Epstein zeta functions and Bloch-Wigner dilogarithm

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Introduction

Denote \([a, b, c]\) the primitive definite positive quadratic form

\[
Q(x, y) = ax^2 + bxy + cy^2, \quad a, b, c \text{ integers}
\]

d \equiv b^2 - 4ac < 0 \text{ its discriminant, } d \equiv 0 \text{ or } 1 \text{ modulo } 4

Define the associate Epstein function

\[
\zeta_Q(s) = \zeta_{[a,b,c]}(s) := \sum_{m,n} \frac{1}{(am^2 + bmn + cn^2)^s}.
\]
Zagier’s conjecture

Pour tout $m \geq 2$

$$\zeta_Q(m) := |\text{disc}(Q)|^{-1/2} \pi^{-m} \zeta_Q(m)$$

is a $\mathbb{Q}$-linear combination of values of the $m$-th polylogarithm for algebraic numbers.

Aim of the talk: Give the expression of $\tilde{\zeta}_Q(2)$ in terms of the Bloch-Wigner dilogarithm for

$$Q \in \{ [1, 0, 15], [3, 0, 5], [2, 1, 2], [1, 1, 4], [1, 0, 30], [3, 0, 10], [5, 0, 6], [2, 0, 15] \}$$

These results were obtained along my computations of Mahler measure of K3-hypersurfaces.
Theorem

\[ + \frac{6\sqrt{15}}{\pi^3} \sum_{m,k} \left( \frac{1}{(m^2 + 15k^2)^2} - \frac{1}{(3m^2 + 5k^2)^2} \right) \]

\[ + \left( \frac{1}{(2m^2 + mk + 2k^2)^2} - \frac{1}{(m^2 + mk + 4k^2)^2} \right) = \frac{8}{5} d_3 \]

\[ \frac{36 \times 4}{8\pi^3} \sqrt{\frac{5}{6}} \sum_{k} \left( \frac{1}{(k^2 + 30m^2)^2} + \frac{1}{(3k^2 + 10m^2)^2} \right) \]

\[ - \left( \frac{1}{(6k^2 + 5m^2)^2} + \frac{1}{(2k^2 + 15m^2)^2} \right) = \frac{14}{5} d_3 \]

\[ d_3 := \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) = L'(\chi_{-3}, -1) = \frac{2\sqrt{3}}{\pi^3} \sum_{m,k} \frac{1}{(m^2 + 3k^2)^2} \]
Denote $D$ the Bloch-Wigner dilogarithm

$$D(x) := \Im Li_2(x) + \log |x| \arg(1 - x)$$

and

$$\Sigma D := D\left(\frac{7 + \sqrt{-15}}{8}\right) + 2D\left(\frac{-1 + \sqrt{-15}}{4}\right) - \frac{2}{3}D\left((\frac{-1 + \sqrt{-15}}{4})^3\right)$$

**Theorem**

$$\tilde{\zeta}_{[2,1,2]}(2) = \frac{2}{15^2} \Sigma D + \frac{1}{15 \times 25} D(j)$$

$$\tilde{\zeta}_{[1,1,4]}(2) = \frac{2}{15^2} \Sigma D - \frac{1}{15 \times 25} D(j)$$

$$\tilde{\zeta}_{[1,0,15]}(2) = \frac{1}{12 \times 15} \Sigma D + \frac{13}{50 \times 15} D(j)$$

$$\tilde{\zeta}_{[3,0,5]}(2) = \frac{1}{12 \times 15} \Sigma D - \frac{13}{50 \times 15} D(j)$$
Sketch of proofs

Some ideas for the proof of the second part of the first theorem

\[ H(d) := \{ \text{equivalent classes of forms of discriminant } d \} \]

\[ h(d) = \#H(d) \]

\[(a, b, c) \sim (a', b', c') \iff \exists p, q, r, s, \quad ps - qr = 1, \text{ such that} \]

\[ ax^2 + bxy + cy^2 = a'(px + qy)^2 + b'(px + qy)(rx + sy) + c'(rx + sy)^2 \]

\[(a, b, c) \text{ reduced} \iff \]

\[ -a < b \leq a \leq c \quad \text{with } b \leq 0 \quad \text{if} \quad a = c \]

In fact \( h(d) \) is just the number of reduced forms of discriminant \( d \).

In our case

\[ d = -120 \quad H(d) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \quad h(-120) = 4 \]
\[ \theta_3(q) = \sum_{n=-\infty}^{+\infty} q^{n^2} \]

We use
- a result of K. Williams (1999) which reduces, for \( d = -120 \), to

\[ \theta_3(q^5)\theta_3(q^6) + \theta_3(q^2)\theta_3(q^{15}) + \theta_3(q^{10})\theta_3(q^3) + \theta_3(q)\theta_3(q^{30}) \]

\[ = 4 + \sum_{n \geq 1} 2 \left( \frac{-120}{n} \right) \frac{q^n}{1 - q^n} \]

série de Lambert

- a formula explained in Zucker and Mac Phedran (arxiv 0708.1224vi)

\[ \zeta_{[a,0,b]}(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \sum_{l=1}^{\infty} e^{-(am^2 t + bn^2 t)} dt \]

\[ = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} [\theta_3(q^a)\theta_3(q^b) - 1] dt \]
Thus we get

\[ \zeta_{[5,0,6]}(s) + \zeta_{[2,0,15]}(s) + \zeta_{[10,0,3]}(s) + \zeta_{[1,0,30]}(s) = 2\zeta(s)L_{-120}(s). \]

But we need

\[ S(s) = \zeta_{[5,0,6]}(s) + \zeta_{[2,0,15]}(s) \quad S_1(s) = \zeta_{[10,0,3]}(s) + \zeta_{[1,0,30]}(s) \]

We want to apply the theorem

**Theorem**

Let \( m \) be positive and prime to \( d \). The number \( \phi(m) \) of a representative system of positive, primitive, integral forms of discriminant \( d \) (a single form being chosen from each class) is

\[ w \sum \left( \frac{d}{\mu} \right) \]

where \( \mu \) ranges over all positive divisors of \( m \) and \( w \) defined by

\[ w = \begin{cases} 
2 & \text{if } d < -4 \\
4 & \text{if } d = -4 \\
6 & \text{if } d = -3. 
\end{cases} \]
Let \( f = [5, 0, 6], \ g = [2, 0, 15], \ h = [3, 0, 10], \ l = [1, 0, 30] \) and
\[
N(f, m) = \# \{ \text{representations of the integer } m \text{ by the form } f \}.
\]

We must take in account all the relations of type \( N(f, 3m) = N(g, m) \). So
\[
S = \sum_{n \geq 1} \frac{N(f, n) + N(g, n)}{n^s}
\]
\[
= \sum_{(n, 30) = 1} \frac{\phi(n)}{2n^s} (A_+ + A_-) + \sum_{(n, 30) = 1} \left( \frac{n}{3} \right) \frac{\phi(n)}{2n^s} (A_- - A_+).
\]
\[
A_+ + A_- = \frac{1}{(1 - \frac{1}{2^s})(1 - \frac{1}{3^s})(1 - \frac{1}{5^s})}
\]
\[
A_- - A_+ = \frac{-1}{(1 + \frac{1}{2^s})(1 - \frac{1}{3^s})(1 + \frac{1}{5^s})}.
\]
Finally

\[ S = \zeta(s)L_{-120}(s) - L_{-3}(s)L_{40}(s) \]

\[ S_1 = \zeta(s)L_{-120}(s) + L_{-3}(s)L_{40}(s) \]

\[ S_1(s) - S(s) = 2L_{-3}(s)L_{40}(s). \]

With the formula

\[ L_k(2s) = \frac{(-1)^{s-1}2^{2s-1}\pi^{2s}}{\sqrt{k}} \sum_{n=1}^{k} \chi_k(n) \frac{B_{2s}(1 - \frac{n}{k})}{(2s)!} \]

we get the proof of the second formula of the theorem 1. (The first one is proved in a paper to appear in CMB)
Proof of the second theorem
It uses essentially a formula of Humbert-Gangl.
The idea is to associate to an imaginary quadratic field an hyperbolic manifold $\mathbb{M}^3$ in the hyperbolic plane.
And by means of triangulations with ideal tetrahedra, its volume, related to the zeta function of the field, can be expressed as a sum of Bloch-Wigner dilogarithms on algebraic numbers.