APÉRY-FERMI PENCIL OF $K3$-SURFACES AND THEIR 2-ISOGENIES

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1. Introduction

The Apéry-Fermi pencil $\mathcal{F}$ is realised with the equations

$$X+\frac{1}{X}+Y+\frac{1}{Y}+Z+\frac{1}{Z}=k, \quad k\in\mathbb{Z},$$

and taking $k=s+\frac{1}{s}$, is seen as the Fermi threefold $\mathcal{Z}$ with compactification denoted $\bar{\mathcal{Z}}$ [PS].
The projection $\pi_s:\bar{\mathcal{Z}}\rightarrow\mathbb{P}^1(s)$ is called the Fermi fibration. In their paper [PS], Peters and Stienstra proved that for $s\notin\{0,\infty,\pm1,3\pm2\sqrt{2},-3\pm2\sqrt{2}\}$ the fibers of the Fermi fibration are $K3$ surfaces with the Néron-Severi lattice of the generic fiber isometric to $E_8(-1)\oplus E_8(-1)\oplus U\oplus\langle-12\rangle$ and transcendental lattice isometric to $T=U\oplus\langle 12\rangle$ (U denotes the hyperbolic lattice and E_8 the unimodular lattice of rank 8). Hence this family appears as a family of M_6-polarized $K3$-surfaces Y_k with period $t\in\mathcal{H}$. And we deduce from a result of Dolgachev [D] the following property. Let $E_t=\mathbb{C}/\mathbb{Z}+t\mathbb{Z}$ and $E'_t=\mathbb{C}/\mathbb{Z}+(-\frac{1}{6t})\mathbb{Z}$ be the corresponding pair of isogenous elliptic curves. Then there exists a canonical involution τ on Y_k such that $Y_k/(\tau)$ is birationally isomorphic to the Kummer surface $E_t\times E'_t/(\pm1)$.
This result is linked to the Shioda-Inose structure of $K3$-surfaces with Picard number 19 and 20 described first by Shioda and Inose [SI] and extended by Morrison [M].
As observed by Elkies [E], the base of the pencil of $K3$ surfaces can be identified with the elliptic modular curve $X_0(6)/\langle w_2,w_3\rangle$. Indeed it can be derived from Peters and Stienstra [PS].
In [Shio], Shioda considers the problem whether every Shioda-Inose structure can be extended to a sandwich. More precisely Shioda proved a "Kummer sandwich theorem" that is, for an elliptic $K3$-surface X (with a section) with two II*-fibres, there exists a unique Kummer surface $S=Km(C_1\times C_2)$ with two rational maps of degree 2, $X\rightarrow S$ and $S\rightarrow X$ where C_1 and C_2 are elliptic curves.
In van Geemen-Sarti [G], Koike [Ko] and Schütt [Sc], sandwich Shioda-Inose structures are constructed via elliptic fibrations with 2-torsion sections.
Recently Bertin and Lecacheux [BL] found all the elliptic fibrations of a singular member Y_2 of $\mathcal{F}$ (i.e. of Picard number 20) and observed that many of its elliptic fibrations are endowed with 2-torsion sections. Thus a question arises: are the corresponding 2-isogenies between Y_2 and this new $K3$-surface S_2 all Morrison-Nikulin meaning that S_2 is Kummer? Observing also that the Shioda's Kummer sandwiching between a $K3$ surface S and its Kummer K is in fact a 2-isogeny between two elliptic fibrations of S and K, we extended the above question to the generic member Y_k of the family $\mathcal{F}$ and obtained the following results.

Theorem 1.1. *Suppose Y_k is a generic $K3$ surface of the family with Picard number* 19.
Let $\pi:Y_k\rightarrow\mathbb{P}^1$ be an elliptic fibration with a torsion section of order 2 which defines an involution i of Y_k (Van Geemen-Sarti involution) then the quotient Y_k/i is either the Kummer surface K_k associated to Y_k given by its Shioda-Inose structure or a surface S_k with transcendental lattice $T_{S_k}=\langle-2\rangle\oplus\langle2\rangle\oplus\langle6\rangle$ and Néron-Severi lattice $NS(S_k)=U\oplus E_8(-1)\oplus E_7(-1)\oplus\langle(-2)\rangle\oplus\langle(-6)\rangle$, which is not a Kummer surface by a result of Morrison [M]. *The $K3$ surface S_k is the Hessian $K3$ surface of a general cubic surface with 3 nodes studied by Dardanelli and van Geemen* [DG]. *Thus,*

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π leads to an elliptic fibration either of K_k or of S_k. Moreover there exist some genus 1 fibrations $\theta : K_k \to \mathbb{P}^1$ without section such that their Jacobian variety satisfies $J_\theta(K_k) = S_k$.
More precisely, among the elliptic fibrations of Y_k (up to automorphisms) 12 of them have a two-torsion section. And only 7 of them possess a Morisson-Nikulin involution i such that $Y_k/i = K_k$.

Theorem 1.2. *In the Apéry-Fermi pencil, the $K3$-surface Y_2 is singular, meaning that its Picard number is 20. Moreover Y_2 has many more 2-torsion sections than the generic $K3$ surface Y_k; hence among its 19 Van Geemen-Sarti involutions, 13 of them are Morrison-Nikulin involutions, 5 are symplectic automorphisms of order 2 (self-involutions) and one exchanges two elliptic fibrations of Y_2.*
The specializations to Y_2 of the 7 Morrison-Nikulin involutions of a generic member Y_k are verified among the 13 Morrison-Nikulin involutions of Y_2, as proved in a general setting by Schütt [Sc]. *The specializations of the 5 involutions between Y_k and the $K3$-surface S_k are among the 6 Van Geemen-Sarti involutions of Y_2 which are not Morrison-Nikulin.*

This theorem provides an example of a Kummer surface K_2 defined by the product of two isogenous elliptic curves (actually the same elliptic curve of j-invariant equal to 8000), having many fibrations of genus one whose Jacobian surface is not a Kummer surface. A similar result but concerning a Kummer surface defined by two non-isogenous elliptic curves has been exhibited by Keum [K].
Throughout the paper we use the following result [Si]. If E denotes an elliptic fibration with a 2-torsion point $(0,0)$:

$$E : y^2 = x^3 + Ax^2 + Bx,$$

the quotient curve $E/\langle(0,0)\rangle$ has a Weierstrass equation of the form

$$E/\langle(0,0)\rangle : y^2 = x^3 - 2Ax^2 + (A^2 - 4B)x.$$

The paper is organised as follows.
In section 2 we recall the Kneser-Nishiyama method and use it to find all the 27 elliptic fibrations of a generic $K3$ of the family $\mathcal{F}$. In section 3, using Elkies's method of "2-neighbors" [El], we exhibit an elliptic parameter giving a Weierstrass equation of the elliptic fibration. The results are summarized in Table 2. Thus we obtain all the Weierstrass equations of the 12 elliptic fibrations with 2-torsion sections. Their 2-isogenous elliptic fibrations are computed in section 5 with their Mordell-Weil groups and discriminants. Section 4 recalls generalities about Nikulin involutions and Shioda-Inose structure. Section 5 is devoted to the proof of Theorem 1.1 while section 6 is concerned with the proof of Theorem 1.2.
In the last section 7, using a theorem of Boissière, Sarti and Veniani [BSV], we explain why Theorem 1.2 cannot be generalised to the other singular K3 surfaces of the family.
Computations were performed using partly the computer algebra system PARI [PA] and mostly the computer algebra system MAPLE and the Maple Library "Elliptic Surface Calculator" written by Kuwata [Ku1].

2. Elliptic fibrations of the family

We refer to [BL], [Sc-Shio] for definitions concerning lattices, primitive embeddings, orthogonal complement of a sublattice into a lattice. We recall only what is essential for understanding this section and section 5.2.

2.1. **Discriminant forms.** Let L be a non-degenerate lattice. The **dual lattice** L^* of L is defined by

$$L^* := \mathrm{Hom}(L, \mathbb{Z}) = \{x \in L \otimes \mathbb{Q}/ \ b(x,y) \in \mathbb{Z} \text{ for all } y \in L\}$$

and the **discriminant group** G_L by

$$G_L := L^*/L.$$

This group is finite if and only if L is non-degenerate. In the latter case, its order is equal to the absolute value of the lattice determinant $\mid \det(G(e)) \mid$ for any basis e of L. A lattice L is **unimodular** if G_L is trivial.

Let G_L be the discriminant group of a non-degenerate lattice L. The bilinear form on L extends naturally to a $\mathbb{Q}$-valued symmetric bilinear form on L^* and induces a symmetric bilinear form

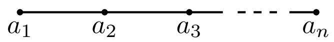
$$b_L : G_L \times G_L \to \mathbb{Q}/\mathbb{Z}.$$

If L is even, then b_L is the symmetric bilinear form associated to the quadratic form defined by

$$\begin{array}{rcl} q_L : G_L & \to & \mathbb{Q}/2\mathbb{Z} \\ q_L(x+L) & \mapsto & x^2 + 2\mathbb{Z}. \end{array}$$

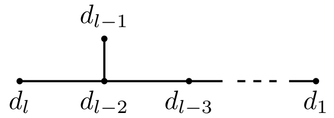
The latter means that $q_L(na) = n^2 q_L(a)$ for all $n \in \mathbb{Z}$, $a \in G_L$ and $b_L(a, a') = \frac{1}{2}(q_L(a+a') - q_L(a) - q_L(a'))$, for all $a, a' \in G_L$, where $\frac{1}{2} : \mathbb{Q}/2\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ is the natural isomorphism. The pair $\boldsymbol{(G_L, b_L)}$ (resp. $\boldsymbol{(G_L, q_L)}$) is called the **discriminant bilinear** (resp. **quadratic**) **form** of L.
The **lattices** $A_n = \langle a_1, a_2, \ldots, a_n \rangle$ $(n \geq 1)$, $D_l = \langle d_1, d_2, \ldots, d_l \rangle$ $(l \geq 4)$, $E_p = \langle e_1, e_2, \ldots, e_p \rangle$ $(p = 6, 7, 8)$ defined by the following **Dynkin diagrams** are called the **root lattices**. All the vertices a_j, d_k, e_l are roots and two vertices a_j and a'_j are joined by a line if and only if $b(a_j, a'_j) = 1$. We use Bourbaki's definitions [Bou]. The discriminant groups of these root lattices are given below.

$\mathbf{A_n}$, $\mathbf{G_{A_n}}$



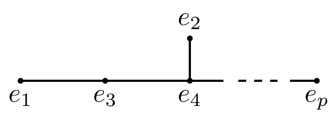
Set
$[1]_n = \frac{1}{n+1} \sum_{j=1}^{n} (n - j + 1) a_j$
then $A_n^* = \langle A_n, [1]_n \rangle$ and
$G_{A_n} = A_n^*/A_n \simeq \mathbb{Z}/(n+1)\mathbb{Z}$.
$q_{A_n}([1]_n) = -\frac{n}{n+1}$.

$\mathbf{D_l}$, $\mathbf{G_{D_l}}$.



Set
$[1]_{D_l} = \frac{1}{2}\left(\sum_{i=1}^{l-2} i d_i + \frac{1}{2}(l-2) d_{l-1} + \frac{1}{2} l d_l\right)$
$[2]_{D_l} = \sum_{i=1}^{l-2} d_i + \frac{1}{2}(d_{l-1} + d_l)$
$[3]_{D_l} = \frac{1}{2}\left(\sum_{i=1}^{l-2} i d_i + \frac{1}{2} l d_{l-1} + \frac{1}{2}(l-2) d_l\right)$
then $D_l^* = \langle D_l, [1]_{D_l}, [3]_{D_l} \rangle$,
$G_{D_l} = D_l^*/D_l = \langle [1]_{D_l} \rangle \simeq \mathbb{Z}/4\mathbb{Z}$ if l is odd,
$G_{D_l} = D_l^*/D_l = \langle [1]_{D_l}, [2]_{D_l} \rangle \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ if l is even.
$q_{D_l}([1]_{D_l}) = -\frac{l}{4}$, $q_{D_l}([2]_{D_l}) = -1$, $b_{D_l}([1], [2]) = -\frac{1}{2}$.

$\mathbf{E_p}$, $\mathbf{G_{E_p}}$ $p = 6, 7, 8$.



Set
$[1]_{E_6} := \eta_6 = -\frac{1}{3}(2e_1 + 3e_2 + 4e_3 + 6e_4 + 5e_5 + 4e_6)$ and
$[1]_{E_7} := \eta_7 = -\frac{1}{2}(2e_1 + 3e_2 + 4e_3 + 6e_4 + 5e_5 + 4e_6 + 3e_7)$,
then $E_6^* = \langle E_6, \eta_6 \rangle$, $E_7^* = \langle E_7, \eta_7 \rangle$ and $E_8^* = E_8$.
$G_{E_6} = E_6^*/E_6 \simeq \mathbb{Z}/3\mathbb{Z}$, $G_{E_7} = E_7^*/E_7 \simeq \mathbb{Z}/2\mathbb{Z}$,
$q_{E_6(\eta_6)} = -\frac{4}{3}$ $q_{E_7(\eta_7)} = -\frac{3}{2}$.

Let L be a Niemeier lattice (i.e. an unimodular lattice of rank 24). Denote L_{root} its root lattice. We often write $L = Ni(L_{\text{root}})$. Elements of L are defined by the glue code composed with glue vectors. Take for example $L = Ni(A_{11}D_7E_6)$. Its glue code is generated by the glue vector $[1, 1, 1]$ where the first 1 means $[1]_{A_{11}}$, the second 1 means $[1]_{D_7}$ and the third 1 means $[1]_{E_6}$. In the glue code $\langle [1, (0, 1, 2)] \rangle$, the notation $(0, 1, 2)$ means any circular permutation of $(0, 1, 2)$. Niemeier lattices, their root lattices and glue codes used in the paper are given in Table 1 (glue codes are taken from Conway and Sloane [Co]).

2.2. **Kneser-Nishiyama technique.** We use the Kneser-Nishiyama method to determine all the elliptic fibrations of Y_k. For further details we refer to [Nis], [Sc-Shio], [BL], [BGL]. In [Nis], [BL], [BGL] only singular $K3$ (i.e. of Picard number 20) are considered. In this paper we follow [Sc-Shio] we briefly recall.

Let $T(Y_k)$ be the transcendental lattice of Y_k, that is the orthogonal complement of $NS(Y_k)$ in $H^2(Y_k, \mathbb{Z})$ with respect to the cup-product. The lattice $T(Y_k)$ is an even lattice of rank $r = 22 - 19 =$

L_{root}	L/L_{root}	glue vectors
E_8^3	(0)	0
$D_{16}E_8$	$\mathbb{Z}/2\mathbb{Z}$	$\langle[1,0]\rangle$
$D_{10}E_7^2$	$(\mathbb{Z}/2\mathbb{Z})^2$	$\langle[1,1,0],[3,0,1]\rangle$
$A_{17}E_7$	$\mathbb{Z}/6\mathbb{Z}$	$\langle[3,1]\rangle$
D_{24}	$\mathbb{Z}/2\mathbb{Z}$	$\langle[1]\rangle$
D_{12}^2	$(\mathbb{Z}/2\mathbb{Z})^2$	$\langle[1,2],[2,1]\rangle$
D_8^3	$(\mathbb{Z}/2\mathbb{Z})^3$	$\langle[1,2,2],[1,1,1],[2,2,1]\rangle$
$A_{15}D_9$	$\mathbb{Z}/8\mathbb{Z}$	$\langle[2,1]\rangle$
E_6^4	$(\mathbb{Z}/3\mathbb{Z})^2$	$\langle[1,(0,1,2)]\rangle$
$A_{11}D_7E_6$	$\mathbb{Z}/12\mathbb{Z}$	$\langle[1,1,1]\rangle$
D_6^4	$(\mathbb{Z}/2\mathbb{Z})^4$	$\langle$even permutations of $[0,1,2,3]\rangle$
$A_9^2D_6$	$\mathbb{Z}/10\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$	$\langle[2,4,0],[5,0,1],[0,5,3]\rangle$
$A_7^2D_5^2$	$\mathbb{Z}/8\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}$	$\langle[1,1,1,2],[1,7,2,1]\rangle$

TABLE 1. Some Niemeier lattices and their glue codes [Co]

3 and signature $(2,1)$. Since $t=r-2=1$, $T(Y_k)[-1]$ admits a primitive embedding into the following indefinite unimodular lattice:

$$T(Y_k)[-1]\hookrightarrow U\oplus E_8$$

where U denotes the hyperbolic lattice and E_8 the unimodular lattice of rank 8. Define M as the orthogonal complement of a primitive embedding of $T(Y_k)[-1]$ in $U\oplus E_8$. Since

$$T(Y_k)[-1]=\begin{pmatrix}0&0&-1\\0&-12&0\\-1&0&0\end{pmatrix},$$

it suffices to get a primitive embedding of (-12) into E_8. From Nishiyama [Nis] we find the following primitive embedding:

$$v=\langle 9e_2+6e_1+12e_3+18e_4+15e_5+12e_6+8e_7+4e_8\rangle\hookrightarrow E_8,$$

giving $(v)^{\perp}_{E_8}=A_2\oplus D_5$. Now the primitive embedding of $T(Y_k)[-1]$ in $U\oplus E_8$ is defined by $U\oplus v$; hence $M=(U\oplus v)^{\perp}_{U\oplus E_8}=A_2\oplus D_5$. By construction, this lattice is negative definite of rank $t+6=1+6=r+4=3+4=26-\rho(Y_k)=7$ with discriminant form $q_M=-q_{T(Y_k)[-1]}=q_{T(Y_k)}=-q_{NS(Y_k)}$. Hence M takes exactly the shape required for Nishiyama's technique.
All the elliptic fibrations come from all the primitive embeddings of $M=A_2\oplus D_5$ into all the Niemeier lattices L. Since M is a root lattice, a primitive embedding of M into L is in fact a primitive embedding into L_{root}. Whenever the primitive embedding is given by a primitive embedding of A_2 and D_5 in two different factors of L_{root}, or for the primitive embedding of M into E_8, we use Nishiyama's results [Nis]. Otherwise we have to determine the primitive embeddings of M into D_l for $l=8,9,10,12,16,24$. This is done in the following lemma.

Lemma 2.1. *We obtain the following primitive embeddings.*

(1)

$$A_2\oplus D_5=\langle d_8,d_6,d_7,d_5,d_4,d_1,d_2\rangle\hookrightarrow D_8$$
$$\langle d_8,d_6,d_7,d_5,d_4,d_1,d_2\rangle^{\perp}_{D_8}=\langle 2d_1+4d_2+6d_3+6d_4+6d_5+6d_6+3d_7+3d_8\rangle=(-12)$$

(2)

$$A_2\oplus D_5=\langle d_9,d_7,d_8,d_6,d_5,d_3,d_2\rangle\hookrightarrow D_9$$
$$\langle d_9,d_7,d_8,d_6,d_5,d_3,d_2\rangle^{\perp}_{D_9}=\langle d_9+d_8+2d_7+2d_6+2d_5+2d_4+d_3-d_1,d_3+2d_2+3d_1\rangle$$

with Gram matrix $\begin{pmatrix}-4&6\\6&-12\end{pmatrix}$ *of determinant* 12.

(3)

$$A_2 \oplus D_5 = \langle d_n, d_{n-2}, d_{n-1}, d_{n-3}, d_{n-4}, d_{n-7}, d_{n-6}\rangle \hookrightarrow D_n, n \geq 10$$

$$\langle d_n, d_{n-2}, d_{n-1}, d_{n-3}, d_{n-4}, d_{n-7}, d_{n-6}\rangle^{\perp}_{D_n} =$$

$$\langle a = d_n + d_{n-1} + 2(d_{n-2} + ... + d_2) + d_1, d_{n-6} + 2d_{n-7} + 3d_{n-8}, d_{n-9}, ..., d_1\rangle$$

$$((A_2 \oplus D_5)^{\perp}_{D_n})_{root} = D_{n-8}.$$

We have also the relation $2.[2]_{D_n} = a + d_1$, *a being the above root.*

Theorem 2.1. *There are* 27 *elliptic fibrations on the generic* $K3$ *surface of the Apéry-Fermi pencil (i.e. with Picard number* 19*). They are derived from all the non isomorphic primitive embeddings of* $A_2 \oplus D_5$ *into the various Niemeier lattices. Among them,* 4 *fibrations have rank* 0*, precisely with the type of singular fibers and torsion.*

$A_{11}2A_22A_1$	$6-$ *torsion*
E_6D_{11}	$0-$ *torsion*
$E_7A_5D_5$	$2-$ *torsion*
$E_8E_6A_3$	$0-$ *torsion.*

The list together with the rank and torsion is given in Table 2.

Proof. The torsion groups can be computed as explained in [BL] or [BGL]. Let us recall briefly the method.
Denote ϕ a primitive embedding of $M = A_2 \oplus D_5$ into a Niemeier lattice L. Define $W = (\phi(M))^{\perp}_{L}$ and $N = (\phi(M))^{\perp}_{L_{\text{root}}}$. We observe that $W_{\text{root}} = N_{\text{root}}$. Thus computing N then N_{root} we know the type of singular fibers. Recall also that the torsion part of the Mordell-Weil group is

$$\overline{W_{\text{root}}}/W_{\text{root}}(\subset W/N)$$

and can be computed in the following way [BGL]: let $l + L_{\text{root}}$ be a non trivial element of L/L_{root}. If there exist $k \neq 0$ and $u \in L_{\text{root}}$ such that $k(l+u) \in N_{\text{root}}$, then $l + u \in W$ and the class of l is a torsion element.
We use also several facts.

(1) If the rank of the Mordell-Weil group is 0, then the torsion group is equal to W/N. Hence fibrations #1($A_3E_6E_8$), #3($D_{11}E_6$), #7($D_5A_5E_7$), #20($A_{11}2A_12A_2$) have respective torsion groups (0), (0), $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/6\mathbb{Z}$.
(2) If there is a singular fiber of type E_8, then the torsion group is (0). Hence the fibrations #1, #2 and #6 have no torsion.
(3) Using lemma 2.2 below and the shape of glue vectors we prove that fibrations #11, #18, #21, #22, #25, #27 have no torsion.

Lemma 2.2. *Suppose* A_2 *primitively embedded in* A_n, $A_2 = \langle a_1, a_2\rangle \hookrightarrow A_n$. *Then for all* $k \neq 0$, $k[1]_{A_n} \notin ((A_2)^{\perp}_{A_n})_{root}$.

Proof. It follows from the fact that $[1]_{A_n}$ is not orthogonal to a_1. □

(4) Using lemma 2.3 below and the shape of glue vectors we can determine the torsion for elliptic fibrations #5, #10, #13, #15 #23.

Lemma 2.3. *Suppose* A_2 *primitively embedded in* D_l, $A_2 = \langle d_l, d_{l-2}\rangle \hookrightarrow D_l$. *Then* $2.[2]_{D_l} \in ((A_2)^{\perp}_{D_l})_{root}$ *but there is no* k *satisfying* $k.[i]_{D_l} \in ((A_2)^{\perp}_{D_l})_{root}, i = 1, 3$.

Proof. It follows from Nishiyama [Nis]:

$$(A_2)^{\perp}_{D_l} = \langle y, x_4, d_{l-4}, ..., d_1\rangle$$

L_{root}	L/L_{root}			type of Fibers	Rk	Tors.
E_8^3	(0)					
	#1	$A_2 \subset E_8$	$D_5 \subset E_8$	$E_6A_3E_8$	0	(0)
	#2	$A_2 \oplus D_5 \subset E_8$		E_8E_8	1	(0)
$D_{16}E_8$	$\mathbb{Z}/2\mathbb{Z}$					
	#3	$A_2 \subset E_8$	$D_5 \subset D_{16}$	E_6D_{11}	0	(0)
	#4	$A_2 \oplus D_5 \subset E_8$		D_{16}	1	$\mathbb{Z}/2\mathbb{Z}$
	#5	$D_5 \subset E_8$	$A_2 \subset D_{16}$	A_3D_{13}	1	(0)
	#6	$A_2 \oplus D_5 \subset D_{16}$		E_8D_8	1	(0)
$D_{10}E_7^2$	$(\mathbb{Z}/2\mathbb{Z})^2$					
	#7	$A_2 \subset E_7$	$D_5 \subset D_{10}$	$E_7A_5D_5$	0	$\mathbb{Z}/2\mathbb{Z}$
	#8	$A_2 \subset E_7$	$D_5 \subset E_7$	$A_5A_1D_{10}$	1	$\mathbb{Z}/2\mathbb{Z}$
	#9	$A_2 \oplus D_5 \subset D_{10}$		$E_7E_7A_1A_1$	1	$\mathbb{Z}/2\mathbb{Z}$
	#10	$D_5 \subset E_7$	$A_2 \subset D_{10}$	$A_1D_7E_7$	2	(0)
$A_{17}E_7$	$\mathbb{Z}/6\mathbb{Z}$					
	#11	$D_5 \subset E_7$	$A_2 \subset A_{17}$	A_1A_{14}	2	(0)
D_{24}	$\mathbb{Z}/2\mathbb{Z}$					
	#12	$A_2 \oplus D_5 \subset D_{24}$		D_{16}	1	(0)
D_{12}^2	$(\mathbb{Z}/2\mathbb{Z})^2$					
	#13	$A_2 \subset D_{12}$	$D_5 \subset D_{12}$	D_9D_7	1	(0)
	#14	$A_2 \oplus D_5 \subset D_{12}$		D_4D_{12}	1	$\mathbb{Z}/2\mathbb{Z}$
D_8^3	$(\mathbb{Z}/2\mathbb{Z})^3$					
	#15	$A_2 \subset D_8$	$D_5 \subset D_8$	$D_5A_3D_8$	1	$\mathbb{Z}/2\mathbb{Z}$
	#16	$A_2 \oplus D_5 \subset D_8$		D_8D_8	1	$\mathbb{Z}/2\mathbb{Z}$
$A_{15}D_9$	$\mathbb{Z}/8\mathbb{Z}$					
	#17	$A_2 \oplus D_5 \subset D_9$		A_{15}	2	$\mathbb{Z}/2\mathbb{Z}$
	#18	$D_5 \subset D_9$	$A_2 \subset A_{15}$	D_4A_{12}	1	(0)
E_6^4	$(\mathbb{Z}/3\mathbb{Z})^2$					
	#19	$A_2 \subset E_6$	$D_5 \subset E_6$	$A_2A_2E_6E_6$	1	$\mathbb{Z}/3\mathbb{Z}$
$A_{11}D_7E_6$	$\mathbb{Z}/12\mathbb{Z}$					
	#20	$A_2 \subset E_6$	$D_5 \subset D_7$	$A_2A_2A_1A_1A_{11}$	0	$\mathbb{Z}/6\mathbb{Z}$
	#21	$A_2 \subset A_{11}$	$D_5 \subset D_7$	$A_8A_1A_1E_6$	1	(0)
	#22	$A_2 \subset A_{11}$	$D_5 \subset E_6$	A_8D_7	2	(0)
	#23	$D_5 \subset E_6$	$A_2 \subset D_7$	$A_{11}D_4$	2	$\mathbb{Z}/2\mathbb{Z}$
D_6^4	$(\mathbb{Z}/2\mathbb{Z})^4$					
	#24	$A_2 \subset D_6$	$D_5 \subset D_6$	$A_3D_6D_6$	2	$\mathbb{Z}/2\mathbb{Z}$
$A_9^2D_6$	$\mathbb{Z}/2 \times \mathbb{Z}/10$					
	#25	$D_5 \subset D_6$	$A_2 \subset A_9$	A_6A_9	2	(0)
$A_7^2D_5^2$	$\mathbb{Z}/4 \times \mathbb{Z}/8$					
	#26	$D_5 \subset D_5$	$A_2 \subset D_5$	$A_1A_1A_7A_7$	1	$\mathbb{Z}/4\mathbb{Z}$
	#27	$D_5 \subset D_5$	$A_2 \subset A_7$	$D_5A_4A_7$	1	(0)

TABLE 2. The elliptic fibrations of the Apéry-Fermi family

with $y = d_l + 2d_{l-1} + 2d_{l-2} + d_{l-3}$ and $x_4 = d_l + d_{l-1} + 2(d_{l-2} + d_{l-3} + ... + d_2) + d_1$ and Gram matrix

$$L^3_{l-3} = \begin{pmatrix} -4 & -1 & 0 & ...0 \\ -1 & & & \\ 1 & D_{l-3} & & \\ . & & & \\ 0 & & & \end{pmatrix}.$$

Moreover $((A_2)^{\perp}_{D_l})_{\text{root}} = \langle x_4, d_{l-4}, ..., d_1\rangle$. From there we compute easily the relation $2.[2]_{D_l} = x_4 + d_{l-4} + 2(d_{l-5} + ... + d_1)$. The last assertion follows from the fact that $[i]_{D_l}$ is not orthogonal to A_2. □

We now give some examples showing the method in detail.

2.2.1. *Fibration #17.* It comes from a primitive embedding of $A_2 \oplus D_5$ into D_9 giving a primitive embedding of $A_2 \oplus D_5$ into $Ni(A_{15}D_9)$ with glue code $\langle[2,1]\rangle$. Since by lemma 2.1(2) $N_{\text{root}} = A_{15}$, among the elements $k.[2,1]$, only $4.[2,1] = [8, 4.1 \in D_9]$ satisfies $2.[8, 0+u] \in N_{\text{root}} = A_{15}$ with $u = 4.1$. Hence the torsion group is $\mathbb{Z}/2\mathbb{Z}$.

2.2.2. *Fibration #19.* It comes from a primitive embedding of $A_2 = \langle e_1, e_3\rangle$ into $E_6^{(1)}$ and $D_5 = \langle e_2, e_3, e_4, e_5, e_6\rangle$ into $E_6^{(2)}$ giving a primitive embedding of $A_2 \oplus D_5$ into $Ni(E_6^4)$. In that case $Ni(E_6^4)/E_6^4 \simeq (\mathbb{Z}/3\mathbb{Z})^2$ and the glue code is $\langle[1,(0,1,2)]\rangle$. Moreover $(D_5)^{\perp}_{E_6} = 3e_2 + 4e_1 + 5e_3 + 6e_4 + 4e_5 + 2e_6 = a$, $(A_2)^{\perp}_{E_6} = \langle e_2, y\rangle \oplus \langle e_5, e_6\rangle$ with $y = 2e_2 + e_1 + 2e_3 + 3e_4 + 2e_5 + e_6$. From the relation

$$[1]_{E_6} = -\frac{1}{3}(2e_1 + 3e_2 + 4e_3 + 6e_4 + 5e_5 + 4e_6)$$

we get

$$-3.[1]_{E_6} = a - 2e_1 - e_3 + e_5 + 2e_6 \in E_6$$

$$-3.[1]_{E_6} = 2y - e_2 + e_5 + 2e_6 \in (A_2)^{\perp}_{E_6}$$

we deduce that only $[1,0,1,2]$, $[2,0,2,1]$, $[0,0,0,0]$ contribute to the torsion thus the torsion group is $\mathbb{Z}/3\mathbb{Z}$.

2.2.3. *Fibration #10.* The embeddings of $A_2 = \langle d_{10}, d_8\rangle$ into D_{10} and $D_5 = \langle e_2, e_3, e_4, e_5, e_6\rangle$ into $E_7^{(1)}$ lead to a primitive embedding of $A_2 \oplus D_5$ into $Ni(D_{10}E_7^2)$ satisfying $Ni(D_{10}E_7^2)/(D_{10}E_7^2) \simeq (\mathbb{Z}/2\mathbb{Z})^2$ with glue code $\langle[1,1,0],[3,0,1]\rangle$. We deduce from lemma 2.3 that no glue vector can contribute to the torsion which is therefore (0).

2.2.4. *Fibration #18.* The embeddings of $A_2 = \langle a_1, a_2\rangle$ into A_{15} and $D_5 = \langle d_9, d_7, d_8, d_6, d_5\rangle$ into D_9 lead to a primitive embedding of $A_2 \oplus D_5$ into $Ni(A_{15}D_9)$ satisfying $Ni(A_{15}D_9)/(A_{15}D_9) \simeq (\mathbb{Z}/8\mathbb{Z})$ with glue code $\langle[2,1]\rangle$. We deduce from lemma 2.2 that no glue vector can contribute to the torsion which is therefore (0).

2.2.5. *Fibration #8.* The primitive embeddings of $A_2 = \langle e_1, e_3\rangle$ into $E_7^{(1)}$ and $D_5 = \langle e_2, e_3, e_4, e_5, e_6\rangle$ into $E_7^{(2)}$ lead to a primitive embedding of $A_2 \oplus D_5$ into $Ni(D_{10}E_7^2)$ satisfying $Ni(D_{10}E_7^2)/(D_{10}E_7^2) \simeq (\mathbb{Z}/2\mathbb{Z})^2$ with glue code $\langle[1,1,0],[3,0,1]\rangle$. From Nishiyama [Nis] we get $(A_2)^{\perp}_{E_7^{(1)}} = \langle e_2, y, e_7, e_6, e_5\rangle \simeq A_5$ with $y = 2e_2 + e_1 + 2e_3 + 3e_4 + 2e_5 + e_6$ and $(D_5)^{\perp}_{E_7} = \langle(-4), e_2 + e_3 + 2(e_4 + e_5 + e_6 + e_7) = (-2)\rangle$. Hence $N = D_{10} \oplus A_5 \oplus (-4) \oplus A_1$ and $W_{\text{root}} = N_{\text{root}} = D_{10} \oplus A_5 \oplus A_1$. Now

$$-2\eta_7 = -2.[1]_{E_7} = 2y - e_2 + e_5 + 2e_6 + 3e_7 \in ((A_2)^{\perp}_{E_7})_{\text{root}}$$

and for all $k \neq 0$, $k.[1]_{E_7} \notin (D_5)^{\perp}_{E_7}$. Hence only the generator $[1,1,0]$ can contribute to the torsion group which is therefore $\mathbb{Z}/2\mathbb{Z}$.

2.2.6. *Fibration #24.* The primitive embeddings $A_2 = \langle d_6, d_4\rangle$ into $D_6^{(1)}$ and $D_5 = \langle d_6, d_5, d_4, d_3, d_2\rangle$ into $D_6^{(2)}$ give a primitive embedding of $A_2 \oplus D_5$ into $L = Ni(D_6^4)$ with $L/L_{\text{root}} \simeq (\mathbb{Z}/2\mathbb{Z})^4$ and glue code $\langle$ even permutations of $[0,1,2,3]\rangle$. From Nishiyama [Nis] we get $(A_2)^{\perp}_{D_6} = \langle y = 2d_5 + d_6 + 2d_4 + d_3, x_4 = d_5 + d_6 + 2(d_4 + d_3) + d_2, d_2, d_1\rangle$, $((A_2)^{\perp}_{D_6})_{\text{root}} = \langle x_4, d_2, d_1\rangle \simeq A_3$ and $(D_5)^{\perp}_{D_6} = \langle x_6'\rangle = \langle d_5 + d_6 + 2(d_4 + d_3 + d_2 + d_1) = (-4)\rangle$. We deduce $N_{\text{root}} = A_3 \oplus D_6 \oplus D_6$. From the relations $2.[2]_{D_6} = x_4 + d_2 + 2d_1$ and $2.[3]_{D_6} = y + x_4 + d_2 + d_1$ we deduce that the glue vectors having 1, 2, 3 or 0 in the first position may belong to W. From the relation $2.[2]_{D_6} = x_6'$ we deduce that only glue vectors with 2 or 0 in the second position may belong to W. Finally only the glue vectors $[0,2,3,1]$, $[1,0,3,2]$, $[1,2,0,3]$, $[2,0,1,3]$, $[2,2,2,2]$, $[3,0,2,1]$, $[3,2,1,0]$, $[0,0,0,0]$ belong to W. Since y and x_6' are not roots, only glue vectors with 0 or 2 in the first position and 0 in the second position may contribute to torsion that is $[2,0,1,3]$, $[0,0,0,0]$. Hence the torsion group is $\mathbb{Z}/2\mathbb{Z}$.

2.2.7. *Fibration* #26. The primitive embeddings of $A_2 = \langle d_5, d_3\rangle$ into $D_5^{(1)}$ and D_5 into $D_5^{(2)}$ give a primitive embedding into $L = Ni(A_7^2 D_5^2)$ with $L/L_{\text{root}} \simeq \mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ and glue code $\langle [1,1,1,2],[1,7,2,1]\rangle$. From Nishiyama we get $(A_2)^{\perp}_{D_5} = \langle y, x_4, d_1\rangle$ with $y = 2d_4 + d_5 + 2d_3 + d_2$, $x_4 = d_5 + d_4 + 2d_3 + 2d_2 + d_1$ and Gram matrix $M_2^4 = \begin{pmatrix} -4 & -1 & 1 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{pmatrix}$ of determinant 12. We also deduce $N_{\text{root}} = E_7^2 A_1^2$, $2.[2]_{D_5} = x_4 + d_1 \in ((A_2)^{\perp}_{D_5})_{\text{root}}$. Moreover neither $k.[1]_{D_5}$ nor $k.[3]_{D_5}$ belongs to $(A_2)^{\perp}_{D_5}$. Thus only glue vectors with 2 or 0 in the third position can belong to W and eventually contribute to torsion, that is $[2,2,2,0]$, $[4,4,0,0]$, $[6,6,2,0]$, $[2,6,0,2]$, $[6,2,0,2]$, $[4,0,2,2]$, $[0,4,2,2]$, $[0,0,0,0]$. Since there is no $u_4 \in D_5$ satisfying $2.(2+u_4) = 0$ or $4.(2+u_4) = 0$, glue vectors with the last component equal to 2 cannot satisfy $k(l+u) \in N_{\text{root}}$ with $l \in L$ and $u \in L_{\text{root}} = A_7^2 D_5^2$. Hence only the glue vectors generated by $\langle [2,2,2,0]\rangle$ contribute to torsion and the torsion group is therefore $\mathbb{Z}/4\mathbb{Z}$.

□

3. Weierstrass Equations for all the elliptic fibrations of Y_k

The method can be found in [BL], [El]. We follow also the same kind of computations used for Y_2 given in [BL]. We give only explicit computations for 4 examples, #19, #2, #9, and #16. For #2 and #9 it was not obvious to find a rational point on the quartic curve. All the results are given in Table 3. For the 2 or 3-neighbor method [El] we give in the third column the starting fibration and in the forth the elliptic parameter. The terms in the elliptic parameter refer to the starting fibration.

3.1. **Fibration #19.** We take $u = \frac{XY}{Z}$ as a parameter of an elliptic fibration and with the birational transformation

$$x = -u(1+uZ)(u+Y), \quad y = u^2\left((u+Y)(uY-1)Z + Y\left(Y+2u+k\right)-1\right)$$

we obtain a Weierstrass equation

$$y^2 + ukyx + u^2\left(u^2+uk+1\right)y = x^3,$$

where the point $(x=0, y=0)$ is a 3-torsion point and the point $\left(-u^2, -u^2\right)$ is of infinite order.
The singular fibers are of type IV^* $(u = 0, \infty)$, I_3 $\left(u^2+uk+1=0\right)$ and
I_1 $\left(27u^2 - k(k^2-27)u + 27 = 0\right)$. Moreover if $k = s + \frac{1}{s}$ the two singular fibers of type I_3 are above $u = -s$ and $\frac{-1}{s}$.

3.2. **Fibration #2.** Using the 3-neighbor method from fibration #19 we construct a new fibration with a fiber of type II^* and the parameter $m = \frac{ys}{(u+s)^2}$. Then we obtain a cubic C_m in w, u, with $x = w\,(u+s)$

$$C_m : (s+u)\,m^2 + u\left(s^2w + u^2s + w + u\right)m - w^3s^2 = 0.$$

From some component of the fiber of type I_3 at $u = -s$ we obtain the rational point on C_m : $\omega_m = \left(u_1 = \frac{ms-1}{s-m}, w_1 = \frac{m\left(s^2-1\right)}{s(s-m)}\right)$ which is not a flex point. The first stage is to obtain a quartic equation $Qua : y^2 = ax^4 + bx^3 + cx^2 + dx + e^2$. First we observe that ω_m is on the line $w = u + \frac{1}{s}$, so we replace w by K with $w = u + \frac{1}{s} + K$ and $u = u_1 + T$. The transformation $K = WT$ gives an equation of degree two in T, with constant term $fW + g$ where f and g belong to $\mathbb{Q}(s,m)$. With the change variable $Wf + g = x$ we have an equation $M(x)T^2 + N(x)T + x = 0$. The discriminant of the quadratic equation in T is $N(x)^2 - 4xM(x)$, a polynomial of degree 4 in x and constant term a square. Easily we obtain the form Qua.
From the quartic form, setting $y = e + \frac{dx}{2e} + x^2X'$, $x = \frac{8c^3X' - 4ce^2 + d^2}{Y'}$ we get

$$Y'^2 + 4e\left(dX' - be\right)Y + 4e^2\left(8e^3X' - 4ce^2 + d^2\right)\left(X'^2 - a\right) = 0.$$

Finally the following Weierstrass equation follows from standard transformation where we replace m by t

$$
\begin{aligned}
&Y^2 - X^3 + \frac{1}{3}t^4(s^2+1)(s^6+219s^4-21s^2+1)X \\
&- \frac{2t^5}{27}\left(-864s^5t^2+(s^4+14s^2+1)(s^8-548s^6+198s^4-44s^3+1)t-864s^5\right)=0,
\end{aligned}
$$

with a section Φ of height 12 corresponding to $\left(8e^3X'-4ce^2+d^2\right)=0$ and $Y'=0$. The coordinates of Φ, too long, are omitted but we can follow the previous computation to obtain it.
Writing the above form as

$$y^2 = x^3 - 3\alpha x + \left(t+\frac{1}{t}\right) - 2\beta$$

we recover the values of the j invariants of the two elliptic curves for the Shioda-Inose structure (see paragraph 4.5.1 and 4.1 below).

3.3. **Fibration #9.** Let $g=\frac{XY}{Z^2}$. Eliminating X and writing $Y=ZU$ we obtain an equation of bidegree 2 in U and Z. If $k=s+\frac{1}{s}$ there is a rational point $U=-1$, $Z=-\frac{s}{g}$ on the previous curve. By standard transformations we get a Weierstrass equation

$$y^2 = x^3 + \frac{1}{4}g^2\left(s^4+14s^2+1\right)x^2 + s^2g^3\left(g+s^2\right)\left(gs^2+1\right)x$$

and a rational point

$$x = \frac{s^2\left(g-1\right)^2\left(g+s^2\right)\left(s^2g+1\right)}{\left(s^2-1\right)^2},$$

$$y = \frac{1}{2}\frac{s^2\left(g^2-1\right)\left(g+s^2\right)\left(s^2g+1\right)\left(2g^2s^2+g\left(s^4-6s^2+1\right)+2s^2\right)}{\left(s^2-1\right)^3}.$$

The singular fibers are of type $2III^*\,(\infty,0)\,,\ 2I_2\left(-s^2,-\frac{1}{s^2}\right),\ 4I_1$.

3.4. **Fibration #16.** Using the fibration #9 we consider the parameter $t=\frac{x}{g(g+s^2)}$ and obtain a Weierstrass equation

$$Y^2 = X^3 + \left(4t\left(t^2+s^2\right)+t^2\left(s^4+14s^2+1\right)\right)X^2 + 16s^6t^4X.$$

The singular fibers are of type $I_4^*\,(\infty,0)\,,\ 4I_1$.

4. Nikulin involutions and Shioda-Inose structure

4.1. **Background.** Let X be a $K3$ surface.
The second cohomology group, $H^2(X,\mathbb{Z})$ equipped with the cup product is an even unimodular lattice of signature $(3,19)$. The period lattice of a surface denoted T_X is defined by

$$T_X = S_X^{\perp} \subset H^2(X,\mathbb{Z})$$

where S_X is the Néron-Severi group of X. The lattice $H^2(X,\mathbb{Z})$ admits a Hodge decomposition of weight two

$$H^2(X,\mathbb{C}) \simeq H^{2,0}\oplus H^{1,1}\oplus H^{0,2}.$$

Similarly, the period lattice T_X has a Hodge decomposition of weight two

$$T_X\otimes\mathbb{C} \simeq T^{2,0}\oplus T^{1,1}\oplus T^{0,2}.$$

An isomorphism between two lattices that preserves their bilinear forms and their Hodge decomposition is called a Hodge isometry.
An automorphism of a $K3$ surface X is called *symplectic* if it acts on $H^{2,0}(X)$ trivially. Such automorphisms were studied by Nikulin in [N2] who proved that a symplectic involution i (*Nikulin involution*) has eight fixed points and that the minimal resolution $Y\to X/\langle i\rangle$ of the eight nodes is again a $K3$ surface.

	Weierstrass Equation	From	Param.
#1	$y^2+tkyx+t^2k(t+1)y=x^3-t^4(t+1)^3$ $II^*(\infty), IV^*(0), I_4(-1), 2I_1$ $r=0$		$\frac{Y(X+Z)^2(Z+Y)}{XZ^3}$
#2	$y^2=x^3-\frac{1}{3}t^4\left(s^2+1\right)\left(s^6+219*s^4-21s^2+1\right)x$ $+\frac{2}{27}t^5(-864s^5t^2+(s^4+14s^2+1)(s^8-548s^6+198s^4-44s^2+1)t-864s^5)$ $2II^*(\infty,0), 4I_1$ $x_P=\Phi$	#19	$\frac{ys}{(s+t)^2}$
#3	$y^2=x^3+\frac{1}{4}t\left(4t^2s^4+\left(s^4-10s^2+1\right)t+12\right)x^2$ $-t^2\left(2ts^2-3\right)x+t^3$ $I_7^*(\infty), IV^*(0), 3I_1$ $r=0$	#7	$\frac{x}{s^4t^2}$
#4	$y^2=x^3+$ $\left(\frac{1}{2}t^3-\frac{1}{24}\left(s^2+1\right)\left(s^6+219s^4-21s^2+1\right)t+\frac{1}{216}\left(s^8-548s^6+198s^4-44s^2+1\right)\left(s^4+14s^2+1\right)\right)x^2$ $+16s^{10}x$ $I_{12}^*(\infty), 6I_1$ x_P	#2	$\frac{x}{2t^2}$
#5	$y^2-k(t+1)yx+ky=x^3+\left(t^3-3\right)x^2+3x-1$ $I_9^*(\infty), I_4(0), 5I_1$ $x_P=0$	#1	$\frac{x}{t^2}$
#6	$y^2=x^3+\left(\frac{1}{4}t^2\left(s^4+14s^2+1\right)+t^3s^2\right)x^2$ $+t^4s^2\left(s^4+1\right)x+t^5s^6$ $I_4^*(\infty), II^*(0), 4I_1$ x_P	#9	$\frac{x}{(t+s^2)(ts^2+1)}$
#7	$y^2=x^3+\frac{1}{4}t\left(t\left(s^4-10s^2+1\right)+8s^4\right)x^2-t^2s^2\left(t-s^2\right)^3x$ $III^*(\infty), I_1^*(0), I_6\left(s^2\right), 2I_1$ $r=0$	#15	$\frac{x}{t}$
#8	$y^2-k(t-1)yx=x(x-1)\left(x-t^3\right)$ $I_6^*(\infty), I_6(0), I_2(1), 4I_1$ $x_P=1$		$\frac{(X+Z)(Y+Z)}{XZ}$
#9	$y^2=x^3+\frac{1}{4}t^2\left(s^4+14s^2+1\right)x^2+t^3s^2\left(t+s^2\right)\left(ts^2+1\right)x$ $2III^*(\infty,0), 2I_2\left(-s^2,-\frac{1}{s^2}\right), 2I_1$ $x_P=\frac{s^2(t-1)^2\left(t+s^2\right)\left(ts^2+1\right)}{\left(s^2-1\right)^2}$		$\frac{XY}{Z^2}$
#10	$y^2+t\left(s^2+1\right)\left(x+t^2s^2\right)y=\left(x-t^3s^2\right)\left(x^2+t^3s^4\right)$ $I_3^*(\infty), III^*(0), I_2(-1), 4I_1$ $x_{P_1}=t^3s^2, \quad x_{P_2}=0$		$\frac{XY}{(Y+Z)Z}$
#11	$y^2+t\left(st-1-s^2\right)yx-s^3y=x^2\left(x+s\left(t\left(s^2-1\right)-s\left(s^2+1\right)\right)\right)$ $I_{15}(\infty), I_2(s), 7I_1$ $x_{P_1}=st, \quad x_{P_2}=-s^3t+s^2\left(s^2+1\right)$	#8	$\frac{y+sx}{xt}$
#12	$y^2=x^3+t\left(t^2s^2+\frac{1}{4}\left(s^4+14s^2+1\right)+\left(s^4+1\right)\right)x^2$ $-\left(2t^2s^4+\frac{1}{2}s^2\left(s^4+14s^2+1\right)t+s^2\left(s^4+1\right)\right)x+ts^6+\frac{1}{4}s^4\left(s^4+14s^2+1\right)$ $I_{12}^*(\infty), 6I_1$ x_P	#14	$\frac{x}{t^2}+\frac{s^2}{t}$
#13	$y^2=x^3+\frac{1}{4}t\left(4t^2+\left(s^4-10s^2+1\right)t+4s^4\right)x^2$ $+2t^3s^4\left(t-s^4\right)x+t^5s^8$ $I_5^*(\infty), I_3^*(0), 4I_1$ $x_P=-ts^4$	#15	$\frac{x}{t^2}$

TABLE 3. Weierstrass equations of the elliptic fibrations of Y_k

We have then the rational quotient map $p : X \to Y$ of degree 2. The transcendental lattices T_X and T_Y are related by the chain of inclusions

$$2T_Y \subseteq p^*T_X = T_X(2) \subseteq T_Y,$$

which preserves the quadratic forms and the Hodge structures.

No	Weierstrass Equation	From	Param.
#14	$y^2=x^3+\left(t^3\left(s^4+1\right)+\frac{1}{4}t^2\left(s^4+14s^2+1\right)+ts^2\right)x^2+s^4t^6x$ $I_0^*\left(\infty\right), I_8^*\left(0\right), 4I_1\left(-\frac{1}{4},-4\frac{s^2}{\left(s^2-1\right)^2},..\right)$ $x_P=\frac{s^4(2t+1)^2}{\left(s^2-1\right)^2}$	#9	$\frac{x}{t\left(t+s^2\right)\left(ts^2+1\right)}$
#15	$\left(y-tx\right)\left(y-s^2tx\right)=x\left(x-ts^2\right)\left(x-ts^2\left(t+1\right)^2\right)$ $I_4^*\left(\infty\right), I_1^*\left(0\right), I_4\left(-1\right), 3I_1\left(\frac{1}{4}\left(\frac{s^2-1}{s}\right)^2,..\right)$ $x_P=s^2t$		$\frac{(XY+1)Z}{X}$
#16	$y^2=x^3+t\left(4\left(t^2+s^2\right)+t\left(s^4+14s^2+1\right)\right)x^2+16s^6t^4x$ $I_4^*\left(\infty,0\right), 4I_1$ $x_P=\frac{-4ts^6(t+1)^2}{\left(t+s^2\right)^2}$	#9	$\frac{x}{t\left(t+s^2\right)}$
#17	$y^2-\frac{1}{2}\left(s^4+14s^2+1-s^2t^2\right)yx=x\left(x-4s^2\right)\left(x-4s^6\right)$ $I_{16}\left(\infty\right), 8I_1\left(\pm\frac{s^2\pm4s-1}{s},..\right)$ $x_{P_1}=4s^2;\quad x_{P_2}=\frac{4s^4\left(ts+s^2-1\right)^2}{\left(ts+1-s^2\right)^2}$	#16	$\frac{y}{ts^2x}$
#18	$y^2+\left(-t^2+\left(s^2-1\right)t-2s^2\right)yx+s^4t^2y=x^2\left(x-s^4\right)$ $I_{13}\left(\infty\right), I_0^*\left(0\right), 5I_1$ $x_P=0$	#15	$\frac{y-tx}{t\left(x-ts^2\right)}$
#19	$y^2+ktyx+t^2\left(t^2+tk+1\right)y=x^3$ $2IV^*\left(\infty,0\right), 2I_3\left(-s,-\frac{1}{s}\right), 2I_1$ $x_P=-t^2$		$\frac{XY}{Z}$
#20	$y^2-yx\left(t^2-kt+1\right)=x\left(x-1\right)\left(x+t^2-tk\right)$ $I_{12}\left(\infty\right), 2I_2\left(0,k\right), 4I_1\left(s,\frac{1}{s},..\right)$ $r=0$		$X+Y+Z$
#21	$y^2=x^3+\frac{1}{4}t^2\left(t^2+2\left(s^2-1\right)t+\left(s^4-10s^2+1\right)\right)x^2$ $+\frac{1}{2}t^3s^4\left(t-\left(s^2-1\right)\right)x+\frac{1}{4}s^8t^4$ $I_9\left(\infty\right), IV^*\left(0\right), 2I_2\left(1,-s^2\right), 3I_1$ $x_P=s^2t^2$	#15	$\frac{y-s^2x-ts^4\left(t^2-1\right)}{x-ts^2(t+1)^2}$
#22	$y^2+\left(t\left(1-s^2\right)+s^2\right)yx+t^3s^2y=x\left(x-s^2t\right)\left(x+t^2s^2\left(1-t\right)\right)$ $I_3^*\left(\infty\right), I_9\left(0\right), 6I_1$ $x_{P_1}=1,\quad x_{P_2}=s^2t$		$\frac{Z(XYZ+s)}{1+YZ}$
#23	$y^2+\left(2t^2-tk+1\right)yx=x\left(x-t^2\right)\left(x-t^4\right)$ $I_0^*\left(\infty\right), I_{12}\left(0\right), 6I_1\left(\frac{1}{k\pm2},..\right)$ $x_{P_1}=t^2,\quad x_{p_2}=\frac{(tk-1)^2}{k^2-4}$		$\frac{1}{X+Y}$
#24	$y^2+\left(s^2+1\right)tyx=x\left(x-t^2s^2\right)\left(x-s^2t\left(t+1\right)^2\right)$ $2I_2^*\left(\infty,0\right), I_4\left(-1\right), 4I_1$ $x_{P_1}=t+1;\quad x_{P_2}=t^2s^2$		$\frac{Z}{Y}$
#25	$y^2+\left(s+t\right)\left(ts+1\right)yx+t^2s^2\left(t\left(s^2-1\right)+s\right)y=$ $x\left(x-st\right)\left(x-t^2s\left(t-s\right)\right)$ $I_7(\infty), I_{10}(0), 7I_1$ $x_{P_1}=ts;\quad x_{P_2}=-t^2s^2$		$\frac{Y-s}{XY+sZ}$
#26	$y^2+(ts-1)(t-s)xy=x\left(x-t^2s^2\right)^2$ $I_8\left(\infty,0\right), I_2\left(s,\frac{1}{s}\right), 4I_1$ $x_P=ts$		Z
#27	$y^2-\left(t\left(s^2-1\right)+s^2\right)yx+t^3s^2\left(t+1\right)y=$ $x^2\left(x+t^2s^2\left(t+1\right)\right)$ $I_1^*\left(\infty\right), I_8\left(0\right), I_5\left(-1\right)4I_1$ $x_P=0$		$\frac{Z-s}{X+Y}$

TABLE 4. Weierstrass equations of the elliptic fibrations of Y_k

$F_6: l_1 = l_2 = 3+2\sqrt(2)$	$m_6 = \frac{x_1}{x_2}$	$2I_2^*, I_4, 4I_2$
$F_8: l_1 = 3+2\sqrt(2), l_2 = \frac{1}{l_1}$	$m_8 = \frac{(x_2-l_2)(x_1-x_2)}{l_2(l_2-1)x_1(x_1-1)}$	$III^*, I_2^*, I_4, I_2, I_1$
$G_8: l_1 = 3+2\sqrt(2), l_2 = l_1$	m_8	$III^*, I_3^*, 3I_2$
$F_5: l_1 = l_2 = 3+2\sqrt(2)$	$m_5 = \frac{(x_1-x_2)(l_2(x_1-l_1)+(l_1-1)x_2)}{(l_2x_1-x_2)(x_1-l_1+(l_1-1)x_2)}$	$I_6^*, I_4, 4I_2.$

In this paper, $K3$ surfaces are given as elliptic surfaces. If we have a 2-torsion section τ, we consider the symplectic involution i (*Van Geemen-Sarti involution*) given by the fiberwise translation by τ. In this situation, the rational quotient map $X \to Y$ is just an isogeny of degree 2 between elliptic curves over $\mathbb{C}(t)$, and we have a rational map $Y \to X$ of degree 2 as the dual isogeny.

Notation We consider the fibration $\#n$ of Y_k with a Weierstrass equation $E^n : y^2 = x^3 + A(t)x^2 + B(t)x$ and the two-torsion point $T = (0,0)$. We will call $\#n - i$ the elliptic fibration $E^n/\langle(0,0)\rangle$ of the elliptic surface Y_k/i if i denotes the translation by T.

4.2. **Fibrations of some Kummer surfaces.** Let E_l be an elliptic curve with invariant j, defined by a Weierstrass equation in the Legendre form

$$E_l : y^2 = x(x-1)(x-l).$$

Then l satisfies the equation $j = 256\frac{(1-l+l^2)^3}{l^2(l-1)^2}$. For a fixed j the six values of l are given by l or $\frac{1}{l}, 1-l, \frac{l-1}{l}, \frac{-1}{l-1}, \frac{l-1}{l}$.

Consider the Kummer surface K given by $E_{l_1} \times E_{l_2}/\pm 1$ and choose as equation for K

$$x_1(x_1-1)(x_1-l_1)t^2 = x_2(x_2-1)(x_2-l_2).$$

Following [Ku] we can construct different elliptic fibrations. In the general case we can consider the three elliptic fibrations F_i of K defined by the elliptic parameters m_i, with corresponding types of singular fibers

$F_6:$	$m_6 = \frac{x_1}{x_2}$	$2I_2^*, 4I_2$
$F_8:$	$m_8 = \frac{(x_2-l_2)(x_1-x_2)}{l_2(l_2-1)x_1(x_1-1)}$	$III^*, I_2^*, 3I_2, I_1$
$F_5:$	$m_5 = \frac{(x_1-x_2)(l_2(x_1-l_1)+(l_1-1)x_2)}{(l_2x_1-x_2)(x_1-l_1+(l_1-1)x_2)}$	$I_6^*, 6I_2.$

In the special case when $E_1 = E_2$ and $j_1 = j_2 = 8000$ we obtain the following fibrations f_8 (III^*, I_2^*, I_4, I_2, I_1) with $l_1 = 3+2\sqrt{2}$, $l_2 = 3-2\sqrt{2}$ and g_8 (III^*, I_3^*, $3I_2$) with $l_1 = l_2 = 3+2\sqrt{2}$.

4.3. **Nikulin involutions and Kummer surfaces.**

Proposition 4.1. *Consider a family $S_{a,b}$ of $K3$ surfaces with an elliptic fibration, a two torsion section defining an involution i and two singular fibers of type I_4^*,*

$$S_{a,b} : Y^2 = X^3 + \left(t + \frac{1}{t} + a\right)X^2 + b^2X.$$

Then the $K3$ surface $S_{a,b}/i$ is the Kummer surface $(E_1 \times E_2)/(\pm Id)$ where the j_i invariants of the elliptic curves E_i, $i = 1,2$ are given by the formulae

$$j_1j_2 = 4096\frac{\left(a^2 - 3 + 12b^2\right)^3}{b^2}$$

$$(j_2 - 1728)(j_1 - 1728) = \frac{1024a^2(2a^2 - 9 - 72b^2)^2}{b^2}.$$

Proof. Recall that if E_i, $i = 1,2$, are two elliptic curves in the Legendre form

$$E_i : y^2 = x(x-1)(x-l_i),$$

the Kummer surface K

$$K : (E_1 \times E_2)/(\pm Id)$$

is defined by the following equation

$$x_1\,(x_1-1)\,(x_1-l_1)\,t^2 = x_2\,(x_2-1)\,(x_2-l_2)\,.$$

The Kummer surface K admits an elliptic fibration with parameter $u=m_6=\frac{x_1}{x_2}$ and Weierstrass equation H_u

$$H_u: Y^2 = X\,(X-u\,(u-1)\,(ul_2-l_1))\,(X-u\,(u-l_1)\,(l_2u-1))\,.$$

The 2-isogenous curve $S_{a,b}/\langle(0,0)\rangle$ has the following Weierstrass equation

$$Y^2 = X\left(X-t\left(t^2+(a-2b)\,t+1\right)\right)\left(X-t\left(t^2+(a+2b)\,t+1\right)\right)$$

with two singular fibers of type I_2^* above 0 and ∞.
We easily prove that $S_{a,b}/\langle(0,0)\rangle$ and H_u are isomorphic on the field $\mathbb{Q}\left(\sqrt{w_2}\right)$ where

$$l_1 = w_1'w_2 = \frac{w_2}{w_1}\ ,\quad l_2 = \frac{1}{w_1'w_2'} = w_1w_2 \text{ and } \ t = w_1u,$$

w_1, w_1' and w_2, w_2' being respectively the roots of polynomials $t^2+(a-2b)\,t+1$ and $t^2+(a+2b)\,t+1$.
Recall that the modular invariant j_i of the elliptic curve E_i is linked to l_i by the relation

$$j_i = 256\frac{\left(1-l_i+l_i^2\right)^3}{l_i^2\,(1-l_i)^2}.$$

By elimination of w_1 and w_2, it follows the relations between j_1 and j_2

$$j_1j_2 = 4096\frac{\left(a^2-3+12b^2\right)^3}{b^2}$$
$$(j_2-1728)\,(j_1-1728) = \frac{1024a^2(2a^2-9-72b^2)^2}{b^2}.$$

□

In the Fermi family, the $K3$ surface Y_k has the fibration #16 with two singular fibers I_4^*, a 2-torsion point and Weierstrass equation

$$y^2 = x^3 + x^2t\left(4\left(t^2+s^2\right)+t\left(s^4+14s^2+1\right)\right)+16t^4s^6x.$$

Taking

$$y = y't^3\left(2\sqrt{s}\right)^3,\quad x = x't^2\left(2\sqrt{s}\right)^2 \text{ and } t = t's,$$

we obtain the following Weierstrass equation

$$y'^{é} = x'^3 + \left(t' + \frac{1}{t'} + \frac{1}{4}\frac{s^4+14s^2+1}{s}\right) + s^4x'.$$

By the previous proposition with $a=\frac{1}{4}\frac{s^4+14s^2+1}{s}$, $\quad b=s^2$, we derive the corollary below.

Corollary 4.1. *The surface obtained with the* 2*-isogeny of kernel* $\langle(0,0)\rangle$ *from fibration #16, is the Kummer surface associated to the product of two elliptic curves of* j*-invariants* j_1, j_2 *satifying*

$$j_1j_2 = \frac{\left(s^2+1\right)^3\left(s^6+219s^4-21s^2+1\right)^3}{s^{10}}$$
$$\left(j_1-12^3\right)\left(j_2-12^3\right) = \frac{(s^4+14s^2+1)^2(s^8-548s^6+198s^4-44s^2+1)^2}{s^{10}}.$$

Remark 4.1. *If* $s=1$ *we find* $j_1=j_2=8000$.

Remark 4.2. *If* $b=1$ *we obtain the family of surfaces studied by Narumiya and Shiga,* [Na]. *Morover if* $a=\frac{9}{4}$ *(resp.* 4*) we find the two modular surfaces associated to the modular groups* $\Gamma_1(7)$ *(resp.* $\Gamma_1(8)$*). In these two cases we get* $j_1=j_2=-3375$ *(resp.* $j_1=j_2=8000$*).*

Remark 4.3. *With the same method we can consider a family of $K3$ surfaces with Weierstrass equations*

$$E_v : Y^2 + XY - (v + \frac{1}{v} - k)Y = X^3 - (v + \frac{1}{v} - k)X^2,$$

singular fibers of type $2I_1^*, 2I_4, 2I_1$ and the point $P_v = (0,0)$ of order 4. The elliptic curve $E'_v = E_v/\langle 2P_v\rangle$ has singular fibers of type $2I_2^*, 4I_2$. An analog computation gives $E'_v \equiv (E_1 \times E_2) / (\pm Id)$ and

$$j_1 j_2 = (256k^2 - 16k - 767)^3$$
$$\left(j_1 - 12^3\right)\left(j_2 - 12^3\right) = (32k-1)^2(128k^2 - 8k - 577)^2.$$

4.4. **Shioda-Inose structure.**

Definition 4.1. *A $K3$ surface X has a Shioda-Inose structure if there is a symplectic involution i on X with rational quotient map $X \overset{p}{\to} Y$ such that Y is a Kummer surface and p^* induces a Hodge isometry $T_X(2) \simeq T_Y$.*
Such an involution i is called a Morrison-Nikulin involution.

An equivalent criterion is that X admits a (Nikulin) involution interchanging two orthogonal copies of $E_8(-1)$ in $NS(X)$, where $E_8(-1)$ is the unique unimodular even negative-definite lattice of rank 8.
Or even more abstractly: $2E_8 \hookrightarrow NS\,(X)$.
Applying this criterion to fibrations #17 and #8 and the Van Geemen-Sarti involution we get the following result.

Proposition 4.2. *The translation by the two torsion point of fibration #17 and #8 endowes Y_k with a Shioda-Inose structure.*

Fibration #17 has a fiber of type I_{16} at $t = \infty$. The idea [G] is to use the components $\Theta_{-2}, \Theta_{-1}, \Theta_0$, Θ_1, $\Theta_2, \Theta_3, \Theta_4$ of I_{16} and the zero section to generate a lattice of type E_8. The two-torsion section intersects Θ_8 and the translation by the two-torsion point on the fiber I_{16} transforms Θ_n in Θ_{n+8}. The translation maps the lattice E_8 on an another disjoint E_8 lattice and defines a Shioda-Inose structure.
For fibration #8, the fiber above $t = 0$ is of type I_6 and the section of order 2 specialises to the singular point $(0,0)$. Then after a blow up, it will not meet the 0-component. If we denote $\Theta_{0,i}$, $0 \leq i \leq 5$, the six components, then the zero section meets $\Theta_{0,0}$ and the 2-torsion section meets $\Theta_{0,3}$. The translation by the 2-torsion section induces the permutation $\Theta_{0,i} \to \Theta_{0,i+3}$.
The fiber above $t = \infty$ is of type I_6^*. The simple components are denoted $\Theta_{\infty,0}, \Theta_{\infty,1}$ and $\Theta_{\infty,2}, \Theta_{\infty,3}$; the double components are denoted C_i with $0 \leq i \leq 6$ and $\Theta_{\infty,0}.C_0 = \Theta_{\infty,1}.C_0 = 1$; $\Theta_{\infty,2}.C_6 = \Theta_{\infty,3}.C_6 = 1$. Then the 2-torsion section intersects $\Theta_{\infty,2}$ or $\Theta_{\infty,3}$ and the translation by the 2-torsion section induces the transposition $C_i \longleftrightarrow C_{6-i}$.
The class of the components $C_0, C_1, C_2, \Theta_{\infty,0}, \Theta_{\infty,1}$, the zero section, $\Theta_{0,0}$ and $\Theta_{0,1}$ define a $E_8\,(-1)$. The Nikulin involution defined by the two torsion section maps this $E_8\,(-1)$ to another copy of $E_8\,(-1)$ orthogonal to the first one; so the Nikulin involution is a Morrison-Nikulin involution.

4.5. **Base change and van Geemen-Sarti involutions.** If a $K3$-surface X has an elliptic fibration with two fibers of type II^*, this fibration can be realised by a Weierstrass equation of type

$$y^2 = x^3 - 3\alpha x + (h + 1/h - 2\beta).$$

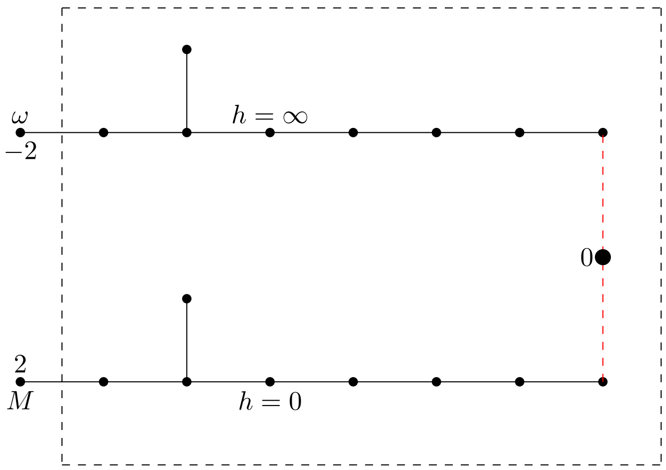
Moreover Shioda [Shio] deduces the “Kummer sandwiching”, $\mathrm{K} \to S \to \mathrm{K}$, identifying the Kummer $\mathrm{K} = E_1 \times E_2/\pm 1$ with the help of the j-invariants of the two elliptic curves E_1, E_2 and giving the following elliptic fibration of K

$$y^2 = x^3 - 3\alpha x + (t^2 + 1/t^2 - 2\beta).$$

This can be viewed as a base change of the fibration of X.

4.5.1. *Alternate elliptic fibration.* We shall now use an alternate elliptic fibration ([Sc-Shio] example 13.6) to show that this construction is indeed a 2-isogeny between two elliptic fibrations of S and K. In the next picture we consider a divisor D of type I_{12}^* composed of the zero section 0 and the components of the II^* fibers enclosed in dashed lines. The far double components of the II^* fibres can be chosen as sections of the new fibration. Take ω as the zero section. The other one is a two-torsion point since the function h has a double pole on ω and a double zero at M. It is the function $'x'$ in a Weierstrass equation. More precisely with the new parameter $u = x$ and the variables $Y = yh$ and $X = h$, we obtain the Weierstrass equation

$$Y^2 = X^3 + (u^3 - 3\alpha u - 2\beta)X^2 + X.$$



In this equation, if we substitute $X(= h)$ by t^2, we obtain an equation in W, t with $Y = Wt^2$, which is the equation for the 2-isogenous elliptic curve. Indeed the birational transformation

$$y = 4\,Y + 4\,U^3 + 2\,UA, x = 2\,\frac{Y + U^3}{U}$$

with inverse

$$U = 1/2\,\frac{y}{x + A}, Y = 1/8\,\frac{\left(-y^2 + 2\,x^3 + 4\,x^2A + 2\,xA^2\right)y}{\left(x + A\right)^3}$$

transforms the curve $Y^2 = U^6 + AU^4 + BU^2$ in the Weierstrass form

$$y^2 = (x + A)(x^2 - 4B).$$

This is an equation for the 2-isogenous curve of the curve $Y^2 = X^3 + AX^2 + BX$ [Si]. On the curve $Y^2 = U^6 + AU^4 + BU^2$, the involution $U \mapsto -U$ means adding the two-torsion point $(x = -A, y = 0)$. Using this above process with $A = (u^3 - 3\alpha u - 2\beta)$, the 2-isogenous curve E_u has a Weierstrass equation

$$Y^2 = (X + (u^3 - 3\alpha u - 2\beta))(X^2 - 4)$$

with singular fibers of type $I_6^*, 6I_2$.
The coefficients α and β can be computed using the j-invariants

$$\alpha^3 = J_1J_2;\;\; \beta^2 = (1 - J_1)(1 - J_2);\;\; j_i = 1728J_i.$$

If the elliptic curve is put in the Legendre form $y'^2 = x'(x' - 1)(x' - l)$ then $j = 256\frac{(1-l+l^2)^3}{l^2(l-1)^2}$, so

$$\begin{aligned}\alpha^3 =& \frac{16}{729}\,\frac{(1 - l_1 + l_1^2)^3(1 - l_2 + l_2^2)^3}{l_1^2(l_1 - 1)^2 l_2^2(l_2 - 1)^2}\\ \beta =& \frac{1}{27}\,\frac{(2l_1 - 1)(l_1 - 2)(2l_2 - 1)(l_2 - 2)(l_1 + 1)(l_2 + 1)}{l_1l_2(l_1 - 1)(l_2 - 1)}.\end{aligned}$$

On the Kummer surface $E_1 \times E_2/\pm 1$ of equation

$$X_1\,(X_1-1)\,(X_1-l_1)\,Z^2 = X_2\,(X_2-1)\,(X_2-l_2)$$

we consider an elliptic fibration (case $\mathcal{J}_5$ of [Ku]) with the parameter $z=\frac{(l_2X_1-X_2)(X_1-l_1+X_2(l_1-1))}{X_2(X_1-1)}$ (in fact $z=-\frac{l_1(l_2-1)}{m_5-1}$ cf. 4.2) and obtain the Weierstrass equation

$$Y^2{=}(X-2l_1l_2(l_1-1)(l_2-1))(X+2l_1l_2(l_1-1)(l_2-1))$$
$$\big(X+4z^3+4(-2l_1l_2+l_1+l_2+1)z^2+4(l_1l_2-1)(l_1l_2-l_1-l_2)z+2l_1l_2(l_1-1)(l_2-1)\big).$$

Substituting $z=w-\frac{1}{3}\,(-2l_1l_2+l_1+l_1+1)$ it follows

$$Y^2 = (X-2l_1l_2(l_1-1)(l_2-1))(X+2l_1l_2(l_1-1)(l_2-1))$$
$$\big(X+4w^3-\tfrac{4}{3}\big(l_2^2-l_2+1\big)\big(l_1^2-l_1+1\big)w+\tfrac{2}{27}(l_2-2)(2l_2-1)(l_1-2)(2l_1-1)(l_2+1)(l_1+1)\big).$$

Up to an automorphism of this Weierstrass form we recover the equation of E_u.
The previous results can be used to show the following proposition

Proposition 4.3. *The translation by the two torsion point of the elliptic fibration #4 gives to Y_k a Shioda-Inose structure.*

5. Proof of Theorem 1.1

We consider an elliptic fibration $\#n$ of Y_k with a two torsion section.
From the Shioda-Tate formula (cf. e.g. [Shio], Corollary 1.7]) we have the relation

$$12=\frac{|\Delta|\prod m_v^{(1)}}{|\mathrm{Tor}|^2}$$

where Δ is the determinant of the height-matrix of a set of generators of the Mordell-Weil group, $m_v^{(1)}$ the number of simple components of a singular fiber and $|\mathrm{Tor}|$ the order of the torsion group of Mordell-Weil group. This formula allows us to determine generators of the Mordell-Weil group except for fibration #4. Using the 2-isogeny we determine also the Mordell-Weil group of $\#n$-i. The discriminant is either 12×2 or 12×8.

Proposition 5.1. *The translation by the two torsion point of the fibration #16 gives to Y_k a Shioda-Inose structure.*

From the previous Proposition 4.1, the translation by the two torsion point of #16 gives to the quotient a Kummer structure. The fibration #16 is of rank one, its Mordell-Weil group is generated by P and the two torsion point. By computation we can see that the Mordell-Weil group of the 2-isogenous curve on $E(\mathbb{C}\,(t))$ is generated by $p\,(P)$ and torsion sections. So we can compute the discriminant of the Néron-Severi group which is 12×8. The second condition, $T_X(2)\simeq T_Y$, is then verified.

Remark 5.1. *The $K3$ surface of Picard number 20 given with the elliptic fibration*

$$Y^2=X^3-\left(t+\frac{1}{t}-\frac{3}{2}\right)X^2+\frac{1}{16}X$$

or

$$y^2=x^3-1/2\,t\,\big(2\,t^2+2-3\,t\big)\,x^2+1/16\,t^4x$$

has rank 1. The Mordell-Weil group is generated by $(0,0)$ and $P=(x=\frac{1}{4},y=\frac{(t-1)^2}{8})$. The determinant of the Néron-Severi group is equal to 12. By computation we have $p\,(P)=2Q$ with $Q=\big(t(t-1)(t^2-t+1),-t^3(t-1)(t^2-t+1)\big)$ of height $\frac{3}{4}$. The determinant of the Néron Severi group of the 2-isogenous curve is then 12 not 12×2^2. So the involution induced by the two-torsion point is not a Nikulin-Morrison involution. Moreover the 2-isogenous elliptic curve is a fibration of the Kummer surface $E\times E/\pm 1$ where $j(E)=0$.

For fibrations $\#n$-i with discriminant of the transcendental lattice 12×8 we prove that we have the Shioda-Inose structure in the following way: from corollary 4.1 this is true for #16-i, from Proposition 4.3 this is true for #4-i and from Proposition 4.2 this is true for #17-i, #8-i. The other fibrations $\#n$-i can be obtained by 2- or 3-neighbor method from #16-i, #8-i or #17-i. The results are given in the Table 5. In the second column are written the Weierstrass equations for the $\#n$ elliptic fibration and its 2-isogenous fibration, singular fibers and the x-coordinates of generators of the Mordell lattice of $\#n$-i. In the third column we give the starting fibration for the 2- or 3-neighbor method and in the last column the parameter used from the starting fibration.

5.1. **The $K3$ surface S_k.** For the remaining fibrations, (discriminant 12×2), using also the 2- or 3-neighbor method, they are proved to lie on the same surface S_k. Except for the case #7 the results are collected in the Table 6 with the same format. The case #7 needs an intermediate fibration explained in the next paragraph.
Starting with fibration #7-i and using the parameter $m_7 = \frac{y}{xt(t-s^2)}$ it follows the Weierstrass equation

$$Y^2+2\left(m_7^2s^2-2\right)YX-16m_7^4s^4Y=\left(X-8m_7^2s^2\right)$$
$$\left(X+8m_7^2s^2\right)\left(X+m_7^2\left(s^4-6s^2+1\right)-4\right)$$

with singular fibers $I_8\left(\infty\right), IV^*\left(0\right), 8I_1$.
Then the parameter $m_{15}=\frac{Y}{\left(X+8m_7^2s^2\right)}$ leads to the fibration #15-i.
For the last part of Theorem 1.1 we give properties of S_k. First we prove that S_k is the Jacobian variety of some genus 1 fibrations of K_k.
Starting with the fibration #26-i and Weierstrass equation

$$y^2=x\left(x+4t^2s^2\right)\left(x+\frac{1}{4}\left(t-s\right)^2\left(ts-1\right)^2\right)$$

the new parameter $m:=\frac{y}{t\left(x+\frac{1}{4}(t-s)^2(ts-1)^2\right)}$ defines an elliptic fibration of #26-i with Weierstrass equation

$$E_m: Y^2-m\left(s^2+1\right)YX=X\left(X-s^2m^2\right)\left(X+\frac{1}{4}\left(2m-s\right)^2\left(2m+s\right)^2\right)$$

and singular fibers are of type $4I_4\left(0,\pm\frac{1}{2}s,\infty\right), 8I_1$.
Then setting as new parameter $n=\frac{X}{m^2}$, it follows a genus one curve in m and Y. Its equation, of degree 2 in Y, can be transformed in

$$w^2=-16n(-n+s^2)m^4+n(s^4\left(8+n\right)-10ns^2+n\left(1+4n\right))m^2-ns^4(-n+s^2).$$

Let us recall the formulae giving the jacobian of a genus one curve defined by the equation $y^2 = ax^4+bx^3+cx^2+dx+e$. If $c_4=2^4(12ae-3bd+c^2)$ and $c_6=2^5(72ace-27ad^2-27b^2e+9bcd-2c^3)$, then the equation of the Jacobian curve is

$$\bar{y}^2=\bar{x}^3-27c_4\bar{x}-54c_6.$$

In our case we obtain

$$y^2=x\left(x+n^3s^2-\frac{1}{4}n^2\left(s^2-1\right)^2\right)$$
$$\left(x+n^3s^2-\frac{1}{4}\left(s^2-4s-1\right)\left(s^2+4s-1\right)n^2+4ns^2\right),$$

which is precisely the fibration #15-i.

No	Weierstrass Equation	From	Param.
#4	see Prop 9		
#8	$y^2 - k(t-1)yx = x(x-1)(x-t^3)$ $I_6^*(\infty), I_6(0), I_2(1), 4I_1$ $y^2 = x^3 + \frac{1}{2}(4t^3 - t^2k^2 + 2tk^2 + 4 - k^2)x^2$ $+\frac{1}{16}(t-1)^2\left(4t^2 + t(4-k^2) + (k-2)^2\right)$ $\left(4t^2 + t(4-k^2) + (k+2)^2\right)x$ $I_3^*(\infty), I_3(0), I_4(1), 4I_2$ $x_Q = -\frac{1}{4}(t-1)\left(4t^2 + t(4-k^2) + (k-2)^2\right)$		
#16	$y^2 = x^3 + t(4(t^2+s^2) + t(s^4+14s^2+1))x^2 + 16s^6t^4x$ $2I_4^*(\infty,0), 4I_1$ $y^2 = x(x - t(4(t^2+s^2) + t(s^4+14s^2+1+8s^3)))$ $(x - t(4(t^2+s^2) + t(s^4+14s^2+1-8s^3)))$ $2I_2^*(\infty,0), 4I_2$ $x_Q = \frac{t^2((t^2+s^2)(3+s^2)+t(-s^4+8s^2+1))^2}{(t+1)^2(t+s^2)^2}$		
#17	$y^2 - \frac{1}{2}(s^4+14s^2+1-s^2t^2)yx = x(x-4s^2)(x-4s^6)$ $I_{16}(\infty), 8I_1\left(\pm\frac{s^2\pm 4s-1}{s}, ..\right)$ $y^2 = x(x - (t^2s^2 - (s^4+14s^2+1) \pm 8s(s^2+1)))$ $(x - (ts+s^2\pm 4s-1)(ts-s^2\pm 4s+1))$ $I_8(\infty), 8I_2\left(\pm\frac{s^2\pm 4s-1}{s}, ..\right)$ $x_{Q_1} = \frac{1}{16}(ts+s^2-4s-1)(ts-s^2+4s+1)$ $(t^2s^2 - (s^4+14s^2+1) + 8s(s^2+1))$ $x_{Q_2} = \frac{1}{16}\frac{(s-1)^2}{(s+1)^2}(ts+s^2+4s-1)$ $(ts-s^2+4s+1)(t^2s^2 - (s^4+14s^2+1) - 8s(s^2+1))$		
#23	$y^2 + (2t^2 - tk + 1)yx = x(x-t^2)(x-t^4)$ $I_0^*(\infty), I_{12}(0), 6I_1\left(\frac{1}{k\pm 2}, ..\right)$ $y^2 = x\left(x + \frac{1}{4}(t(k-2)-1)(4t^2 - (k+2)t + 1)\right)$ $\left(x + \frac{1}{4}(t(k+2)-1)(4t^2 - (k-2)t + 1)\right)$ $I_0^*(\infty), I_6(0), 6I_2\left(\frac{1}{k\pm 2}, ..\right)$ $x_{Q_1} = \frac{-1}{4}(4t^2 - (k-2)t + 1)(t(k-2)-1);$ $x_{Q_2} = \frac{-1}{4}\frac{k-2}{k+2}(t(k+2)-1)(4t^2 - (k-2)t + 1)$	#8 #26 #24	$\frac{y - y_{2Q} + \frac{k}{2}(x - x_{2Q})}{t(x - x_{2Q})}$ $\frac{2y - (t-s)(ts-1)x}{t(x - ts(ts-1)^2)}$ $\frac{(y - y_{2Q_2}) + \frac{s^2+1}{2}(x - x_{2Q_2})}{(t+1)(x - x_{2Q_2})}$
#24	$y^2 + (s^2+1)tyx = x(x - t^2s^2)\left(x - s^2t(t+1)^2\right)$ $2I_2^*(\infty,0), I_4(-1), 4I_1$ $y^2 = x^3 + \frac{1}{2}t(4t^2s^2 - t(s^4-10s^2+1) + 4s^2)x^2$ $+\frac{1}{16}t^2\left(4t^2s^2 + \left(8s^2 - (s-1)^4\right)t + 4s^2\right)$ $\left(4t^2s^2 + \left(8s^2 - (s+1)^4\right)t + 4s^2\right)x$ $2I_1^*(\infty,0), 5I_2(-1,)$ $x_{Q_1} = \frac{1}{4}(2t^2s^2 + t(s^2-1) - 2)^2;$ $x_{Q_2} = -\frac{1}{4}t\left(4t^2s^2 + \left(8s^2 - (s+1)^4\right)t + 4s^2\right)$		
#26	$y^2 + (ts-1)(t-s)xy = x(x - t^2s^2)^2$ $2I_8(\infty,0), I_2\left(s, \frac{1}{s}\right), 4I_1$ $y^2 = x(x + 4t^2s^2)\left(x + \frac{1}{4}(t-s)^2(st-1)^2\right)$ $I_4\left(\infty, 0, s, \frac{1}{s}\right), 4I_2$ $x_{Q_1} = ts(ts-1)^2$		

TABLE 5. Fibrations with discriminant 12×8 (Fibrations of the Kummer K_k)

No	Weierstrass Equation	From	Param.
#7	$y^2 = x^3 + \frac{1}{4}t\left(t\left(s^4-10s^2+1\right)+8s^4\right)x^2 - t^2s^2\left(t-s^2\right)^3 x$ $III^*(\infty), I_1^*(0), I_6\left(s^2\right), 2I_1$ $y^2 = x^3 - \frac{1}{2}t\left(t\left(s^4-10s^2+1\right)+8s^4\right)x^2$ $+\frac{1}{16}t^3\left(64t^2s^2+\left(s^8-20s^6-90s^4-20s^2+1\right)t+16s^4\left(s^2+1\right)^2\right)x$ $III^*(\infty), I_2^*(0), I_3\left(s^2\right), 2I_2$		
#9	$y^2 = x^3 + \frac{1}{4}\left(s^4+14s^2+1\right)t^2x^2 + t^3s^2\left(s^2+t\right)\left(ts^2+1\right)x$ $2III^*(\infty,0), 2I_2\left(-s^2,-\frac{1}{s^2}\right), 2I_1$ $y^2 = x^3 - \frac{1}{2}\left(s^4+14s^2+1\right)t^2x^2$ $-\frac{1}{16}t^3\left(64t^2s^4+t\left(-s^8+36s^6-198s^4+36s^2-1\right)+64s^4\right)x$ $2III^*(\infty,0), 2I_1\left(-s^2,-\frac{1}{s^2}\right), 2I_2$ $x_Q = \frac{1}{4}\frac{(t+1)^2\left(2t^2s^2+t\left(s^4-6s^2+1\right)+2s^2\right)^2}{\left(s^2-1\right)^2(t-1)^2}$	#20	$\frac{y}{(ts-1)^4}$
#14	$y^2 = x^3 + \left(t^3\left(s^4+1\right)+\frac{1}{4}t^2\left(s^4+14s^2+1\right)+ts^2\right)x^2 + t^6s^4x$ $I_0^*(\infty), I_8^*(0), 4I_1\left(\frac{-1}{4}, \frac{-4s^2}{\left(s^2-1\right)^2}, ..\right)$ $y^2 = x\left(x-\frac{1}{4}\left(s^2+1\right)^2t^3-\frac{1}{4}\left(s^4+14s^2+1\right)t^2-ts^2\right)$ $\left(x-\frac{1}{4}\left(s^2-1\right)^2t^3-\frac{1}{4}\left(s^4+14s^2+1\right)t^2-ts^2\right)$ $I_0^*(\infty), I_4^*(0), 4I_2\left(\frac{-1}{4}, \frac{-4s^2}{\left(s^2-1\right)^2}, ..\right)$ $x_Q = \frac{1}{4}t^2\left(s^2-1\right)^2(4t+1)$	#15	$\frac{t^2s^2}{x+t^3s^2-\frac{1}{4}t^2\left(s^2-1\right)^2}$
#15	$(y-tx)\left(y-s^2tx\right) = x\left(x-ts^2\right)\left(x-ts^2(t+1)^2\right)$ $I_4^*(\infty), I_1^*(0), I_4(-1), 3I_1\left(\frac{1}{4}\left(\frac{s^2-1}{s}\right)^2, ..\right)$ $y^2 = x\left(x+t^3s^2-\frac{1}{4}t^2(s^2-1)^2\right)$ $\left(x+t^3s^2-\frac{1}{4}(s^2-4s-1)(s^2+4s-1)t^2+4ts^2\right)$ $I_2^*(\infty,0), 4I_2\left(-1, \frac{1}{4}\left(\frac{s^2-1}{s}\right)^2, ..\right)$ $x_Q = \frac{1}{4}t^2(s^2-1)^2$	#20	$\frac{y}{x(t-s)^2}$
#20	$y^2 - (t^2s-\left(s^2+1\right)t+3s)yx - s^2(t-s)(ts-1)y = x^3$ $I_{12}(\infty), 2I_3\left(s, \frac{1}{s}\right), 2I_2\left(0, \frac{s^2+1}{s}\right), 2I_1$ $y^2 + \left(t^2s-(s^2+1)t-3s\right)yx - s(t-s)^2(ts-1)^2y = x^3$ $3I_6\left(\infty, s, \frac{1}{s}\right), 2I_2, 2I_1\left(0, \frac{s^2+1}{s}\right)$		

TABLE 6. Fibrations with discriminant 12×2 (Fibrations of S_k)

Remark 5.2. *Using the new parameter* $p = \frac{Y}{m^2\left(X+\frac{1}{4}(2m-s)^2(2m+s)^2\right)}$ *another result can be derived from* E_m *leading to*

$$E_p : Y^2 - 2s(2p-1)(2p+1)YX = X\left(X+64s^2p^2\right)\left(X+(2sp+1)(2sp-1)(s+2p)(s-2p)\right),$$

with singular fibers $2I_0^*, 4I_2, 4I_1$. *From* E_p *and the new parameter* $k = \frac{X}{p^2}$ *we obtain a genus one fibration whose jacobian is #14-i.*
Starting from the fibration $\#26-i$, *the parameter* $q = \frac{x}{t^2}$ *leads to a genus one fibration whose jacobian is the fibration #20-i.*

5.2. **Transcendental and Néron-Severi lattices of the surface S_k.**

Lemma 5.1. *The five fibrations* $\#7-i$, $\#9-i$, $\#14-i$, $\#15-i$, $\#20-i$ *are fibrations of the same* $K3$ *surface* S_k *with transcendental lattice*

$$T_{S_k} = \langle(-2)\rangle \oplus \langle 2\rangle \oplus \langle 6\rangle$$

and Néron-Severi lattice

$$NS(S_k) = U \oplus E_8(-1) \oplus E_7(-1) \oplus \langle(-2)\rangle \oplus \langle(-6)\rangle.$$

Moreover these fibrations specialise in fibrations of Y_2 for $k = 2$.

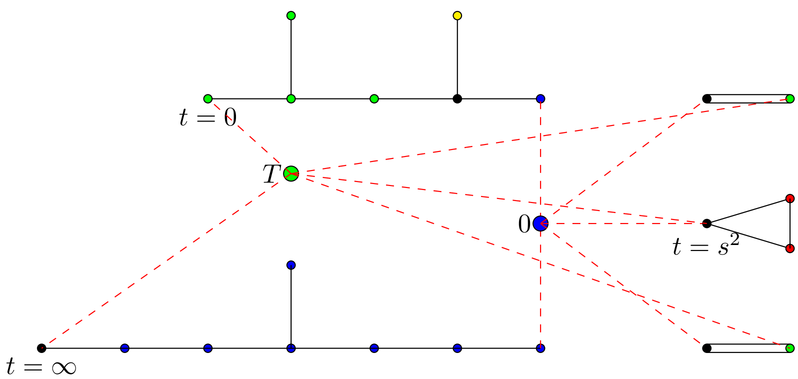
Proof. These five fibrations are respectively the fibrations given in Table 6 and recalled below with the type of their singular fibers, their rank and torsion group:

$\#7-i$	$2A_1A_2D_6E_7$	rk 0	$\mathbb{Z}/2\mathbb{Z}$
$\#9-i$	$2A_12E_7$	rk 1	$\mathbb{Z}/2\mathbb{Z}$
$\#14-i$	$4A_1D_4D_8$	rk 1	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$
$\#15-i$	$4A_12D_6$	rk 1	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$
$\#20-i$	$2A_13A_5$	rk 0	$\mathbb{Z}/6\mathbb{Z}.$

We already know from the above results that they are fibrations of S_k but we give below another proof of this fact. Denote S_k the $K3$ surface defined by the elliptic fibration #7-i with Weierstrass equation given in Table 6 and draw the graph of the singular fibers, the zero and two-torsion sections of the elliptic fibration

$$Y^2 = X^3 + \left(\left(-1/2\,s^4 + 5\,s^2 - 1/2\right)t^2 - 4\,s^4 t\right)X^2$$
$$\left(4\,s^2t^5 + \left(1/16\,s^8 - \frac{45}{8}\,s^4 - 5/4\,s^2 - 5/4\,s^6 + 1/16\right)t^4 + s^4\left(s^2+1\right)^2 t^3\right)X$$

with singular fibers $III^*(\infty), I_2^*(0), I_3(s^2), 2I_1(t_1, t_2)$.



With the parameter $m = \frac{X}{t}$ we obtain another fibration with singular fibers $II^*(\infty)$ (in blue), $I_2^*(0)$ (in green), $I_3(\frac{1}{4}s^2(s^2-1)^2)$ (part of it in red), $I_2(4s^4)$, $I_1(\sigma_0)$ (in yellow), where $\sigma_0 = -\frac{(s^2-6s+1)(s^2+6s+1)(s^2+1)^4}{1728s^2}$.

This new fibration Σ_k has no torsion, rank 0, Weierstrass equation

$$\begin{aligned} y^2 &= x^3 + 2m((-s^4+10s^2-1)m + 2s^4(s^2+1)^2)x^2 \\ &\quad + \left(m - 4s^4\right)m^3\left(\left(s^8 - 20s^6 - 90s^4 - 20s^2 + 1\right)\right)x + 256m^5s^2(m-4s^4)^2 \end{aligned}$$

and Néron-Severi group

$$NS(\Sigma_k) = U \oplus E_8 \oplus D_6 \oplus A_2 \oplus A_1.$$

By Morrison ([M], Corollary 2.10 ii), the Néron-Severi group of an algebraic $K3$ surface X with $12 \leq \rho(X) \leq 20$ is uniquely determined by its signature and discriminant form. Thus we compute $q_{NS(S_k)}$ with the help of the fibration Σ_k. From

$$D_6^*/D_6 = \langle[1]_{D_6}, [3]_{D_6}\rangle \quad \text{and} \quad q_{D_6}([1]_{D_6}) = q_{D_6}([3]_{D_6}) = -\frac{3}{2},$$

we deduce the discriminant form, since $b_{D_6}([1]_{D_6},[3]_{D_6})=0$,

$$\begin{aligned}(G_{NS(S_k)},q_{NS(S_k)}) &= \mathbb{Z}/2\mathbb{Z}(-\frac{3}{2})\oplus\mathbb{Z}/2\mathbb{Z}(-\frac{3}{2})\oplus\mathbb{Z}/3\mathbb{Z}(-\frac{2}{3})\oplus\mathbb{Z}/2\mathbb{Z}(-\frac{1}{2}) \mod. 2\mathbb{Z}\\ &= \mathbb{Z}/2\mathbb{Z}(\frac{1}{2})\oplus\mathbb{Z}/6\mathbb{Z}(-\frac{1}{6})\oplus\mathbb{Z}/2\mathbb{Z}(-\frac{1}{2}).\end{aligned}$$

From Morrison ([M] Theorem 2.8 and Corollary 2.10) there is a unique primitive embedding of $NS(S_k)$ into the $K3$-lattice $\Lambda=E_8(-1)^2\oplus U^3$, whose orthogonal is by definition the transcendental lattice T_{S_k}. Now from Nikulin([Nik] Proposition 1.6.1), it follows

$$G_{NS(S_k)}\simeq(G_{NS(S_k)})^{\perp}=G_{T_{S_k}},\quad q_{T_{S_k}}=-q_{NS(s_k)}.$$

In other words the discriminant form of the transcendental lattice is

$$(G_{T_{S_k}},q_{T_{S_k}})=\mathbb{Z}/2\mathbb{Z}(-\frac{1}{2})\oplus\mathbb{Z}/6\mathbb{Z}(\frac{1}{6})\oplus\mathbb{Z}/2\mathbb{Z}(\frac{1}{2}).$$

From this last relation we prove that $T_{S_k}=\langle-2\rangle\oplus\langle6\rangle\oplus\langle2\rangle$. Denoting T' the lattice $T'=\langle-2\rangle\oplus\langle6\rangle\oplus\langle2\rangle$, we observe that T' and T_{S_k} have the same signature and discriminant form. Since $|\det(T')|=24$ is small, there is only one equivalence class of forms in a genus, meaning that such a transcendental lattice is, up to isomorphism, uniquely determined by its signature and discriminant form ([Co] p. 395).

Now computing a primitive embedding of T_{S_k} into Λ, since by Morrison ([M] Corollary 2.10 i) this embedding is unique, its orthogonal provides $NS(S_k)$. Take the primitive embedding $\langle(-2)\rangle=\langle e_2\rangle\hookrightarrow E_8$, $\langle2\rangle=\langle u_1+u_2\rangle\hookrightarrow U$, $\langle6\rangle=\langle u_1+3u_2\rangle\hookrightarrow U$, (u_1,u_2) denoting a basis of U. Hence we deduce

$$NS(S_k)=U\oplus E_8(-1)\oplus E_7(-1)\oplus(-2)\oplus(-6).$$

Using their Weierstrass equations and a 2-neighbor method [El], it was proved in the previous subsection that all the fibrations #7-i, #9-i, #14-i, #15-i, #20-i are on the same $K3$-surface. We can recover this result, since we know the transcendental lattice, using the Kneser-Nishiyama method.

In that purpose, embed $T_{S_k}(-1)$ into $U\oplus E_8$ in the following way: $(-2)\oplus(-6)$ primitively embedded in E_8 as in Nishiyama ([Nis] p. 334) and $\langle2\rangle=\langle u_1+u_2\rangle\hookrightarrow U$. We obtain $M=(T_{S_k}[-1])^{\perp}_{U\oplus E_8}=A_1\oplus A_1\oplus A_5$. Now all the elliptic fibrations of S_k are obtained from the primitive embeddings of M into the various Niemeier lattices, as explained in section 2.

We identify some of these elliptic fibrations with fibrations #7-i, #9-i, #14-i, #15-i, #20-i in exhibiting their torsion and infinite sections as explained in Bertin-Lecacheux [BL1], computing contributions and heights using [Sc-Shio] p. 51-52. Using the Weierstrass equations given in Table 6, we compute the different local contributions and heights. Finally the identification is performed using [Sc-Shio] (11.9).

5.2.1. Take the primitive embedding into $Ni(D_{10}E_7^2)$, given by $A_5=\langle e_2,e_4,e_5,e_6,e_7\rangle\hookrightarrow E_7$ and $A_1^2=\langle d_{10},d_7\rangle\hookrightarrow D_{10}$.

Since $(A_5)^{\perp}_{E_7}=A_2$ and $(A_1^2)^{\perp}_{D_{10}}=A_1\oplus A_1\oplus D_6$, it follows $N=N_{\text{root}}=2A_1A_2D_6E_7$, $\det N=24\times4$, thus the rank is 0 and the torsion group $\mathbb{Z}/2\mathbb{Z}$. Hence this fibration can be identified with the elliptic fibration #7-i.

5.2.2. The primitive embedding is into $Ni(D_{10}E_7^2)$, given by

$$A_5\oplus A_1^2=$$
$$\langle d_{10},d_8,d_7,d_6,d_5,d_{10}+d_9+2(d_8+d_7+d_6+d_5+d_4)+d_3,d_3\rangle\hookrightarrow D_{10}.$$

We get

$$(A_5\oplus A_1^2)^{\perp}_{D_{10}}=(-6)\oplus\langle x\rangle\oplus\langle d_1\rangle=(-6)\oplus A_1\oplus A_1$$

with

$$x=d_9+d_{10}+2(d_8+d_7+d_6+d_5+d_4+d_3+d_2)+d_1$$

and

$$(-6)=3d_9+2d_{10}+4d_8+3d_7+2d_6+d_5.$$

$0+2F+$	Cont. A_1	Cont. A_1	Cont. A_1	Cont. A_1	Cont. D_4	Cont. D_6	ht.
$[0,[3],[3]]$	0	0	1/2	1/2	1	2	0
$[[2],[2]-(d_1+d_2),[1]]$	1/2	1/2	0	0	1	2	0
$[[2],[1]-(d_6+d_8),[2]]$	1/2	1/2	1/2	1/2	1	1	0
$[[2]-(d_1+d_2),[1]-(d_6+d_7+d_8),[2]]$	0	0	1/2	0	1	1	3/2

Table 7. Contributions and heights of the sections of 5.1.3

Thus $N_{\text{root}} = A_1A_1E_7^2$ and the rank of the fibration is 1. Since $2[2]_{D_{10}} = x+d_1$ and there is no other relation with $[1]_{D_{10}}$ or $[3]_{D_{10}}$, among the glue vectors $\langle[1,1,0\rangle, \langle[3,0,1]\rangle$ generating $Ni(D_{10}E_7^2)$, only $\langle[2,1,1]\rangle$ contributes to torsion.
Hence the torsion group is $\mathbb{Z}/2\mathbb{Z}$. Moreover the 2-torsion section is

$$2F+0+[[2],[1],[1]]$$

with height $4-(1/2+1/2+3/2+3/2)=0$. The infinite section is

$$3F+0+[(-6),0,0]$$

with height 6. Hence this fibration can be identified with the fibration #9-i.

5.2.3. The primitive embedding is into $Ni(D_8^3)$, given by $A_5 = \langle d_8,d_6,d_5,d_4,d_3\rangle \hookrightarrow D_8^{(1)}$ and $A_1^2 = \langle d_8,d_1\rangle \hookrightarrow D_8^{(2)}$. We compute $(A_5)^{\perp}_{D_8} = (-6)\oplus\langle x_1=(-2)\rangle\oplus\langle d_1\rangle$ with $x_1 = d_7+d_8+2(d_6+d_5+d_4+d_3+d_2)+d_1$

$$\begin{aligned}(A_1^2)^{\perp}_{D_8} = \quad & \langle d_7\rangle\oplus\langle x_1 = d_7+d_8+2(d_6+d_5+d_4+d_3+d_2)+d_1\rangle\\ & \oplus\langle d_5,d_4,x_3=d_7+d_8+2d_6+d_5,d_3\rangle = A_1\oplus A_1\oplus D_4.\end{aligned}$$

We deduce $N_{\text{root}} = 4A_1D_4D_8$ (hence the fibration has rank 1) and the relations

(1) $$2[2]_{D_8} = x_1+d_1$$

(2) $$2([2]_{D_8}-(d_1+d_2)) = x_3+2d_3+2d_4+d_5$$

(3) $$2[3]_{D_8} = x_1+2x_3+d_3+2d_4+d_5+d_7$$

(4) $$2([1]_{D_8}-(d_6+d_8)) = x_1+x_3+d_3+2d_4+2d_5+d_7$$

(5) $$2([1]_{D_8}-(d_6+d_7+d_8)) = 2x_3+3d_5+4d_4+3d_3+2d_2+d_1-d_7.$$

Thus, among the glue vectors $\langle[1,2,2],[1,1,1],[2,2,1]\rangle$ generating the Niemeier lattice, only vectors $\langle[0,3,3],[2,1,2]\rangle$ contribute to torsion and the torsion group is $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$.
From relations (1) to (5) we deduce the various contributions and heights of the following sections (see Table 5.1.3).
Hence this fibration can be identified with the fibration #14-i.

5.2.4. The primitive embedding is into $Ni(D_8^3)$ and given by

$$A_5 = \langle d_8,d_6,d_5,d_4,d_3\rangle \hookrightarrow D_8^{(1)} \quad A_1 = \langle d_8\rangle \hookrightarrow D_8^{(2)} \quad A_1 = \langle d_8\rangle \hookrightarrow D_8^{(3)}.$$

As previously $(A_5)^{\perp}_{D_8} = \langle(-6)\rangle\oplus\langle x_1\rangle\oplus\langle d_1\rangle$; we get also $\langle d_8\rangle^{\perp}_{D_8} = \langle d_7\rangle\oplus\langle x_4 = d_7+d_8+2d_6+d_5,d_5,d_4,d_3,d_2,d_1\rangle = A_1\oplus D_6$. Hence $N_{\text{root}} = 4A_12D_6$, and the rank is 1. Moreover it follows the relations

(6) $$2[2]_{D_8} = x_1+d_1$$

(7) $$2[2]_{D_8} = x_3+d_5+2d_4+2d_3+2d_2+2d_1$$

(8) $$2([1]_{D_8}-(d_5+d_6+d_7+d_8)) = 2x_3+d_5+4d_4+3d_3+2d_2+d_1-d_7$$

(9) $$2[3]_{D_8} = 3x_3+d_7+2d_5+4d_4+3d_3+2d_2+d_1 \in A_1\oplus D_6.$$

$0+2F+$	Cont. A_1	Cont. A_1	Cont. A_1	Cont. D_6	Cont. A_1	Cont. D_6	ht.
[0,[3],[3]]	0	0	1/2	1+1/2	1/2	1+1/2	0
[[2],[1]−$(d_5+d_6+d_7+d_8)$,[2]]	1/2	1/2	1/2	1+1/2	0	1	0
[[2],[2],[1]−$(d_5+d_6+d_7+d_8)$]	1/2	1/2	0	1	1/2	1+1/2	0
[[3],0,[3]]	0	1/2	0	0	1/2	1+1/2	3/2

Table 8. Contributions and heights of the sections of 5.1.4

We deduce that among the glue vectors generating $Ni(D_8^3)$, only $\langle [0,3,3],[2,1,2]\rangle$ contribute to torsion. So the torsion group is $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$. From relations (6) to (9) we deduce the various contributions and heights of the following sections (see Table 5.1.4).
Hence this fibration can be identified with the fibration #15-i.

5.2.5. The primitive embedding is onto $Ni(A_5^4D_4)$ given by $A_5\hookrightarrow A_5$, $A_1\oplus A_1=\langle d_4,d_1>\rangle\hookrightarrow D_4$. Since $\langle d_4,d_1\rangle_{D_4}^{\perp}=A_1^2$, we get $N=N_{\text{root}}=3A_52A_1$; thus the rank of the fibration is 0 and since $\det(N)=24\times 6^2$, the torsion group is $\mathbb{Z}/6\mathbb{Z}$.
This fibration can be identified with the fibration #20-i.

□

Remark 5.3. *From fibration* $\#20-i$ *the surface* S_k *appears to be a double cover of the rational elliptic modular surface associated to the modular groupe* $\Gamma_0(6)$ *given in Beauville's paper* [Beau]

$$(x+y)(y+z)(z+x)(t-s)(ts-1)=8sxyz.$$

6. Proof of Theorem 1.2

We recall first on Table 9 the results obtained by Bertin and Lecacheux in [BL]. The notation $\#17(18-m)$ for example refers for #17 to the generic case when relevant and for $(18-m)$ to notations used in Bertin-Lecacheux [BL].
Comparing to the fibrations of the family you remark more elliptic fibrations with 2-torsion sections on Y_2. All the corresponding involutions are denoted τ. Some of them are specialisations for $s=1$ of the generic ones. Those generic which are Morrison-Nikulin still remain Morrison-Nikulin for Y_2 by a Schütt's lemma [Sc], namely $\#4-\tau$, $\#8-\tau$, $\#16-\tau$, $\#17-\tau$, $\#23-\tau$, $\#24a)-\tau$, $\#26-\tau$. Others (($\#5-\tau$, $\#8\text{bis}-\tau$, $\#10-\tau$, $\#15<(p,0>)-\tau$, $\#24\,b-\tau$, $\#24\,c-\tau$) are specific to K_2 and cannot be deduced from elliptic fibrations of the generic Kummer. To identify them we have to use the distinguished property of Y_2, that is Y_2 is a singular $K3$ with Picard number 20.
Hence Y_2 inherits of a Shioda-Inose structure, that is the quotient of Y_2 by an involution is isomorphic to a Kummer surface K_2 realized from the product of CM elliptic curves [SI], [SM] provided in the following way.
Since the transcendental lattice of Y_2 is $\mathbb{T}(Y_2)=\begin{pmatrix}2&0\\0&4\end{pmatrix}=\begin{pmatrix}2a&b\\b&2c\end{pmatrix}$
we get $b^2-4ac=-8$, $\tau_1=\frac{-b+\sqrt{b^2-4ac}}{2a}$, $\tau_2=\frac{b+\sqrt{b^2-4ac}}{2}$, hence $\tau=\tau_1=\tau_2=i\sqrt{2}$.
We deduce $K_2=E\times E/\pm 1$ with $E=\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$. Since $j(i\sqrt{2})=8000$ [C], we obtain a Weierstrass equation of E, namely $Y^2=X(X^2+4X+2)$, which endowes K_2 with an elliptic fibration. The fact that the two CM elliptic curves are equal and satisfy $j(E)=8000$ can be obtained also by specialisation from the Shioda-Inose structure of the family. (see 4.1 Remark 4.1).
The elliptic curve E can be also put in the Legendre form:

$$E\quad y^2=x(x-1)(x-l),$$

l satisfying the equation $j=8000=\frac{256(1-l+l^2)^3}{l^2(l-1)^2}$. Thus $l=3\pm 2\sqrt{2}$ or $l=-2\pm 2\sqrt{2}$ or $l=\frac{1\pm\sqrt{2}}{2}$.

Proposition 6.1. *The elliptic fibrations* $\#8bis-\tau$ *and* $\#10-\tau$ *are elliptic fibrations of* K_2.

L_{root}	L/L_{root}			Fibers	R	Tor.
E_8^3	(0)					
	$\#1(11-f)$	$A_1 \subset E_8$	$D_5 \subset E_8$	$E_7A_3E_8$	0	(0)
	$\#2(13-h)$	$A_1 \oplus D_5 \subset E_8$		$A_1E_8E_8$	1	(0)
E_8D_{16}	$\mathbb{Z}/2\mathbb{Z}$					
	$\#3(30-\phi)$	$A_1 \subset E_8$	$D_5 \subset D_{16}$	E_7D_{11}	0	(0)
	$\#4(16-o)$	$A_1 \oplus D_5 \subset E_8$		A_1D_{16}	1	$\mathbb{Z}/2\mathbb{Z}$
	$\#5(17-q)$	$D_5 \subset E_8$	$A_1 \subset D_{16}$	$A_3A_1D_{14}$	0	$\mathbb{Z}/2\mathbb{Z}$
	$\#6(25-\delta)$	$A_1 \oplus D_5 \subset D_{16}$		$E_8A_1D_9$	0	(0)
$E_7^2D_{10}$	$(\mathbb{Z}/2\mathbb{Z})^2$					
	$\#7(29-\beta)$	$A_1 \subset E_7$	$D_5 \subset D_{10}$	$E_7D_6D_5$	0	$\mathbb{Z}/2\mathbb{Z}$
	$\#8(9-r)$	$A_1 \subset E_7$	$D_5 \subset E_7$	$D_6A_1D_{10}$	1	$\mathbb{Z}/2\mathbb{Z}$
	$\#8\text{bis}(24-\psi)$	$A_1 \oplus D_5 \subset E_7$		E_7D_{10}	1	$\mathbb{Z}/2\mathbb{Z}$
	$\#9(12-g)$	$A_1 \oplus D_5 \subset D_{10}$		$E_7E_7A_1A_3$	0	$\mathbb{Z}/2\mathbb{Z}$
	$\#10(10-e)$	$D_5 \subset E_7$	$A_1 \subset D_{10}$	$A_1A_1D_8E_7$	1	$\mathbb{Z}/2\mathbb{Z}$
E_7A_{17}	$\mathbb{Z}/6\mathbb{Z}$					
	$(21-c)$	$A_1 \oplus D_5 \subset E_7$		A_{17}	1	$\mathbb{Z}/3\mathbb{Z}$
	$\#11(19-n)$	$D_5 \subset E_7$	$A_1 \subset A_{17}$	A_1A_{15}	2	(0)
D_{24}	$\mathbb{Z}/2\mathbb{Z}$					
	$\#12(23-i)$	$A_1 \oplus D_5 \subset D_{24}$		A_1D_{17}	0	(0)
D_{12}^2	$(\mathbb{Z}/2\mathbb{Z})^2$					
	$\#13(26-\pi)$	$A_1 \subset D_{12}$	$D_5 \subset D_{12}$	$A_1D_{10}D_7$	0	$\mathbb{Z}/2\mathbb{Z}$
	$\#14(22-u)$	$A_1 \oplus D_5 \subset D_{12}$		$A_1D_5D_{12}$	0	$\mathbb{Z}/2\mathbb{Z}$
D_8^3	$(\mathbb{Z}/2\mathbb{Z})^3$					
	$\#15(6-p)$	$A_1 \subset D_8$	$D_5 \subset D_8$	$A_1D_6A_3D_8$	0	$(\mathbb{Z}/2)^2$
	$\#16(14-t)$	$A_1 \oplus D_5 \subset D_8$		$A_1D_8D_8$	1	$\mathbb{Z}/2\mathbb{Z}$
D_9A_{15}	$\mathbb{Z}/8\mathbb{Z}$					
	$\#17(18-m)$	$A_1 \oplus D_5 \subset D_9$		$A_1A_1A_1A_{15}$	0	$\mathbb{Z}/4\mathbb{Z}$
	$\#18(28-\alpha)$	$D_5 \subset D_9$	$A_1 \subset A_{15}$	D_4A_{13}	1	(0)
E_6^4	$(\mathbb{Z}/3\mathbb{Z})^2$					
	$\#19(8-b)$	$A_1 \subset E_6$	$D_5 \subset E_6$	$A_5E_6E_6$	1	$\mathbb{Z}/3\mathbb{Z}$
$A_{11}E_6D_7$	$\mathbb{Z}/12\mathbb{Z}$					
	$\#20(7-w)$	$A_1 \subset E_6$	$D_5 \subset D_7$	$A_5A_1A_1A_{11}$	0	$\mathbb{Z}/6\mathbb{Z}$
	$\#21(27-\mu)$	$A_1 \subset A_{11}$	$D_5 \subset D_7$	$A_9A_1A_1E_6$	1	(0)
	$(20-j)$	$A_1 \oplus D_5 \subset D_7$		$A_{11}E_6A_1$	0	$\mathbb{Z}/3\mathbb{Z}$
	$\#22(15-l)$	$A_1 \subset A_{11}$	$D_5 \subset E_6$	A_9D_7	2	(0)
	$\#23(2-k)$	$D_5 \subset E_6$	$A_1 \subset D_7$	$A_{11}A_1D_5$	1	$\mathbb{Z}/4\mathbb{Z}$
D_6^4	$(\mathbb{Z}/2\mathbb{Z})^4$					
	$\#24(5-d)$	$A_1 \subset D_6$	$D_5 \subset D_6$	$A_1D_4D_6D_6$	1	$(\mathbb{Z}/2)^2$
$D_6A_9^2$	$\mathbb{Z}/2 \times \mathbb{Z}/10$					
	$\#25(3-v)$	$D_5 \subset D_6$	$A_1 \subset A_9$	A_7A_9	2	(0)
$D_5^2A_7^2$	$\mathbb{Z}/4 \times \mathbb{Z}/8$					
	$\#26(1-s)$	$D_5 \subset D_5$	$A_1 \subset D_5$	$A_1A_3A_7A_7$	0	$\mathbb{Z}/8\mathbb{Z}$
	$\#27(4-a)$	$D_5 \subset D_5$	$A_1 \subset A_7$	$D_5A_5A_7$	1	(0)

TABLE 9. The elliptic fibrations of Y_2

Proof. It follows from the 4.2 fibration f_8 that the fibration $\#10-\tau$ with Weierstrass equation

$$Y^2 = X^3 - 2U^2(U-1)X^2 + U^3(U+1)^2(U-4)X,$$

singular fibers $III^*(0)$, $I_2^*(\infty)$, $I_4(-1)$, $I_2(4)$, $I_1(-1/2)$, and $\mathbb{Z}/2\mathbb{Z}$-torsion is an elliptic fibration of K_2. Similarly from the 4.2 fibration g_8, we deduce that the elliptic fibration $\#10-\tau$ with Weierstrass equation

$$Y^2 = X^3 + 2(t+5t^2)X^2 + t^2(4t+1)(t^2+6t+1)X,$$

singular fibers $III^*(\infty)$, $I_3^*(0)$, $3I_2(-1/4, t^2+6t+1=0)$, and $\mathbb{Z}/2\mathbb{Z}$-torsion is an elliptic fibration of K_2. □

To achieve the proof of Theorem 1.2 we need also the following lemma.

Lemma 6.1. *The Kummer K_2 has exactly 4 extremal elliptic fibrations given by Shimada Zhang [SZ] with the type of their singular fibers and their torsion group*

(1) E_7 A_7 A_3 A_1 $\mathbb{Z}/2\mathbb{Z}$,
(2) D_9 A_7 A_1 A_1, $\mathbb{Z}/2\mathbb{Z}$,
(3) D_6 D_5 A_7, $\mathbb{Z}/2\mathbb{Z}$,
(4) A_7 A_3 A_3 A_3 A_1 A_1, $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$.

From lemma (6.1 (2)) we obtain the fibration $\#5 - \tau$ and from lemma (6.1 (3)) the fibration $\#15 < (p, 0) > -\tau$.
We notice also that fibrations $\#17 - \tau$ and $\#26 - \tau$ obtained by specialisation are also fibration (4) of lemma (6.1) and fibration $\#23 - \tau$, by a 2-neighbor process of parameter $m = \frac{X}{k^2(k^2+4)}$ gives fibration (3) of lemma (6.1).
Finally, by a 2-neighbor process of parameter $m = \frac{X}{d^2(d+1)}$, fibrations $\#24b) - \tau$ and $\#24c) - \tau$ gives fibration $\#16 - \tau$, hence are elliptic fibrations of K_2.

Corollary 6.1. *As a byproduct of the proof we get Weierstrass equations for extremal fibrations of lemma (6.1) (2), (3), (4).*

The table of symplectic automorphisms of order 2 (self involutions) results from an easy computation.

7. 2-isogenies and isometries

Theorem 1.2, where the 2-isogenous $K3$ surfaces of Y_2 are either its Kummer K_2 or Y_2 itself, cannot be generalised to all the other singular $K3$ surfaces of the Apéry-Fermi family. The reason is the relation with a Theorem of Boissière, Sarti and Veniani [BSV], telling when p-isogenies (p prime) between complex projective $K3$ surfaces X and Y define isometries between their rational transcendental lattices $T_{X,\mathbb{Q}}$ and $T_{Y,\mathbb{Q}}$. (These lattices are isometric if there exists $M \in \mathrm{Gl}(n, \mathbb{Q})$ satisfying $T_{X,\mathbb{Q}} = M^t T_{Y,\mathbb{Q}} M$. Let us recall the part of their Theorem related to 2-isogenies.

Theorem 7.1. [BSV] *Let $\gamma : X \to Y$ be a 2-isogeny between complex projective $K3$ surfaces X and Y. Then $rk(T_{Y,\mathbb{Q}}) = rk(T_{X,\mathbb{Q}}) =: r$ and*

(1) *If r is odd, there is no isometry between $T_{Y,\mathbb{Q}}$ and $T_{X,\mathbb{Q}}$.*
(2) *If r is even, there exists an isometry between $T_{Y,\mathbb{Q}}$ and $T_{X,\mathbb{Q}}$ if and only if $T_{Y,\mathbb{Q}}$ is isometric to $T_{Y,\mathbb{Q}}(2)$. This property is equivalent to the following: for every prime number q congruent to 3 or 5 modulo 8, the q-adic valuation $\nu_q(\det T_Y)$ is even.*

As a Corollary we deduce the following result.

Theorem 7.2. *Among the singular $K3$ surfaces of the Apéry-Fermi family defined for k rational integer, only Y_2 and Y_{10} possess symplectic automorphisms of order 2 ("self 2-isogenies").*

Proof. The singular $K3$-surfaces of the Apéry-Fermi family defined for k rational integer are

$$Y_0, \quad Y_2, \quad Y_3, \quad Y_6, \quad Y_{10}, \quad Y_{18}, \quad Y_{102}, \quad Y_{198}.$$

This list has been computed numerically by Boyd [Boy]. Using the notation [SZ], that is writing the transcendental lattice $T_Y = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ as $T_Y = [a\ b\ c]$ we get:

$$T_{Y_0} = [4\ 2\ 4] \qquad T_{Y_2} = [2\ 0\ 4] \qquad T_{Y_6} = [2 \quad 0 \quad 12].$$

They are obtained by specialisation of fibration #20 for $k = 0$, 2 and 6. For $k = 0$ the elliptic fibration has rank 0 and singular fibers of type I_{12}, I_4, $2I_3$. For $k = 2$, the transcendental lattice is already known. For $k = 6$, the elliptic fibration has rank 0 and type of singular fibers I_{12}, I_3, $2I_2$. Now using Shimada-Zhang table [SZ], we derive the previous announced transcendental lattices.
The transcendental lattices T_{Y_3} and $T_{Y_{18}}$ were computed in the paper [BFFLM]. With the method used there, we can compute the transcendental lattices of Y_{10}, Y_{102} and Y_{198}. We obtain:

$$T_{Y_3} = [2\ 1\ 8] \qquad T_{Y_{10}} = [6 quad; 0 \quad 12] \qquad T_{Y_{18}} = [10 \quad 0 \quad 12]$$

No	Weierstrass Equation	From or to
#4	$y^2 = x^3 + (o^3 - 5o^2 + 2)x^2 + x$ $I_{12}^*(\infty), I_2(0), 4I_1(1, 5, o^2 - 4o - 4)$ $Y^2 = X(X - o^3 + 5o^2)(X - o^3 + 5o^2 - 4)$ $I_6^*(\infty), I_4(0), 4I_2(1, 5, o^2 - 4o - 4)$	Spec.$\#4 - i$
#5	$y^2 = x^3 + (q^3 + q^2 + 2q - 2)x^2 + (1 - 2q)x$ $I_{10}^*(\infty), I_4(0), I_2(\frac{1}{2}), 2I_1(q^2 + 2q + 5)$ $Y^2 = X^3 - 2(q^3 + q^2 + 2q - 2)X^2 + Xq^4(q^2 + 2q + 5)$ $I_5^*(\infty), I_8(0), 2I_2(q^2 + 2q + 5), I_1(\frac{1}{2})$	lemma(6.1)(2)
#8	$y^2 = x^3 - r(r^2 - r + 2)x^2 + r^3x$ $I_6^*(\infty), I_2^*(0), I_2(1), 2I_1(\pm 2i)$ $Y^2 = X^3 + 2r(r^2 - r + 2)X^2 + (r - 1)^2r^2(r^2 + 4)X$ $I_3^*(\infty), I_1^*(0), I_4(1), 2I_2(\pm 2i)$	Spec.$\#8 - i$
#8bis	$y^2 = x^3 - (\psi + 5\psi^2)x^2 - \psi^5x$ $I_6^*(\infty), III^*(0), 3I_1(-\frac{1}{4}, \psi^2 + 6\psi + 1)$ $Y^2 = X^3 + 2(5\psi^2 + \psi)X^2 + X\psi^2(4\psi + 1)(\psi^2 + 6\psi + 1)$ $III^*(\infty), I_3^*(0), 3I_2(-(\frac{1}{4}), \psi^2 + 6\psi + 1)$	lemma(6.1)
#10	$y^2 = x(x^2 - e^2(e - 1)x + e^3(2e + 1))$ $I_4^*(\infty), III^*(0), 2I_2(-1, -\frac{1}{2}), I_1(4)$ $Y^2 = X^3 + 2e^2(e - 1)X^2 + e^3(e - 4)(e + 1)^2X$ $I_2^*(\infty), III^*(0), I_4(-1), I_2(4), I_1(-\frac{1}{2})$	lemma(6.1)
#15 $< (p, 0) >$	$y^2 = x(x - p)(x - p(p + 1)^2)$ $I_4^*(\infty), I_2^*(0), I_4(-1), I_2(-2)$ $Y^2 = X^3 - p(p^2 + 2p - 1)X^2 - p^3(p + 2)X$ $I_2^*(\infty), I_1^*(0), I_8(-1), I_1(-2)$	lemma(6.1)(3)
#16	$y^2 = x^3 + t(t^2 + 4t + 1)x^2 + t^4x$ $2I_4^*(\infty, 0), I_2(-1), 2I_1(t^2 + 6t + 1 = 0)$ $Y^2 = X^3 - 2t(t^2 + 4t + 1)X^2 + t^2(t + 1)^2(t^2 + 6t + 1)X$ $2I_2^*(\infty, 0), I_4(-1), 2I_2(t^2 + 6t + 1)$	Spec.$\#16 - i$
#17	$y^2 = x(x^2 + x(\frac{1}{4}(m^2 - 4)^2 - 2) + 1)$ $I_{16}(\infty), 3I_2(0, \pm 2), 2I_1(\pm 2\sqrt{2})$ $Y^2 = X(X - \frac{1}{4}m^4 + 2m^2)(X - \frac{1}{4}m^4 + 2m^2 - 4)$ $I_8(\infty), 3I_4(0, \pm 2), 2I_2(\pm 2\sqrt{2})$	lemma(6.1)(4) Spec.$\#17 - i$
#23	$y^2 = x^3 + x^2(\frac{1}{4}k^4 - k^3 + k^2 - 2k) + k^2x$ $I_{12}(\infty), I_1^*(0), I_2(2), 3I_1(4, \pm 2i)$ $Y^2 = X^3 - (\frac{1}{2}k^4 - 2k^3 - 4k)X^2 + \frac{k^3(k-4)(k^2+4)(k-2)^2}{16}X$ $I_6(\infty), I_2^*(0), I_4(2), 3I_2(4, \pm 2i)$	$\frac{X}{k^2(k^2+4}$ to lemma(6.1)(3)
#24 $a)\langle(0, 0)\rangle$ $b)\langle(d + d^2, 0)\rangle$ $c)\langle(d^2 + d^3, 0)\rangle$	$y^2 = x(x - (d + d^2))(x - (d^3 + d^2))$ $2I_2^*(\infty, 0), I_0^*(-1), I_2(1)$ $a) : Y^2 = X^3 + 2d(d + 1)^2X^2 + d^2(d^2 - 1)^2X$ $2I_1^*(\infty, 0), I_0^*(-1), I_4(1)$ $b) : Y^2 = X^3 + 2d(d + 1)(d - 2)X^2 + d^4(d + 1)^2X$ $I_1^*(\infty), I_4^*(0), I_0^*(-1), I_1(1)$ $c) : Y^2 = X^3 - 2d(d + 1)(2d - 1)X^2 + Xd^2(d + 1)^2$	$a)$Spec.$\#24 - i$ $b)m = X/d^2(d + 1)$ to $\#16 - \tau$ $c)$ similar to $b)$
#26	$y^2 = x^3 + x^2(\frac{1}{4}(s - 1)^4 - 2s^2) + s^4x$ $2I_8(\infty, 0), I_4(1), I_2(-1), 2I_1(3 \pm 2\sqrt{2})$ $Y^2 = X(X - \frac{1}{4}(s - 1)^4 + 4s^2)(X - \frac{1}{4}(s - 1)^4)$ $I_8(1), 3I_4(0, -1, \infty), 2I_2(3 \pm 2\sqrt{2})$	Spec.$\#26 - i$ lemma(6.1)(4)

TABLE 10. Morrison-Nikulin involutions of Y_2 (fibrations of K_2)

$$T_{Y_{102}} = [12 \quad 0 \quad 26] \qquad [T_{Y_{198}} = [12 \quad 0 \quad 34].$$

Applying Bessière, Sarti and Veniani's Theorem, we conclude that only Y_2 and Y_{10} may have self isogenies. By Theorem 1.2, Y_2 has self isogenies. We shall prove that Y_{10} satisfies the same property.

No	Weierstrass Equation
#7	$y^2 = x^3 + 2\beta^2(\beta-1)x^2 + \beta^3(\beta-1)^2x$ $I_2^*(\infty), III^*(0), I_1^*(-1)$ $Y^2 = X^3 - 4\beta^2(\beta-1)X^2 + 4\beta^3(\beta-1)^3X$ $I_1^*(\infty), III^*(0), I_2^*(1)$
#9	$y^2 = x^3 + 4g^2x^2 + g^3(g+1)^2x$ $2III^*(\infty,0), I_4(-1), I_2(1)$ $Y^2 = X^3 - 8g^2X^2 - 4g^3(g-1)^2X$ $2III^*(\infty,0), I_4(1), I_2(-1)$
#13	$y^2 = x^3 + x^2\pi(\pi^2 - 2\pi - 2) + \pi^2(2\pi+1)x$ $I_6^*(\infty), I_3^*(0), I_2(-1/2), I_1(4)$ $Y^2 = X^3 - 2X^2\pi(\pi^2 - 2\pi - 2) + \pi^5(\pi-4)X$ $I_6^*(0), I_3^*(\infty), I_2(4), I_1(-1/2)$
#14	$y^2 = x^3 + u(u^2+4u+2)x^2 + u^2x$ $I_8^*(\infty), I_1^*(0), I_2(-2), I_1(-4)$ $Y^2 = \left(X - u\,(u-2)^2\right) X\,(X - 4\,u)$ $I_4^*(\infty), I_2^*(0), I_4(-2), I_2(-4)$
#15 $a) < (0,0) >$ $b) < (p(p+1)^2, 0) >$	$y^2 = x(x-p)(x-p(p+1)^2)$ $I_4^*(\infty), I_2^*(0), I_4(-1), I_2(-2)$ $a) : Y^2 = X(X + 4p + p^3 + 4p^2)(X + p^3)$ $I_4^*(0), I_2^*(\infty), I_4(-2), I_2(-1)$ $b) : Y^2 = X^3 - 2p(2p^2 + 4p + 1)X^2 + p^2X$ $I_8^*(\infty), I_1^*(0), I_2(-1), I_1(-2)$
#20	$y^2 = x^3 - (2 - w^2 - \frac{1}{4}w^4)x^2 - (w^2-1)x$ $I_{12}(\infty), I_6(0), 2I_2(\pm 1), 2I_1(\pm 2i\sqrt{2})$ $Y^2 = X^3 + 2(2 - w^2 - \frac{1}{4}w^4)X^2 + \frac{1}{16}w^6(w^2+8)X$ $I_{12}(0), I_6(\infty), 2I_2(\pm 2i\sqrt{2}), 2I_1(\pm 1)$

TABLE 11. Self involutions of Y_2

Consider the following elliptic fibration of rank 0 of Y_{10} (other interesting properties of Y_{10} will be studied in a forthcoming paper):

$$y^2 = x^3 + x^2(9(t+5)(t+3) + (t+9)^2) - xt^3(t+5)^2$$

with singular fibers $III^*(\infty)$, $I_6(0)$, $I_4(-5)$, $I_3(-9)$, $I_2(-4)$ and 2-torsion. Its 2-isogenous curve has a Weierstrass equation

$$Y^2 = X^3 + X^2(-20t^2 - 180t - 432) + 4X(t+4)^2(t+9)^3$$

with singular fibers $III^*(\infty)$, $I_6(-9)$, $I_4(-4)$, $I_3(0)$, $I_2(-5)$, rank 0 and 2-torsion. Hence this 2-isogeny defines an automorphism of order 2 of Y_{10} given by $x = -\frac{X}{2}$, $y = \frac{iY}{2\sqrt{2}}$.

□

Moreover we observe that

$$T_{Y_2} = [2\ 0\ 4], \qquad T_{Y_2,\mathbb{Q}} = [2\ 0\ 1],$$
$$T_{K_2} = [4\ 0\ 8], \qquad T_{K_2,\mathbb{Q}} = [2\ 0\ 1],$$

Similarly

$$T_{Y_{10},\mathbb{Q}} = [6\ 0\ 3], \qquad T_{K_{10},\mathbb{Q}} = [3\ 0\ 6].$$

Hence we suspect some relations between the transcendental lattices of K_{Y_i} and of S_{Y_i} for singular Y_i. We give some examples of such relations in the following proposition.

Proposition 7.1. *Even if the 2-isogenies from Y_0, Y_6 are not isometries, the following rational transcendental lattices satisfy the relations*

(1) $T_{K_0,\mathbb{Q}} = T_{S_0,\mathbb{Q}}$,
(2) $T_{K_6,\mathbb{Q}} = T_{S_6,\mathbb{Q}}$,

(3) $\det(T_{K_3})=\det(T_{S_3})$.

Proof. (1) For $k=0$ we get two elliptic fibrations of rank 0, namely #20 and #8. The fibration #8-i gives a rank 0 elliptic fibration of K_0 with Weierstrass equation

$$y^2=x^3+2x^2(t^3+1)+x(t-1)^2(t^2+t+1)^2,$$

type of singular fibers D_7, $3A_3$, A_2, 4-torsion and $T_{K_0}=[8\ 4\ 8]$. On the other end the fibration #20-i gives a rank 0 elliptic fibration of S_0

$$y^2=x(x-\frac{1}{4}(t-3I)(t+I)^3)(x-\frac{1}{4}(t+3I)(t-I)^3)$$

with type of singular fibers $3A_5$ $(\infty,\pm I)$, $3A_1$ $(0,\pm 3I)$, $\mathbb{Z}/2\times\mathbb{Z}/6$-torsion, the 3-torsion points being $(\frac{1}{4}(t^2+1)^2,\pm\frac{1}{2}(t^2+1)^2)$. Hence, by Shimada-Zhang 's list $T_{S_0}=[2\ 0\ 6]$. Now we can easily deduce the relation

$$\begin{pmatrix}1/2 & 0\\ 1/2 & -1\end{pmatrix}\begin{pmatrix}8 & 4\\ 4 & 8\end{pmatrix}\begin{pmatrix}1/2 & 1/2\\ 0 & -1\end{pmatrix}=\begin{pmatrix}2 & 0\\ 0 & 6\end{pmatrix}.$$

(2) For $k=6$ the elliptic fibration #20 has rank 0 and $\#20-i$ gives a rank 0 elliptic fibration of S_6:

$$y^2=x^3+x^2(-\frac{t^4}{2}+6t^3-21t^2+18t+\frac{3}{2})+x\frac{(t-3)^2}{16}(t^2-6t+1)^3,$$

with singular fibers $2I_6(t^2-6t+1=0)$, $I_6(\infty)$, $I_4(3)$, $2I_1(0,6)$, and $\mathbb{Z}/6\mathbb{Z}$-torsion. Using Shimada-Zhang's list [SZ], we find $T_{S_6}=[4\ 0\ 6]$. Since

$$T_{K_6}=\begin{pmatrix}4 & 0\\ 0 & 24\end{pmatrix}\underset{\mathbb{Q}}{\sim}\begin{pmatrix}1 & 0\\ 0 & 6\end{pmatrix}$$

and

$$T_{S_6}=\begin{pmatrix}4 & 0\\ 0 & 6\end{pmatrix}\underset{\mathbb{Q}}{\sim}\begin{pmatrix}1 & 0\\ 0 & 6\end{pmatrix}$$

we get straightforward

$$T_{K_6,\mathbb{Q}}=T_{S_6,\mathbb{Q}}.$$

(3) Consider the elliptic fibration #20 of Y_3 with Weierstrass equation

$$y^2=x^3+\frac{1}{4}(t^4-6t^3+15t^2-18t-3)x^2-t(t-3)x,$$

singular fibers $I_{12}(\infty)$, $2I_3(t^2-3t+1=0)$, $2I_2(0,3)$, $2I_2(t^2-3t+9=0)$, rank 1 and 6-torsion. The infinite section $P_3=(t,-\frac{1}{2}t(t^2-3t+3))$, of height $\frac{5}{4}$ generates the free part of the Mordell-Weil group, since $\det(T_{Y_3})=15$ by the previous theorem and by the Shioda-Tate formula

$$\det(T_{Y_3})=\frac{5}{4}\frac{12\times 3^2\times 2^2}{6^2}=15.$$

Its 2-isogenous curve has Weierstrass equation

$$y^2=x^3+(-\frac{1}{2}t^4+3t^3-\frac{15}{2}t^2+9t+\frac{3}{2})x^2+\frac{1}{16}(t^2-3t+9)(t^2-3t+1)x,$$

singular fibers $3I_6(\infty,t^2-3t+1=0)$, $2I_2(t^2-3t+9=0)$, $2I_1(3,0)$, rank 1 and 6-torsion. The section Q_3 image by the 2-isogeny of the infinite section P_3 is an infinite section of height $\frac{5}{2}$. Since neither Q_3 nor $Q_3+(0,0)$ are 2-divisible, the section Q_3 generates the free part of the Mordell-Weil group. Hence by the Shioda-Tate formula, it follows

$$\det(T_{S_3})=\frac{5}{2}\frac{6^3\times 2^2}{6^2}=60=\det(T_{K_3}).$$

□

Remark 7.1. *The Kummer surface K_0 is nothing else than the Schur quartic* [BSV] *(section 6.3) with equation*

$$x^4 - xy^3 = z^4 - zt^3.$$

References

[Beau] A. Beauville, *Les familles stables de courbes elliptiques sur $\mathbb{P}^1$ admettant 4 fibres singulières*, C.R.Acad.Sc. Paris, 294 (1982), 657–660.

[BGL] M.J. Bertin, A. Garbagnati, R. Hortsch, O. Lecacheux, M. Mase, C. Salgado, U. Whitcher, *Classifications of Elliptic Fibrations of a Singular K3 Surface*, in Women in Numbers Europe, 17–49, Research Directions in Number Theory, Association for Women in Mathematics Series, Springer, 2015.

[BFFLM] M.J. Bertin, A. Feaver, J. Fuselier, M. Lalìn, M. Manes, *Mahler measure of some singular $K3$-surfaces*, Contemporary Math. 606, CRM Proceedings, Women in Numbers 2, Research Directions in Number Theory, 149–169.

[BL] M.J. Bertin, O. Lecacheux, *Elliptic fibrations on the modular surface associated to* $\Gamma_1(8)$, in Arithmetic and geometry of K3 surfaces and Calabi-Yau threefolds, 153–199, Fields Inst. Commun., **67**, Springer, New York, 2013.

[BL1] M.J. Bertin, O. Lecacheux, *Automorphisms of certain Niemeier lattices and elliptic fibrations,* Albanian Journal of Mathematics Volume 11, Number 1 (2017) 13-34.

[BSV] S. Boissière, A. Sarti, D. Veniani, *On prime degree isogenies between $K3$ surfaces,* Rendiconti del Circolo Matematico di Palermo Series 2, Volume 66, Issue 1, (2017), 3-18.

[Bou] N. Bourbaki, *Groupes et algèbres de Lie*, Chap.4, 5, 6, Masson, Paris (1981).

[Boy] D. Boyd, *Private communication.*

[C] P. Comparin, A. Garbagnati, *Van Geemen-Sarti involutions and elliptic fibrations on K3 surfaces double cover of $\mathbb{P}^2$,* J. Math. Soc. Japan Volume 66, Number 2 (2014), 479–522.

[Co] J. H. Conway, N. J. A. Sloane, *Sphere packings, lattices and groups*, Grundlehren der Mathematischen Wissenshaften 290, Second Edition, Springer-Verlag, (1993).

[D] I.V. Dolgachev, *Mirror symmetry for lattice polarized $K3$ surfaces*, Journal of Mathematical Sciences, 81:3, (1996), 2599-2630.

[DG] E. Dardanelli, B. van Geemen, *Hessians and the moduli space of cubic surfaces*, in Algebraic Geometry Korean-Japan Conference in Honor of I. Dolgachev'60th Birthday 2004, Contemp. Math. 422 (2007) 17–36. AMS.

[E] N. Elkies, *Private communication.*

[El] N. Elkies, A. Kumar, *$K3$ surfaces and equations for Hilbert modular surfaces,* Algebra Number Theory, Volume 8, Number 10 (2014), 2297–2411.

[G] B. van Geemen, A. Sarti, *Nikulin Involutions on $K3$ Surfaces*, Math. Z. 255 (2007),731–753.

[K] J. Keum, *A note on elliptic $K3$ surfaces*, Transactions of the American Mathematical Society, Vol. 352, No. 5 (2000), 2077–2086.

[Ko] K. Koike, *Elliptic $K3$ surfaces admitting a Shioda-Inose structure*, Commentarii Mathematici Universitatis Sancti Pauli Vol 61 N 1 (2012), 77–86.

[Ku] M. Kuwata, T. Shioda, *Elliptic parameters and defining equations for elliptic fibrations on a Kummer surface*, Advanced Studies in Pure Mathematics 50, 2008, Algebraic Geometry in East Asia - Hanoi 2005, 177-215.

[Ku1] M. Kuwata, Maple Library Elliptic Surface Calculator, http://c-faculty.chuo-u.ac.jp/$\sim$ kuwata/ESC.php.

[M] D. Morrison, *On $K3$ surfaces with large Picard number*, Invent. Math. 75 (1984), 105–121.

[Na] N. Narumiya, H. Shiga, *The mirror map for a family of $K3$ surfaces induced from the simplest* 3*-dimensional reflexive polytope,* Proceedings on Moonshine and related topics (Montréal, QC, 1999), CRM Proc. Lecture Notes, 30 (2001), 139–161, Amer. Math. Soc.

[Nik] V. V. Nikulin, *Integral symmetric bilinear forms and some of their applications*, Math. USSR Izv. **14**, No. 1 (1980), 103–167.

[N2] V. V. Nikulin, *Finite automorphism groups of Kahler surfaces of type K3*, Proc. Moscow Math. Soc. 38 (1979), 75–137.

[Nis] K.-I. Nishiyama, *The Jacobian fibrations on some $K3$ surfaces and their Mordell-Weil groups*, Japan. J. Math. **22** (1996), 293–347.

[PA] The PARI Group, Bordeaux GP/PARI version 2.7.3, 2015, http://pari.math.u-bordeaux.fr.

[PS] C. Peters, J. Stienstra, *A pencil of $K3$-surfaces related to Apéry's recurrence for $\zeta(3)$ and Fermi surfaces for potential zero*, Arithmetic of Complex Manifolds (Erlangen, 1988) (W.-P. Barth & H. Lange, eds.), Lecture Notes in Math., vol. 1399, Springer, Berlin (1989), 110–127.

[Sc] M. Schütt, *Sandwich theorems for Shioda-Inose structures*, Izvestiya Mat. 77 (2013), 211–222.

[Sc-Shio] M. Schütt, T. Shioda, *Elliptic surfaces*, Algebraic geometry in East Asia - Seoul 2008, Adv. Stud. Pure Math. **60** (2010), 51–160.

[Shio] T. Shioda, *Kummer sandwich theorem of certain elliptic $K3$ surfaces*, Proc. Japan Acad. **82**, Ser. A (2006), 137–140.

[Shio1] T. Shioda *On elliptic modular surfaces* J. Math. Soc. Japan 24 (1972), no 1, 20–59.

[SI] T. Shioda, H. Inose, *On singular $K3$ surfaces*, in: Baily, W. L. Jr., Shioda, T. (eds), Complex analysis and algebraic geometry, Iwanami Shoten, Tokyo (1977), 119–136.

[SM] T. Shioda, N. Mitani, *Singular abelian surfaces and binary quadratic forms*, in: Classification of algebraic varieties and compact complex manifolds, Lect. Notes in Math. 412 (1974).

[SZ] I. Shimada, D. Q. Zhang, *Classification of extremal elliptic $K3$ surfaces and fundamental groups of open $K3$ surfaces*, Nagoya Math. J. **161** (2001), 23–54.

[Si] J. Silverman, The arithmetic of Elliptic Curves, Graduate texts in mathematics Springer 106 (1986).

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