Epstein zeta functions and Bloch-Wigner dilogarithm

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Introduction

Denote $[a, b, c]$ the primitive definite positive quadratic form

$$Q(x, y) = ax^2 + bxy + cy^2, \qquad \text{a,b,c integers}$$

$d = b^2 - 4ac < 0$ its discriminant, $d \equiv 0 \quad \text{or} \quad 1 \quad \text{modulo} \quad 4$
Define the associate Epstein function

$$\zeta_Q(s) = \zeta_{[a,b,c]}(s) := \sum_{m,n}{}' \frac{1}{(am^2 + bmn + cn^2)^s}.$$

Zagier's conjecture

Pour tout $m \geq 2$

$$\tilde{\zeta_Q}(m) := \mid \mathrm{disc}(Q) \mid^{-1/2} \pi^{-m} \zeta_Q(m)$$

is a $\mathbb{Q}$-linear combination of values of the m-th polylogarithm for algebraic numbers.

Aim of the talk: Give the expression of $\tilde{\zeta_Q}(2)$ in terms of the Bloch-Wigner dilogarithm for

$$Q \in$$

$$\{[1,0,15],[3,0,5],[2,1,2],[1,1,4],[1,0,30],[3,0,10],[5,0,6],[2,0,15]\}$$

These results were obtained along my computations of Mahler measure of K3-hypersurfaces.

Theorem

$$+\frac{6\sqrt{15}}{\pi^3}\sum_{m,k}{}' \left(\frac{1}{(m^2+15k^2)^2}-\frac{1}{(3m^2+5k^2)^2}\right)$$

$$+\left(\frac{1}{(2m^2+mk+2k^2)^2}-\frac{1}{(m^2+mk+4k^2)^2}\right)=\frac{8}{5}d_3$$

$$\frac{36\times 4}{8\pi^3}\sqrt{\frac{5}{6}}\sum{}' \left(\frac{1}{(k^2+30m^2)^2}+\frac{1}{(3k^2+10m^2)^2}\right)$$

$$-\left(\frac{1}{(6k^2+5m^2)^2}+\frac{1}{(2k^2+15m^2)^2}\right)=\frac{14}{5}d_3$$

$$d_3:=\frac{3\sqrt{3}}{4\pi}L(\chi_{-3},2)=L'(\chi_{-3},-1)=\frac{2\sqrt{3}}{\pi^3}\sum{}' \frac{1}{(m^2+3k^2)^2}$$

Denote D the Bloch-Wigner dilogarithm

$$D(x) := \Im Li_2(x) + \log \mid x \mid \arg(1-x)$$

and

$$\Sigma D := D(\frac{7+\sqrt{-15}}{8}) + 2D(\frac{-1+\sqrt{-15}}{4}) - \frac{2}{3}D((\frac{-1+\sqrt{-15}}{4})^3)$$

Theorem

$$\tilde{\zeta}_{[2,1,2]}(2) = \frac{2}{15^2}\Sigma D + \frac{1}{15 \times 25}D(j)$$

$$\tilde{\zeta}_{[1,1,4]}(2) = \frac{2}{15^2}\Sigma D - \frac{1}{15 \times 25}D(j)$$

$$\tilde{\zeta}_{[1,0,15]}(2) = \frac{1}{12 \times 15}\Sigma D + \frac{13}{50 \times 15}D(j)$$

$$\tilde{\zeta}_{[3,0,5]}(2) = \frac{1}{12 \times 15}\Sigma D - \frac{13}{50 \times 15}D(j)$$

Sketch of proofs

Some ideas for the proof of the second part of the first theorem

$$H(d) := \{\text{equivalent classes of forms of discriminant } d\}$$

$$h(d) = \#H(d)$$

$$(a, b, c) \sim (a', b', c') \Leftrightarrow \exists p, q, r, s, \qquad ps - qr = 1, \text{such that}$$

$$ax^2 + bxy + cy^2 = a'(px + qy)^2 + b'(px + qy)(rx + sy) + c'(rx + sy)^2$$

$$(a, b, c) \text{ reduced} \iff$$

$$-a < b \leq a \leq c \qquad \text{with } b \leq 0 \quad \text{if} \quad a = c$$

In fact $h(d)$ is just the number of reduced forms of discriminant d.
In our case

$$d = -120 \qquad H(d) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \qquad h(-120) = 4$$

$$\theta_3(q) = \sum_{n=-\infty}^{+\infty} q^{n^2}$$

We use

- a result of K. Williams (1999) which reduces, for $d = -120$, to

$$\theta_3(q^5)\theta_3(q^6) + \theta_3(q^2)\theta_3(q^{15}) + \theta_3(q^{10})\theta_3(q^3) + \theta_3(q)\theta_3(q^{30})$$
$$= 4 + \underbrace{\sum_{n\geq 1} 2\left(\frac{-120}{n}\right)\frac{q^n}{1-q^n}}_{\text{série de Lambert}}$$

- a formula explained in Zucker and Mac Phedran (arxiv 0708.1224vi)

$$\zeta_{[a,0,b]}(s) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\sum{}' e^{-(am^2t+bn^2t)}dt$$
$$= \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}[\theta_3(q^a)\theta_3(q^b) - 1]dt$$

Thus we get

$$\zeta_{[5,0,6]}(s) + \zeta_{[2,0,15]}(s) + \zeta_{[10,0,3]}(s) + \zeta_{[1,0,30]}(s) = 2\zeta(s)L_{-120}(s).$$

But we need

$$S(s) = \zeta_{[5,0,6]}(s) + \zeta_{[2,0,15]}(s) \qquad S_1(s) = \zeta_{[10,0,3]}(s) + \zeta_{[1,0,30]}(s)$$

We want to apply the theorem

Theorem

Let m be positive and prime to d. The number $\phi(m)$ of a representative system of positive, primitive, integral forms of discriminant d (a single form being chosen from each class) is $w\sum\left(\frac{d}{\mu}\right)$ where μ ranges over all positive divisors of m and w defined by

$$w = \begin{cases} 2 & \text{if } d < -4 \\ 4 & \text{if } d = -4 \\ 6 & \text{if } d = -3. \end{cases}$$

Let $f=[5,0,6]$, $g=[2,0,15]$, $h=[3,0,10]$, $l=[1,0,30]$ and

$$N(f,m)=\#\{\text{representations of the integer } m \text{ by the form } f\}$$

We must take in account all the relations of type $N(f,3m)=N(g,m)$. So

$$S=\sum_{n\geq 1}\frac{N(f,n)+N(g,n)}{n^s}$$

$$=\sum_{(n,30)=1}\frac{\phi(n)}{2n^s}(A_+ + A_-)+\sum_{(n,30)=1}\left(\frac{n}{3}\right)\frac{\phi(n)}{2n^s}(A_- - A_+).$$

$$A_+ + A_- = \frac{1}{(1-\frac{1}{2^s})(1-\frac{1}{3^s})(1-\frac{1}{5^s})}$$

$$A_- - A_+ = \frac{-1}{(1+\frac{1}{2^s})(1-\frac{1}{3^s})(1+\frac{1}{5^s})}.$$

Finally

$$S = \zeta(s)L_{-120}(s) - L_{-3}(s)L_{40}(s)$$

$$S_1 = \zeta(s)L_{-120}(s) + L_{-3}(s)L_{40}(s)$$

$$S_1(s) - S(s) = 2L_{-3}(s)L_{40}(s).$$

With the formula

$$L_k(2s) = \frac{(-1)^{s-1}2^{2s-1}\pi^{2s}}{\sqrt{k}} \sum_{n=1}^{k} \chi_k(n) \frac{B_{2s}(1-\frac{n}{k})}{(2s)!}$$

we get the proof of the second formula of the theorem 1. (The first one is proved in a paper to appear in CMB)

Proof of the second theorem

It uses essentially a formula of Humbert-Gangl.

The idea is to associate to an imaginary quadratic field an hyperbolic manifold $\mathbb{M}^3$ in the hyperbolic plane.

And by means of triangulations with ideal tetrahedra, its volume, related to the zeta function of the field, can be expressed as a sum of Bloch-Wigner dilogarithms on algebraic numbers.