$. From the relation $2[2]_{D_9} = x_6$ we deduce that only glue vectors with 2 or 0 in the second position may belong to $W$. Finally only the glue vectors $[0, 2, 3, 1], [1, 0, 3, 2], [1, 2, 0, 3], [2, 0, 1, 3], [2, 2, 2, 2], [3, 0, 2, 1], [3, 2, 1, 0], [0, 0, 0, 0]$ belong to $W$. Since $y$ and $x_6$ are not roots, only glue vectors with 0 or 2 in the first position and 0 in the second position may contribute to torsion that is $[2, 0, 1, 3], [0, 0, 0, 0]$. Hence the torsion group is $\mathbb{Z}/2\mathbb{Z}$. 

APÉRY-FERMI PENCIL OF K3-SURFACES AND THEIR 2-ISOGENIES
2.2.7. Fibration #26. The primitive embeddings of $A_2 = (d_5, d_3)$ into $D_5^{(1)}$ and $D_5$ into $D_5^{(2)}$ give a primitive embedding into $L = N(\mathbb{A}_2^2 D_5^2)$ with $L/L_{\text{root}} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ and glue code $\langle 1, 1, 1, 2 \rangle$. From Nishiyama we get $(A_2)_{D_5} = \{y, x_1, d_1\}$ with $y = 2d_3 + d_5 + 2d_3 + d_5$.

$x_4 = d_5 + d_4 + 2d_3 + 2d_2 + d_4$ and Gram matrix $M_2^+ = \begin{pmatrix} -4 & -1 & 1 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{pmatrix}$ of determinant 12. We also deduce $N_{\text{root}} = E_2 A_4^2, 2, \{2 \} D_5 = x_4 + d_1 \in ((A_2)_{D_5})_{\text{root}}$. Moreover neither $k, [1] D_5$ nor $k, [3] D_5$ belongs to $(A_2)_{D_5}$. Thus only glue vectors with 2 or 0 in the third position can belong to $W$ and eventually contribute to torsion, that is $[2, 2, 2, 0]$, $[4, 4, 0, 0]$, $[6, 6, 2, 0]$, $[2, 6, 6, 2]$, $[6, 2, 0, 2]$, $[4, 0, 2, 2]$, $[0, 4, 2, 2]$, $[0, 0, 0, 0]$. Since there is no $u_4 \in D_5$ satisfying $2(2 + u_4) = 0$ or $4(2 + u_4) = 0$, glue vectors with the last component equal to 2 cannot satisfy $k(l + u) \in N_{\text{root}}$ with $l \in L$ and $u \in L_{\text{root}} = A_2^2 D_5^2$.

Hence only the glue vectors generated by $\langle [2, 2, 2, 0] \rangle$ contribute to torsion and the torsion group is therefore $\mathbb{Z}/4\mathbb{Z}$.

\end{proof}

3. Weierstrass Equations for all the elliptic fibrations of $Y_k$

The method can be found in [BL, E]. We follow also the same kind of computations used for $Y_2$ given in [BL]. We give only explicit computations for 4 examples, #19, #2, #9, and #16. For #2 and #9 it was not obvious to find a rational point on the quartic curve. All the results are given in Table 3. For the 2 or 3-neighbor method [E] we give in the third column the starting fibration and in the forth the elliptic parameter. The terms in the elliptic parameter refer to the starting fibration.

3.1. Fibration #19. We take $u = \frac{XY}{Z}$ as a parameter of an elliptic fibration and with the birational transformation

$$x = -u(1 + uZ)(u + Y), \quad y = u^2((u + Y)(uY - 1)Z + Y(Y + 2u + k) - 1)$$

we obtain a Weierstrass equation

$$y^2 + ukyx + u^2(u^2 + uk + 1)y = x^3,$$

where the point $(x = 0, y = 0)$ is a 3-torsion point and the point $(-u^2, -u^2)$ is of infinite order.

The singular fibers are of type $IV^\ast$ ($u = 0, \infty$), $I_3(u^2 + uk + 1 = 0)$ and $I_1(27u^2 - k(k^2 - 27)u + 27 = 0)$. Moreover if $k = s + \frac{4}{3}$ the two singular fibers of type $I_3$ are above $u = -s$ and $\frac{-2}{3}$.\[\]

3.2. Fibration #2. Using the 3-neighbor method from fibration #19 we construct a new fibration with a fiber of type $II^\ast$ and the parameter $m = \frac{4s}{(u + s)^2}$. Then we obtain a cubic $C_m$ in $w, u, \omega_m = \left(\begin{array}{c} u_1 = \frac{m - s - 1}{s - m}, w_1 = \frac{n(s - 1)}{s(s - m)} \end{array}\right)$ which is not a flex point. The first stage is to obtain a quartic equation $Qua: y^2 = ax^4 + bx^3 + cx^2 + dx + e$. First we observe that $\omega_m$ is on the line $w = u + \frac{1}{s}$, so we replace $w$ by $K$ with $w = u + \frac{1}{s} + K$ and $u = u_1 + T$. The transformation $K = WT$ gives an equation of degree two in $T$, with constant term $fW + g$ where $f$ and $g$ belong to $\mathbb{Q}(s, m)$. With the change variable $Wy + g = x$ we have an equation $M(x)^2 + N(x)X + x = 0$. The discriminant of the quadratic equation in $T$ is $N(x)^2 - 4xM(x)$, a polynomial of degree 4 in $x$ and constant term a square. Easily we obtain the form $Qua$.

From the quartic form, setting $y = e + \frac{d}{2A} + x^2X', \quad x = \frac{8e^3X' - 4ec^2 + d^2}{Y^2}$ we get

$$Y'^2 + 4e(4dx - be)Y + 4e^2(8e^3X' - 4ec^2 + d^2)\left(X'^2 - a\right) = 0.$$ 

Finally the following Weierstrass equation follows from standard transformation where we replace $m$ by $t$.
$Y^2 - X^3 + \frac{1}{3}t^4 (s^2 + 1)(s^6 + 219s^4 - 21s^2 + 1)X$
\[- \frac{2r^5}{27} (-864s^5t^2 + (s^4 + 14s^2 + 1)(s^8 - 548s^6 + 198s^4 - 44s^3 + 1)t - 864s^5) = 0,
\]
with a section $\Phi$ of height 12 corresponding to \((8e^3X' - 4ex^2 + d^2) = 0\) and $Y' = 0$. The coordinates of $\Phi$, too long, are omitted but we can follow the previous computation to obtain it.

Writing the above form as
\[
y^2 = x^3 - 3ax + \left( t + \frac{1}{t} \right) - 2\beta
\]
we recover the values of the $j$ invariants of the two elliptic curves for the Shioda-Inose structure (see paragraph 4.5.1 and 4.1 below).

3.3. **Fibration #9.** Let $g = \frac{XY}{2x}$. Eliminating $X$ and writing $Y = ZU$ we obtain an equation of bidegree 2 in $U$ and $Z$. If $k = s + \frac{1}{2}$ there is a rational point $U = -1, Z = -\frac{\sqrt{3}}{g}$ on the previous curve. By standard transformations we get a Weierstrass equation
\[
y^2 = x^3 + \frac{1}{4}g^2 (s^4 + 14s^2 + 1) x^2 + s^2g^3 (g + s^2) (g^2s^2 + 1) x
\]
and a rational point
\[
x = \frac{s^2 (g - 1)^2 (g + s^2) (s^2g + 1)}{(s^2 - 1)^2},
\]
\[
y = \frac{1}{2} s^2 (g^2 - 1) (g + s^2) (s^2g + 1) (2g^2s^2 + g (s^4 - 6s^2 + 1) + 2s^2)}{(s^2 - 1)^3).
\]
The singular fibers are of type $2III^* (\infty, 0), 2I_2 \left(-s^2, -s + \frac{1}{2}\right), 4I_1$.

3.4. **Fibration #16.** Using the fibration #9 we consider the parameter $t = \frac{x}{g(s + x)}$ and obtain a Weierstrass equation
\[
Y^2 = X^3 + (4t (t^2 + s^2) + t^2 (s^4 + 14s^2 + 1)) X^2 + 16s^6t^4 X.
\]
The singular fibers are of type $I_4^* (\infty, 0), 4I_1$.

4. **Nikulin involutions and Shioda-Inose structure**

4.1. **Background.** Let $X$ be a $K3$ surface.

The second cohomology group, $H^2(X, \mathbb{Z})$ equipped with the cup product is an even unimodular lattice of signature $(3, 19)$. The period lattice of a surface denoted $T_X$ is defined by
\[
T_X = S_X^\perp \subset H^2(X, \mathbb{Z})
\]
where $S_X$ is the Néron-Severi group of $X$. The lattice $H^2(X, \mathbb{Z})$ admits a Hodge decomposition of weight two
\[
H^2(X, \mathbb{C}) \simeq H^{2,0} \oplus H^{1,1} \oplus H^{0,2}.
\]
Similarly, the period lattice $T_X$ has a Hodge decomposition of weight two
\[
T_X \otimes \mathbb{C} \simeq T^{2,0} \oplus T^{1,1} \oplus T^{0,2}.
\]
An isomorphism between two lattices that preserves their bilinear forms and their Hodge decomposition is called a Hodge isometry.

An automorphism of a $K3$ surface $X$ is called symplectic if it acts on $H^{2,0}(X)$ trivially. Such automorphisms were studied by Nikulin in [N2] who proved that a symplectic involution $i$ (Nikulin involution) has eight fixed points and that the minimal resolution $Y \to X/(i)$ of the eight nodes is again a $K3$ surface.
We have then the rational quotient map $p : X \to Y$ of degree 2. The transcendental lattices $T_X$ and $T_Y$ are related by the chain of inclusions
$$2T_Y \subseteq p^*T_X = T_X(2) \subseteq T_Y,$$
which preserves the quadratic forms and the Hodge structures.

<table>
<thead>
<tr>
<th>#</th>
<th>Weierstrass Equation</th>
<th>From</th>
<th>Param.</th>
</tr>
</thead>
<tbody>
<tr>
<td>#1</td>
<td>$y^2 + 1 + 6x^2 + t^3 (t + 1) + y = x^3 - t^3 (t + 1)^3$</td>
<td>$Y(\infty, \frac{219 + 219s - 21x^2 + 1}{21})$</td>
<td>$x^2$</td>
</tr>
<tr>
<td>#2</td>
<td>$y^2 = x^3 + \frac{24}{11} l 3 t (s^2 - 10 x^2 + 1) + (1 + 12) x^2$</td>
<td>$I_1^2(\infty, 0), 4I_1$</td>
<td>$x_8 + (x + 1)^2$</td>
</tr>
<tr>
<td>#3</td>
<td>$y^2 = x^3 + \frac{1}{4} l (2t^3 + (s^2 - 10 x^2 + 1) + t^3) x^2$</td>
<td>$I_1^2(\infty, 0), 4I_1$</td>
<td>$x_8$</td>
</tr>
<tr>
<td>#4</td>
<td>$y^2 = x^3 + \frac{1}{4} l (2t^3 + (s^2 - 10 x^2 + 1) + t^3) x^2$</td>
<td>$I_1^2(\infty, 0), 4I_1$</td>
<td>$x_8$</td>
</tr>
<tr>
<td>#5</td>
<td>$y^2 = x^3 + \frac{1}{4} l (2t^3 + (s^2 - 10 x^2 + 1) + t^3) x^2$</td>
<td>$I_1^2(\infty, 0), 4I_1$</td>
<td>$x_8$</td>
</tr>
<tr>
<td>#6</td>
<td>$y^2 = x^3 + \frac{1}{4} l (2t^3 + (s^2 - 10 x^2 + 1) + t^3) x^2$</td>
<td>$I_1^2(\infty, 0), 4I_1$</td>
<td>$x_8$</td>
</tr>
<tr>
<td>#7</td>
<td>$y^2 = x^3 + \frac{1}{4} l (2t^3 + (s^2 - 10 x^2 + 1) + t^3) x^2$</td>
<td>$I_1^2(\infty, 0), 4I_1$</td>
<td>$x_8$</td>
</tr>
<tr>
<td>#8</td>
<td>$y^2 = x^3 + \frac{1}{4} l (2t^3 + (s^2 - 10 x^2 + 1) + t^3) x^2$</td>
<td>$I_1^2(\infty, 0), 4I_1$</td>
<td>$x_8$</td>
</tr>
<tr>
<td>#9</td>
<td>$y^2 = x^3 + \frac{1}{4} l (2t^3 + (s^2 - 10 x^2 + 1) + t^3) x^2$</td>
<td>$I_1^2(\infty, 0), 4I_1$</td>
<td>$x_8$</td>
</tr>
<tr>
<td>#10</td>
<td>$y^2 = x^3 + \frac{1}{4} l (2t^3 + (s^2 - 10 x^2 + 1) + t^3) x^2$</td>
<td>$I_1^2(\infty, 0), 4I_1$</td>
<td>$x_8$</td>
</tr>
<tr>
<td>#11</td>
<td>$y^2 = x^3 + \frac{1}{4} l (2t^3 + (s^2 - 10 x^2 + 1) + t^3) x^2$</td>
<td>$I_1^2(\infty, 0), 4I_1$</td>
<td>$x_8$</td>
</tr>
<tr>
<td>#12</td>
<td>$y^2 = x^3 + \frac{1}{4} l (2t^3 + (s^2 - 10 x^2 + 1) + t^3) x^2$</td>
<td>$I_1^2(\infty, 0), 4I_1$</td>
<td>$x_8$</td>
</tr>
<tr>
<td>#13</td>
<td>$y^2 = x^3 + \frac{1}{4} l (2t^3 + (s^2 - 10 x^2 + 1) + t^3) x^2$</td>
<td>$I_1^2(\infty, 0), 4I_1$</td>
<td>$x_8$</td>
</tr>
</tbody>
</table>

Table 3. Weierstrass equations of the elliptic fibrations of $Y_h$
<table>
<thead>
<tr>
<th>No</th>
<th>Weierstrass Equation</th>
<th>From</th>
<th>Param.</th>
</tr>
</thead>
<tbody>
<tr>
<td>#14</td>
<td>( y^2 = x^3 + (t^4 (s^4 + 1) + \frac{1}{4} t^2 (s^4 + 14 s^2 + 1) + ts) x^2 + ts^2 p x )</td>
<td></td>
<td>( \frac{x}{t(t+s^2)(1+s^2)} )</td>
</tr>
<tr>
<td></td>
<td>( I_0^2(\infty), I_0^2(0), 4l_1 \left( \frac{t + \frac{3}{4} - \frac{4}{(s^2-1)^2}}{t(s^2-1)^2} \right) )</td>
<td></td>
<td>( \frac{1}{4l_1} )</td>
</tr>
<tr>
<td>#15</td>
<td>( y^2 = (y-\frac{t}{3}x^2) (y-\frac{t}{3}x^2) )</td>
<td></td>
<td>( \frac{x}{t(t+s^2)} )</td>
</tr>
<tr>
<td></td>
<td>( x_p = \frac{s^2 t}{(s^2-1)^2} )</td>
<td></td>
<td>( \frac{1}{X+Y} )</td>
</tr>
<tr>
<td>#16</td>
<td>( y^2 = x^3 + (t^4 (s^4 + 1) + \frac{1}{4} t^2 (s^4 + 14 s^2 + 1) + ts) x^2 + ts^2 p x )</td>
<td></td>
<td>( \frac{x}{t(t+s^2)} )</td>
</tr>
<tr>
<td></td>
<td>( I_0^2(\infty), 4l_1 \left( \frac{t + \frac{3}{4} - \frac{4}{(s^2-1)^2}}{t(s^2-1)^2} \right) )</td>
<td></td>
<td>( \frac{1}{4l_1} )</td>
</tr>
<tr>
<td>#17</td>
<td>( y^2 - \frac{t}{3} (s^4 + 14 s^2 + 1 - s) t^2 y = x (x - 4 s^2) (x - 4 s^2) )</td>
<td></td>
<td>( \frac{y}{t s^2} )</td>
</tr>
<tr>
<td></td>
<td>( I_0^2(\infty), 4l_1 \left( \frac{t + \frac{3}{4} - \frac{4}{(s^2-1)^2}}{t(s^2-1)^2} \right) )</td>
<td></td>
<td>( \frac{1}{X+Y} )</td>
</tr>
<tr>
<td>#18</td>
<td>( y^2 + (-t^4 + (s^4 - 1) t^2 + ts^2) y = x^2 (x - s^2) )</td>
<td></td>
<td>( \frac{y-tz}{t(x-s^2)} )</td>
</tr>
<tr>
<td></td>
<td>( I_{13}(\infty), I_0^2(0), 5l_1 )</td>
<td></td>
<td>( \frac{1}{X+Y} )</td>
</tr>
<tr>
<td>#19</td>
<td>( y^2 + 4 t x y + t^2 (t^2 + t + 1) y = x^3 )</td>
<td></td>
<td>( X + Y + Z )</td>
</tr>
<tr>
<td></td>
<td>( \frac{2t^2}{I_{12}(\infty), 2l_2(0,0), 4l_1(s, \frac{1}{2} s, \ldots)} )</td>
<td></td>
<td>( \frac{1}{X+Y} )</td>
</tr>
<tr>
<td>#20</td>
<td>( y^2 - y x (t^4 - k t + 1) = x (x - t) (x + t^4 - t k) )</td>
<td></td>
<td>( \frac{1}{X+Y} )</td>
</tr>
<tr>
<td></td>
<td>( I_{12}(\infty), 2l_2(0,0), 4l_1(s, \frac{1}{2} s, \ldots) )</td>
<td></td>
<td>( \frac{1}{X+Y} )</td>
</tr>
<tr>
<td>#21</td>
<td>( y^2 = x^3 - \frac{1}{2} t^2 (t^2 + 2 (s^4 - 1) t + (s^4 - 10 s^2 + 1) x^2) )</td>
<td></td>
<td>( \frac{Z}{X+Y+Z} )</td>
</tr>
<tr>
<td></td>
<td>( \frac{1}{2l_1^2(\infty), 4l_1(0,0), 3l_1(1, -s^2), 3l_1} )</td>
<td></td>
<td>( \frac{1}{X+Y} )</td>
</tr>
<tr>
<td>#22</td>
<td>( y^2 + (t^2 (1 - s^2) + s^2) y^2 + t^4 s^2 y = x (x - s^2 t) (x + t^4 s^2 (1 - t)) )</td>
<td></td>
<td>( X + Y + Z )</td>
</tr>
<tr>
<td></td>
<td>( I_0^2(\infty), I_0^2(0), 6l_1 )</td>
<td></td>
<td>( \frac{1}{X+Y} )</td>
</tr>
<tr>
<td></td>
<td>( x_{p_1} = 1, x_{p_2} = s^2 t )</td>
<td></td>
<td>( \frac{1}{X+Y} )</td>
</tr>
<tr>
<td>#23</td>
<td>( y^2 + (2 t^2 - t k + 1) y x = x (x - t^2) (x - t^4) )</td>
<td></td>
<td>( \frac{1}{X+Y} )</td>
</tr>
<tr>
<td></td>
<td>( I_0^2(\infty), I_{12}(0), 6l_1 \left( \frac{1}{4 s^2 \ldots} \right) )</td>
<td></td>
<td>( \frac{1}{X+Y} )</td>
</tr>
<tr>
<td></td>
<td>( x_{p_1} = t^2, x_{p_2} = \frac{(tk-1)^2}{4 s^2} )</td>
<td></td>
<td>( \frac{1}{X+Y} )</td>
</tr>
<tr>
<td>#24</td>
<td>( y^2 + (s^2 + 1) t y x = x (x - t^2) (x - s^2 t (t + 1)) )</td>
<td></td>
<td>( \frac{Z}{X} )</td>
</tr>
<tr>
<td></td>
<td>( 2I_0^2(\infty), I_1(1), 4l_1 )</td>
<td></td>
<td>( \frac{Z}{X} )</td>
</tr>
<tr>
<td></td>
<td>( x_{p_1} = t + 1, x_{p_2} = t^2 s^2 )</td>
<td></td>
<td>( \frac{Z}{X} )</td>
</tr>
<tr>
<td>#25</td>
<td>( y^2 + (s + t) (t s + 1) y x + t^4 s^2 (t^2 s^2 - 1) x + s y = x (x - s) (x - t^2 s (t - s)) )</td>
<td></td>
<td>( \frac{Z}{X+Y} )</td>
</tr>
<tr>
<td></td>
<td>( I_1(\infty), I_{12}(0), 7l_1 )</td>
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<td>( \frac{Z}{X+Y} )</td>
</tr>
<tr>
<td></td>
<td>( x_{p_1} = ts, x_{p_2} = -t^2 s^2 )</td>
<td></td>
<td>( \frac{Z}{X+Y} )</td>
</tr>
<tr>
<td>#26</td>
<td>( y^2 + (ts - 1)(t - s) x y = x (x - t^2 s^2) )</td>
<td></td>
<td>( Z )</td>
</tr>
<tr>
<td></td>
<td>( I_0^2(\infty), I_2(s, \frac{1}{2} s), 4l_1 )</td>
<td></td>
<td>( Z )</td>
</tr>
<tr>
<td></td>
<td>( x_p = 0 )</td>
<td></td>
<td>( Z )</td>
</tr>
<tr>
<td>#27</td>
<td>( y^2 - (t (s^2 - 1) + s^2) y x + t^4 s^2 (t + 1) y = )</td>
<td></td>
<td>( \frac{Z}{X+Y} )</td>
</tr>
<tr>
<td></td>
<td>( x (x + t^2 s^2 (t + 1)) )</td>
<td></td>
<td>( \frac{Z}{X+Y} )</td>
</tr>
<tr>
<td></td>
<td>( I_0^2(\infty), I_0(0), I_5(-1) 4l_1 )</td>
<td></td>
<td>( \frac{Z}{X+Y} )</td>
</tr>
</tbody>
</table>

Table 4. Weierstrass equations of the elliptic fibrations of \( Y_k \)
Proposition 4.1. Let $K$ be an elliptic curve with invariant $j$, defined by a Weierstrass equation in the Legendre form

\[ E_l : y^2 = x(x - 1)(x - l). \]

Then $l$ satisfies the equation $j = 256\left(\frac{1+4t^2}{(1-t)^2}\right)$. For a fixed $j$ the six values of $l$ are given by $l$ or $\frac{1}{l}$, $1 - l$, $\frac{1}{1 - l}$, and $\frac{1}{1 - \frac{1}{l}}$.

Consider the Kummer surface $K$ given by $E_{l_1} \times E_{l_2} / \pm 1$ and choose as equation for $K$

\[ x_1(x_1 - 1)(x_1 - l_1)^2 = x_2(x_2 - 1)(x_2 - l_2). \]

Following [Ku] we can construct different elliptic fibrations. In the general case we can consider the three elliptic fibrations $F_i$ of $K$ defined by the elliptic parameters $m_i$, with corresponding types of singular fibers

\[
\begin{align*}
    F_0 : m_0 &= \frac{x_1}{(x_2 - l_2)(x_1 - x_2)} & 2I_2^*, 4I_2 \\
    F_8 : m_8 &= \frac{(x_2 - l_2)(x_1 - x_2)}{(x_2 - 1)(x_1 - 1)} & 3I_2^*, 3I_2, I_1 \\
    F_5 : m_5 &= \frac{(x_2 - l_2)(x_1 - l_1)^2}{(l_2 x_1 - x_2)(x_1 - l_1)(l_1 - 1)x_2} & I_6^*, 6I_2.
\end{align*}
\]

In the special case when $E_1 = E_2$ and $j_1 = j_2 = 8000$ we obtain the following fibrations $F_8$ ($III^*$, $I_2^*$, $I_4$, $I_2$, $I_1$) with $l_1 = 3 + 2\sqrt{2}$, $l_2 = 3 - 2\sqrt{2}$ and $g_8$ ($III^*$, $I_1^*$, $3I_2$) with $l_1 = l_2 = 3 + 2\sqrt{2}$.

4.3. Nikulin involutions and Kummer surfaces.

Proposition 4.1. Consider a family $S_{a,b}$ of K3 surfaces with an elliptic fibration, a two torsion section defining an involution $\iota$ and two singular fibers of type $I_1^*$.

\[ S_{a,b} : Y^2 = X^3 + \left(t + \frac{1}{l} + a\right)X^2 + b^2X. \]

Then the K3 surface $S_{a,b}/\iota$ is the Kummer surface $(E_1 \times E_2)/(\pm I_d)$ where the $j_i$ invariants of the elliptic curves $E_i$, $i = 1, 2$ are given by the formulae

\[ j_1j_2 = \frac{4096(a^2 - 3 + 12b^2)^3}{b^2}. \]

\[ (j_2 - 1728)(j_1 - 1728) = \frac{1024a^2(2a^2 - 9 - 72b^2)^2}{b^2}. \]

Proof. Recall that if $E_i$, $i = 1, 2$, are two elliptic curves in the Legendre form

\[ E_i : y^2 = x(x - 1)(x - l_i), \]

the Kummer surface $K$

\[ K : (E_1 \times E_2)/(\pm I_d) \]
The 2-isogenous curve $S$ is defined by the following equation
\[ x_1 (x_1 - 1) (x_1 - l_1) t^2 = x_2 (x_2 - 1) (x_2 - l_2). \]

The Kummer surface $K$ admits an elliptic fibration with parameter $u = m_0 = \frac{a}{x_2}$ and Weierstrass equation $H_u$
\[ H_u : Y^2 = X (X - u (a - 1) (u_2 - l_1)) (X - u (u - 1) (l_2 u - 1)). \]

The 2-isogenous curve $S_{a,b}/((0,0))$ has the following Weierstrass equation
\[ Y^2 = X (X - t (t^2 + (a - 2b) t + 1)) (X - t (t^2 + (a + 2b) t + 1)) \]
with two singular fibers of type $I_1^2$ above 0 and $\infty$.

We easily prove that $S_{a,b}/((0,0))$ and $H_u$ are isomorphic on the field $\mathbb{Q}(\sqrt{w_2})$ where
\[ l_1 = w_1' w_2 = \frac{w_2}{w_1}, \quad l_2 = \frac{1}{w_1' w_2} = w_1 w_2 \text{ and } t = w_1 u, \]
$w_1$, $w_1'$ and $w_2$, $w_2'$ being respectively the roots of polynomials $t^2 + (a - 2b) t + 1$ and $t^2 + (a + 2b) t + 1$.

Recall that the modular invariant $j_i$ of the elliptic curve $E_i$ is linked to $l_i$ by the relation
\[ j_i = 256 \frac{(1 - l_i + l_2^3)^3}{l_i^2 (1 - l_i)^2}. \]

By elimination of $w_1$ and $w_2$, it follows that $j_1 j_2 = 4096 \frac{(a^2 - 3 + 12b^2)^3}{b^2} (j_2 - 1728) (j_1 - 1728) = \frac{1024 a^2 (2a^2 - 9 - 72b^2)^2}{b^2}.

\[ \square \]

In the Fermi family, the $K_3$ surface $Y_b$ has the fibration #16 with two singular fibers $I_1^2$, a 2-torsion point and Weierstrass equation
\[ y^2 = x^3 + x^2 t (4 (t^2 + s^2) + t (s^4 + 14s^2 + 1)) + 16t^4 s^6 x. \]

Taking
\[ y = y' t^3 (2 \sqrt{3})^3, \quad x = x' t^2 (2 \sqrt{3})^2 \text{ and } t = t' s, \]
we obtain the following Weierstrass equation
\[ y' = x'^3 + \left( t' + \frac{1}{t'} + \frac{1}{4} \frac{s^4 + 14s^2 + 1}{s} \right) + s^4 x'. \]

By the previous proposition with $a = \frac{1}{4} s^4 + 14s^2 + 1$, $b = s^2$, we derive the corollary below.

**Corollary 4.1.** The surface obtained with the 2-isogeny of kernel $((0,0))$ from fibration #16, is the Kummer surface associated to the product of two elliptic curves of $j$-invariants $j_1, j_2$ satisfying
\[ j_1 j_2 = \frac{(s^2 + 1)^3 (s^6 + 219s^4 - 21s^2 + 1)^3}{s^{10}}. \]

\[ (j_1 - 12^3) (j_2 - 12^3) = \frac{(s^4 + 14s^2 + 1)^2 (s^8 - 548s^6 + 198s^4 - 44s^2 + 1)^2}{s^{10}}.

\[ \text{Remark 4.1.} \text{ If } s = 1 \text{ we find } j_1 = j_2 = 8000. \]

\[ \text{Remark 4.2.} \text{ If } b = 1 \text{ we obtain the family of surfaces studied by Narumiya and Shiga, } \text{[Na]. Moreover if } a = \frac{9}{4} \text{ (resp. 4) we find the two modular surfaces associated to the modular groups } \Gamma_1(7) \text{ (resp. } \Gamma_1(8)). \text{ In these two cases we get } j_1 = j_2 = -3375 \text{ (resp. } j_1 = j_2 = 8000). \]
Remark 4.3. With the same method we can consider a family of K3 surfaces with Weierstrass equations

\[ E_v : Y^2 + XY - \left( v + \frac{1}{v} - k \right) Y = X^3 - \left( v + \frac{1}{v} - k \right) X^2, \]

singular fibers of type $2I^*_1, 2I_1$ and the point $P_v = (0, 0)$ of order 4. The elliptic curve $E_v' = E_v / \langle 2P_v \rangle$ has singular fibers of type $2I^*_3, 4I_2$. An analog computation gives $E_v' \equiv (E_1 \times E_2) / (\pm \text{Id})$ and

\[
\begin{align*}
    j_1 j_2 &= (256k^2 - 16k - 767)^3, \\
    (j_1 - 12)^6 (j_2 - 12)^6 &= (32k - 1)^2 (128k^2 - 8k - 577)^2.
\end{align*}
\]

4.4. Shioda-Inose structure.

Definition 4.1. A K3 surface $X$ has a Shioda-Inose structure if there is a symplectic involution $i$ on $X$ with rational quotient map $X \overset{p}{\rightarrow} Y$ such that $Y$ is a Kummer surface and $p^*$ induces a Hodge isometry $T_X(2) \cong T_Y$.

Such an involution $i$ is called a Morrison-Nikulin involution.

An equivalent criterion is that $X$ admits a (Nikulin) involution interchanging two orthogonal copies of $E_8(-1)$ in $NS(X)$, where $E_8(-1)$ is the unique unimodular even negative-definite lattice of rank 8.

Or even more abstractly: $2E_8 \hookrightarrow NS(X)$.

Applying this criterion to fibrations #17 and #8 and the Van Geemen-Sarti involution we get the following result.

Proposition 4.2. The translation by the two torsion point of fibration #17 and #8 endows $Y_k$ with a Shioda-Inose structure.

Fibration #17 has a fiber of type $I_1$ at $t = \infty$. The idea [1] is to use the components $\Theta_{-2}, \Theta_{-1}, \Theta_0, \Theta_1, \Theta_2, \Theta_3, \Theta_4$ of $I_{16}$ and the zero section to generate a lattice of type $E_8$. The two-torsion section intersects $\Theta_8$ and the translation by the two-torsion point on the fiber $I_{16}$ transforms $\Theta_8$ in $\Theta_{8+8}$. The translation maps the lattice $E_8$ on another disjoint $E_8$ lattice and defines a Shioda-Inose structure.

For fibration #8, the fiber above $t = 0$ is of type $I_6$ and the section of order 2 specialises to the singular point $(0, 0)$. Then after a blow up, it will not meet the 0-component. If we denote $\Theta_{0,i}$, $0 \leq i \leq 5$, the six components, then the zero section meets $\Theta_{0,0}$ and the 2-torsion section meets $\Theta_{0,3}$. The translation by the 2-torsion section induces the permutation $\Theta_{0,i} \mapsto \Theta_{0,i+3}$.

The fiber above $t = \infty$ is of type $I^*_6$. The simple components are denoted $\Theta_{\infty,0}, \Theta_{\infty,1}$ and $\Theta_{\infty,2}, \Theta_{\infty,3}$; the double components are denoted $C_i$ with $0 \leq i \leq 6$ and $\Theta_{\infty,0} C_0 = \Theta_{\infty,1} C_0 = 1$; $\Theta_{\infty,2} C_6 = \Theta_{\infty,3}$. The translation by the 2-torsion section induces the transposition $C_i \leftrightarrow C_{6-i}$.

The class of the components $C_0, C_1, C_2, \Theta_{\infty,0}, \Theta_{\infty,1}$, the zero section, $\Theta_{0,0}$ and $\Theta_{0,1}$ define a $E_8(-1)$. The Nikulin involution defined by the two torsion section maps this $E_8(-1)$ to another copy of $E_8(-1)$ orthogonal to the first one; so the Nikulin involution is a Morrison-Nikulin involution.

4.5. Base change and van Geemen-Sarti involutions. If a K3-surface $X$ has an elliptic fibration with two fibers of type $II^*$, this fibration can be realised by a Weierstrass equation of type

\[ y^2 = x^3 - 3ax + (h + 1/h - 2\beta). \]

Moreover Shioda [Shio] deduces the “Kummer sandwiching”, $K \rightarrow S \rightarrow K$, identifying the Kummer $K = E_1 \times E_2 \pm 1$ with the help of the $j$-invariants of the two elliptic curves $E_1, E_2$ and giving the following elliptic fibration of $K$

\[ y^2 = x^3 - 3ax + (t^2 + 1/t^2 - 2\beta). \]

This can be viewed as a base change of the fibration of $X$. 

4.5.1. Alternate elliptic fibration. We shall now use an alternate elliptic fibration ([Sc-Shi] example 13.6) to show that this construction is indeed a 2-isogeny between two elliptic fibrations of $S$ and $K$. In the next picture we consider a divisor $D$ of type $I^*_2$ composed of the zero section 0 and the components of the $II^*$ fibers enclosed in dashed lines. The far double components of the $II^*$ fibres can be chosen as sections of the new fibration. Take $\omega$ as the zero section. The other one is a two-torsion point since the function $h$ has a double pole on $\omega$ and a double zero at $M$. It is the function $'x'$ in a Weierstrass equation. More precisely with the new parameter $u = x$ and the variables $Y = y h$ and $X = h$, we obtain the Weierstrass equation

\[ Y^2 = X^3 + (u^3 - 3\alpha u - 2\beta)X^2 + X. \]

In this equation, if we substitute $X (= h)$ by $t^2$, we obtain an equation in $W, t$ with $Y = W t^2$, which is the equation for the 2-isogenous elliptic curve. Indeed the birational transformation

\[ y = 4Y + 4U^3 + 2UA, \quad x = 2 \frac{Y + U^3}{U} \]

with inverse

\[ U = \frac{1}{2} \frac{y}{x + A}, \quad Y = \frac{1}{8} \left( -y^2 + 2x^3 + 4x^2A + 2xA^2 \right) y \]

transforms the curve $Y^2 = U^6 + AU^4 + BU^2$ in the Weierstrass form

\[ y^2 = (x + A)(x^2 - 4B). \]

This is an equation for the 2-isogenous curve of the curve $Y^2 = X^3 + AX^2 + BX$ ([Sh]). On the curve $Y^2 = U^6 + AU^4 + BU^2$, the involution $U \mapsto -U$ means adding the two-torsion point $(x = -A, y = 0)$. Using this above process with $A = (u^3 - 3\alpha u - 2\beta)$, the 2-isogenous curve $E_u$ has a Weierstrass equation

\[ Y^2 = (X + (u^3 - 3\alpha u - 2\beta))(X^2 - 4) \]

with singular fibers of type $I^*_6, 6I_2$.

The coefficients $\alpha$ and $\beta$ can be computed using the $j$-invariants

\[ \alpha^3 = J_1J_2; \quad \beta^2 = (1 - J_1)(1 - J_2); \quad j_i = 1728J_i. \]

If the elliptic curve is put in the Legendre form $y'^2 = x'(x'^2 - 1)(x' - l)$ then

\[ j = \frac{256(1-l^2)^3}{l^4(l-1)^2}, \]

so

\[ \alpha^3 = \frac{16}{729} \frac{(1-l_1 + l_2^2)^3(1 - l_2^2 + l_1^2)^3}{l_1^2(l_1 - 1)l_2^2(l_2 - 1)^2}, \]

\[ \beta = \frac{1}{27} \frac{(2l_1 - 1)(l_1 - 2)(2l_2 - 1)(l_2 - 2)(l_1 + 1)(l_2 + 1)}{l_1l_2(l_1 - 1)(l_2 - 1)}. \]
On the Kummer surface $E_1 \times E_2/\pm 1$ of equation

$$X_1(X_1-1)(X_1-l_1)Z^2 = X_2(X_2-1)(X_2-l_2)$$

we consider an elliptic fibration (case $J_5$ of [Kil]) with the parameter

$$z = \frac{(l_2X_1-X_2)(X_1-l_1+X_2(l_1-1))}{X_2(X_1-l_1)}$$

(in fact $z = -\frac{1}{4}(l_1-1)$ cf. 4.2) and obtain the Weierstrass equation

$$Y^2 = (X-2l_1l_2(l_1-1)(l_2-1))(X+2l_1l_2(l_1-1)(l_2-1))$$

$$X + 4z^3 + 4(2l_1l_2+1)z^2 +4(l_1-1)(l_1-l_2)z+2l_1l_2(l_1-1)(l_2-1).$$

Substituting $z = w - \frac{1}{8}(-2l_1l_2 + l_1 + l_1 + 1)$ it follows

$$Y^2 = (X-2l_1l_2(l_1-1)(l_2-1))(X+2l_1l_2(l_1-1)(l_2-1))$$

$$X + 4w^3 - \frac{3}{4}(l_1^2_l_2 + 1)(l_1^2 - l_1 + 1)w + \frac{3}{8}(l_2^2 - 2)(2l_1 - 1)(l_1 - 1)(l_1 - 1)(l_2 + 1).$$

Up to an automorphism of this Weierstrass form we recover the equation of $E_4$.

The previous results can be used to show the following proposition

**Proposition 4.3.** The translation by the two torsion point of the elliptic fibration #4 gives to $Y_k$ a Shioda-Inose structure.

5. **Proof of Theorem 1.1**

We consider an elliptic fibration #n of $Y_k$ with a two torsion section.

From the Shioda-Tate formula (cf. e.g. [Shi], Corollary 1.7) we have the relation

$$12 = \frac{\Delta \prod m^{(1)}_w}{|\text{Tor}|^2}$$

where $\Delta$ is the determinant of the height-matrix of a set of generators of the Mordell-Weil group, $m^{(1)}_w$ the number of simple components of a singular fiber and $|\text{Tor}|$ the order of the torsion group of Mordell-Weil group. This formula allows us to determine generators of the Mordell-Weil group except for fibration #4. Using the 2-isogeny we determine also the Mordell-Weil group of #n-i. The discriminant is either $12 \times 2$ or $12 \times 8$.

**Proposition 5.1.** The translation by the two torsion point of the fibration #16 gives to $Y_k$ a Shioda-Inose structure.

From the previous Proposition 4.1, the translation by the two torsion point of #16 gives to the quotient a Kummer structure. The fibration #16 is of rank one, its Mordell-Weil group is generated by $P$ and the two torsion point. By computation we can see that the Mordell-Weil group of the 2-isogenous curve on $E(C(t))$ is generated by $p(P)$ and torsion sections. So we can compute the discriminant of the Néron-Severi group which is $12 \times 8$. The second condition, $T_X(2) \simeq T_Y$, is then verified.

**Remark 5.1.** The $K3$ surface of Picard number 20 given with the elliptic fibration

$$Y^2 = X^3 - \left(\frac{x + \frac{1}{4} - \frac{3}{2}}{t} \right) X^2 + \frac{1}{16} X$$

or

$$y^2 = x^3 - 1/2 t \left(2 t^2 + 2 - 3 t \right) x^2 + 1/16 t^4 x$$

has rank 1. The Mordell-Weil group is generated by $(0,0)$ and $P = (x = \frac{1}{4}, y = \frac{(t-1)^2}{8})$. The determinant of the Néron-Severi group is equal to 12. By computation we have $p(P) = 2Q$ with $Q = (t(t-1)(t^2 - t + 1), -t^2(t-1)(t^2 - t + 1))$ of height $\frac{1}{4}$. The determinant of the Néron Severi group of the 2-isogenous curve is then 12 not $12 \times 2^2$. So the involution induced by the two-torsion point is not a Nikulin-Morrison involution. Moreover the 2-isogenous elliptic curve is a fibration of the Kummer surface $E \times E/\pm 1$ where $j(E) = 0$. 

For fibrations #n-i with discriminant of the transcendental lattice $12 \times 8$ we prove that we have the Shioda-Inose structure in the following way: from corollary 4.1 this is true for #16-i, from Proposition 4.3 this is true for #4-i and from Proposition 4.2 this is true for #17-i, #8-i. The other fibrations #n-i can be obtained by 2- or 3-neighbor method from #16-i, #8-i or #17-i. The results are given in the Table [8]. In the second column are written the Weierstrass equations for the #n elliptic fibration and its 2-isogenous fibration, singular fibers and the $x$-coordinates of generators of the Mordell lattice of #n-i. In the third column we give the starting fibration for the 2- or 3-neighbor method and in the last column the parameter used from the starting fibration.

5.1. The $K3$ surface $S_k$. For the remaining fibrations, (discriminant $12 \times 2$), using also the 2- or 3-neighbor method, they are proved to lie on the same surface $S_k$. Except for the case #7 the results are collected in the Table [9] with the same format. The case #7 needs an intermediate fibration explained in the next paragraph.

Starting with fibration #7-i and using the parameter $m = \frac{2}{x(1-s^2)}$ it follows the Weierstrass equation

$$Y^2 + 2 \{m_7^2 s^2 - 2\} Y X - 16 m_7^2 s^4 Y = (X - 8 m_7^2 s^2) (X + 8 m_7^2 s^2) (X + m_7^2 (s^4 - 6s^2 + 1) - 4)$$

with singular fibers $I_k(\infty), IV^*(0), 8I_1$.

Then the parameter $m_{15} = \frac{2}{Y(X+8m_7^2s^2)}$ leads to the fibration #15-i.

For the last part of Theorem 1.1 we give properties of $S_k$. First we prove that $S_k$ is the Jacobian variety of some genus 1 fibrations of $K_k$.

Starting with the fibration #26-i and Weierstrass equation

$$y^2 = x (x + 4t^2 s^2) \left( x + \frac{1}{4} (t - s)^2 (ts - 1)^2 \right)$$

the new parameter $m := \frac{2}{t(x + \frac{1}{4} (t - s)^2 (ts - 1)^2)}$ defines an elliptic fibration of #26-i with Weierstrass equation

$$E_m : Y^2 - m \{ s^2 + 1 \} Y X = X (X - s^2 m^2) \left( X + \frac{1}{4} (2m - s)^2 (2m + s)^2 \right)$$

and singular fibers are of type $4I_4(0, \pm \frac{1}{2}s, \infty), 8I_1$.

Then setting as new parameter $n = \frac{2}{m}$, it follows a genus one curve in $m$ and $Y$. Its equation, of degree 2 in $Y$, can be transformed in

$$w^2 = -16n(-n + s^2)m^4 + n(s^4(8 + n) - 10ns^2 + n(1 + 4n))m^2 - n^4(-n + s^2).$$

Let us recall the formulae giving the jacobian of a genus one curve defined by the equation $y^2 = ax^4 + bx^3 + cx^2 + dx + e$. If $c_3 = 2^4(12ae - 3bd + c^2)$ and $c_6 = 2^5(72ace - 27ad^2 - 27b^2e + 9bed - 2c^3)$, then the equation of the Jacobian curve is

$$\bar{y}^2 = \bar{x}^3 - 27c_4\bar{x} - 54c_6.$$

In our case we obtain

$$y^2 = x \left( x + n^3 s^2 - \frac{1}{4} n^2 (s^2 - 1)^2 \right)$$

$$\left( x + n^3 s^2 - \frac{1}{4} (s^2 - 4s - 1) (s^2 + 4s - 1) n^2 + 4ns^2 \right),$$

which is precisely the fibration #15-i.
<table>
<thead>
<tr>
<th>No.</th>
<th>Weierstrass Equation</th>
<th>From</th>
<th>Param.</th>
</tr>
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<tr>
<td>#8</td>
<td>( y^2 - k(t-1) y = x (x - 1 { x - t^2 } ) )</td>
<td>( I_{12}^0(\infty), I_0(0), I_2(1), 4I_4 )</td>
<td>( x_Q = \frac{1}{2} (t - 1) (4t^2 + 4t - 4k^2) )</td>
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<tr>
<td>#16</td>
<td>( y^2 = x^3 + (4(t^2 + t^2) + t(s^4 + 14s^2 + 1)) x^2 + 16s^4 t^4 x )</td>
<td>( 2I_2^0(\infty, 0), 4I_4 )</td>
<td>( x_Q = t^2 (t^2 + 4s^2) (s^4 + 14s^2 + 1) )</td>
</tr>
<tr>
<td>#17</td>
<td>( y^2 - \frac{1}{2} (s^4 + 14s^2 + 1 - s^4 t^4) y = x (x - 4s^2) (x - 4s^4) )</td>
<td>( I_6(\infty), 8I_1( \pm \frac{t^2 + 4s^2 - 1}{s} , .. ) )</td>
<td>( x_Q = \frac{1}{8} t(2s^2 - 4s) (ts - s^2 + 4s + 1) )</td>
</tr>
<tr>
<td>#23</td>
<td>( y^2 + (2t^2 - 6t + 1) y = x (x - t^2) (x - t^3) )</td>
<td>( I_6^0(\infty), I_1(0), 6I_2 ) ( 1 + \frac{t}{t^2 + 4s^2} )</td>
<td>( x_Q = \frac{1}{16} t(4t^2 - 2k + t + 1) (t(k + 2) - 1) )</td>
</tr>
<tr>
<td>#24</td>
<td>( y^2 + (s^2 + 1) y x = x (x - t^2 s^2) (x - s^2 t (t + 1)^2) )</td>
<td>( 2I_2^0(\infty, 0), I_4(-1), 4I_4 )</td>
<td>( x_Q = \frac{1}{2} (4t^2 - 4ks^2 + 4s^2 - (s + 1)^4) t + 10s^2 )</td>
</tr>
<tr>
<td>#26</td>
<td>( y^2 = x (x + 1) (t - s) xy = x (x - t^2 s^2)^4 )</td>
<td>( 2I_2^0(\infty, 0), I_2(s, \pm 1), 4I_4 )</td>
<td>( x_Q = \frac{1}{8} t^4 (4s^2 - 2k + 2)^2 )</td>
</tr>
</tbody>
</table>

| Table 5. Fibrations with discriminant 12 \times 8 (Fibrations of the Kummer \( K_k \)) |
### APÉRY-FERMI PENCIL OF K3-SURFACES AND THEIR 2-ISOGENIES

#### Table 6. Fibrations with discriminant \( 12 \times 2 \) (Fibrations of \( S_k \))

<table>
<thead>
<tr>
<th>No</th>
<th>Weierstrass Equation</th>
<th>From</th>
<th>Param.</th>
</tr>
</thead>
</table>
| #7 | \( y^2 = x^3 + \frac{1}{4} t (t (x^4 - 10x^2 + 1) + 8x^3) x^2 + t^2 s^2 (t - s^2)^2 x \)  
\( 2II^{+}(∞), I^+_1(0), I^+_2(s^2), 2I_1 \) | \( y \) | \( t/x \) |
| #9 | \( y^2 = x^3 + \frac{1}{4} (s^4 + 14s^2 + 1) t^2 x^2 + t^2 s^4 (s^2 + t) (t^2 + 1) x \)  
\( 2II^{+}(∞), 2I_1 (−s^2, −t) \)  
\( 2II^{+}(∞), 2I_1 (−s^2, −t) \)  
\( x_Q = \frac{1}{4} t^3 (t^2 x^2 + t (s^2 - 1)^2 x^2) \) | \( y \) | \( x/1 \) |
| #14 | \( y^2 = x (x - \frac{1}{4} (s^2 + 1)^2 t^3 - \frac{1}{4} (s^4 + 14s^2 + 1) t^2 - ts^2) \)  
\( 2I^+_1(∞), I^+_1(0), 4I_2 \)  
\( 2I^+_1(∞), I^+_1(0), 4I_2 \)  
\( x_Q = \frac{1}{4} t^2 (s^2 - 1)^2 (4t + 1) \) | \( y \) | \( x/1 \) |
| #15 | \( (y - tx) (y - s^2tx) = x (x - ts^2) (x - tx^2 (t + 2)^2) \)  
\( 2I^+_1(∞), I^+_1(0), 4I_2 (-1, \frac{1}{4} (s^2 - 1)^2 \ldots) \)  
\( 2I^+_1(∞), I^+_1(0), 4I_2 (-1, \frac{1}{4} (s^2 - 1)^2 \ldots) \)  
\( x_Q = \frac{1}{4} t^2 (s^2 - 1)^2 \) | \( y \) | \( x/1 \) |
| #20 | \( y^2 - (t^3 s - (s^2 + 1) t + 3s) yx - s^2 (t - s) (t s - 1) y = x^3 \)  
\( I^{+}_2(∞), 2I_1 (s, \frac{1}{2}), 2I_2 (0, \frac{1}{2} s^2), 2I_1 \)  
\( 3I_2 (∞), 2I_1 (s, \frac{1}{2}), 2I_2 (0, \frac{1}{2} s^2) \) | \( y \) | \( x/1 \) |

#### Remark 5.2.
Using the new parameter \( p = \frac{Y}{m^2(X + \frac{1}{4} (2m - s)^2 (2m + s)^2)} \), another result can be derived from \( E_m \) leading to

\[
E_p : Y^2 - 2s (2p - 1) (2p + 1) Y X = X (X + 64s^2 p^2) (X + (2sp + 1) (2sp - 1) (s + 2p) (s - 2p)),
\]

with singular fibers \( 2I^+_1, 4I_2, 4I_1 \). From \( E_p \) and the new parameter \( k = \frac{X}{F} \), we obtain a genus one fibration whose Jacobian is #14-i. Starting from the fibration #26 - i, the parameter \( q = \frac{X}{F} \) leads to a genus one fibration whose Jacobian is the fibration #20-i.

### 5.2. Transcendental and Néron-Severi lattices of the surface \( S_k \).

#### Lemma 5.1.
The five fibrations #7 - i, #9 - i, #14 - i, #15 - i, #20 - i are fibrations of the same K3 surface \( S_k \) with transcendental lattice

\[
T_{S_k} = (-2) \oplus (2) \oplus (6)
\]
and Néron-Severi lattice

\[ \text{NS}(S_k) = U \oplus E_8(-1) \oplus E_7(-1) \oplus ((-2)) \oplus ((-6)). \]

Moreover these fibrations specialise in fibrations of \( Y_2 \) for \( k = 2 \).

**Proof.** These five fibrations are respectively the fibrations given in Table 6 and recalled below with the type of their singular fibers, their rank and torsion group:

\[
\begin{align*}
\#7 - i & : 2A_1A_2D_0E_7 \quad \text{rk} \ 0 \quad \mathbb{Z}/2\mathbb{Z} \\
\#9 - i & : 2A_12E_7 \quad \text{rk} \ 1 \quad \mathbb{Z}/2\mathbb{Z} \\
\#14 - i & : 4A_1D_1D_8 \quad \text{rk} \ 1 \quad \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \\
\#15 - i & : 4A_12D_6 \quad \text{rk} \ 1 \quad \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \\
\#20 - i & : 2A_13A_5 \quad \text{rk} \ 0 \quad \mathbb{Z}/6\mathbb{Z}.
\end{align*}
\]

We already know from the above results that they are fibrations of \( S_k \) but we give below another proof of this fact. Denote \( S_0 \) the \( K3 \) surface defined by the elliptic fibration \#7-i with Weierstrass equation given in Table 6 and draw the graph of the singular fibers, the zero and two-torsion sections of the elliptic fibration

\[
Y^2 = X^3 + \left( \frac{-1/2}{s^4} + 5s^2 - 1/2 \right) t^2 - 4s^4 t \right) X^2 \\
\left( 4s^2t^5 + \frac{1}{16} s^8 - 5/4s^2 - 5/4s^6 + 1/16 \right) t^4 + s^4 \left( s^2 + 1 \right) t^3 \right) X
\]

with singular fibers \( I^*(\infty), I_2^*(0), I_3(s^2), 2I_1(t_1, t_2). \)

With the parameter \( m = \frac{1}{12} \) we obtain another fibration with singular fibers \( I^*(\infty) \) (in blue), \( I_2^*(0) \) (in green), \( I_3(s^2-1) (s^2-1)^2 \) (part of it in red), \( I_2(4s^4), I_1(\sigma_0) \) (in yellow), where \( \sigma_0 = \frac{(s^2-6s+1)(s^2+6s+1)(s^4+1)}{12s^4} \).

This new fibration \( \Sigma_k \) has no torsion, rank 0, Weierstrass equation

\[
y^2 = x^3 + 2m((-s^4 + 10s^2 - 1)m + 2s^4(s^2 + 1)^2)x^2 \\
+ (m-4s^4) m^3 \left( (s^8 - 20s^6 - 90s^4 - 20s^2 + 1) \right) x + 256m^5 s^2 (m-4s^4)^2
\]

and Néron-Severi group \( \text{NS}(\Sigma_k) = U \oplus E_8 \oplus D_6 \oplus A_2 \oplus A_1 \).

By Morrison ([M], Corollary 2.10 ii), the Néron-Severi group of an algebraic \( K3 \) surface \( X \) with \( 12 \leq \rho(X) \leq 20 \) is uniquely determined by its signature and discriminant form. Thus we compute \( q_{\text{NS}(S_k)} \) with the help of the fibration \( \Sigma_k \).

\[
D_6^*/D_6 = [(1)D_6, [3]D_6] \quad \text{and} \quad q_{D_6}([1]D_6) = q_{D_6}([3]D_6) = -\frac{3}{2}.
\]
we deduce the discriminant form, since \(b_D([1]_{D_6}, [3]_{D_6}) = 0\),
\[
(G_{NS(S_k)}, q_{NS(S_k)}) = \mathbb{Z}/22\langle -\frac{3}{2} \rangle \oplus \mathbb{Z}/2\mathbb{Z}\langle -\frac{3}{2} \rangle \oplus \mathbb{Z}/3\mathbb{Z}\langle -\frac{1}{2} \rangle \mod 2\mathbb{Z}
\]
\[
= \mathbb{Z}/22\langle \frac{1}{2} \rangle \oplus \mathbb{Z}/6\mathbb{Z}\langle -\frac{1}{6} \rangle \oplus \mathbb{Z}/2\mathbb{Z}\langle -\frac{1}{2} \rangle.
\]
From Morrison [M] Theorem 2.8 and Corollary 2.10 there is a unique primitive embedding of \(NS(S_k)\) into the K3-lattice \(E_6(-1) \oplus U^3\), whose orthogonal is by definition the transcendental lattice \(T_{S_k}\). Now from Nikulin [Nik] Proposition 1.6.1, it follows
\[
G_{NS(S_k)} \cong (G_{NS(S_k)})^\perp = \langle G_{T_{S_k}}, q_{T_{S_k}} \rangle = -q_{NS(S_k)}.
\]
In other words the discriminant form of the transcendental lattice is
\[
(G_{T_{S_k}}, q_{T_{S_k}}) = \mathbb{Z}/22\langle -\frac{1}{2} \rangle \oplus \mathbb{Z}/6\mathbb{Z}\langle \frac{1}{6} \rangle \oplus \mathbb{Z}/2\mathbb{Z}\langle \frac{1}{2} \rangle.
\]
From this last relation we prove that \(T_{S_k} = \langle -2 \rangle \oplus \langle 6 \rangle \oplus \langle 2 \rangle\). Denoting \(T'\) the lattice \(T' = \langle -2 \rangle \oplus \langle 6 \rangle \oplus \langle 2 \rangle\), we observe that \(T'\) and \(T_{S_k}\) have the same signature and discriminant form. Since \(|\det(T')| = 24\) is small, there is only one equivalence class of forms in a genus, meaning that such a transcendental lattice is, up to isomorphism, uniquely determined by its signature and discriminant form (KKN p. 395).

Now computing a primitive embedding of \(T_{S_k}\) into \(\Lambda\), since by Morrison [M] Corollary 2.10 i) this embedding is unique, its orthogonal provides \(NS(S_k)\). Take the primitive embedding \(\langle -2 \rangle = (e_2) \mapsto E_8\), \(\langle 2 \rangle = (\langle u_1 + u_2 \rangle \mapsto U, \langle 6 \rangle = \langle u_1 + 3u_2 \rangle \mapsto U\), \((u_1, u_2)\) denoting a basis of \(U\). Hence we deduce
\[
NS(S_k) = U \oplus E_8(-1) \oplus E_7(-1) \oplus (-2) \oplus (-6).
\]
Using their Weierstrass equations and a 2-neighbor method [El], it was proved in the previous subsection that all the fibrations \#7-i, \#9-i, \#14-i, \#15-i, \#20-i are on the same K3-surface. We can recover this result, since we know the transcendental lattice, using the Kneser-Nishiyama method.

In that purpose, embed \(T_{S_k}(-1)\) into \(U \oplus E_8\) in the following way: \((-2) \oplus (-6)\) primitively embedded in \(E_8\) as in Nishiyama [Ni] p. 334 and \(\langle 2 \rangle = \langle u_1 + u_2 \rangle \mapsto U\). We obtain \(M = (T_{S_k}[-1])^\perp_{U \oplus E_8} = A_1 \oplus A_1 \oplus A_5\). Now all the elliptic fibrations of \(S_k\) are obtained from the primitive embeddings of \(M\) into the various Niemier lattices, as explained in section 2.

We identify some of these elliptic fibrations with fibrations \#7-i, \#9-i, \#14-i, \#15-i, \#20-i in exhibiting their torsion and infinite sections as explained in Bertin-Lecacheux [BL1], computing contributions and heights using [Sc-Shio] p. 51-52. Using the Weierstrass equations given in Table 5.2.1, we compute the different local contributions and heights. Finally the identification is performed using [Sc-Shio] (11.9).

5.2.1. Take the primitive embedding into \(Ni(D_{10}E_7^2)\), given by \(A_5 = \langle e_2, e_4, e_6, e_7 \rangle \mapsto E_7\) and \(A_7^2 = \langle d_{10}, d_7 \rangle \mapsto D_{10}\).

Since \((A_3)^2_{D_1} = A_2\) and \((A_7^2)^2_{D_{10}} = A_1 \oplus A_1 \oplus D_6\), it follows \(N = N_{\text{root}} = 2A_1A_2D_6E_7\), det \(N = 24 \times 4\), thus the rank is 0 and the torsion group \(\mathbb{Z}/2\mathbb{Z}\). Hence this fibration can be identified with the elliptic fibration \#7-i.

5.2.2. The primitive embedding is into \(Ni(D_{10}E_7^2)\), given by
\[
A_5 \oplus A_7^2 = \langle d_{10}, d_8, d_7, d_6, d_5, d_{10} + d_9 + 2(d_8 + d_7 + d_6 + d_5 + d_4) + d_3, d_4 \rangle \mapsto D_{10}.
\]
We get
\[
(A_5 \oplus A_7^2)^2_{D_{10}} = \langle -6 \rangle \oplus \langle x \rangle \oplus \langle d_4 \rangle = \langle -6 \rangle \oplus A_1 \oplus A_1
\]
with
\[
x = d_9 + d_{10} + 2(d_8 + d_7 + d_6 + d_5 + d_4 + d_3 + d_2) + d_1
\]
and
\[
\langle -6 \rangle = 3d_9 + 2d_{10} + 4d_8 + 3d_7 + 2d_6 + d_5.
\]
Thus \( N_{\text{root}} = A_1 A_4 E_7^2 \) and the rank of the fibration is 1. Since \( 2[2]_{D_{10}} = x + d_4 \) and there is no other relation with \([1]_{D_{10}}\) or \([3]_{D_{10}},\) among the glue vectors \([\langle 1, 1, 0 \rangle, \langle 3, 0, 1 \rangle]\) generating \( Ni(D_{10}E_7^2),\) only \([\langle 2, 1, 1 \rangle]\) contributes to torsion.

Hence the torsion group is \( \mathbb{Z}/2\mathbb{Z}.\) Moreover the 2-torsion section is

\[
2F + 0 + [\langle 2, 1, 1 \rangle]
\]

with height \( 4 - (1/2 + 1/2 + 3/2 + 3/2) = 0.\) The infinite section is

\[
3F + 0 + [\langle -6, 0, 0 \rangle]
\]

with height 6. Hence this fibration can be identified with the fibration \#9-i.

5.2.3. The primitive embedding is into \( Ni(D_8^3),\) given by \( A_5 = \langle d_8, d_6, d_5, d_4, d_3 \rangle \hookrightarrow D_8^{(1)} \) and \( A_1 = \langle d_8, d_4 \rangle \hookrightarrow D_8^{(3)}.\) We compute \( (A_5)_{D_8}^\perp = (\langle -6 \rangle \oplus \langle x_1 = (-2) \rangle) \oplus \langle d_1 \rangle \) with \( x_1 = d_7 + d_8 + 2(d_6 + d_5 + d_4 + d_3 + d_2 + d_1) \)

\[
(A_1)_{D_8}^\perp = \langle d_7 \rangle \oplus \langle x_1 = d_7 + d_8 + 2(d_6 + d_5 + d_4 + d_3 + d_2 + d_1) \rangle
\]

\[
\oplus \langle d_5, d_4, x_3 = d_7 + d_8 + 2d_6 + d_5, d_3 \rangle = A_1 \oplus A_1 \oplus D_4.
\]

We deduce \( N_{\text{root}} = 4A_1 D_4 D_8\) (hence the fibration has rank 1) and the relations

(1) \( 2[2]_{D_8} = x_1 + d_1 \)

(2) \( 2(2)_{D_8} - (d_1 + d_2) = x_3 + 2d_3 + 2d_4 + d_5 \)

(3) \( 2[3]_{D_8} = x_1 + 2x_3 + d_3 + 2d_4 + d_5 + d_7 \)

(4) \( 2(1)_{D_8} - (d_6 + d_8) = x_1 + x_3 + d_3 + 2d_4 + 2d_5 + d_7 \)

(5) \( 2(1)_{D_8} - (d_6 + d_8) = x_1 + x_3 + d_3 + 2d_4 + 2d_5 + d_7 \)

Thus, among the glue vectors \([\langle 1, 2, 2 \rangle, \langle 1, 1, 1 \rangle, \langle 2, 1, 2 \rangle]\) generating the Niemeier lattice, only vectors \([\langle 0, 3, 3 \rangle, \langle 2, 1, 2 \rangle]\) contribute to torsion and the torsion group is \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.

From relations (1) to (5) we deduce the various contributions and heights of the following sections (see Table 5.1.3).

Hence this fibration can be identified with the fibration \#14-i.

5.2.4. The primitive embedding is into \( Ni(D_8^3),\) and given by

\[
A_5 = \langle d_8, d_6, d_5, d_4, d_3 \rangle \hookrightarrow D_8^{(1)} \quad A_1 = \langle d_8 \rangle \hookrightarrow D_8^{(2)} \quad A_1 = \langle d_8 \rangle \hookrightarrow D_8^{(3)}.
\]

As previously \( (A_5)_{D_8}^\perp = (\langle -6 \rangle \oplus \langle x_1 \rangle) \oplus \langle d_1 \rangle;\) we get also \( (d_8)_{D_8}^\perp = \langle d_7 \rangle \oplus \langle x_4 = d_7 + d_8 + 2d_6 + d_5, d_4, d_3, d_2, d_1 \rangle = A_1 \oplus D_6.\) Hence \( N_{\text{root}} = 4A_1 2D_6,\) and the rank is 1. Moreover it follows the relations

(6) \( 2[2]_{D_8} = x_1 + d_1 \)

(7) \( 2[2]_{D_8} = x_3 + d_5 + 2d_4 + 2d_3 + 2d_2 + 2d_1 \)

(8) \( 2(1)_{D_8} - (d_5 + d_6 + d_7 + d_8) = 2x_3 + d_5 + 4d_4 + 3d_3 + 2d_2 + d_1 - d_7 \)

(9) \( 2[3]_{D_8} = 3x_3 + d_7 + 2d_5 + 4d_4 + 3d_3 + 2d_2 + d_1 \in A_1 \oplus D_6.\)
The elliptic curve 
\[ E \]
specialisation from the Shioda-Inose structure of the family. (see 4.1 Remark 4.1).

\[ j \]
equation of

\[ K \]
We deduce

\[ Ni \]
Proposition 6.1.
The elliptic fibrations

\[ l \]
satisfying the equation

\[ torsion. \] So the torsion group is

\[ Z \]
5.2.5. The primitive embedding is onto

\[ H \]
Hence this fibration can be identified with the fibration #15-i.

Comparing to the fibrations of the family you remark more elliptic fibrations with 2-torsion sections
notations used in Bertin-Lecacheux [BL].

\[ − \]
−

\[ #17(18 \]
−

\[ τ \]

given in Beauville's paper [Beau] (x + y)(y + z)(z + x)(t − s)(ts − 1) = 8sxyz.

6. Proof of Theorem 1.2

We recall first on Table [BL] the results obtained by Bertin and Lecacheux in [BL]. The notation #17(18 − m) for example refers for #17 to the generic case when relevant and for (18 − m) to notations used in Bertin-Lecacheux [BL].

Comparing to the fibrations of the family you remark more elliptic fibrations with 2-torsion sections on Y_2. All the corresponding involutions are denoted \( τ \). Some of them are specialisations for \( s = 1 \) of the generic ones. Those generic which are Morrison-Nikulin still remain Morrison-Nikulin for Y_2 by a Schütt’s lemma [Sc], namely #4 − τ, #8 − τ, #16 − τ, #17 − τ, #23 − τ, #24a − τ, #26 − τ. Others ((#5 − τ, #8bis − τ, #10 − τ, #15 < (p, 0 >) − τ, #24b − τ, #24c − τ) are specific to K_2 and cannot be deduced from elliptic fibrations of the generic Kummer. To identify them we have to use the distinguished property of Y_2, that is Y_2 is a singular K3 with Picard number 20.

Hence Y_2 inherits of a Shioda-Inose structure, that is the quotient of Y_2 by an involution is isomorphic to a Kummer surface K_2 realized from the product of CM elliptic curves [Sh], [SM] provided in the following way.

Since the transcendental lattice of Y_2 is

\[ T(Y_2) = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} \]

we get

\[ b^2 − 4ac = −8, \tau_1 = \frac{−b + \sqrt{b^2−4ac}}{2a}, \tau_2 = \frac{b + \sqrt{b^2−4ac}}{2a}, \] hence \( \tau_1 = \tau_2 = i\sqrt{2} \).

We deduce K_2 = E x E/±1 with E = \( \mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z}) \). Since \( j(i\sqrt{2}) = 8000 \) [C], we obtain a Weierstrass equation of E, namely

\[ Y^2 = X(X^2 + 4X + 2), \]

which endows K_2 with an elliptic fibration. The fact that the two CM elliptic curves are equal and satisfy \( j(E) = 8000 \) can be obtained also by specialisation from the Shioda-Inose structure of the family. (see [4.1] Remark 4.1).

The elliptic curve E can be also put in the Legendre form:

\[ E \]
y^2 = x(x − 1)(x − l),

\[ l \]
satisfying the equation

\[ j = 8000 = \frac{256(k − (l) + l)}{k^2(l − 1)^2}. \] Thus \( l = 3 ± 2\sqrt{2} \) or \( l = −2 ± 2\sqrt{2} \) or \( l = \frac{1 + \sqrt{2}}{2} \).

**Proposition 6.1.** The elliptic fibrations #8bis − τ and #10 − τ are elliptic fibrations of K_2.

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<thead>
<tr>
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<th></th>
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<tbody>
<tr>
<td>[0,3],[3]</td>
<td>0</td>
<td>0</td>
<td>1/2</td>
<td>1+1/2</td>
<td>1/2</td>
<td>1+1/2</td>
<td>0</td>
</tr>
<tr>
<td>([2],[1]−(d_1+d_4,d_4,d_3)]</td>
<td>1/2</td>
<td>1/2</td>
<td>1/2</td>
<td>1+1/2</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>([2],[2],[1]−(d_1+d_4,d_4,d_3)]</td>
<td>1/2</td>
<td>1/2</td>
<td>0</td>
<td>1</td>
<td>1/2</td>
<td>1+1/2</td>
<td>0</td>
</tr>
<tr>
<td>([3],[0],[3])</td>
<td>0</td>
<td>1/2</td>
<td>0</td>
<td>0</td>
<td>1/2</td>
<td>1+1/2</td>
<td>3/2</td>
</tr>
</tbody>
</table>

**Table 8.** Contributions and heights of the sections of 5.1.4
\( L_{\text{root}} \) & \( L/L_{\text{root}} \) & Fibers & \( R \) & Tor. \\
\hline
\( E_8 \) & (0) & & & \\
\hline
#1(11 - \( f \)) & \( A_1 \subset E_8 \) & \( D_5 \subset E_8 \) & \( E_7A_1E_8 \) & 0 (0) \\
#2(13 - h) & \( A_1 \oplus D_5 \subset E_8 \) & \( A_1E_8A_1 \) & 1 (0) \\
\hline
\( E_8D_{16} \) & \( \mathbb{Z}/2\mathbb{Z} \) & & & \\
\hline
#3(30 - \( \phi \)) & \( A_1 \subset E_8 \) & \( D_5 \subset D_{16} \) & \( E_7D_{11} \) & 0 (0) \\
#4(16 - o) & \( A_1 \oplus D_5 \subset E_8 \) & \( A_1D_{16} \) & 1 \( \mathbb{Z}/2\mathbb{Z} \) \\
#5(17 - q) & \( D_5 \subset E_8 \) & \( A_1 \subset D_{16} \) & \( A_1A_1D_{14} \) & 0 \( \mathbb{Z}/2\mathbb{Z} \) \\
#6(25 - \( d \)) & \( A_1 \oplus D_5 \subset D_{16} \) & \( E_8A_1D_9 \) & 0 (0) \\
\hline
\( E_6^4D_{10} \) & \( \mathbb{Z}/2\mathbb{Z}^4 \) & & & \\
\hline
#7(29 - \( b \)) & \( A_1 \subset E_7 \) & \( D_5 \subset D_{10} \) & \( E_7D_5D_5 \) & 0 \( \mathbb{Z}/2\mathbb{Z} \) \\
#8(9 - \( r \)) & \( A_1 \subset E_7 \) & \( D_5 \subset E_7 \) & \( D_6A_1D_{10} \) & 1 \( \mathbb{Z}/2\mathbb{Z} \) \\
#8bis(24 - \( \psi \)) & \( A_1 \oplus D_5 \subset E_7 \) & \( E_7D_{10} \) & 1 \( \mathbb{Z}/2\mathbb{Z} \) \\
#9(12 - \( g \)) & \( A_1 \oplus D_5 \subset D_{10} \) & \( E_7E_1A_3 \) & 0 \( \mathbb{Z}/2\mathbb{Z} \) \\
#10(10 - \( e \)) & \( D_5 \subset E_7 \) & \( A_1 \subset D_{10} \) & \( A_1A_1D_7E_7 \) & 1 \( \mathbb{Z}/2\mathbb{Z} \) \\
\hline
\( E_7A_{17} \) & \( \mathbb{Z}/6\mathbb{Z} \) & & & \\
\hline
#(21 - c) & \( A_1 \oplus D_5 \subset E_7 \) & \( A_{17} \) & 1 \( \mathbb{Z}/3\mathbb{Z} \) \\
#11(19 - \( n \)) & \( D_5 \subset E_7 \) & \( A_1 \subset A_{17} \) & \( A_{1}A_{15} \) & 2 (0) \\
\hline
\( D_{24} \) & \( \mathbb{Z}/2\mathbb{Z} \) & & & \\
#12(13 - \( i \)) & \( A_1 \oplus D_5 \subset D_{24} \) & \( A_1D_{17} \) & 0 (0) \\
\hline
\( D_{72}^+ \) & \( \mathbb{Z}/2\mathbb{Z} \) & & & \\
\hline
#13(26 - \( \pi \)) & \( A_1 \subset D_{12} \) & \( D_5 \subset D_{12} \) & \( A_1D_{10}D_7 \) & 0 \( \mathbb{Z}/2\mathbb{Z} \) \\
#14(22 - \( a \)) & \( A_1 \oplus D_5 \subset D_{12} \) & \( A_1D_9D_{12} \) & 0 \( \mathbb{Z}/2\mathbb{Z} \) \\
\hline
\( D_{24}^+ \) & \( \mathbb{Z}/2\mathbb{Z} \) & & & \\
\hline
#15(6 - \( p \)) & \( A_1 \subset D_9 \) & \( D_5 \subset D_8 \) & \( A_1D_6A_4D_8 \) & 0 \( \mathbb{Z}/2\mathbb{Z} \) \\
#16(14 - \( t \)) & \( A_1 \oplus D_5 \subset D_8 \) & \( A_1D_6D_8 \) & 1 \( \mathbb{Z}/2\mathbb{Z} \) \\
\hline
\( D_{9A_{15}} \) & \( \mathbb{Z}/8\mathbb{Z} \) & & & \\
\hline
#17(18 - \( m \)) & \( A_1 \oplus D_5 \subset D_9 \) & \( A_1A_1A_1A_{15} \) & 0 \( \mathbb{Z}/4\mathbb{Z} \) \\
#18(28 - \( a \)) & \( D_5 \subset D_9 \) & \( A_1 \subset A_{17} \) & \( D_2A_{13} \) & 1 (0) \\
\hline
\( E_6 \) & \( \mathbb{Z}/3\mathbb{Z} \) & & & \\
#19(8 - \( b \)) & \( A_1 \subset E_6 \) & \( D_5 \subset E_6 \) & \( A_6E_6E_6 \) & 1 \( \mathbb{Z}/3\mathbb{Z} \) \\
\hline
\( A_{11}E_6D_{17} \) & \( \mathbb{Z}/12\mathbb{Z} \) & & & \\
\hline
#20(7 - \( w \)) & \( A_1 \subset E_6 \) & \( D_5 \subset D_7 \) & \( A_5A_1A_1A_{11} \) & 0 \( \mathbb{Z}/6\mathbb{Z} \) \\
#21(27 - \( \mu \)) & \( A_1 \subset A_{11} \) & \( D_5 \subset D_7 \) & \( A_5A_1A_1E_6 \) & 1 (0) \\
#22(15 - \( i \)) & \( A_1 \subset A_{11} \) & \( D_5 \subset E_6 \) & \( A_6D_7 \) & 2 (0) \\
#23(2 - \( k \)) & \( D_5 \subset E_6 \) & \( A_1 \subset D_7 \) & \( A_1A_1A_6 \) & 1 \( \mathbb{Z}/4\mathbb{Z} \) \\
\hline
\( D_{44}^+ \) & \( \mathbb{Z}/2\mathbb{Z} \) & & & \\
\hline
#24(5 - \( d \)) & \( A_1 \subset D_6 \) & \( D_5 \subset D_6 \) & \( A_1D_4D_4D_6 \) & 1 \( \mathbb{Z}/2\mathbb{Z} \) \\
\hline
\( D_{6A_5}^+ \) & \( \mathbb{Z}/2 \times \mathbb{Z}/10 \) & & & \\
#25(3 - \( v \)) & \( D_5 \subset D_6 \) & \( A_1 \subset A_9 \) & \( A_7A_9 \) & 2 (0) \\
\hline
\( D_{44}^2A_4^3 \) & \( \mathbb{Z}/4 \times \mathbb{Z}/2 \) & & & \\
\hline
#26(1 - \( s \)) & \( D_5 \subset D_7 \) & \( A_1 \subset D_7 \) & \( A_1A_4A_7A_7 \) & 0 \( \mathbb{Z}/8\mathbb{Z} \) \\
#27(4 - \( a \)) & \( D_5 \subset D_7 \) & \( A_1 \subset A_7 \) & \( D_2A_5A_7 \) & 1 (0) \\
\hline

Table 9. The elliptic fibrations of \( Y_2 \)

**Proof.** It follows from the 4.2 fibration \( f_8 \) that the fibration \#10 - \( \tau \) with Weierstrass equation

\[
Y^2 = X^3 - 2U^2(U - 1)X^2 + U^3(U + 1)^2(U - 4)X,
\]

singular fibers \( III'*(0) \), \( I_5^*(\infty) \), \( I_4(-1) \), \( I_2(4) \), \( I_1(-1/2) \), and \( \mathbb{Z}/2\mathbb{Z} \)-torsion is an elliptic fibration of \( K_2 \). Similarly from the 4.2 fibration \( g_8 \), we deduce that the elliptic fibration \#10 - \( \tau \) with Weierstrass equation

\[
Y^2 = X^3 + 2(t + 5t^2)X^2 + t^3(4t + 1)(t^2 + 6t + 1)X,
\]

singular fibers \( III'*(\infty) \), \( I_5^*(0) \), \( 3I_2(-1/4, t^2 + 6t + 1 = 0) \), and \( \mathbb{Z}/2\mathbb{Z} \)-torsion is an elliptic fibration of \( K_2 \).

To achieve the proof of Theorem 1.2 we need also the following lemma.
Lemma 6.1. The Kummer $K_2$ has exactly 4 extremal elliptic fibrations given by Shimada Zhang [SZ] with the type of their singular fibers and their torsion group

(1) $E_7 A_7 A_3 A_1 Z/2Z$,
(2) $D_9 A_7 A_1, Z/2Z$,
(3) $D_6 D_5 A_7, Z/2Z$,
(4) $A_7 A_3 A_3 A_1 A_1, Z/2Z \times Z/4Z$.

From lemma (6.1 (2)) we obtain the fibration #5 - $\tau$ and from lemma (6.1 (3)) the fibration #15 < $(p, 0) > - \tau$.

We notice also that fibrations #17 - $\tau$ and #26 - $\tau$ obtained by specialisation are also fibration (4) of lemma (6.1) and fibration #23 - $\tau$, by a 2-neighbor process of parameter $m = X_{(g(t+1))}$ gives fibration (3) of lemma (6.1).

Finally, by a 2-neighbor process of parameter $m = X_{(g(t+1))}$, fibrations #24b - $\tau$ and #24c - $\tau$ gives fibration #16 - $\tau$, hence are elliptic fibrations of $K_2$.

Corollary 6.1. As a byproduct of the proof we get Weierstrass equations for extremal fibrations of lemma (6.7), (2), (3), (4).

The table of symplectic automorphisms of order 2 (self involutions) results from an easy computation.

7. 2-ISOGENIES AND ISOMETRIES

Theorem 1.2, where the 2-isogenous $K3$ surfaces of $Y_2$ are either its Kummer $K_2$ or $Y_2$ itself, cannot be generalised to all the other singular $K3$ surfaces of the Apéry-Fermi family. The reason is the relation with a Theorem of Boissière, Sarti and Veniani [BSV], telling when $p$-isogenies ($p$ prime) between complex projective $K3$ surfaces $X$ and $Y$ define isometries between their rational transcendental lattices $T_{X, Q}$ and $T_{Y, Q}$. (These lattices are isometric if there exists $M \in \text{Gl}(n, Q)$ satisfying $T_{X, Q} = M^t T_{Y, Q} M$. Let us recall the part of their Theorem related to 2-isogenies.

Theorem 7.1. [BSV] Let $\gamma : X \to Y$ be a 2-isogeny between complex projective $K3$ surfaces $X$ and $Y$. Then $\text{rk}(T_{Y, Q}) = \text{rk}(T_{X, Q}) =: r$ and

(1) If $r$ is odd, there is no isometry between $T_{Y, Q}$ and $T_{X, Q}$.
(2) If $r$ is even, there exists an isometry between $T_{Y, Q}$ and $T_{X, Q}$ if and only if $T_{Y, Q}$ is isometric to $T_{Y, Q}(2)$. This property is equivalent to the following: for every prime number $q$ congruent to 3 or 5 modulo 8, the $q$-adic valuation $\nu_q(\text{det} T_Y)$ is even.

As a Corollary we deduce the following result.

Theorem 7.2. Among the singular $K3$ surfaces of the Apéry-Fermi family defined for $k$ rational integer, only $Y_2$ and $Y_{198}$ possess symplectic automorphisms of order 2 ("self 2-isogenies").

Proof. The singular $K3$-surfaces of the Apéry-Fermi family defined for $k$ rational integer are

$$Y_0, \quad Y_2, \quad Y_3, \quad Y_6, \quad Y_{10}, \quad Y_{18}, \quad Y_{102}, \quad Y_{198}.$$ 

This list has been computed numerically by Boyd [Boy]. Using the notation [SZ], that is writing the transcendental lattice $T_Y = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ as $T_Y = [a \ b \ c]$ we get:

$$T_{Y_6} = [4 \ 2 \ 4] \quad T_{Y_2} = [2 \ 0 \ 4] \quad T_{Y_6} = [2 \ 0 \ 12].$$

They are obtained by specialisation of fibration #20 for $k = 0$, 2 and 6. For $k = 0$ the elliptic fibration has rank 0 and singular fibers of type I$_{12}$, I$_4$, 2I$_3$. For $k = 2$, the transcendental lattice is already known. For $k = 6$, the elliptic fibration has rank 0 and type of singular fibers I$_{12}$, I$_4$, 2I$_2$. Now using Shimada-Zhang table [SZ], we derive the previous announced transcendental lattices.

The transcendental lattices $T_{Y_3}$ and $T_{Y_{18}}$ were computed in the paper [BFLLM]. With the method used there, we can compute the transcendental lattices of $Y_{10}$, $Y_{102}$ and $Y_{198}$. We obtain:

$$T_{Y_3} = [2 \ 1 \ 8] \quad T_{Y_{10}} = [6\text{quad}; 0 \ 12] \quad T_{Y_{18}} = [10 \ 0 \ 12].$$
Weierstrass Equation

<table>
<thead>
<tr>
<th>No</th>
<th>From or to</th>
</tr>
</thead>
<tbody>
<tr>
<td>#4</td>
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</tr>
<tr>
<td>#5</td>
<td>lemma6.1(2)</td>
</tr>
<tr>
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<td>Spec.#8 - i</td>
</tr>
<tr>
<td>#8bas</td>
<td>lemma6.1(3)</td>
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<tr>
<td>#10</td>
<td>Spec.#16 - i</td>
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<tr>
<td>#15</td>
<td>Spec.#17 - i</td>
</tr>
<tr>
<td>#16</td>
<td>lemma6.1(4)</td>
</tr>
<tr>
<td>#23</td>
<td>lemma6.1(3)</td>
</tr>
<tr>
<td>#24</td>
<td>Spec.#24 - i</td>
</tr>
<tr>
<td>#26</td>
<td>Spec.#26 - i</td>
</tr>
</tbody>
</table>

Table 10. Morrison-Nikulin involutions of $Y_2$ (fibrations of $K_2$)

\[
T_{Y_{102}} = \begin{bmatrix} 12 & 0 & 26 \\ \end{bmatrix} \quad T_{Y_{128}} = \begin{bmatrix} 12 & 0 & 34 \end{bmatrix}.
\]

Applying Bessière, Sarti and Veniani’s Theorem, we conclude that only $Y_2$ and $Y_{10}$ may have self isogenies. By Theorem 1.2, $Y_2$ has self isogenies. We shall prove that $Y_{10}$ satisfies the same property.
Weierstrass Equation

<table>
<thead>
<tr>
<th>No</th>
<th>Proposition 7.1. Even if the</th>
</tr>
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<tbody>
<tr>
<td>#7</td>
<td>( Y^2 = x^3 - 2\beta \beta'\beta^{-1}x^2 + \beta'\beta^{-1}x )</td>
</tr>
<tr>
<td></td>
<td>( Y^2 = X^3 - 4\beta'\beta^{-1}(2\beta - 1)X^2 + 4\beta'\beta^{-1}(\beta - 1)X )</td>
</tr>
<tr>
<td></td>
<td>( Y^2 = x^3 + 4\beta'\beta^{-1}x^2 + \beta'\beta^{-1}(y + 1)x )</td>
</tr>
<tr>
<td></td>
<td>( Y^2 = X^3 - 8\beta'\beta^{-1}X^2 - 4\beta'\beta^{-1}X )</td>
</tr>
<tr>
<td></td>
<td>( Y^2 = X^3 - 2\beta'\beta^{-1}x^2 + \beta'\beta^{-1}(\beta - 2) + \beta'\beta^{-1}(2\beta + 1)x )</td>
</tr>
<tr>
<td>#9</td>
<td>( Y^2 = X^3 - 2\beta'\beta^{-1}x^2 + \beta'\beta^{-1}(\beta - 2) + \beta'\beta^{-1}(\beta - 4)x )</td>
</tr>
<tr>
<td>#13</td>
<td>( Y^2 = X^3 - u(2u - 2)X(X - 4u) )</td>
</tr>
<tr>
<td>#14</td>
<td>( Y^2 = x(x - p)(x - p + 1)^2 )</td>
</tr>
<tr>
<td>#15</td>
<td>( Y^2 = X(X + 4p + p^3 + 4p^3)(X + p^3) )</td>
</tr>
<tr>
<td>#20</td>
<td>( Y^2 = X^3 - 2\beta'\beta^{-1}(p(2\beta + 1)X^2 + p^3X) )</td>
</tr>
</tbody>
</table>

Table 11. Self involutions of \( Y_2 \)

Consider the following elliptic fibration of rank 0 of \( Y_{10} \) (other interesting properties of \( Y_{10} \) will be studied in a forthcoming paper):

\[ y^2 = x^3 + x^2(9t + 5)(t + 3) + (t + 9)^2 - xt^5(t + 5)^2 \]

with singular fibers \( 3I^3(\infty), I_0(4), I_0(-5), I_3(-9), I_2(-4) \) and 2-torsion. Its 2-isogenous curve has a Weierstrass equation

\[ Y^2 = X^3 + X^2(-20t^2 - 180t - 432) + 4X(t + 4)^2(t + 9)^3 \]

with singular fibers \( 3I^3(\infty), I_0(-9), I_4(-4), I_3(0), I_2(-5) \), rank 0 and 2-torsion. Hence this 2-isogeny defines an automorphism of order 2 of \( Y_{10} \) given by \( x = -\frac{3}{2}, y = \frac{3}{2\sqrt{2}} \).

Moreover we observe that

\[ T_{Y_2} = [2 0 4], \quad T_{Y_{2},Q} = [2 0 1], \]
\[ T_{K_2} = [4 0 8], \quad T_{K_{2,Q}} = [2 0 1], \]

Similarly

\[ T_{Y_{10},Q} = [6 0 3], \quad T_{K_{10},Q} = [3 0 6]. \]

Hence we suspect some relations between the transcendental lattices of \( K_Y \) and of \( S_Y \) for singular \( Y_i \). We give some examples of such relations in the following proposition.

**Proposition 7.1.** Even if the 2-isogenies from \( Y_0, Y_6 \) are not isometries, the following rational transcendental lattices satisfy the relations

1. \( T_{K_0,Q} = T_{S_0,Q} \),
2. \( T_{K_{6,Q}} = T_{S_{6,Q}} \).
(3) \( \det(T_{K_3}) = \det(T_{S_3}). \)

**Proof.** (1) For \( k = 0 \) we get two elliptic fibrations of rank 0, namely \#20 and \#8. The fibration \#8-i gives a rank 0 elliptic fibration of \( K_0 \) with Weierstrass equation
\[
y^2 = x^3 + 2x^2(t^3 + 1) + x(t - 1)^2(t^2 + t + 1)^2,
\]
with type of singular fibers \( D_2, 3A_3, A_2, 4 \)-torsion and \( T_{K_0} = [8 4 8] \). On the other end the fibration \#20-i gives a rank 0 elliptic fibration of \( S_0 \)
\[
y^2 = x(x - \frac{1}{4}(t - 3I)(t + I)^3)(x - \frac{1}{4}(t + 3I)(t - I)^3)
\]
with type of singular fibers \( 3A_3 \) or \( 3A_1 \) \((\infty, \pm I)\), \( \mathbb{Z}/2 \times \mathbb{Z}/6 \)-torsion, the \( 3 \)-torsion points being \((\frac{1}{4}(t^2 + 1)^2, \pm \frac{1}{2}(t^2 + 1)^2)\). Hence, by Shimada-Zhang’s list \( T_{S_0} = [2 0 6] \). Now we can easily deduce the relation
\[
\begin{pmatrix}
1/2 & 0 \\
1/2 & -1
\end{pmatrix}
\begin{pmatrix}
8 & 4 \\
4 & 8
\end{pmatrix}
\begin{pmatrix}
1/2 & 1/2 \\
0 & -1
\end{pmatrix} =
\begin{pmatrix}
2 & 0 \\
0 & 6
\end{pmatrix}.
\]

(2) For \( k = 6 \) the elliptic fibration \#20 has rank 0 and \#20 - \( i \) gives a rank 0 elliptic fibration of \( S_6 \):
\[
y^2 = x^3 + x^2(-\frac{t^4}{2} + 6t^3 - 21t^2 + 18t + \frac{3}{2}) + x\frac{(t - 3)^2}{16}(t^2 - 6t + 1)^3,
\]
with singular fibers \( 2I_4(t^2 - 6I + 1 = 0), I_4(\infty), I_4(3), 2I_1(0, 6), \) and \( \mathbb{Z}/6\mathbb{Z} \)-torsion. Using Shimada-Zhang’s list \( [SZ] \), we find \( T_{S_6} = [4 0 6] \). Since
\[
T_{K_6} = \begin{pmatrix} 4 & 0 \\ 0 & 24 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}
\]
and
\[
T_{S_6} = \begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}
\]
we get straightforward
\[
T_{K_6, \mathbb{Q}} = T_{S_6, \mathbb{Q}}.
\]

(3) Consider the elliptic fibration \#20 of \( Y_3 \) with Weierstrass equation
\[
y^2 = x^3 + \frac{1}{4}(t^4 - 6t^3 + 15t^2 - 18t - 3)x^2 - t(t - 3)x,
\]
singular fibers \( I_4(\infty), 2I_2(t^2 - 3I + 1 = 0), 2I_2(0, 3), 2I_2(t^2 - 3I + 9 = 0), \) rank 1 and 6-torsion. The infinite section \( P_3 = (t, -\frac{1}{2}(t^2 - 3I + 3)) \), of height \( \frac{5}{4} \) generates the free part of the Mordell-Weil group, since \( \det(T_{Y_3}) = 15 \) by the previous theorem and by the Shioda-Tate formula
\[
\det(T_{Y_3}) = \frac{512 \times 3^2 \times 2^2}{6^2} = 15.
\]
Its 2-isogenous curve has Weierstrass equation
\[
y^2 = x^3 + (-\frac{1}{2}t^4 + 3t^3 - \frac{15}{2}t^2 + 9t + \frac{3}{2})x^2 + \frac{1}{16}(t^2 - 3t + 9)(t^2 - 3t + 1)x,
\]
singular fibers \( 3I_4(\infty, t^2 - 3I + 1 = 0), 2I_2(t^2 - 3I + 9 = 0), 2I_1(3, 0), \) rank 1 and 6-torsion. The section \( Q_3 \) image by the 2-isogeny of the infinite section \( P_3 \) is an infinite section of height \( \frac{5}{4} \). Since neither \( Q_{1i} \) nor \( Q_{1i} + (0, 0) \) are 2-divisible, the section \( Q_3 \) generates the free part of the Mordell-Weil group. Hence by the Shioda-Tate formula, it follows
\[
\det(T_{S_3}) = \frac{5 \times 6^3 \times 2^2}{2 \times 6^2} = 60 = \det(T_{K_3}).
\]
\( \square \)
**Remark 7.1.** The Kummer surface $K_0$ is nothing else than the Schur quartic \[BSV\] (section 6.3) with equation
\[
x^4 - xy^3 = z^4 - zt^3.
\]

**References**


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