I INTRODUCTION

Lehmer's " On factorization of certain cyclotomic functions" (1933)

was searching large prime numbers.

The complexity of the method is related to the growth of the Mahler's measure of P

$$M(P) = \prod_{P(\alpha)=0} max(|\alpha|, 1)$$

for P monic with integer coefficients.

Lehmer's polynomial

$$X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1$$

has the smallest known measure 1.1762...

Lehmer's polynomial is a Salem polynomial (i. e. irreducible, monic, with integer coefficients, one root inside the unit disk, one root outside and some on.)

So a Salem polynomial is reciprocal and cuts the 1-torus $\mathbb{T}^1$.

The logarithmic Mahler's measure of a polynomial P

$$m(P) = \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} log \mid P(x_1, \cdots, x_n) \mid \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}$$

is related to the Mahler's measure by

$$M(P) = exp(m(P)).$$

By Jensen's formula, if $P \in \mathbb{Z}[X]$ is monic, then

$$M(P) = \prod_{P(\alpha)=0} max(\mid \alpha \mid, 1).$$

Boyd's limit formula (1981)

$$m(P(x, x^N)) \longrightarrow m(P(x, y))$$

is a hope to get small measures in one variable from small measures in two variables.

$M(P_1) = M((x+1)y^2+(x^2+x+1)y+x(x+1)) = 1.25$

$M(P_2) = M(y^2 + (x^2 + x + 1)y + x^2) = 1.28..$

are the smallest measures in two variables.

Notice that P_1 and P_2 cut the 2-torus respectively in (j, ij^2) and (j, ij).

Question: what about reciprocal polynomials in 3 variables and cutting the 3-torus?

Boyd and Mossinghoff (2002) found

$$M(P_3) = M(X + \frac{1}{X} + Y + \frac{1}{Y} + Z + \frac{1}{Z} + 1)$$

$$= 1.4483035845491699038...$$

using an explicit formula of Bertin.

Polynomials P_1, P_2 define elliptic curves and P_3 defines a quartic surface in $\mathbb{P}^3$ that are Calabi-Yau varieties.

Definition: A smooth projective variety X (over $\mathbb{C}$, $\mathbb{Q}$ or a number field) of dimension d is a Calabi-Yau variety if

1) $H^i(X, O_X) = 0$ for $0 < i < d$

2) $K_X := \Lambda^d \Omega^1_X \simeq O_X$

Thus

$p_g(X) := dim H^0(X, K_X) = dim(H^d(X, O_X) = 1.$

If $d = 1$, 1) is empty and 2)$\Rightarrow$, that if X has a rational point, X is an elliptic curve.

If $d = 2$, $H^1(X, O_X) = 0$ and $p_g(X) = 1 \Rightarrow$ that a Calabi-Yau in dimension 2 is a $K3$ surface:

for example Kummer surfaces, quartics in $\mathbb{P}^3$, double coverings of $\mathbb{P}^2$ branched along a sextic.

Explicit formulae: The first were given by Deninger (1997)

1) $m(P_2) =? \frac{15}{4\pi^2} L(E,2) = L'(E,0)$

with E elliptic curve of conductor 15 defined by P_2.

2) $m(P_2)$ is an Eisenstein-Kronecker series of the elliptic curve E more or less an elliptic regulator.

Many examples of 1) by Boyd and of 2) by R-Villegas.

EXPLICIT FORMULAE FOR CALABI-YAU IN DIMENSION 1

Let

$$P_k(x,y) = (x+y+1)(x+1)(y+1) + kxy$$

the family given by Beauville isomorphic to elliptic curves with rational 5-torsion.

But $\mathcal{H}/\Gamma_0(N)^*$ is the moduli space of (E, C_N) of elliptic curves with cyclic isogeny modulo the Fricke involution.

$$\Gamma_0(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sl_2(\mathbb{Z}) \ / \ c \equiv O \ \ (N)\}$$

and

$$\Gamma_0(N)^* = \langle \Gamma_0(N), w_N \rangle$$

where

$$w_N = \begin{pmatrix} 0 & -\frac{1}{\sqrt{N}} \\ \sqrt{N} & 0 \end{pmatrix}$$

is the Fricke involution.

So Beauville's family is modular in the following sense.

Let $\mathcal{F}$ be a fundamental domain for $\Gamma_1(5)$, there is a unique $\tau \in \mathcal{F}$, such that

$$\begin{array}{rcl} -\frac{1}{k} & = & t(\tau) \\ t(\tau) & = & q\prod_{n=1}^{\infty}(1-q^n)^{5(\frac{n}{5})}, \qquad q = e^{2\pi\imath\tau} \\ & = & q - 5q^2 + 15q^3 - 30q^4 + 40q^5 + \ldots \end{array}$$

$(\frac{n}{5})$ being Legendre's symbol.

Theorem 1. *(Bertin) Let $k \in \mathbb{Z}$ such that P_k does not vanish on $\mathbb{T}^2$ ($k \notin [-12, 0]$), then*

$$m(P_k) = \Re(-2\pi i\tau + (1 - \frac{i}{2})\sum_{n\geq 1}\sum_{d|n}\chi(d)d^2\frac{e^{2\pi in\tau}}{n}$$

$$+(1 + \frac{i}{2})\sum_{n\geq 1}\sum_{d|n}\bar{\chi}(d)d^2\frac{e^{2\pi in\tau}}{n}).$$

where χ is the odd quadratic character of conductor 5 satisfying $\chi(2) = i$.

Theorem 2. *(Bertin) With the previous notations, $m(P_k)$ can be expressed as an Eisenstein-Kronecker series:*

$$m(P_k) = \Re\left(\frac{5^2 \Im\tau}{2\pi^2} \sum\nolimits'_{m,n} \frac{C(\chi)\chi(n) + \bar{C}(\bar{\chi})\bar{\chi}(n)}{(5m\tau + n)^2(5m\bar{\tau} + n)}\right)$$

where, if $c(\chi)$ is the Gauss sum for the character χ,

$$C(\chi) = \left(-\frac{1}{4} + \frac{i}{2}\right)c(\bar{\chi}).$$

Sketch of proofs

Let $m(P_k) = m(k)$.

-
$$m(k) = \Re(\tilde{m}(k)),$$

-
$$\tilde{m}(k) =$$
$$\frac{1}{(2\pi i)^2}\int_{\mathbb{T}^2} \log(k + \frac{(x+y+1)(x+1)(y+1)}{xy})\frac{dx}{x}\frac{dy}{y}$$
so

-
$$\tilde{m}'(k) = \frac{1}{(2\pi i)^2}\int_{\mathbb{T}^2} (\frac{1}{k + \frac{(x+y+1)(x+1)(y+1)}{xy}})\frac{dx}{x}\frac{dy}{y}.$$
Then

- $\tilde{m}'(k)$ is a period of the elliptic curve associated to P_k, thus a solution of the Picard-Fuchs differential equation of the family.

  Now, if

- $$(x+y+1)(x+1)(y+1) - \frac{1}{t}xy = 0$$

  by Verrill, the corresponding P-F equation is

  $$t(t^2+11t-1)y''+(3t^2+22t-1)y'+(t+3)y = 0$$

  and a solution is

  $$f = \frac{\eta(5\tau)^{5/2}}{(t(\tau)\eta(\tau))^{1/2}} = 1+3q+4q^2+2q^3+q^4+...$$

  where $t(\tau)$ is given above. So

$$f(t) = \sum_{n\geq 0}(\sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)^2\left(\begin{array}{c} n+k \\ k \end{array}\right)t^n.$$

Comparing their q-developments, we deduce

$$\tilde{m}'(k) = -tf$$

and

$$d\tilde{m} = -f\frac{dt}{t} = -f\frac{t'(q)dq}{t}. \in \mathcal{M}_3(\Gamma_1(5))$$

Let $L_\chi(q) \in \mathcal{M}_3(\Gamma)$,

$$L_\chi(q) = \sum_{n\geq 1}(\sum_{d|n}\chi(d)d^2)q^n,$$

Now

$$-fq\frac{dt}{tdq} = -1 + (1 - \frac{i}{2})L_\chi + (1 + \frac{i}{2})L_{\bar{\chi}}.$$

- Finally, by integration between q and $+\infty$, we get the formula.

For the proof of theorem 2 we express $m(k)$ in terms of a function $K(e^{2\pi i\tau})$, real, periodic of period 5, which can be developped in a Fourier series, following an idea given in Weil "Elliptic functions according to Eisenstein and Kronecker".

The elliptic regulator

Let K be a field. By Matsumoto, $K_2(K)$ can be described in terms of symbols $\{f, g\}$, f and $g \in K^*$ and relations.

For example, if v is a discrete valuation on K with maximal ideal $\mathcal{M}$ and residual field k, Tate's tame symbol

$$(x, y)_v \equiv (-1)^{v(x)v(y)} \frac{x^{v(y)}}{y^{v(x)}} \qquad (mod \mathcal{M})$$

defines a homomorphism

$$\lambda_v : K_2(F) \to k^*.$$

Let E an elliptic curve on $\mathbb{Q}$ and $\mathbb{Q}(E)$ its rational function field. To any $P \in E(\bar{\mathbb{Q}})$ is associated a valuation on $\mathbb{Q}(E)$ that gives the homomorphism

$$\lambda_P : K_2(\mathbb{Q}(E)) \to \mathbb{Q}(P)^*$$

and the exact sequence

$$0 \to K_2(E)\otimes\mathbb{Q} \to K_2(\mathbb{Q}(E))\otimes\mathbb{Q} \xrightarrow{\lambda} \coprod_{P\in E(\bar{\mathbb{Q}})} \mathbb{Q}(P)^*\otimes\mathbb{Q} -$$

By definition $K_2(E)$ is modulo torsion

$$K_2(E) \simeq \ker\lambda = \cap_P \ker\lambda_P \subset K_2(\mathbb{Q}(E)).$$

By a theorem due to Villegas, under some hypothesis, if $P \in \mathbb{Q}[x^{\pm}, y^{\pm}]$ defines a smooth curve C, we get

$$\{x, y\} \in K_2(C)$$

. In particuliar, if

$$P(x, y) = (x + y + 1)(x + 1)(y + 1) + xy$$

we get

$$\{x, y\} \in K_2(E)$$

. Let f et g dans $\mathbb{Q}(E)^*$and define

$$\eta(f, g) = \log \mid f \mid d\text{arg}g - \log \mid g \mid d\text{arg}f.$$

Definition The elliptic regulator r of E is given by

$$\begin{array}{rcl} r : K_2(E) & \rightarrow & \mathbb{R} \\ \{f, g\} & \mapsto & \frac{1}{2\pi} \int_\gamma \eta(f, g) \end{array}$$

for a suitable loop γ.

But P does not cut the torus and when x describes the unit circle, one root of P, say $y_1(x)$ satisfies

$$\mid y_1(x) \mid < 1$$

and $(x, y_1(x))$ is a suitable loop on E. So

$$\begin{aligned} m(P) &= \frac{1}{(2\pi i)^2} \int_{|x|=1} \int_{|y|=1} \log \mid P_1(x, y) \mid \frac{dx}{x}\frac{dy}{y} \\ &= -\frac{1}{2\pi i} \int_{|x|=1} \log \mid y_1(x) \mid \frac{dx}{x}, \end{aligned}$$

from Jensen's formula and

$$m(P) = \frac{-1}{2\pi i} \int_{\sigma_1} \log \mid y_1 \mid \frac{dx}{x}$$

$$= \frac{1}{2\pi} \int_{\sigma_1} \eta(x, y) = \pm r(\{x, y\}.$$

Analytic expression of the regulator

Bloch gave an other expression of the regulator

$$
\begin{array}{ccccc}
K_2(E)\otimes\mathbb{Q} & \to & K_2(\mathbb{Q}(E))\otimes\mathbb{Q} & \to & \mathbb{R} \\
 & & \{f,g\} & \mapsto & \frac{\Im\tau^2}{\pi^2}\sum_{i,j} a_i b_j K_{2,1}(
\end{array}
$$

where

$$K_{2,1}(t) := \sum_{\gamma\in L,\gamma\neq 0} \frac{< t,\gamma >}{\gamma^2\bar{\gamma}}$$

and

$$< t,\gamma > := \exp(\pi\frac{t\bar{\gamma}-\bar{t}\gamma}{\Im\tau}).$$

Hence the importance of getting $m(P)$ as an Eisenstein-Kronecker series.

EXPLICIT FORMULAE FOR CALABI-YAU IN DIMENSION 2

Let

$$P_k = X + \frac{1}{X} + Y + \frac{1}{Y} + Z + \frac{1}{Z} - k$$

and

$$Q_k = X + \tfrac{1}{X} + Y + \tfrac{1}{Y} + Z + \tfrac{1}{Z}$$
$$+ XY + \frac{1}{XY} + ZY + \frac{1}{ZY} + XYZ + \frac{1}{XYZ} - k.$$

These polynomials define families of $K3$ hypersurfaces.

Theorem 3. *1) Let* $k = t + \frac{1}{t}$ *and define*

$$t = \frac{\eta(\tau)^6 \eta(6\tau)^6}{\eta(2\tau)^6 \eta(3\tau)^6} = q^{1/2} - 6q^{3/2} + 15q^{5/2} - 20q^{7/2} + \ldots$$

with η *Dedekind eta function*

$$\eta(\tau) = e^{\frac{\pi i\tau}{12}} \prod_{n\geq 1}(1 - e^{2\pi i n\tau}).$$

Then

$$m(P_k) =$$

$$\Re\{-\pi i\tau + \sum_{n\geq 1}(\sum_{d|n} d^3)(\frac{4q^n}{n} - \frac{16q^{2n}}{2n} + \frac{36q^{3n}}{3n} - \frac{144q^{6n}}{6n})\}.$$

2) If $k = -(t + \frac{1}{t}) - 2$ *and*

$$t = \frac{\eta(3\tau)^4\eta(12\tau)^8\eta(2\tau)^{12}}{\eta(\tau)^4\eta(4\tau)^8\eta(6\tau)^{12}}.$$

Then

$$m(Q_k) = \Re$$

$$\{-2\pi i\tau + \sum_{n\geq 1}(\sum_{d|n} d^3)(\frac{-2q^n}{n} + \frac{32q^{2n}}{2n} + \frac{18q^{3n}}{3n} - \frac{288q^{6n}}{6n})\}$$

Theorem 4. *With the previous notations, we can express the measure in terms of Eisenstein-Kronecker series*

1)

$$m(P_k) = \frac{\Im\tau}{8\pi^3} \sideset{}{'}\sum_{m,\kappa}$$
$$-\Re\frac{2\times 4}{(m\tau+\kappa)^3(m\bar{\tau}+\kappa)} + \frac{4}{(m\tau+\kappa)^2(m\bar{\tau}+\kappa)^2}$$
$$+\Re\frac{2\times 16}{(2m\tau+\kappa)^3(2m\bar{\tau}+\kappa)} + \frac{16}{(2m\tau+\kappa)^2(2m\bar{\tau}+\kappa)^2}$$
$$-\Re\frac{2\times 36}{(3m\tau+\kappa)^3(3m\bar{\tau}+\kappa)} + \frac{36}{(3m\tau+\kappa)^2(3m\bar{\tau}+\kappa)^2}$$
$$+\Re\frac{2\times 144}{(6m\tau+\kappa)^3(6m\bar{\tau}+\kappa)} + \frac{144}{(6m\tau+\kappa)^2(6m\bar{\tau}+\kappa)^2}$$

2)

$$m(Q_k) = \frac{\Im\tau}{8\pi^3} \sum_{m,\kappa}{}'$$

$$\Re\frac{2\times 2}{(m\tau+\kappa)^3(m\bar{\tau}+\kappa)} + \frac{2}{(m\tau+\kappa)^2(m\bar{\tau}+\kappa)^2}$$

$$-\,\Re\frac{2\times 32}{(2m\tau+\kappa)^3(2m\bar{\tau}+\kappa)} + \frac{32}{(2m\tau+\kappa)^2(2m\bar{\tau}+\kappa)^2}$$

$$-\,\Re\frac{2\times 18}{(3m\tau+\kappa)^3(3m\bar{\tau}+\kappa)} + \frac{18}{(3m\tau+\kappa)^2(3m\bar{\tau}+\kappa)^2}$$

$$+\,\Re\frac{2\times 288}{(6m\tau+\kappa)^3(6m\bar{\tau}+\kappa)} + \frac{288}{(6m\tau+\kappa)^2(6m\bar{\tau}+\kappa)^2}$$