Regulators and Mahler measure

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Historic

The first was D.H.Lehmer
" On factorization of certain cyclotomic functions" (1933)
with his famous question (still unsolved): does there exist a monic irreducible polynomial P, non cyclotomic, with integer coefficients such that

$$\Omega(P) := \prod_{P(\alpha)=0} max(|\alpha|, 1) < \Omega(P_0) \simeq 1.1762...$$

where P_0 is the Lehmer's polynomial

$$X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1?$$

In fact

$$\Omega(P) = M(P)$$

the Mahler measure of P (introduced by Mahler in 1962).
The logarithmic Mahler's measure of a polynomial P is

$$m(P) := \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} log \mid P(x_1, \cdots, x_n) \mid \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}$$

and the Mahler's measure

$$M(P) := exp(m(P)).$$

By Jensen's formula, if $P \in \mathbb{Z}[X]$ is monic, then

$$M(P) = \prod_{P(\alpha)=0} max(\mid \alpha \mid, 1).$$

If $A(x, y)$ is in two variables we can write

$$A(x,y) = a_0(y) \prod_{j=1}^{d} (x - x_j(y))$$

with $x_j(y)$ algebraic functions in y.

By Jensen's formula

$$m(A) = m(a_0) + \sum_{j=1}^{d} \frac{1}{2\pi i} \int_{|y|=1} \log^+ \mid x_j(y) \mid \frac{dy}{y}$$

where $\log^+ \mid z \mid = \log \mid z \mid$ if $\mid z \mid \geq 1$ and 0 otherwise.
Defining

$$\eta(x, y) := \log \mid x \mid d \arg y - \log \mid y \mid d \arg x$$

a real differential 1-form on $X \setminus S$ (X the variety defined by the polynomial A, smooth projective completion of Y zero locus of A, S points of X where x or y has a zero or a pole), we get

$$m(A) = m(a_0) + \frac{1}{2\pi} \int_{\gamma} \eta(x, y)$$

γ oriented path on X projecting to $Y \cap \{\mid y \mid = 1, \mid x \mid \geq 1\}$

Then Smyth (1971)

$$M(P) \geq M(X^3 - X - 1) \simeq 1.32..$$

if P is non reciprocal. The obstruction for Lehmer's question is therefore the reciprocal polynomials.

Boyd's limit formula (1981)

$$m(P(x, x^N)) \longrightarrow m(P(x, y))$$

is a hope to get small measures in one variable from small measures in two variables.

$$M((x+1)y^2 + (x^2+x+1)y + x(x+1)) = 1.25...$$

$$M(y^2 + (x^2+x+1)y + x^2) = 1.28..$$

are the smallest known measures in two variables.

At the same time Smyth obtained the first explicit Mahler measures:

$$m(x+y+1) = L'(\chi_{-3}, -1) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2)$$

$$m(x+y+z+1) = \frac{7}{2\pi^2}\zeta(3)$$

Boyd meets Deninger (Calgary CMS Summer meeting (1996))

The result is Deninger's guess (1996) proved in 2011 by Rogers and Zudilin, then again by Zudilin in 2013

$$m(x + \frac{1}{x} + y + \frac{1}{y} + 1) = \frac{15}{4\pi^2} L(E,2) =: L'(E,0) = b_{15}$$

The elliptic curve E (algebraic closure of the zero set of the polynomial) is $15a8$ (Cremona's notation) of conductor 15 defined by

$$Y^2 + XY + Y = X^3 + X^2$$

Its L-series is given by the modular form

$$f_{15A}(z) = \eta(z)\eta(3z)\eta(5z)\eta(15z)$$

The polynomial

$$P = x + \frac{1}{x} + y + \frac{1}{y} + 1$$

is tempered

"Tempered" means the roots of all the face polynomials of the Newton polygon of P are roots of unity.

The polynomial

$$Y^2 + XY + Y - (X^3 + X^2)$$

is also tempered.

Very important to obtain formulas "à la Deninger".

Just after Deninger's guess, Boyd obtained a lot of conjectures based on numerical computations.
He studied families of 'tempered polynomials' mostly reciprocal defining elliptic curves, comparing their Mahler measures and their L-series.
The most famous family is p_k

$$m(p_k) = m\left(x + \frac{1}{x} + y + \frac{1}{y} + k\right) \stackrel{?}{=} s_k \frac{N_k}{4\pi^2} L(E_{(k)}, 2) = s_k b_k$$

s_k is a rational number (often integer), $E_{(k)}$ is the elliptic curve, algebraic closure of the zero set of the polynomial.

Results for the family p_k

(B.=Brunault, L.=Lalin, M.=Mellit, R-V.=Rodriguez-Villegas, R.=Rogers, S.=Samart, Z.=Zudilin)

k	s_k	N_k	Proofs from
1	1	15	R.-Z. (2011), Z. (2013)
2	1	24	Z. (2013)
3	2	21	B. (April 2015), L.-S.-Z. (July 2015)
5	6	15	?
6	1/2	120	
7	1/2	231	
8	4	24	R.-L. (2008)
9	1/2	195	
10	-1/8	840	
11	-1/8	1155	
12	2	48	B. (April 2015)

Results: Family p_k (continued)

k	s_k	N_k	Proofs from
i	2	17	Z. (2013)
2i	1	40	Z. (2013)
$\sqrt{2}$	$1/4$	56	Z. (2013)
$4/\sqrt{2}$	1	32	R.-V. (1999) CM
$4\sqrt{2}$	1	64	R.-V. (1999) CM

Results: other families

In the family

$$P_k = x^3 + y^3 + 1 - kxy$$

Mellit preprint (2009) arxiv (2012) $m_{-1} = 2b_{14},\ m_5 = 7b_{14}$
In the family

$$P_k = (x+1)y^2 + (x^2 + kx + 1)y + x(x+1)$$

Mellit preprint (2009) arxiv (2012) $m_1 = b_{14},\ m_{-5} = 6b_{14},\ m_{10} = 10b_{14}$
In the family

$$P_k = y^2 + kxy + y - x^3$$

Brunault arxiv (april 2015) $m_{-1} = 2b_{14},\ m_{-2} = b_{35},\ m_{-3} = b_{54}$

A new result (Bertin August 2015)

Related also to the family

$$P_k(x,y) = (x+1)y^2 + (x^2+kx+1)y + x(x+1)$$

Boyd conjectured the two formulae

$$m_4 = 3b_{20} \quad \text{and} \quad m_{-2} = 2b_{20}.$$

In fact, E^4 is isomorphic to the curve $20a2$ $[0,1,0,-1,0]$, E^{-2} is isomorphic to the curve $20a1$ $[0,1,0,4,4]$, 2-isogenous to $20a2$.
The corresponding modular form on $\Gamma_0(20)$ thus giving the L-series is

$$f_{20A} = \eta(2z)^2\eta(10z)^2 = q - 2q^3 - q^5 + 2q^7 + q^9 + 2q^{13} + 2q^{15} \ldots.$$

The main ingredients are regulators and modular units.

Some comments

For CM elliptic curves in Boyd's families Rodriguez-Villegas proved the conjectures using Eisenstein-Kronecker series.
For example,

$$\begin{aligned} \mathrm{m}\left(x+\frac{1}{x}+y+\frac{1}{y}+k\right) &= \Re\left(\frac{16\Im\tau}{\pi^2}\sum_{m,n}{}' \frac{\chi_{-4}(m)}{(m+n4\tau)^2(m+n4\bar{\tau})}\right) \\ &= \Re\left(-\pi\tau+2\sum_{n=1}^{\infty}\sum_{d\mid n}\chi_{-4}(d)d^2\frac{q^n}{n}\right) \end{aligned}$$

with

$$q=e^{2\pi i\tau}=q\left(\frac{16}{k^2}\right)=\exp\left(-\pi\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1,1-\frac{16}{k^2}\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1,\frac{16}{k^2}\right)}\right)$$

Beilinson's result, Zagier's conjecture

For elliptic modular curves E, Beilinson proved

$$L(E,2) = \pi D^E(\xi), \quad \xi \in \mathbb{Z}[E(\mathbb{C})]_{\text{tors}}$$

For a general elliptic curve E, Zagier conjectured

$$L(E/\mathbb{Q},2) \stackrel{?}{=} \pi D^E(\xi), \quad \xi \in \mathbb{Z}[E(\bar{\mathbb{Q}})]^{\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})}$$

“tempered” and the K_2 of the elliptic curve

Let X be a smooth projective algebraic curve defined over $\mathbb{C}$ and let $\mathbb{C}(X)$ be its function field. Let x, $y \in \mathbb{C}(X)$ be two non-constant rational functions and let $S \subset X$ be the set of zeros and poles of x or y. The image of the rational map $(x, y) : X \setminus S \to \mathbb{C}^* \times \mathbb{C}^*$ is of dimension 1; let $A \in \mathbb{C}[x, y]$ be a defining equation.

$$\{x, y\} \in K_2(X) \otimes \mathbb{Q} \Leftrightarrow A \text{“tempered”}$$

(since the “Tame symbol” is related to the zeros of the face polynomials by CCGLS's paper at Inventiones (1994) that uses Puiseux's expansions)
CCGLS=Cooper, Culler, Gillet, Long, Shalen “Plane curves associated to character varieties of 3-manifolds”

Integral expression of the regulator

The regulator r can be expressed as an integral

$$\begin{array}{rccc} r: & K_2(E) & \to & \mathbb{C} \\ & \{f,g\} & \mapsto & \frac{1}{2\pi}\int_\gamma \eta(f,g) \end{array}$$

with

$$\eta(f,g) = \log|f| d(\arg g) - \log|g| d(\arg f),$$

f and $g \in \mathbb{Q}(E)$ and γ closed path not going through zeros and poles of f and g and generating the subgroup of cycles $H_1(E,\mathbb{Z})^-$

The Mahler measure can be expressed as a regulator if we can prove that the path of integration in the expression of the Mahler measure belongs to $H_1(E, \mathbb{Z})^-$.
This is precisely the case for the polynomial P_{-2}.
Set $P_{-2}(x_2, y_2)$ the polynomial

$$P_{-2}(x_2, y_2) = (x_2 + 1)y_2^2 + (x_2^2 - 2x_2 + 1)y_2 + x_2(x_2 + 1).$$

Then

$$2m_{-2} = \pm r(\{x_2, y_2\}).$$

The diamond operator

Let $\mathbb{Z}[\langle P\rangle]$ the subgroup of $\mathbb{Z}[E(\mathbb{Q})]$ generated by $P \in E(\mathbb{Q})$ and $\mathbb{Z}[E(\mathbb{Q})]^-$ its quotient by the relation $cl(-P) = -cl(P)$.
Define

$$\begin{array}{rccc} \diamond : & \mathbb{Z}[\langle P\rangle] \times \mathbb{Z}[\langle P\rangle] & \to & \mathbb{Z}[E(\mathbb{Q})]^- \\ & ((f),(g)) & \mapsto & (f)\diamond(g) = \sum_{m,n} a_n b_m cl((n-m)P) \end{array}$$

$(f) = \sum_{n\in\mathbb{Z}} a_n[nP]$,$(g) = \sum_{n\in\mathbb{Z}} b_n[nP]$

The elliptic dilogarithm (introduced by Bloch)

E elliptic curve on $\mathbb{Q}$
On $E(\mathbb{C})$, we have the representations

$$\begin{array}{ccccc} E(\mathbb{C}) & \stackrel{\sim}{\to} & \mathbb{C}/(\mathbb{Z}\tau+\mathbb{Z}) & \stackrel{\sim}{\to} & \mathbb{C}^*/q^{\mathbb{Z}} \\ (\mathfrak{P}(u),\mathfrak{P}'(u)) & \mapsto & u(mod\Lambda) & \mapsto & z=\exp 2\pi iu \end{array}$$

The elliptic dilogarithm D^E is

$$D^E(P)=\sum_{n=-\infty}^{+\infty} D(q^n z)$$

where D denotes the Bloch-Wigner dilogarithm.
$(D(z)=\Im(Li_2^{[c]}(z)+\log\mid z\mid \log^{[c]}(1-z))$

Bloch's Theorem: regulator and elliptic dilogarithm

Theorem

Let f and g functions on the elliptic curve E with divisors elements of $\mathbb{Z}[\langle P\rangle]$ such that $\{f,g\} \in K_2(E)$, then

$$\pi r(\{f,g\}) = D^E((f) \diamond (g))$$

Touafek's results

Some years later (2008), in his thesis (not published in extenso), Touafek considered the elliptic curve E_2 defined by the equation

$$Y_2^2 + 2X_2Y_2 + 2Y_2 = (X_2 - 1)^3$$

exhibited the isomorphisms between E_2, 20a1 and E^{-2}, remarked that

$$\{X_2, Y_2\} \in K_2(E_2) \otimes \mathbb{Q}$$

$$\{x_2, y_2\} \in K_2(E^2) \otimes \mathbb{Q}$$

and used Bloch's theorem to derive the equality

$$r(\{X_2, Y_2\}) = r(\{x_2, y_2\})$$

and conjectured their common value $4b_{20}$.

Proof of Touafek's conjecture: modular units

The idea is the parametrization by modular units. (Brunault, Mellit, Zudilin).
Recall that a modular unit is a modular function whose all zeros and poles are cusps, for example certain quotient of eta functions for $\Gamma_0(20)$.
We proved the lemma

Lemma

The elliptic curve E_2 defined by

$$Y_2^2 + 2X_2Y_2 + 2Y_2 = (X_2 - 1)^3$$

is isomorphic to the curve '20a1' $[0, 1, 0, 4, 4]$ *in Cremona's classification and can be parametrized by eta quotients, modular units on $X_0(20)$. More precisely*

$$X_2 = \frac{\eta(4\tau)^4}{\eta(20\tau)^4}\frac{\eta(10\tau)^2}{\eta(2\tau)^2}$$
$$Y_2 = -\frac{\eta(4\tau)}{\eta(\tau)}\frac{\eta(5\tau)^5}{\eta(20\tau)^5}$$

Modular units

Let us recall first the definition of the modular unit g_a:

$$g_a(\tau) := q^{NB(a/N)/2} \prod_{\substack{n \geq 1 \\ n \equiv a \bmod N}} (1-q^n) \prod_{\substack{n \geq 1 \\ n \equiv -a \bmod N}} (1-q^n)$$

Now it follows from the definition of a modular unit:

$$X_2 = \left(\frac{g_4 g_8}{g_2 g_6}\right)^2$$
$$Y_2 = -\frac{g_5^4 g_{10}^2}{g_1 g_2 g_3 g_6 g_7 g_9}$$

Main ingredient: Brunault-Mellit's theorem (proof by Zudilin)

Theorem

For integers a, b, c with ac and bc not divisible by N, we have the formula

$$\int_{c/N}^{i\infty} \eta(g_a, g_b) = \frac{1}{4\pi} L(f(\tau) - f(i\infty), 2)$$

where $f(\tau) = f_{a,b;c}(\tau)$, $f_{a,b;c} := e_{a,bc}e_{b,-ac} - e_{a,-bc}e_{b,ac}$ *and*

$$e_{a,b}(\tau) = \frac{1}{2}\left(\frac{1+\zeta_N^a}{1-\zeta_N^a} + \frac{1+\zeta_N^b}{1-\zeta_N^b}\right) + \sum_{m,\, n \geq 1} (\zeta_N^{am+bn} - \zeta_N^{-(am+bn)}) q^{mn}$$

$\zeta_N := \exp(2\pi i/N)$, $q := \exp(2\pi i \tau)$.

How to choose c: the path of integration

If $\alpha,\ \beta \in \mathcal{H}^*$ satisfy $\beta = M(\alpha)$, $M \in \Gamma_0(N)$ (α and β are said equivalent under the action of $\Gamma_0(N)$).
Any smooth path (for instance a geodesic path) projects to a closed path in the quotient space $X_0(N)$, hence determines an integral homology class in $H_1(X_0(N), \mathbb{Z})$, which depends only on α and β and not on the path chosen. In fact the class depends only on the matrix M. This homology class is denoted by the modular symbol $\{\alpha, \beta\}_{\Gamma_0(N)}$. Conversely, every homology class $\gamma \in H_1(X_0(N), \mathbb{Z})$ can be represented by such a modular symbol $\{\alpha, \beta\}_{\Gamma_0(N)}$.
For $f \in S_2(\Gamma_0(N))$,

$$< \gamma, f >:= \int_\gamma 2\pi i f(z) dz = 2\pi i \int_\alpha^\beta f(z) dz$$

is called a period of the cusp form f.
Elements of $H_1^-(X_0(N), \mathbb{R})$ are identified by

$$< \gamma, f > \in i\mathbb{R} \iff \gamma \in H_1^-(X_0(N), \mathbb{R}).$$

Recall also that by the Manin-Drinfeld theorem, the rational homology $H_1(X_0(N), \mathbb{Q})$ is generated by paths between cusps.
The closed path of integration γ generating $H_1(E, \mathbb{Z})^-$ in the expression of the regulator becomes under the parametrization a closed path generator of $H_1^-(X_0(20), \mathbb{Z})$, hence an appropriate modular symbol we can compute using Sage. We can take the closed path $\{-3/20, 3/20\}$ and apply theorem B-M-Z. So

$$\begin{aligned} & r(\{X_2, Y_2\}) \\ &= \frac{1}{2\pi}\left(\int_{-3/20}^{i\infty} - \int_{3/20}^{i\infty}\right)\eta(\left(\frac{g_4 g_8}{g_2 g_6}\right)^2, \frac{g_5^4 g_{10}^2}{g_1 g_2 g_3 g_6 g_7 g_9}) \\ &= \frac{1}{4\pi^2} 4 \times 20 L(f, 2) \end{aligned}$$

f is the newform of conductor 20

$$f(q) = q - 2q^3 - q^5 + 2q^7 + q^9 + \dots$$

The end

We have just proved Touafek's conjecture

$$r(\{X_2, Y_2\}) = \frac{1}{2\pi^2} 40 L(f, 2) = 4 b_{20}$$

and previously it was obtained

$$r(\{X_2, Y_2\}) = r(\{x_2, y_2\})$$

$$2 m_{-2} = \pm r(\{x_2, y_2\}).$$

We deduce Boyd's conjecture

$$m_{-2} = m(P_{-2}) = 2 b_{20}$$

where $b_{20} = \frac{20}{4\pi^2} L(E^{-2}, 2)$.

Proof of the second conjecture

Similarly, Touafek considered the isomorphic curves E^4 defined by

$$(x_1+1)y_1^2+(x_1^2+4x_1+1)y_1+x_1(x_1+1)=0$$

and the elliptic curve E_1 defined by

$$Y_1^2+2X_1Y_1-X_1^3+X_1=0.$$

Both polynomials are tempered; so the respective regulators $r(\{x_1,y_1\}$ and $r(\{X_1,Y_1\})$ can be defined and from Touafek's computations we can also deduce the equality

$$r(\{x_1,y_1\})=\frac{3}{2}r(\{X_1,Y_1\}).$$

Touafek proved also the relation

$$r(\{X_2,Y_2\})=r(\{X_1,Y_1\}).$$

As previously we get

$$2m_4=r(\{x_1,y_1\}).$$

Finally, it follows

$$\begin{aligned} 2m_4 = r(\{x_1, y_1\}) &= \tfrac{3}{2} r(\{X_1, Y_1\}) \\ &= \tfrac{3}{2} r(\{X_2, Y_2\}) \\ &= \tfrac{3}{2} 4 b_{20} \end{aligned}$$

Conjectures again

Let

$$P_1 = y_1^2(x_1+1)^2 + y_1(2(x_1+1)^2 - 9x_1) + (x_1+1)^2$$
$$P_2 = (x_2+1)^2 y_2^2 + x_2 y_2 + (x_2+1)^2$$

Boyd conjectured

$$m(P_1) \stackrel{?}{=} 4\frac{21}{4\pi^2}L(f_{21},2) = 4b_{21}$$
$$m(P_2) \stackrel{?}{=} \frac{3}{2}\frac{21}{4\pi^2}L(f_{21},2) = \frac{3}{2}b_{21}$$

I proved (2004) that the elliptic curves defined by P_1 and P_2 are both isomorphic to E_1 defined by

$$Y_1^2 + 3X_1Y_1 = X_1(X_1-1)^2,$$

$$\pi r(\{X_1, Y_1\}) = 8D^{E_1}((1,0))$$

and conjectured

$$\pi r(\{X_1, Y_1\}) \stackrel{?}{=} 4b_{21}.$$

This last conjecture can be deduced from Brunault's or Lalin-Samart-Zudilin's results. I propose a variant obtained from discussions with O. Lecacheux.

The curve E_1 is 2-isogenous to $E_0 = 21a_1$ defined by

$$y^2 + xy = x^3 - 4x - 1.$$

$$\begin{aligned} E_0(\mathbb{Q}) &\simeq \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \\ &\simeq \langle P = (5,8)\rangle \times \langle Q = (-2,1)\rangle \end{aligned}$$

We get

$$2P = (2,-1) \quad 3P = (5,-13) \quad P + Q = (-1,-1) \quad 3P + Q = (-1,2)$$

Now choose the following isomorphism from the modular curve $X_0(21)$ into E_0:

$$\begin{array}{ccccc} \text{cusps} & 0 & 1/3 & 1/7 & \infty \\ & \downarrow & \downarrow & \downarrow & \downarrow \\ & 3P & 3P+Q & Q & (0) \end{array}$$

Define the modular units

$$\begin{array}{rcl} (f) & = & 2(Q)-2(0) \quad f=\frac{\eta(7\tau)\eta(3\tau)^3}{\eta(\tau)\eta(21\tau)^3} \\ (g) & = & 4(3P+Q)-4(0) \quad g=\frac{\eta(3\tau)\eta(7\tau)^7}{\eta(\tau)\eta(21\tau)^7} \end{array}$$

Theorem

Let E the elliptic curve defined by the tempered polynomial

$$Y^2-(2X+1)(X-2)Y+(X-1)^4$$

The curves E and E_0 are isomorphic by

$$\begin{aligned} x &= X-2 & X &= x+2 \\ y &= Y-X^2+X+2 & Y &= y+x^2+3x \end{aligned}$$

The curve E is parametrisable by the modular units f and g

$$\begin{aligned} r(\{X,Y\}) &= \frac{1}{2\pi}\int_{-8/21}^{8/21}\eta\Big(\frac{g_3^2 g_6^2 g_9^7}{g_1 g_2 g_4 g_5 g_8 g_{10}},\frac{g_7^6}{g_1 g_2 g_4 g_5 g_8 g_{10}}\Big) \\ &= 4\times\frac{21}{4\pi^2}L(f_{21},2) \\ &= 4b_{21} \\ &= \frac{8}{\pi}(D^{E_0}(P)-D^{E_0}(P+Q)) \end{aligned}$$

Corollary

Using the 2-isogeny and a relation between the elliptic dilogarithms we deduce

$$D^{E_1}((1,0)) = D^{E_0}(P) - D^{E_0}(P+Q)$$

then

$$r(\{X_1, Y_1\}) = 4b_{21}$$

and

$$m(P_1) = 4b_{21}$$