Elliptic fibrations on the modular surface associated to $\Gamma_1(8)$

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Introduction

Motivation:

- ranks of elliptic curves (Lecacheux)
- links between the Mahler measure of K3-surfaces and their L-series (Bertin)

Introduction

The story begins with the family (Y_k) of K3-surfaces

$$(Y_k) \;\; X + \frac{1}{X} + Y + \frac{1}{Y} + Z + \frac{1}{Z} = k$$

For $k = 2, 3, 6, 10, 18, 102, 198$ and some rational k^2, Y_k is a singular $K3$ i.e. with Picard number $\rho = 20$ (Boyd (computational), Schütt)

- Y_2 has transcendental lattice

$$\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}.$$

- with the elliptic fibration of parameter $X + Y + Z = s$, Y_2 comes from the elliptic pencil of Beukers-Stienstra

$$xyz + \tau(x+y)(x+z)(y+z) = 0$$

where $1/\tau = (s-1)^2$

- with parameter s, the singular fibers are of
Dynkin type A_{11}, A_5, $2A_1$
or equivalently of
Kodaira type I_{12}, I_6, $2I_2$, $2I_1$
and

$$\text{Mordell-Weil group: } \mathbb{Z}/6\mathbb{Z}$$

Introduction

- Y_2 carries also the structure of the modular elliptic surface for $\Gamma_1(8)$ with parameter $Z = s$, the singular fibers are of
Dynkin type $2A_7,\ A_3,\ A_1$
i.e. Kodaira type $2I_8,\ I_4,\ I_2,\ 2I_1$
Mordell-Weil group: $\mathbb{Z}/8\mathbb{Z}$

Previous results

Elkies gave a list of 11 $D < 0$ corresponding to a unique $K3$ over $\mathbb{Q}$ with NS of rank 20 and discriminant $-D$ consisting entirely of classes of divisors over $\mathbb{Q}$
For $D = -8$, he obtained a model

$$y^2 = x^3 - 675x + 27(27t - 196 + \frac{27}{t})$$

with $2E_8$ $(= 2II^*)$ fibers at $t = 0$ and $t = \infty$ and A_1 $(= I_2)$ at $t = -1$
M-W has rank 1 and no torsion

Previous results

Schütt proved the existence of $K3$ surfaces of Picard rank 20 over $\mathbb{Q}$ and gave for the discriminant $D = -8$ an elliptic fibration with singular fibers A_3, E_7, E_8 (I_4, III^*, II^*)

Previous results

Shimada & Zhang obtained a list, without equations but with M-W, of extremal $K3$ surfaces.
In particuliar 14 extremal elliptic $K3$ with transcendental lattice

$$\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}.$$

The results (B-L)

Theorem

(B-L) There are 30 *elliptic fibrations with section, distinct up to isomorphism, on the elliptic surface*

$$X + \frac{1}{X} + Y + \frac{1}{Y} + Z + \frac{1}{Z} = 2,$$

listed with the rank and torsion of their Mordell-Weil group.
The list contains 14 *fibrations of rank* 0, 13 *fibrations of rank* 1 *and* 3 *fibrations of rank* 2.
For each fibration, a Weierstrass model is determined.

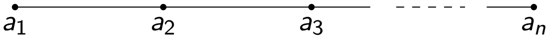
Table of fibrations

parameter	singular fibers	type of reductible fibers	Rank	Torsion
$1-s$	$2I_8, I_4, I_2, 2I_1$	A_1, A_3, A_7, A_7	0	8
$2-k$	$I_1^*, I_{12}, I_2, 3I_1$	A_{11}, A_1, D_5	1	4
$3-v$	$I_8, I_{10}, 6I_1$	A_7, A_9	2	0
$4-a$	$I_8, I_1^*, I_6, 3I_1$	D_5, A_5, A_7	1	0
$5-d$	$2I_2^*, I_2, I_0^*$	$A_1, D_4, 2D_6$	1	2×2
$6-p$	I_2^*, I_4^*, I_2, I_4	A_1, D_6, A_3, D_8	0	2×2
$7-w$	$I_6, I_{12}, 2I_2, 2I_1$	$A_5, A_1, A_1 A_{11}$	0	6
$8-b$	$2IV^*, I_6, 2I_1$	A_5, E_6, E_6	1	3
$9-r$	$I_6^*, I_2^*, I_2, 2I_1$	D_6, A_1, D_{10}	1	0
$10-e$	$III^*, I_4^*, 2I_2, I_1$	A_1, A_1, D_8, E_7	1	2
$11-f$	III^*, II^*, I_4, I_1	E_7, A_3, E_8	0	0
$12-g$	$2III^*, I_4, I_2$	E_7, E_7, A_1, A_3	0	2
$13-h$	$2II^*, I_2, 2I_1$	A_1, E_8, E_8	1	0
$14-t$	$2I_4^*, I_2, 2I_1$	A_1, D_8, D_8	1	2

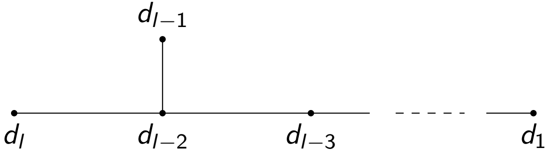
parameter	singular fibers	type of reductible fibers	Rank	Torsion
$15 - l$	$I_{10}, I_3^*, 5I_1$	A_9, D_7	2	0
$16 - o$	$I_{12}^*, I_2, 4I_1$	A_1, D_{16}	1	2
$17 - q$	$I_{10}^*, I_4, I_2, 2I_1$	A_3, A_1, D_{14}	0	2
$18 - m$	$I_{16}, 3I_2, 2I_1$	A_1, A_1, A_1, A_{15}	0	4
$19 - n$	$I_{16}, I_2, 6I_1$	A_1, A_{15}	2	0
$20 - j$	$IV^*, I_{12}, I_2, 2I_1$	A_{11}, E_6, A_1	0	3
$21 - c$	$I_{18}, 6I_1$	A_{17}	1	3
$22 - u$	I_8^*, I_1^*, I_2, I_1	A_1, D_5, D_{12}	0	2
$23 - i$	$I_{13}^*, I_2, 3I_1$	$A_1 D_{17}$	0	0
$24 - \psi$	$III^*, I_6^*, 3I_1$	$E_7 D_{10}$	1	2
$25 - \delta$	$I_5^*, II^*, I_2, 2I_1$	$E_8, A_1 D_9$	0	0
$26 - \pi$	I_3^*, I_6^*, I_2, I_1	A_1, D_{10}, D_7	0	2
$27 - \mu$	$IV^*, I_{10}, 2I_2, 2I_1$	A_9, A_1, A_1, E_6	1	0
$28 - \alpha$	$I_0^*, I_{14}, 4I_1$	D_4, A_{13}	1	0
$29 - \beta$	III^*, I_2^*, I_1^*	E_7, D_6, D_5	0	2
$30 - \phi$	$III^*, I_7^*, 2I_1$	E_7, D_{11}	0	0

Dynkin diagrams of root lattices

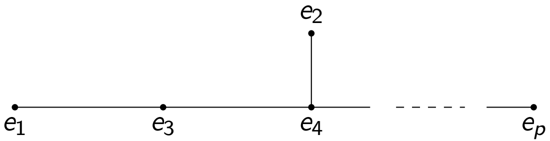
$A_n = \langle a_1, a_2, \ldots, a_n \rangle$



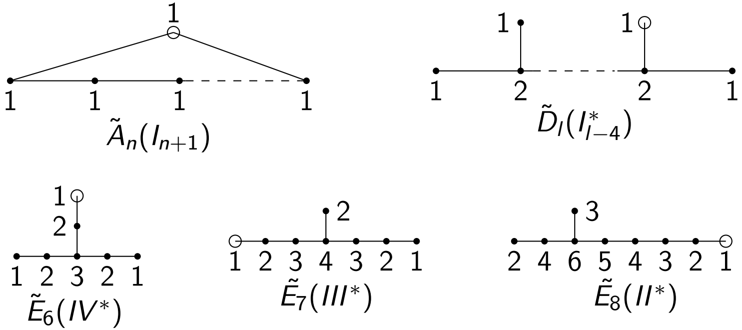
$D_l = \langle d_1, d_2, \ldots, d_l \rangle$



$E_p = \langle e_1, e_2, \ldots, e_p \rangle$



Extended Dynkin diagrams



Some definitions

- The trivial lattice $T(X)$

$$T(X) = < \bar{O}, F > \oplus_{v \in S} T_v$$

where $\bar{O}$ denotes the zero section, F the general fiber, S the points of C corresponding to the singular fibers and T_v the lattice generated by the fiber components except the zero component.

- The frame $W(X)$

$$W(X) = \langle \bar{O}, F \rangle^{\perp} \subset NS(X).$$

The frame $W(X)$ is a negative-definite even lattice of rank $\rho(X) - 2$.

-

$$MWL(X) = W(X)/\overline{W(X)}_{\text{root}} \quad (MW)_{\text{tors}} = \overline{W(X)}_{\text{root}}/W(X)_{\text{root}}$$

$$T(X) = U \oplus W(X)_{\text{root}}.$$

The bar denotes the primitive closure of a set inside another.

Some ingredients in the proof

Fact Up to isomorphism, there is only a finite number of elliptic fibrations

Nishiyama's method

- Idea: embed the frames of all elliptic fibrations into an unimodular lattice of rank 24,i.e. a Niemeier lattice given in the following table.

Niemeier lattices

L_{root}	L/L_{root}	L_{root}	L/L_{root}
E_8^3	(0)	$D_5^{\oplus 2} \oplus A_7^{\oplus 2}$	$\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$
$E_8 \oplus D_{16}$	$\mathbb{Z}/2\mathbb{Z}$	$A_8^{\oplus 3}$	$\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z}$
$E_7^{\oplus 2} \oplus D_{10}$	$(\mathbb{Z}/2\mathbb{Z})^2$	A_{24}	$\mathbb{Z}/5\mathbb{Z}$
$E_7 \oplus A_{17}$	$\mathbb{Z}/6\mathbb{Z}$	$A_{12}^{\oplus 2}$	$\mathbb{Z}/13\mathbb{Z}$
D_{24}	$\mathbb{Z}/2\mathbb{Z}$	$D_4^{\oplus 6}$	$(\mathbb{Z}/2\mathbb{Z})^6$
$D_{12}^{\oplus 2}$	$(\mathbb{Z}/2\mathbb{Z})^2$	$D_4 \oplus A_5^{\oplus 4}$	$\mathbb{Z}/2\mathbb{Z} \oplus (\mathbb{Z}/6\mathbb{Z})^2$
$D_8^{\oplus 3}$	$(\mathbb{Z}/2\mathbb{Z})^3$	$A_6^{\oplus 4}$	$(\mathbb{Z}/7\mathbb{Z})^2$
$D_9 \oplus A_{15}$	$\mathbb{Z}/8\mathbb{Z}$	$A_4^{\oplus 6}$	$(\mathbb{Z}/5\mathbb{Z})^3$
$E_6^{\oplus 4}$	$(\mathbb{Z}/3\mathbb{Z})^2$	$A_3^{\oplus 8}$	$(\mathbb{Z}/4\mathbb{Z})^4$
$E_6 \oplus D_7 \oplus A_{11}$	$\mathbb{Z}/12\mathbb{Z}$	$A_2^{\oplus 12}$	$(\mathbb{Z}/3\mathbb{Z})^6$
$D_6^{\oplus 4}$	$(\mathbb{Z}/2\mathbb{Z})^4$	$A_1^{\oplus 24}$	$(\mathbb{Z}/2\mathbb{Z})^{12}$
$D_6 \oplus A_9^{\oplus 2}$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/10\mathbb{Z}$	0	Λ_{24}

Realization

Let q_L denote the discriminant quadratic form of L.

- Determine an even negative-definite lattice M such that

$$q_M = -q_{NS(X)} \quad \mathrm{rk}(M) + \rho(X) = 26$$

- By Nikulin's theorem, $M \oplus W(X)$ admits a Niemeier lattice L as an overlattice such that the embeddings of M and $W(X)$ into L are primitive and satisfy

$$M_L^{\perp} = W(X), \quad W(X)_L^{\perp} = M.$$

How to get M?

cf. Schütt & Shioda, Elliptic surfaces in Adv. stud. in Pure Math. (2010)
Let $\mathbb{T}(X)$ the transcendental lattice, i.e.

$$\mathbb{T}(X) = NS(X)^{\perp} \subset H^2(X, \mathbb{Z})$$

$$\text{rk}(\mathbb{T}(X)) = r = 22 - \rho(X) \quad \text{signature} = (2, 20 - \rho(X))$$

Let $t = r - 2$.
By Nikulin's theorem on signatures, since $U^t \bigoplus E_8$ is an even unimodular lattice

$$\underbrace{\mathbb{T}(X)[-1]}_{\text{signature}(20-\rho(X),2)} \underset{\text{primitive}}{\hookrightarrow} \underbrace{U^t \oplus E_8}_{(20-\rho(X),20-\rho(X)+8)}$$

And

$$M = \mathbb{T}(X)[-1]^{\perp} \hookrightarrow U^t \oplus E_8$$

By construction, M is a negative definite lattice of rank $2t + 8 - r = r + 4 = 26 - \rho(X)$. Again by Nikulin

$$q_M = -q_{\mathbb{T}(X)[-1]} = q_{\mathbb{T}(X)} = -q_{\mathrm{NS}(X)}.$$

So M has the required shape.
Now, for

$$\mathbb{T}(X) = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$$

we get

$$M = A_1 \oplus D_5.$$

Torsion

Denote $N := M^{\perp}$ into L_{root} and $W = M^{\perp}$ into L.

- Since M satisfies $M_{\text{root}} = M$,

$$M \text{ primitively embedded in } L_{\text{root}} \iff M \text{ primitively embedded in } L$$

$$N_{\text{root}} = W_{\text{root}}$$

- Rank of the Mordell-Weil $= \mathrm{rk}(W) - \mathrm{rk}(W_{\text{root}})$
- $$(MW)_{\text{tors}} = \overline{W}_{\text{root}}/W_{\text{root}}$$
- If $\det N = \det M$, then the Mordell-Weil group is torsion-free.
- If $r = 0$, then the Mordell-Weil group is isomorphic to W/N.
- In general,

$$\overline{W}_{\text{root}}/W_{\text{root}} \subset W/N \subset L/L_{\text{root}}.$$

An example

Let

$$D_5 \oplus A_1 \underset{\text{primitive}}{\hookrightarrow} E_7$$

with the embedding $\langle e_2, e_5, e_4, e_3, e_1 \rangle \oplus \langle e_7 \rangle$ Thus

$$(D_5 \oplus A_1)^{\perp}_{E_7} = \langle 3e_2 + 2e_1 + 4e_3 + 6e_4 + 5e_5 + 4e_6 + 2e_7 \rangle = \langle (-4) \rangle$$

Thus a primitive embedding in $L_{\text{root}} = E_7 A_{17}$ and

$$N = (D_5 \oplus A_1)^{\perp}_{L_{\text{root}}} = \langle (-4) \rangle \oplus A_{17}$$

Now

$$W_{\text{root}} = N_{\text{root}} = A_{17}$$

$$\text{rk} MW = \text{rk} W - \text{rk} W_{\text{root}} = 18 - 17 = 1$$

$$\det(N) = 4 \times 18 = \det(W) \times 9$$

$$W/N \simeq \mathbb{Z}/3\mathbb{Z}$$

Recall

$$\overline{W}_{\text{root}}/W_{\text{root}} \subset W/N \subset L/L_{\text{root}}.$$

Since

$$L/L_{\text{root}} \simeq \langle \eta_7 + 3\alpha_{17} + L_{\text{root}} \rangle \simeq \mathbb{Z}/6\mathbb{Z}$$

with

$$\eta_7 = -\frac{1}{2}(2e_1 + 3e_2 + 4e_3 + 6e_4 + 5e_5 + 4e_6 + 3e_7)$$

$$\alpha_{17} = \frac{1}{18}(17a_1 + 16a_2 + \ldots + a_{17})$$

It follows

$$W/N = \langle 6\alpha_{17} + N \rangle$$

and since $6\alpha_{17} \in \overline{W}_{\text{root}}$

$$\overline{W}_{\text{root}}/W_{\text{root}} \simeq \mathbb{Z}/3\mathbb{Z}$$

Modular surface for $\Gamma_1(8)$

Let

$$X + \frac{1}{X} + Y + \frac{1}{Y} = k$$

the modular surface for $\Gamma_1(4) \cap \Gamma_0(8)$ (Beauville).
Using the birational transformation

$$X = \frac{-U(U-1)}{V} \text{ and } Y = \frac{V}{U-1}$$

with inverse

$$U = -XY \text{ and } V = -Y(XY+1)$$

we obtain the Weierstrass equation

$$V^2 - kUV = U(U-1)^2.$$

The point $Q = (U = 1, V = 0)$ is a $4-$torsion point. If we want A with $2A = Q$ to be a rational point, then $k = -s - 1/s + 2$. It follows

$$V^2 + (s + \frac{1}{s} - 2)UV = U(U-1)^2. \tag{1}$$

Thus the point $(-s, 1)$ is of order 8 on

$$X + \frac{1}{X} + Y + \frac{1}{Y} + s + \frac{1}{s} = 2.$$

Hence, the modular surface Y_2 associated to $\Gamma_1(8)$ with the elliptic fibration

$$(X, Y, Z) \mapsto Z = s$$

and singular fibers I_8 (at $s = 0$ and $s = \infty$), I_4 (at $s = 1$), I_2, $2I_1$.

Proposition

Let X be a $K3$ surface and D an effective divisor on X that has the same type as a singular fiber of an elliptic fibration. Then X admits a unique elliptic fibration with D as a singular fiber.
Moreover, any irreducible curve C on X with $D.C = 1$ induces a section of the elliptic fibration.

If X is a $K3$ surface and

$$\pi : X \to C$$

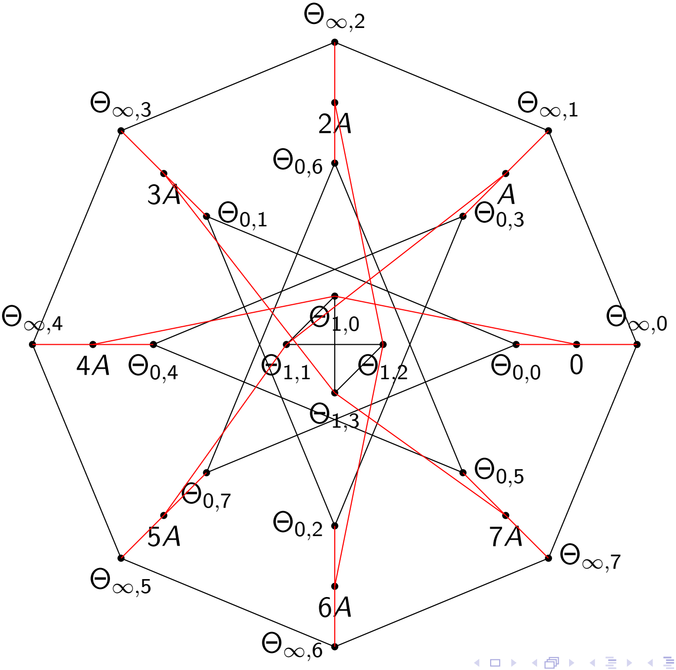
an elliptic fibration, then the curve C is of genus 0 and we define an elliptic parameter as a generator of the function field of C.
Moreover if we have two effective divisors D_1 and D_2 for the same fibration we can choose an elliptic parameter with divisor $D_1 - D_2$.

From a fibration to another fibration

We start with the fibration of parameter s

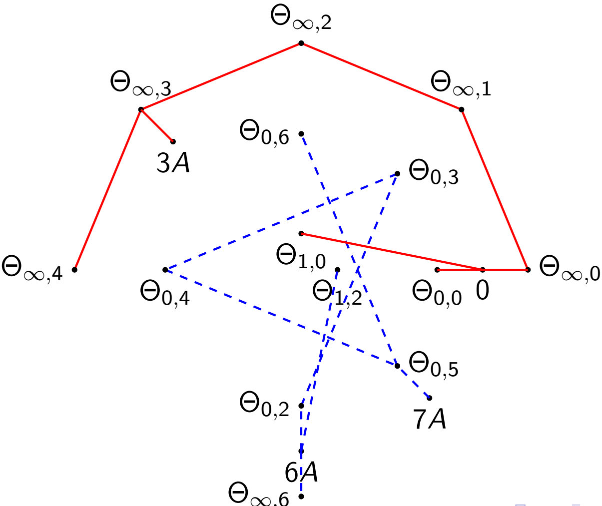
$$E_s : y^2 + (s^2 + 1 - 2s)yx = x(x - s^2)^2$$

and draw the graph of torsion sections and singular fibers for $s = 0, \infty, 1$.


$\Theta_{\infty,2}$
$\Theta_{\infty,3}$
$\Theta_{\infty,1}$
$2A$
$\Theta_{0,6}$
$3A$
A
$\Theta_{0,1}$
$\Theta_{0,3}$
$\Theta_{\infty,4}$
$\Theta_{1,0}$
$\Theta_{\infty,0}$
$4A$
$\Theta_{0,4}$
$\Theta_{1,1}$
$\Theta_{1,2}$
$\Theta_{0,0}$
0
$\Theta_{1,3}$
$\Theta_{0,5}$
$\Theta_{0,7}$
$\Theta_{0,2}$
$5A$
$7A$
$\Theta_{\infty,7}$
$\Theta_{\infty,5}$
$6A$
$\Theta_{\infty,6}$

Fibration of parameter t

On the previous graph , we can see two singular fibers of type I_4^* of another fibration. They correspond to two divisors D_1 and D_2



We write $D_i = \delta_i + \Delta_i$ with $i = 1, 2$ where δ_i is an horizontal divisor and Δ_i a vertical divisor.
There exists a function t_0 on E_s with divisor $\delta_1 - \delta_2$.
We compute divisor of t_0 on Y_2.
So the parameter of the new fibration is $t = t_0 s^a (s-1)^b$ and we can compute a, b.
Using a standard transformation we obtain

$$y^2 = x^3 + t(t^2 + 1 + 4t)x^2 + t^4 x.$$

The singular fibers are

$$\begin{array}{llll} \text{at} & t = 0 & \text{of type} & I_4^* \\ \text{at} & t = \infty & \text{of type} & I_4^* \\ \text{at} & t = -1 & \text{of type} & I_2 \\ \text{at} & t = t_0 & \text{of type} & I_1 \end{array}$$

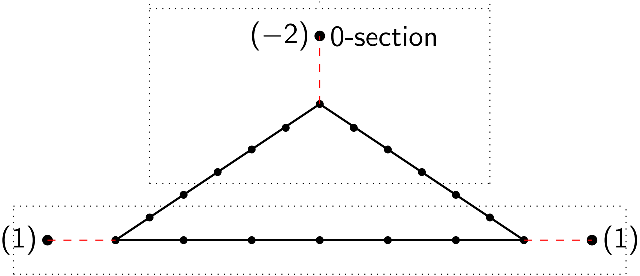
where t_0 is a root of the polynomial $Z^2 + 6Z + 1$.
The Mordell-Weil group is of rank one and the point $(-t^3, 2t^4)$ is of height 1. The torsion group is of order 2.

Fibration of parameter ψ

We start with the fibration of parameter c

$$y^2 + (c^2 + 5)yx + y = x^3$$

with a singular fiber I_{18} for $c = \infty$ and 3-torsion section.



On this graph we see two singular fibers III^* and I_6^*. The function x has the horizontal divisor $-2(0) + P + (-P)$ if P denotes the 3-torsion point and we can take it as the parameter ψ of the new fibration. We get the equation

$$y^2 = x^3 - 5\,x^2\psi^2 - \psi\,x^2 - \psi^5 x$$

.

The singular fibers are

$$\begin{array}{llll} \text{at} & \psi = \infty & \text{of type} & I_6^* \\ \text{at} & \psi = 0 & \text{of type} & III^* \\ \text{at} & \psi = -\frac{1}{4} & \text{of type} & I_1 \\ \text{at} & \psi = \psi_0 & \text{of type} & I_1 \end{array}$$

with ψ_0 root of the polynomial $x^2 + 6x + 1$. The Mordell-Weil group is of rank 1. The torsion-group is of order 2.