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Weyl group actions on the cohomology of character and quiver varieties

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Résumé

Nous étudions la cohomologie de certaines variétés de caractères et de leurs analogues additifs, les variétés de carquois en forme de comète. Ces variétés de caractères classifient les représentations du groupe fondamental d'une surface de Riemann épointée avec monodromie prescrite autour des points. Le polynôme de Poincaré pour la cohomologie d'intersection à support compact est calculé. Des actions de groupes de Weyl sur les escpaces de cohomologie sont également étudiées. Des traces de ces actions apparaissent comme certain coefficients de structure d'une algèbre engendrée par les polynômes de Kostka modifiés.

Mots-clefs : Variétés de caractères, variétés de carquois, groupe de Weyl, cohomologie d'intersection.

Abstract

We study the cohomology of some character varieties and their additive analogous, comet-shaped quiver varieties. Those character varieties classify representations of the fundamental group of a punctured Riemann surface with prescribed monodromies around the punctures. The Poincaré polynomial for compactly supported intersection cohomology is computed. Weyl group actions on the cohomology spaces are also studied. Some traces of those actions are related to particular structure coefficients of an algebra spanned by modified Kostka polynomials.

Keywords: Character varieties, quiver varieties, Weyl group, intersection cohomology.

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Contents

Chapter 1

Introduction

Character varieties studied in this thesis classify rank n local systems over a genus g Riemann surface with k-punctures $(p_j)_{1\leq j\leq k}$. The monodromy around the puncture p_j is imposed to be in the closure $\overline{C}_j$ of a conjugacy class C_j of $\mathrm{GL}_n(\mathbb{C})$. The character variety is an affine variety defined as a geometric invaritant theory quotient:

$$M_{\overline{\mathcal{C}}} := \Big\{(A_1,B_1,\dots,A_g,B_g,X_1,\dots,X_k)\in \mathrm{GL}_n^{2g}\times\overline{C}_1\times\dots\times\overline{C}_k \Big| \\ A_1B_1A_1^{-1}B_1^{-1}\dots A_gB_gA_g^{-1}B_g^{-1}X_1\dots X_k = \mathrm{Id}\Big\}//\mathrm{GL}_n$$

with GL_n acting by overall conjugation. A genericity condition is imposed on the k-uple of conjugacy classes so that the quotient has good properties (see 3.5.2). We study the cohomology of those varieties. As they are not smooth, it is convenient to study their *intersection cohomology*. We compute the Poincaré polynomial for compactly supported intersection cohomology of those character varieties. This Poincaré polynomial encodes the dimension of the compactly supported intersection cohomology spaces $IH_c^r\left(M_{\overline{\mathcal{C}}},\overline{\mathbb{Q}}_l\right)$ as coefficients of a polynomial:

$$P_c\left(M_{\overline{\mathcal{C}}},v\right) := \sum_r \dim IH_c^r\left(M_{\overline{\mathcal{C}}},\overline{\mathbb{Q}}_l\right)v^r$$

When the conjugacy classes are semisimple, they are closed, and the variety $M_{\overline{\mathcal{C}}}$ is smooth. Then the intersection cohomology coincides with the usual cohomology. Cohomology of character varieties has been extensively studied in various context.

1.1 Cohomology of character varieties: state of the art

1.1.1 One puncture with a central monodromy

A first interesting case is when there is only one puncture and the associated monodromy is central. The genericity condition implies that the monodromy is $e^{\frac{2i\pi d}{n}}\mathrm{Id}$ with d,n coprime. Then the character variety is denoted by M_B^d. The index B stands for Betti moduli space. Non-Abelian Hodge theory relates this Betti moduli space to a Dolbeault moduli space M_{Dol}^d. This can be seen as a generalization of Narasimhan-Seshadri [NS65] result relating unitary representations and holomorphic vector bundles. M_{Dol}^d is is the moduli space of stable Higgs bundles of rank

n and degree d. Non-Abelian Hodge correspondence was proved in rank $n = 2$ by Hitchin [Hit87] and Donaldson [Don87]. It was generalized to higher ranks and higher dimensions by Corlette [Cor88] and Simpson [Sim88] see also [Sim92]. The correspondence is obtained as a homeomorphism between moduli spaces by Simpson [Sim94a; Sim94b].

Many computations of the cohomology are performed from the Dolbeault side. First Hitchin [Hit87] computed the Poincaré polynomial in rank $n = 2$. Gothen [Got94] extended the computation to rank $n = 3$. Hausel-Thaddeus [HT03b; HT04] computed the cohomology ring in rank $n = 2$. García-Prada, Heinloth, Schmitt [GHS11] gave a recursive algorithm to compute the motive of the Dolbeault moduli space. They computed an explicit expression in rank $n = 4$. García-Prada, Heinloth [GH13] obtained an explicit formula for y-genus in any rank.

As in the last examples, there exist more precise cohomological information than the Poincaré polynomial. The character varieties are affine, by Deligne [Del71], their cohomology carries a mixed-Hodge structure. The non-Abelian Hodge theory does not preserve this mixed-Hodge structure. Indeed the cohomology of the Dolbeault moduli space is pure contrarily to the cohomology of the affine character variety. De Cataldo-Hausel-Migliorini [CHM12] conjectured that under non-Abelian Hodge correspondence, the weight filtration coincides with a perverse filtration induced by Hitchin fibration. This is known as the $P = W$ conjecture, they proved it in rank $n = 2$. Recently, de Cataldo-Maulik-Shen [CMS19] proved the conjecture for genus $g = 2$ and any rank.

Another interesting aspect of those moduli spaces is the mirror symmetry. Hausel-Thaddeus [HT01; HT03a] conjectured that the moduli space of PGL_n-Higgs bundles and the moduli space of SL_n-Higgs bundles are related by mirror symmetry, see also [Hau04]. This conjecture was proved by Groechenig-Wyss-Ziegler [GWZ17] and a motivic version by Loeser-Wyss [LW21]. Mirror symmetry was also studied in the parabolic case by Biswas-Dey [BD12]. Gothen-Oliveira [GO17] proved a parabolic version of the conjecture, for particular ranks.

An efficient approach to compute cohomological invariant is to count points of algebraic varieties over finite fields. On the Betti side, Hausel and Rodriguez-Villegas [HR08] gave a conjectural formula for the mixed-Hodge polynomial of character varieties with one puncture and a central generic monodromy. They proved the E-polynomial specialization of the conjecture by counting points over finite fields. With a similar approach, Mereb [Mer15] computed the E-polynomial of SL_n character varieties. Hausel [Hau04] also proposed a conjectural formula for the Hodge polynomial of the associated Dolbeault moduli space. Mozgovoy [Moz11] extended this conjecture to the motives of the Dolbeault moduli space.

Schiffmann [Sch16] computed the Poincaré polynomial of the Dolbeault moduli space by counting Higgs bundles over finite fields. In following articles [MS14; MS20] Mozgovoy-Schiffmann extended this counting to twisted Higgs bundles. Chaudouard-Laumon [CL16] counted Higgs bundles using automorphic forms.

Mellit [Mel17b] proved that the formula obtained by Schiffmann [Sch16] is equivalent to the Poincaré polynomial specialization of the conjecture of Hausel and Rodriguez Villegas [HR08].

Fedorov-Soibelman-Soibelman [FSS17] computed the motivic class of the moduli stack of semistable Higgs bundles.

1.1.2 Any number of punctures and arbitrary monodromies

Logares-Muñoz-Newstead [LMN12] computed the E-polynomial of character varieties for SL_2 and small genus $g = 1; 2$. They consider one puncture with any conjugacy class, without the genericity assumption. They also obtained the Hodge numbers in genus $g = 1$. Logares-Muñoz [LM13] extended those results to genus $g = 1$ and two punctures. They computed the E-polynomials and some Hodge numbers. Martínez-Muñoz [MM14a; MM14b] computed the E-polynomial of SL_2-character varieties for any genus and any conjugacy class at the puncture. Martínez [Mar17] then treated the case of PGL_2-character varieties.

Simpson [Sim90] generalized non-Abelian Hodge theory to character varieties with punctures and arbitrary conjugacy classes. The generalization is even larger as it concerns filtered local systems. They correspond to parabolic Higgs bundles on the Dolbeault side. The moduli space of stable parabolic Higgs bundles was constructed algebraically by Yokogawa [Yok93]. The moduli spaces were constructed analytically by Konno [Kon93] for Higgs fields with nilpotent residues and by Nakajima [Nak96]. Those analytic constructions provide the non-Abelian Hodge theory as a diffeomorphism. Biquard-Boalch [BB04] proved a more general wild non-Abelian Hodge theory and constructed the associated moduli spaces. Biquard, García-Prada and Mundet i Riera [BGM15] generalized filtered non-Abelian Hodge theory to a large family of groups.

On the Dolbeault side of this correspondence, Boden-Yokogawa [BY96] computed the Poincaré polynomial of the moduli space of parabolic Higgs bundles, in rank $n = 2$, using Morse theory. García-Prada, Gothen, Muñoz [GGM07] computed the Poincaré polynomial in rank $n = 3$.

Hausel, Letellier and Rodriguez-Villegas [HLR11] made a conjecture for the mixed-Hodge polynomial of character varieties with generic semisimple conjugacy classes at punctures. Counting points of the character variety over finite field they proved the E-polynomial specialization. Chuang-Diaconescu-Pan [CDP14] and Chuang-Diaconescu-Donagi-Pantev [Chu+15] proposed a string theoretic interpretation of the conjecture. This string theoretic approach was also applied to wild character varieties by Diaconescu [Dia17] and Diaconescu-Donagi-Pantev [DDP18]. Another approach uses recursive relations for various genus. It is used by Mozgovoy [Moz11], Carlsson and Rodriguez-Villegas [CR18]. Similarly to this recursive approach, González-Prieto [Gon18] developped a topological quantum field theory associated to character varieties.

Mellit [Mel17a] proved the Poincaré polynomial specialization of the conjecture from [HLR11] by counting parabolic Higgs bundles over finite fields. This result is of the utmost importance for this thesis. This is the starting point of the computation of intersection cohomology of the character variety with the closure of any generic conjugacy classes at punctures. Fedorov-Soibelman-Soibelman [FSS20] computed the motivic class of the moduli stack of semistable parabolic Higgs bundles.

1.2 Intersection cohomology of character varieties

1.2.1 Poincaré polynomial

Letellier [Let13] gave a conjectural formula for the mixed-Hodge polynomial of the character variety $M_{\overline{\mathcal{C}}}$, with any type of generic conjugacy classes at punctures. This formula generalizes the one for semisimple conjugacy classes [HLR11]. It also involves Hausel-Letellier-Villegas kernel $\mathbb{H}_n^{HLV}$. This kernel lies in

$$\mathrm{Sym}\,[X_1] \qquad \mathrm{Sym}\,[X_k]$$

with $\mathrm{Sym}\,[X_j]$ the space of symmetric functions in the infinite set of variable X_j. The definition of the kernel is recalled in 3.6.1, it uses modified Macdonald polynomials. The Poincaré polynomial specialization of Letellier's conjecture is the following formula

$$P_c(M_{\overline{\mathcal{C}}}; v) = v^{d} \ \langle s \ ' ; \mathbb{H}_n^{HLV}(\ \ 1; v)\rangle : \tag{1.1}$$

encodes the Jordan type of the conjugacy classes, see (3.36). d is the dimension of the variety $M_{\overline{\mathcal{C}}}$, the symmetric function $s\ '$ is a variant of Schur functions, it is defined in (3.47). A very interesting feature of this relation is that no matter the k-uple of conjugacy classes, the cohomology is encoded in a single object, the kernel $\mathbb{H}_n^{HLV}$.

Mellit [Mel17a] computed the Poincaré polynomial of character varieties with semisimple conjugacy classes. Let $\boldsymbol{\mathcal{S}} = (S_1; : : : ; S_k)$ a generic k-uple of conjugacy classes. The Jordan type of this k-uple is determined by k partitions $^1; : : : ; ^k$. The parts of the partition ^{j} are the multiplicities of the distinct eigenvalues of S_j. As checked in 3.6.2, Mellit's result is a particular case of the Poincaré polynomial specialization of the conjecture:

$$P_c(M_{\boldsymbol{\mathcal{S}}}; v) = v^{d} \ \langle h \ ; \mathbb{H}_n^{HLV}(\ \ 1; v)\rangle : \tag{1.2}$$

With h the symmetric function

$$h \ := h_{\ 1}[X_1] : : : h_{\ k}[X_k]:$$

The complete symmetric functions $(h\ [X])_{\ 2P_n}$ form a basis of the space of symmetric functions of degree n. The set of partitions of an integer n is denoted by P_n. The transition matrices in the space of symmetric functions are well known, for instance they are in Macdonald book [Mac15]. Hence we can express $s\ '$ in terms of h . To compute the Poincaré polynomial of character varieties with any type of conjugacy classes it is enough to understand the combinatoric relations between those symmetric functions in terms of geometric relation between $M_{\overline{\mathcal{C}}}$ and $M_{\boldsymbol{\mathcal{S}}}$. Letellier obtained such a relation, but between $M_{\overline{\mathcal{C}}}$ and a resolution of singularities of $M_{\overline{\mathcal{C}}}$.

1.2.2 Springer theory and resolution of character varieties

Logares-Martens [LM08] constructed Grothendieck-Springer resolutions for moduli spaces of parabolic Higgs bundles. Letellier [Let13] constructed resolution of singularities of character varieties

$$\widetilde{M}_{\boldsymbol{L};\boldsymbol{P};} \ \ ! \ \ M_{\overline{\mathcal{C}}}:$$

Symplectic resolutions of character varieties were also studied in details by Schedler-Tirelli [ST19]. The construction of $\widetilde{M}_{\mathbf{L},\mathbf{P},\sigma}$ is recalled in 3.5.11, it relies on Springer theory. This theory closely intertwines the geometry of reductive groups with the representation theory of their Weyl groups. A first step in this direction comes from Green [Gre55] who computed the characters of general linear groups over finite fields in terms of symmetric functions. Then Springer [Spr76] proved a correspondence between unipotent conjugacy classes and representations of Weyl groups for any connected reductive group. Following work of Lusztig [Lus81] for the general linear group, Borho-MacPherson [BM83] obtained Springer correspondence in terms of intersection cohomology.

Let us briefly recall their result for the Springer resolution of the unipotent locus in GL_n. Let B the subgroup of upper triangular matrices, U the subgroup of B with 1 on the diagonal. T is the subgroup of diagonal matrices so that $B = TU$. Let $\mathcal{U}$ the set of unipotent elements in GL_n, *i.e.* the set of matrices with all eigenvalues equal to 1. Then $\mathcal{U}$ is stratified by conjugacy classes $(C_\lambda)_{\lambda\in P_n}$ with λ the partition of n with parts specifying the size of the Jordan blocks. Let

$$\widetilde{\mathcal{U}} = \{(X, gB) \in \mathcal{U} \times \mathrm{GL}_n/B \mid g^{-1}Xg \in U\}$$

the projection to the first factor $\widetilde{\mathcal{U}} \to \mathcal{U}$ is a resolution of singularities. Borho-Macpherson approach to Springer theory provides the following relation between cohomology of the resolution $\widetilde{\mathcal{U}}$ and intersection cohomology of the closure of the strata of $\mathcal{U}$

$$H_c^{r+\dim \mathcal{U}}\left(\widetilde{\mathcal{U}}, \overline{\mathbb{Q}}_l\right) = \bigoplus_{\lambda\in P_n} V_\lambda \otimes IH_c^{r+\dim C_\lambda}\left(\overline{C}_\lambda, \overline{\mathbb{Q}}_l\right).$$

V_λ is the irreducible representation of the symmetric group indexed by the partition λ. The indexing is as in Macdonald's book [Mac15], so that $V_{(n)}$ is the trivial representation and $V_{(1^n)}$ the sign. In terms of Poincaré polynomial previous relation becomes

$$v^{-\dim \mathcal{U}} P_c\left(\widetilde{\mathcal{U}}, v\right) = \sum_{\lambda\in P_n} (\dim V_\lambda)\, v^{-\dim C_\lambda} P_c\left(\overline{C}_\lambda, v\right).$$

Interestingly, this relation between $v^{-\dim \mathcal{U}} P_c\left(\widetilde{\mathcal{U}}, v\right)$ and $v^{-\dim C_\lambda} P_c\left(\overline{C}_\lambda, v\right)$ is exactly the base change relation expressing the symmetric function h_{1^n} in terms of Schur functions $(s_\lambda)_{\lambda\in P_n}$

$$h_{1^n} = \sum_{\lambda\in P_n} (\dim V_\lambda)\, s_\lambda.$$

In this simple example, a base change relation between complete symmetric functions and Schur functions has a geometrical interpretation in terms of Springer resolutions.

For character varieties the idea is similar but a more general theory is necessary. It is provided by Lusztig parabolic induction [Lus84; Lus85; Lus86]. Letellier applied this theory to obtain relations between cohomology of the resolution $\widetilde{M}_{\mathbf{L},\mathbf{P},\sigma}$ and intersection cohomology of character varieties $M_{\overline{C}_\lambda,\sigma}$ (see 3.36 and 3.3.1 for the definition of the k-uple of conjugacy classes $C_{\lambda,\sigma}$). This relation is used to prove that various formulations of the conjecture are equivalent [Let11, Proposition 5.7]. In terms of Poincaré polynomial the relation reads

$$v^{-d_\mu} P_c\left(\widetilde{M}_{\mathbf{L},\mathbf{P},\sigma}, t\right) = \sum_{\lambda \preceq \mu} (\dim A_{\lambda,\mu})\, v^{-d_\lambda} P_c\left(M_{\overline{C}_\lambda,\sigma}, v\right). \tag{1.3}$$

This geometric relation is discussed in details in 6.2, it is exactly a combinatoric relation between various basis of symmetric functions:

$$h_{\prime} = \sum_{\preceq} (\dim A_{\prime;}) s : \tag{1.4}$$

It will appear that the Poincaré polynomial of resolution the $\widetilde{M}_{\mathbf{L};\mathbf{P};}$ is equal to the Poincaré polynomial of a character variety with semisimple monodromie $M_{\mathcal{S}}$. Together with Mellit's result (1.2), this implies

$$v^{d} P_c\left(\widetilde{M}_{\mathbf{L};\mathbf{P};}; v\right) = v^{d} P_c(M_{\mathcal{S}}; v) = \left\langle h_{\prime}; \mathbb{H}_n^{HLV}(1; v)\right\rangle$$

Relations (1.3) (1.4) can be inverted so that the Poincaré polynomial of a character variety with any type of monodromies can be expressed as Poincaré polynomial of character varieties with semisimple monodromies. This is exactly what is necessary to obtain the general formula 1.1 from Mellit's result for semisimple conjugacy classes 1.2.

To summarize, computing the Poincaré polynomial for intersection cohomology of character varieties requires three elements:

- Mellit's result for character varieties with semisimple monodromies (1.2).
- Letellier's relation (1.3) between cohomology of the resolution $\widetilde{M}_{\mathbf{L};\mathbf{P};}$ and intersection cohomology of character varieties $M_{\overline{\mathcal{C}}}$.
- Relation between cohomology of the resolution $\widetilde{M}_{\mathbf{L};\mathbf{P};}$ and cohomology of a character variety with semisimple monodromies $M_{\mathcal{S}}$.

The last point is studied in Chapter 6 where a diffeomorphism between the resolution $\widetilde{M}_{\mathbf{L};\mathbf{P};}$ and a character variety with semisimple monodromies $M_{\mathcal{S}}$ is detailed so that the Poincaré polynomial coincide. First the particular case of the sphere with four punctures is studied. Then the character varieties are cubic surfaces given by an explicit equation, the Fricke relation [FK97]. The geometry of cubic surfaces is well-known since Cayley [Cay69], see also Bruce-Wall [BW79] and Manin [Man86]. Smooth projective cubic surfaces in $\mathbb{P}^3$ are obtained as $\mathbb{P}^2$ blow-up in six points. This description gives a direct prove, on the Betti side, that the resolution is diffeomorphic to a character variety with semisimple monodromies.

Constructing the diffeomorphism in the general case requires analytical technics. They are detailed in 6.6.1, they rely on the filtered version of non-Abelian Hodge theory and Riemann-Hilbert correspondence. Those correspondences are due to Simpson [Sim90]. The moduli spaces providing non-Abelian Hodge theory as a diffeomorphism were constructed by Konno [Kon93], Nakajima [Nak96] and Biquard-Boalch [BB04] in the more general setting of wild non-Abelian Hodge theory. Filtered version of Riemann-Hilbert correspondence is described as a diffeomorphism by Yamakawa [Yam08]. A filtered version of non-Abelian Hodge theory was also developped for a large family of groups by Biquard, García-Prada and Mundet i Riera [BGM15]. In Chapter 6 this is used to construct a diffeomorphism between $\widetilde{M}_{\mathbf{L};\mathbf{P};}$ and $M_{\mathcal{S}}$, see Theorem 6.1.3. Finally it is used in 6.2 to prove the Poincaré polynomial specialization of Letellier's conjecture:

Theorem 1.2.1. *Consider a generic k-uple of conjugacy classes $C_{\ ;}$ (notations are introduced in* (3.36)*). the Poincaré polynomial for compactly supported intersection cohomology of the character variety $M_{\overline{C}\ ;}$ is*

$$P_c\left(M_{\overline{C}\ ;}\, ;v\right) = v^{d}\ \left\langle s\ ,;\mathbb{H}_n^{HLV}(\ 1;v)\right\rangle :$$

In addition to provide a combinatorial relation between Poincaré polynomials, a fundamental aspect of Springer theory and Lusztig parabolic induction is the action of Weyl group on cohomology spaces.

1.2.3 Weyl group action on the cohomology of character varieties

The construction of resolutions of character varieties relies on Springer resolutions and Lusztig parabolic induction. Therefore there is a Weyl group action on the cohomology of resolutions of character varieties (see Letellier [Let13]). It is interesting to notice that the Weyl group only acts on the cohomology and not on the variety itsel. Another Weyl group action on the cohomology of character varieties and their resolutions is constructed by Mellit [Mel19]. He constructed a family containing resolutions of character varieties and character varieties with semisimple monodromies. Different fibers of the family have different conjugacy classes prescribed at the k-th puncture, the $k\ \ 1$ first conjugacy classes being fixed and semisimple. With this family, Mellit constructed a monodromic Weyl group action on the cohomology of some character varieties. This action is unified with the Springer action on the cohomology of some resolutions. Both appear as various fibers of an equivariant local system. It is actually difficult to construct this local system. To obtain it, Mellit used subtle cell decomposition of character varieties.

In Chapter 5, following a suggestion of Mellit, we use this family and the Weyl group action to compute the Poincaré polynomial of character varieties with $k\ \ 1$ semisimple monodromies and any conjugacy class prescribed at the last puncture. This result is less general than Chapter 6 where any k-uple of generic conjugacy classes is considered. However, the advantage of this approach is that it remains on the Betti side and avoids the analytic technicality of non-Abelian Hodge theory. Except for Mellit's result about the Poincaré polynomial of character varieties which was obtained from the Dolbeault side.

As explained in previous section, in order to compute the intersection cohomology of character varieties for any conjugacy classes, we construct a diffeomorphism between a resolution $\widetilde{M}_{\boldsymbol{L};\boldsymbol{P};}$ and a character variety with semisimple monodromies $M_{\boldsymbol{\mathcal{S}}}$. This diffeomorphism allows to move the Springer-like Weyl group action on the cohomology of the resolution, to a Weyl group action on the cohomology of the character varieties with semisimple monodromies $M_{\boldsymbol{\mathcal{S}}}$. This action is enough for our purpose of computation of the Poincaré polynomial. Moreover, it also provides the -twisted Poincaré polynomials, *i.e.* the trace of any elements of the Weyl group on the cohomology spaces, see Definition 3.6.6. Considering a k-uple of generic semisimple conjugacy classes $\boldsymbol{\mathcal{S}} = (S_1;\dots;S_k)$, the relative Weyl group is the group permuting eigenvalues with the same multiplicity in a given class S_j. Next theorem is proved in 6.2.2.

Theorem 1.2.2. *For any conjugacy class in the relative Weyl group, the -twisted Poincaré polynomial of the character variety $M_{\mathcal{S}}$ is*

$$P_c \; (M_{\mathcal{S}}; v) := \sum_r \operatorname{tr}\left(\; ; H_c^r(M_{\mathcal{S}}; \overline{\mathbb{Q}}_l)\right) v^r = (\; 1)^{r(\;)} v^d \left\langle \widetilde{h} \; ; \mathbb{H}_n^{HLV}(\; 1; v)\right\rangle :$$

The symmetric functions $\widetilde{h}$ and $r(\;)$ are defined in 3.5.18.

However a more satisfying approach would be to directly construct a monodromic Weyl group action on the cohomology of character varieties with semisimple monodromies. Like the one constructed by Mellit for the k-th monodromy.

1.3 Additive version of character varieties

1.3.1 Comet-shaped quiver varieties

There is an additive version of character varieties. Let $\mathcal{O} = (O_1; : : : ; O_k)$ a k-uple of adjoint orbits in $\mathfrak{gl}_n$ the Lie algebra of GL_n. The additive analogous of character variety is defined as the following GIT quotient

$$Q_{\overline{\mathcal{O}}} := \left\{(A_1; B_1; : : : ; A_g; B_g; X_1; : : : ; X_k) \; 2 \; \mathfrak{gl}_n^{2g} \quad \overline{O}_1 \qquad \overline{O}_k \,\middle|\, \sum_{i=1}^{g} [A_i; B_i] + \sum_{j=1}^{k} X_j = 0\right\} == \mathrm{GL}_n$$

with $[A_i; B_i] := A_i B_i \quad B_i A_i$ the Lie bracket and GL_n acting by overall conjugation. Like in the multiplicative case, a genericity condition is imposed to the eigenvalues of the adjoint orbits (Definition 3.5.8). This condition allows to have a well behaved quotient. Such varieties were studied by Crawley-Boevey [Cra03b; Cra06] in genus $g = 0$, in particular he proved a criteria for non-emptiness. For any genus and semisimple adjoint orbits, they were studied by Letellier, Hausel and Rodriguez-Villegas [HLR11]. Letellier [Let11] generalized to any type of conjugacy classes. Interestingly, the geometry of those varieties is closely related to representation theory of the general linear group over a finite field $\mathrm{GL}_n(\mathbb{F}_q)$ see [Let12].

Many things are easier to study on the additive versions than on the character varieties. For instance the cohomology of those varieties is pure. Therefore, by counting points, Letellier, Hausel and Rodriguez-Villegas [HLR11] and Letellier [Let11] obtained the Poincaré polynomial. This is different to the character variety where only the E-polynomial is obtained by this method.

A fundamental aspect of this additive analogous is the interpretation in terms of Nakajima's quiver varieties introduced in [Nak94]. Because of this interpretation, the varieties $Q_{\overline{\mathcal{O}}}$ are referred to as comet-shaped quiver varieties [HLR11] or crab-shaped quiver varieties for instance by Schedler-Tirelli [ST19].

Weyl group action on the cohomology of Nakajima's quiver varieties were studied by Nakajima [Nak94; Nak00], Lusztig [Lus00] and Maffei [Maf02]. They were used to prove Kac conjecture by Letellier, Hausel, Rodriguez-Villegas [HLR13] and to study unipotent character of $\mathrm{GL}_n(\mathbb{F}_q)$ by Letellier [Let12]. A construction of Weyl group action relies on the hyperkähler structure of Nakajima's quiver varieties. Those varieties can be constructed as hyperkähler quotients as introduced by Hitchin-Karlhede-Lindström-Roček [Hit+87]. The quotients are obtained considering the action of a compact group on a fiber of the hyperkähler moment map. Such moment map is useful as it allows to construct a family containing both resolutions $\widetilde{Q}_{\boldsymbol{L};\boldsymbol{P};}$ and the varieties $Q_{\overline{\mathcal{O}}}$. Then the hyperkähler moment map is a locally trivial fibration over a regular locus. This is the property missing so far for character varieties and which could allow to construct a monodromic Weyl group action in general. This property of the hyperkähler moment map for quiver varieties was known and used by experts such as Nakajima and Maffei. Chapter 2 is devoted to its proof as we could not locate one in the literature. Then in Chapter 4 it is applied to comet shaped quiver varieties in order to have a coherent description of the Springer-like actions and the monodromic action. The combinatorics of the action obtained appears to be rich.

1.3.2 Combinatorics of the Weyl group action on the cohomology of comet-shaped quiver varieties

We study combinatorics aspect of the cohomology of character varieties and their additive analogous. Modified Macdonald polynomial appearing in Hausel-Letellier-Villegas kernel $\mathbb{H}_n^{HLV}$ were introduced by Garsia-Haiman [GH96] as a deformation of Macdonald polynomials [Mac15]. The transition matrix between the modified Macdonald polynomials and the Schur function is formed by the so-called modified Kostka polynomial $\left(\widetilde{K}_{;}(q;t)\right)_{;\,2P_n}$. The fact that they are polynomials in $q;t$ with integer coefficients is far from trivial. It is known as Macdonald conjecture, it is a consequence of the $n!$ conjecture of Garsia-Haiman [GH93], this last conjecture was proved by Haiman [Hai01].

In unpublished notes, Rodriguez-Villegas studied an algebra spanned by modified Kostka polynomial. The structure coefficients $c_{;}(q;t)$ of this algebra are defined by

$$\widetilde{K}_{;}\widetilde{K}_{;} = \sum c_{;}\widetilde{K}_{;} \quad \text{for all} \quad 2\,P_n:$$

Rodriguez-Villegas conjectured that the coefficients $c_{;}$ are actually polynomials in $q;t$ with integer coefficients. Moreover he noticed that they are related to the Hausel-Letellier-Villegas kernel. He studied in particular the coefficients $c^{1^n}_{;}$, they appear as a generalization of the $(q;t)$-Catalan sequence from Garsia-Haiman [GH96]. Rodriguez-Villegas proved that the coefficient $c^{1^n}_{;}$ has an expression similar to the conjecture concerning the mixed Hodge polynomial of character varieties (with genus $g = 0$)

$$c^{1^n}_{;}(q;t) = (\;1)^{n\;\;1}\left\langle s\,[X_1]s\,[X_2]p_n[X_3]h_{(n\;\;1;1)}[X_4];\mathbb{H}_n^{HLV}\left(q^{\frac{1}{2}};t^{\frac{1}{2}}\right)\right\rangle:$$

In Chapter 4 we prove that a specialization of this formula indeed relates the coefficients $c^{1^n}_{;}$ to traces of Weyl group actions on the cohomology of comet-shaped quiver varieties.

Theorem 1.3.1. *Consider a generic 4-uple of adjoint orbits of the following type:*

- O_1 *has one eigenvalue with Jordan type* $^0\,2\,P_n$ *.*
- O_2 *has one eigenvalue with Jordan type* $^0\,2\,P_n$*.*

- *O_3 is semisimple regular i.e. it has n distinct eigenvalues.*
- *O_4 is semisimple with one eigenvalue of multiplicity $n - 1$ and the other of multiplicity 1.*

Then the Weyl group with respect to O_3 is the symmetric group S_n and it acts on the cohomology of $Q_{\overline{\mathcal{O}}}$. Let w a n-cycle in this Weyl group then

$$c^{1^n}_{;}(0;t) = t^{\frac{\dim Q_{\overline{\mathcal{O}}}}{2}} \sum_r \mathrm{tr}\left(w; IH_c^{2r}\left(Q_{\overline{\mathcal{O}}}; \overline{\mathbb{Q}}_l\right)\right) t^r:$$

The coefficient $c^{1^n}_{;}(0;t)$ thus appears as a Poincaré polynomial twisted by an n-cycle.

A similar result (Theorem 6.2.7) relates the coefficients $c^{1^n}_{;}(1;t)$ to a twisted Poincaré polynomial of character varieties. Conjecturaly $c^{1^n}_{;}(q;t)$ is related to a twisted mixed-Hodge polynomial of resolutions of character varieties 4.4.3.

It would be interesting to also find a geometric interpretation of the others coefficients $c_{;}$.

1.4 Plan of the thesis

The second chapter can be read independently of the others. We study the locally trivial property of the hyperkähler moment map for quiver varieties over a regular locus. This result was known and used by expert such as Nakajima [Nak94] and Maffei [Maf02]. We detail the prove here as we could not locate one in the literature. This result is used in Chapter 4.

The third chapter contains reminder of the geometric and combinatoric background behind character varieties and comet-shaped quiver varieties. Most of the notations relative to conjugacy classes, resolutions and Weyl groups are also introduced in this chapter.

In Chapter 4 we study a family of comet-shaped quiver varieties and their resolutions. It relies on the local triviality of the hyperkähler moment map recalled in Chapter 2. As usual in the theory of quiver varieties, this local triviality allows to construct a monodromic Weyl group action on the cohomology of the comet-shaped quiver varieties. We check that the representations obtained in this family are isomorphic to the Springer-like actions. Then those actions are related to particular coefficients of the algebra spanned by Kostka polynomials and Theorem 1.3.1 is proved.

Chapter 5 is devoted to the study of the family of character varieties constructed by Mellit [Mel19]. Following his suggestion, we use the monodromic Weyl group action to compute the Poincaré polynomial for intersection cohomology of character varieties with $k - 1$ monodromies semisimple and any conjugacy class at the last puncture. This is a particular case of Theorem 1.2.1. Except for Mellit's result about the Poincaré polynomial of character varieties with semisimple monodromies, this chapter remains on the Betti side and uses only algebraic tools.

In the last chapter the Poincaré polynomial of character varieties with any generic k-uple of conjugacy classes at punctures is computed, thus proving Theorem 1.2.1. Contrarily to previous chapter, the computation requires analytic methods such as non-Abelian Hodge theory. As a by-product we obtain a Weyl group action on the cohomology of character varieties and an expression for the -twisted Poincaré polynomials: Theorem 1.2.2.

Chapter 2

Trivializations of moment maps

We study various trivializations of moment maps. First in the general framework of a reductive group G acting on a smooth affine variety. We prove that the moment map is a locally trivial fibration over a regular locus of the center of the Lie algebra of H a maximal compact subgroup of G. The construction relies on Kempf-Ness theory [KN79] and Morse theory of the square norm of the moment map studied by Kirwan [Kir84], Ness-Mumford [NM84] and Sjamaar [Sja98]. Then we apply it together with ideas from Nakajima [Nak94] and Kronheimer [Kro89] to trivialize the hyperkähler moment map for Nakajima's quiver varieties. Notice this trivialization result about quiver varieties was known and used by experts such as Nakajima and Maffei but we could not locate a proof in the literature.

2.1 Introduction

2.1.1 Symplectic quotients and GIT quotients of affine varieties

Consider a reductive group G acting on a complex smooth affine variety X. For $\in X(G)$ a linear character, X^{-ss} is the -semistable locus and X^{-s} the -stable locus. Mumford's geometric invariant theory [MF82] provides a quotient

$$X^{-ss} \to X^{-ss}/\!/G.$$

The affine variety X can be embedded in an hermitian vector space W such that the G-action is linear and restricts to a unitary action of a maximal compact subgroup $H \quad G$. The hermitian norm on W is denoted by $||\dots||$. We study the associated real moment map

$$: X \to \mathfrak{h}$$

with $\mathfrak{h}$ the Lie algebra of H. Its definition relies on the choice of a non degenerate scalar product $\langle \dots, \dots \rangle$ on $\mathfrak{h}$ invariant under the adjoint action of H. The real moment map satisfies for all $Y \in \mathfrak{h}$

$$\langle \ (x), Y \rangle = \frac{1}{2}\frac{d}{dt}||\exp(itY).x||^2\Big|_{t=0} \tag{2.1}$$

Thanks to the invariant scalar product, to a linear character is associated an element in $Z(\mathfrak{h})$, the center of the Lie algebra $\mathfrak{h}$, such that for all $Y \in \mathfrak{h}$

$$\langle \ , Y \rangle = id \ _{\mathrm{Id}}(Y).$$

For a pair (;), Kempf-Ness theory [KN79] relates the symplectic quotient (defined by Meyer [Mey73] and Marsden-Weinstein [MW74]) to the GIT quotient, it gives an homeomorphism

$$^{1}(\ )/H \to X^{\ \text{-ss}}//G.$$

We study trivialization of the moment map over a regular locus in the center of the Lie algebra h. First, in Section 2.2, we study the general framework of a unitary action of a compact group on a smooth affine variety. After a reminder of Migliorini's version of Kempf-Ness theory [Mig96], a regular locus in Z(h) is defined. Over this locus the moment map is proved to be a locally trivial fibration. The case of a torus action was treated by Kac-Peterson [KP84]. The construction of the regular locus uses the negative gradient flow of square norm of the moment map studied by Kirwan [Kir84], Ness-Mumford [NM84], Sjamaar [Sja98], Harada-Wilkin [HW08] and Hoskins [Hos13].

Nakajima's quiver varieties introduced in [Nak94] are particular instances of the symplectic quotients studied in Section 2.2. Moreover they are hyperkähler quotients as defined by Hitchin-Karlhede-Lindström-Roček [Hit+87], the construction of those varieties is recalled in Section 2.3. In Section 2.4, the idea of Kronheimer [Kro89] and Nakajima [Nak94] of consecutive use of different complex structures are applied together with techniques from previous sections to prove that the hyperkähler moment map is a locally trivial fibration. This implies in particular that the cohomology of the fibers forms a local system. This later result is used by Nakajima in [Nak94, Section 9] to construct a Weyl group action on the cohomology of quiver varieties. Maffei pursued this construction in [Maf02]. I was informed by Nakajima that the property of the cohomology of the fibers can also be obtained by generalizing Slodowy argument from [Slo80] to quiver varieties. Similar results concerning cohomology of the fibers also exist in the framework of deformations of symplectic quotient singularities in Ginzburg-Kaledin [GK04]. Finally Crawley-Boevey and Van den Bergh [CV04] trivialize the hyperkähler moment map for Nakajima's quiver varieties over complex lines. Nakajima explained to us how to extend their result to quaternionic lines minus a point thanks to the theory of twistor spaces see Theorem 2.4.15.

In the remaining of the introduction the results are stated and the various steps of the constructions are outlined.

2.1.2 Real moment map for the action of a reductive group on an affine variety

In Section 2.2, $H \quad G$ is a maximal compact subgroup acting unitarily on a smooth affine variety X embedded in an hermitian vector space. The differential geometry point of view from Kempf-Ness theory allows to extend the definition of -stability for elements $\ \in X\ (G)^{\mathbb{R}} := X\ (G) \otimes_{\mathbb{Z}} \mathbb{R}$. The correspondence between linear characters and elements in the center of the Lie algebra h thus extends to an isomorphism of R-vector spaces between $X\ (G)^{\mathbb{R}}$ and Z(h).

In 2.2.4 we prove a Lie group variant of Hilbert-Mumford criterion for -stability. It is adapted to the differential geometric point of view of Kempf-Ness theory and the use of real parameters $\ \in X\ (G)^{\mathbb{R}}$. Similar criteria are discussed by Georgoulas, Robbin and Salamon in [GRS13].

Theorem 2.1.1 (Hilbert-Mumford criterion for stability). *Let $\in X(G)^{\mathbb{R}}$ and $x \in X$. The following statements are equivalent*

(i) x *is* -*stable.*

(ii) *For all* $Y \in \mathfrak{h}$, *different from zero, such that* $\lim_{t \to +\infty} \exp(itY).x$ *exists then* $\langle \, , Y \rangle < 0$.

This theorem is applied in 2.3.2 to generalize a result of King [Kin94] characterizing -stability for quiver representations.

The regular locus B^{reg} is introduced in 2.2.5. Its construction relies on the study of the negative gradient flow of the square norm of the moment map from Kirwan [Kir84], Ness-Mumford [NM84], Sjamaar [Sja98], Harada-Wilkin [HW08] and Hoskins [Hos13]. B^{reg} is an open subset of $Z(\mathfrak{h})$ such that for $\in B^{reg}$, one has $X^{-ss} = X^{-s} \neq \emptyset$ and for all $x \in X^{-s}$ the stabilizer of x is trivial. Over the regular locus, the moment map is a locally trivial fibration. A similar fibration when G is a torus follows from a result of Kac-Peterson [KP84]. Let us also mention that with the flow of the norm square in the hermitian space W, Sjamaar [Sja98] constructed a retraction of the 0-stable locus to the fiber over 0 of the moment map.

Theorem 2.1.2. *Let $_0$ in B^{reg}, and U_0 the connected component of B^{reg} containing $_0$. There is a diffeomorphism f such that the following diagram commutes*

$$\begin{array}{ccc} U_0 \quad {}^{1}(_0) & \xrightarrow[f]{} & {}^{1}(U_0) \\ & \searrow & \downarrow \\ & & U_0 \end{array}$$

Moreover f is H equivariant so that the diagram goes down to quotient

$$\begin{array}{ccc} U_0 \quad {}^{1}(_0)/H & \longrightarrow & {}^{1}(U_0)/H \\ & \searrow & \downarrow \\ & & U_0 \end{array}$$

To prove this theorem, first we prove that for any $\in U_0$ and $x \in X^{0\text{-}s}$ there exists a unique $Y(\,, x) \in \mathfrak{h}$ such that $\exp(iY(\,, x)).x \in {}^{1}(\,)$. This is achieved thanks to Migliorini's version of Kempf-Ness theory [Mig96] which applies to affine varieties and real parameters $\in X(G)^{\mathbb{R}}$. Then the map f is defined by

$$f(\,, x) := \exp(iY(\,, x)).x$$

and similarly for its inverse

$$f^{1}(x) = (\,(x), \exp(iY(_0, x)).x).$$

The smoothness of f and its inverse is proved in 2.2.6 with the implicit function theorem.

2.1.3 Nakajima's quiver varieties and hyperkähler moment map

The quiver varieties considered in this thesis were introduced by Nakajima [Nak94]. Let $\tilde{Q}$ be an extended quiver with vertices I_0 and edges $\tilde{\Omega}$, fix a dimension vector $v \in \mathbb{N}^{I_0}$. The space of representations of $\tilde{Q}$ with dimension vector v is

$$\mathrm{Rep}\left(\tilde{Q}, v\right) = \bigoplus_{\alpha \in \tilde{\Omega}} \mathrm{Mat}_{\mathbb{C}}(v_{h(\alpha)}, v_{t(\alpha)}).$$

with $h(\alpha) \in I_0$ the head of the edge α and $t(\alpha) \in I_0$ its tail. This space is acted upon by the group

$$G_v = \left\{ (g_j)_{j \in I_0} \in \prod_{j \in I_0} \mathrm{GL}_{v_j} \;\middle|\; \prod_{j \in I_0} \det(g_j) = 1 \right\}.$$

This action is described in 2.3.1, it restricts to a unitary action of the maximal compact subgroup

$$U_v = \left\{ (g_j)_{j \in I_0} \in \prod_{j \in I_0} U_{v_j} \;\middle|\; \prod_{j \in I_0} \det(g_j) = 1 \right\}$$

with U_{v_j} the group of unitary matrices of size v_j. Denote by $\mathfrak{u}_v$ the Lie algebra of U_v. This is a particular instance of the general situation of Section 2.2: a unitary action of a compact group on a smooth complex affine variety. Let $\theta \in \mathbb{Z}^{I_0}$ such that $\sum_j v_j \theta_j = 0$. Define χ_θ a linear character of G_v by

$$\chi_\theta((g_j)_{j \in I_0}) := \prod_{j \in I_0} \det(g_j)^{\theta_j}. \tag{2.2}$$

For quiver representations, the correspondence between linear characters and elements in the center of $\mathfrak{u}_v$ is easily described: to the character χ_θ is associated the element $(i\theta_j \mathrm{Id}_{v_j})_{j \in I_0} \in Z(\mathfrak{u}_v)$. This element is still denoted by θ, and $Z(\mathfrak{u}_v)$ is identified in this way with a subspace of $\mathbb{R}^{I_0}$.

A well-known theorem from King [Kin94] gives a characterization of θ-stability for quiver representations. In 2.3.2 this result is generalized to real parameters corresponding to elements $\theta \in X^*(G)^{\mathbb{R}}$.

Theorem 2.1.3. *For $\theta \in \mathbb{R}^{I_0}$ such that $\sum_{j \in I_0} \theta_j v_j = 0$ and associated element $\theta \in X^*(G_v)^{\mathbb{R}}$. A quiver respresentation (V, ρ) is θ-stable if and only if for all subrepresentation $W \subset V$*

$$\sum_{j \in I_0} \theta_j \dim W_j < 0,$$

unless $W = V$ or $W = 0$.

The space $\mathrm{Rep}\left(\tilde{Q}, v\right)$ admits three complex structures denoted by I, J and K, they are detailed in 2.4.1. There is a real moment map for each one of this complex structure, they are denoted by μ_I, μ_J and μ_K. They are defined as in equation (2.1), for instance

$$\langle \mu_I(x), Y \rangle = \frac{1}{2} \frac{d}{dt} \|\exp(t.I.Y).x\|^2 \Big|_{t=0}$$

and

$$\langle \mu_J(x), Y \rangle = \frac{1}{2}\frac{d}{dt}\|\exp(t.J.Y).x\|^2\Big|_{t=0}.$$

Together they form the hyperkähler moment map $\mu_{\mathbb{H}} = (\mu_I, \mu_J, \mu_K)$, it takes values in $\mathfrak{u}_v^{3}$.

Nakajima's quiver varieties are constructed for $(\zeta_I, \zeta_J, \zeta_K) \in Z(\mathfrak{u}_v)^{3}$ as quotients of fibers of the hyperkähler moment map.

$$\mathfrak{m}_v(\zeta_I, \zeta_J, \zeta_K) = \mu_{\mathbb{H}}^{-1}(\zeta_I, \zeta_J, \zeta_K)/U_v.$$

The hyperkähler regular locus in $Z(\mathfrak{u}_v)^{3}$ is defined by:

Definition 2.1.4 (Hyperkähler regular locus). *For $w \in \mathbb{N}^{I_0}$ a dimension vector*

$$H_w := \left\{ (\zeta_I, \zeta_J, \zeta_K) \in \left(\mathbb{R}^{I_0}\right)^3 \,\middle|\, \sum_j w_j \zeta_{I,j} = \sum_j w_j \zeta_{J,j} = \sum_j w_j \zeta_{K,j} = 0 \right\}.$$

The regular locus is

$$H_v^{\mathrm{reg}} = H_v \setminus \bigcup_{w<v} H_w \tag{2.3}$$

the union is over dimension vector $w \neq v$ such that $0 \leq w_i \leq v_i$.

In 2.4.3 various trivializations of the hyperkähler moment map are discussed. We prove that the hyperkähler moment map is a locally trivial fibration by consecutive use of constructions of Theorem 2.1.2 for each complex structure and associated moment map. The idea of consecutive use of different complex structures comes from Kronheimer [Kro89] and Nakajima [Nak94].

Theorem 2.1.5 (Local triviality of the hyperkähler moment map). *Over the regular locus H_v^{reg}, the hyperkähler moment map $\mu_{\mathbb{H}}$ is a locally trivial fibration compatible with the U_v-action:*

Any $(\zeta_I, \zeta_J, \zeta_K) \in H_v^{\mathrm{reg}}$ admits an open neighborhood V, and a diffeomorphism f such that the following diagram commutes

$$\begin{array}{ccc} V \times \mu_{\mathbb{H}}^{-1}(\zeta_I, \zeta_J, \zeta_K) & \xrightarrow{f} & \mu_{\mathbb{H}}^{-1}(V) \\ & \searrow & \downarrow \mu_{\mathbb{H}} \\ & & V \end{array}$$

Moreover f is compatible with the U_v-action so that the diagram goes down to quotient

$$\begin{array}{ccc} V \times \mu_{\mathbb{H}}^{-1}(\zeta_I, \zeta_J, \zeta_K)/U_v & \longrightarrow & \mu_{\mathbb{H}}^{-1}(V)/U_v \\ & \searrow & \downarrow p \\ & & V \end{array}$$

A similar trivialization of the hyperkähler moment map over lines is described in [CV04, Lemma 2.3.3]. In Theorem 2.4.15 we provide an extension of their result using twistor spaces as suggested by Nakajima.

Denote by μ the map obtained by taking quotient of the hyperkähler moment map over the regular locus

$$\mu_H^{-1}(H_v^{\mathrm{reg}})/U_v \to H_v^{\mathrm{reg}}.$$

Consider $H^i \mu_* \overline{\mathbb{Q}}_l$, the cohomology sheaves of the derived pushforward of the constant sheaf. As a direct corollary of the local triviality of the hyperkähler moment map, those sheaves are locally constant. Moreover as H_v^{reg} is simply connected, those sheaves are constant. They provide the local system of the cohomology of the fibers.

2.2 Kempf-Ness theory for affine varieties

Kempf-Ness [KN79] relate geometric invariant theory quotients to symplectic quotients. In this section we recall Migliorini's version of this theory [Mig96] which applies to affine varieties and real parameter $\chi \in X^*(G)^{\mathbb{R}}$. Then we prove that the real moment map is a locally trivial fibration over a regular locus.

G is a connected reductive group acting on a smooth affine variety X. The action is assumed to have a trivial kernel.

2.2.1 Characterization of semistability from a differential geometry point of view

For $\chi \in X^*(G)$ a linear character of G, a regular function $f \in \mathbb{C}[X]$ is χ-equivariant if there exists a strictly positive integer r such that $f(g.x) = \chi(g)^r f(x)$ for all $x \in X$.

Definition 2.2.1. *A point $x \in X$ is χ-semistable if there exists a χ-equivariant regular function f such that $f(x) \neq 0$. The set of χ-semistable points is denoted by $X^{\chi\text{-ss}}$.*

A point $x \in X$ is χ-stable if it is χ-semistable and if its orbit $G.x$ is closed in $X^{\chi\text{-ss}}$ and its stabilizer is finite. The set of χ-stable points is denoted by $X^{\chi\text{-s}}$.

The GIT quotient as defined by Mumford [MF82] is denoted by $X^{\chi\text{-ss}} \to X^{\chi\text{-ss}}//G$. A point of this quotient represents a closed G-orbit in $X^{\chi\text{-ss}}$. When working over the field of complex numbers, such quotients are related to symplectic quotients. The affine variety X can be embedded as a closed subvariety of an hermitian space W with hermitian pairing denoted by $p(\ldots,\ldots)$. The embedding can be chosen so that the action of G on X comes from a linear action on W and the action of a maximal compact subgroup $H \subset G$ preserves the hermitian pairing, $p(h.u, h.v) = p(u, v)$ for all $h \in H$ and $u, v \in W$. Then G can be identified with a subgroup of $\mathrm{GL}(W)$. The hermitian pairing induces a symplectic form on the underlying real space

$$\omega(\ldots,\ldots) := \mathrm{Re}\, p(i\ldots,\ldots) \tag{2.4}$$

with i a square root of -1 and Re the real part. The hermitian pairing on the ambient space induces an hermitian metric on X. As X is a smooth subvariety of W, its tangent space is stable under multiplication by i, hence the non-degeneracy of the hermitian metric implies the non degeneracy of the restriction of the symplectic form ω to the tangent space of X and the symplectic form on W restricts to a symplectic form on X. Then the action of G on X induces a symplectic action of H on X.

For $x \in X$ introduce the Kempf-Ness map

$$\phi^{\chi,x} : \begin{array}{ccl} G & \to & \mathbb{R} \\ g & \mapsto & ||g.x||^2 - \log\left(|\chi(g)|^2\right) \end{array}$$

with $||\dots||$ the hermitian norm.

Theorem 2.2.2 ([Mig96] Theorem A.4). *A point $x_0 \in X$ is χ-semistable if and only if there exists in the closure of its orbit a point $x \in \overline{G.x_0}$ such that $\phi^{\chi,x}$ has a minimum at the identity.*

Remark 2.2.3. *Let $X(G)^{\mathbb{R}} := X(G) \otimes_{\mathbb{Z}} \mathbb{R}$, the definiton of $\phi^{\chi,x}$ makes sense not only for linear characters but for any $\chi \in X(G)^{\mathbb{R}}$. It provides the following generalization of the definition of χ-semistability and χ-stability for any $\chi \in X(G)^{\mathbb{R}}$.*

Definition 2.2.4 (Semistable points). *Let $\chi \in X(G)^{\mathbb{R}}$, a point x_0 is χ-semistable if there exists $x \in \overline{G.x_0}$ such that $\phi^{\chi,x}$ has a minimum at the identity.*

A point x_0 is χ-stable if it is χ-semistable, its orbit is closed in $X^{\chi\text{-}ss}$ and its stabilizer is finite.

In the following of this chapter, χ-stability and χ-semistability as well as the notations $X^{\chi\text{-}s}$ and $X^{\chi\text{-}ss}$ always refer to this definition.

2.2.2 Correspondence between linear characters and elements in the center of the Lie algebra of H

The Lie algebra of G is denoted by $\mathfrak{g}$ and the real Lie algebra of H is $\mathfrak{h}$. Fix a non-degenerate scalar product $\langle \dots ; \dots \rangle$ on $\mathfrak{h}$ invariant under the adjoint action.

Proposition 2.2.5 (Polar decomposition). *For all $g \in G$ there exists a unique $(h; Y) \in H \times \mathfrak{h}$ such that $g = h\exp(iY)$ such an expression is called a polar decomposition. This implies for the Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus i\mathfrak{h}$.*

Proof. It follows from [OVG94] Theorem 6.6. □

The first step in Kempf-Ness theory is to associate to a character $\chi \in X(G)$ an element in the center $Z(\mathfrak{h})$ of the Lie algebra $\mathfrak{h}$. As H is compact, its image under a complex character lies in the unit circle. Consider the differential of the character at the identity, it is a $\mathbb{C}$-linear map $d\chi_{\mathrm{Id}} : \mathfrak{g} \to \mathbb{C}$. The inclusion $\chi(H) \subset S^1$ implies for the Lie algebra $d\chi_{\mathrm{Id}}(\mathfrak{h}) \subset i\mathbb{R}$. By $\mathbb{C}$-linearity, $d\chi_{\mathrm{Id}}(i\mathfrak{h}) \subset \mathbb{R}$ and the following map is $\mathbb{R}$-linear

$$d\chi_{\mathrm{Id}}(i\dots) : \begin{array}{ccl} \mathfrak{h} & \to & \mathbb{R} \\ Y & \mapsto & d\chi_{\mathrm{Id}}(iY) \end{array}. \tag{2.5}$$

The invariant scalar product on $\mathfrak{h}$ identifies this linear form with an element of $\mathfrak{h}$ denoted by ξ_χ satisfying for all $Y \in \mathfrak{h}$

$$\langle \xi_\chi ; Y \rangle = i d\chi_{\mathrm{Id}}(Y).$$

Moreover, as the scalar product is invariant for the adjoint action and so is the character χ, the element ξ_χ lies in the center of $\mathfrak{h}$. This construction is $\mathbb{Z}$-linear so that it extends to an $\mathbb{R}$-linear map

$$\xi : \begin{array}{ccl} X(G)^{\mathbb{R}} & \to & Z(\mathfrak{h}) \\ \chi & \mapsto & \xi_\chi \end{array}$$

Proposition 2.2.6. *The* $\mathbb{R}$*-linear map* φ *is an isomorphism from* $X^*(G)^{\mathbb{R}}$ *to* $Z(\mathfrak{h})$*.*

Proof. As G is a complex reductive group $G = Z(G)D(G)$ with $Z(G)$ its center and $D(G)$ its derived subgroup. Then $X^*(G)$ identifies with the set of linear characters of the torus $Z(G)$. Hence $X^*(G)$ is a $\mathbb{Z}$-module of rank the complex dimension of $Z(G)$ so that $\dim_{\mathbb{R}} X^*(G)^{\mathbb{R}} = \dim_{\mathbb{R}} Z(\mathfrak{h})$. It remains to prove that φ is injective. Let χ a linear character such that $d\chi_{\mathrm{Id}}(iY) = 0$ for all $Y \in \mathfrak{h}$. By $\mathbb{C}$-linearity and polar decomposition $d\chi_{\mathrm{Id}} = 0$. Hence for any $g \in G$ the differential at g is also zero $d\chi_g = 0$. As G is connected, χ is the trivial character. □

Remark 2.2.7. *This isomorphism justifies the notation* χ_ξ *for elements in* $X^*(G)^{\mathbb{R}}$*, such elements are uniquely determined by a choice of* $\xi \in Z(\mathfrak{h})$*, moreover*

$$\chi_\xi \chi_{\xi'} = \chi_{\xi+\xi'}.$$

2.2.3 Correspondence between symplectic quotient and GIT quotient

Definition 2.2.8 (Real moment map)**.** *The real moment map* $\mu : X \to \mathfrak{h}$ *is defined thanks to the invariant scalar product* $\langle \ldots, \ldots \rangle$ *by*

$$\langle \mu(x), Y \rangle = \frac{1}{2}\frac{d}{dt}||\exp(itY).x||^2\Big|_{t=0}$$

for all $Y \in \mathfrak{h}$ *and* $x \in X$*. In this section the real moment map is just called the moment map. Later on complex and hyperkähler moment maps are also considered.*

Example 2.2.9. *Assume the compact group* H *is a torus* T*. The ambient space decomposes as an orthogonal direct sum* $W = \bigoplus W_\chi$ *with* χ *linear characters of* T *and*

$$W_\chi = \{x \in W | t.x = \chi(t)w \text{ for all } t \in T\}$$

Similarly to 2.2.2, a character χ *is uniquely determined by an element* ξ_χ *in* $\mathfrak{t}$ *the Lie algebra of* T *such that*

$$i\,d\chi_{\mathrm{Id}}(Y) = \langle \xi_\chi ; Y \rangle.$$

Let A *the finite subset of elements* $\xi \in \mathfrak{t}$ *such that* $W_\xi \neq \{0\}$*. Let us compute* μ_T *the moment map for the torus action. Let* $x = \sum_{\xi \in A} x_\xi$ *in* W*, for* Y *in* $\mathfrak{t}$ *the Lie algebra of* T

$$\begin{aligned}\langle \mu_T(x); Y \rangle &= \frac{1}{2}\frac{d}{dt}||\exp(itY).x||^2\Big|_{t=0} \\ &= \sum_{\xi \in A} i\,d\chi_{\mathrm{Id}}(Y)||x_\xi||^2 \\ &= \left\langle \sum_{\xi \in A} ||x_\xi||^2 \xi ; Y \right\rangle\end{aligned}$$

Therefore the non-degeneracy of the scalar product implies $\mu_T(x) = \sum_{\xi \in A} ||x_\xi||^2 \xi$*. In particular the image of* μ_T *is the cone* $C(A) \subset \mathfrak{t}$ *spanned by positive coefficients combinations of elements* $\xi \in A$*. This example proves to be useful later on.*

Proposition 2.2.10 (Guillemin-Sternberg [GS82]). *$d_x\mu$ the differential of the moment map at x is surjective if and only if the stabilizer of x in H is finite.*

Proof. A computation using the definition of the moment map and the symplectic form gives for $v \in T_xX$ a tangent vector at x and $Y \in \mathfrak{h}$

$$\langle d_x\mu(v), Y\rangle = \omega\left(\frac{d}{dt}\exp(tY).x\Big|_{t=0}, v\right).$$

This relation is often taken as a definition of the moment map. By non degeneracy of the symplectic form ω it implies that Y is orthogonal to the image of $d_x\mu$ if and only if the stabilizer of x contains $\exp(tY)$ for all $t \in \mathbb{R}$. Hence the differential of the moment map is surjective if and only if the stabilizer of x is finite. □

Lemma 2.2.11. *Let $\chi \in X(G)^{\mathbb{R}}$ and $x \in X$, then $\psi^{\chi,x}$ has a minimum at the identity if and only if $\mu(x) = \xi$.*

Moreover if $\psi^{\chi,x}$ has a minimun at the identity and at a point $h\exp(iY)$ with $h \in H$ and $Y \in \mathfrak{h}$, then $\exp(iY).x = x$.

Proof. Up to a shift in the definition of the moment map, this result is [Mig96, Corollary A.7]. The proof is recalled as it is useful for next proposition.

For all $h \in H$ and $g \in G$

$$\psi^{\chi,x}(hg) = \psi^{\chi,x}(g)$$

so that the differential of $\psi^{\chi,x}$ at the identity vanishes on $\mathfrak{h}$. For $Y' + iY \in \mathfrak{h} \oplus i\mathfrak{h}$ this differential is

$$\begin{aligned} d\psi^{\chi,x}_{\mathrm{Id}}(Y' + iY) = d\psi^{\chi,x}_{\mathrm{Id}}(iY) &= \frac{d}{dt}\|\exp(itY).x\|^2\Big|_{t=0} - d\chi_{\mathrm{Id}}(iY) - \overline{d\chi_{\mathrm{Id}}(iY)} \\ &= 2\langle \mu(x), Y\rangle - 2\langle \xi, Y\rangle. \end{aligned}$$

last equality follows from the definition of the moment map μ and the discussion in 2.2.2 defining ξ and proving the reality of $d\chi_{\mathrm{Id}}(iY)$.

So far we proved that $\psi^{\chi,x}$ has a critical point at the identity if and only if $\mu(x) = \xi$, it remains to prove that this critical point is necessarily a minimum. Let $\psi^{\chi,x}$ be critical a the identity and $g \in G$ written in polar form $g = h\exp(iY)$. The action of iY is hermitian so that it can be diagonalized in an orthonormal basis (e_j) such that $iY.e_j = \lambda_j e_j$ with $\lambda_j \in \mathbb{R}$.

$$\begin{aligned} \psi^{\chi,x}(h\exp(iY)) - \psi^{\chi,x}(\mathrm{Id}) &= \psi^{\chi,x}(\exp(iY)) - \psi^{\chi,x}(\mathrm{Id}) \\ &= \sum_j |\exp(\lambda_j)p(e_j;x)|^2 - \log\left(\prod_j \exp(2r_j\lambda_j)\right) \\ &\quad - \sum_j |p(e_j;x)|^2 \end{aligned}$$

with r_j real parameters determined by $\chi \in X(G)^{\mathbb{R}}$. As $\psi^{\chi,x}$ is critical at the identity:

$$0 = \frac{d}{dt}\psi^{\chi,x}(\exp(itY))\Big|_{t=0} = \sum_j \left(2\lambda_j|p(e_j;x)|^2 - 2r_j\lambda_j\right).$$

Combining the two previous equations

$$^{;x}(h\exp(iY)) \quad ^{;x}(\mathrm{Id}) = \sum_j (\exp(2\ \ _j) \quad 2\ \ _j \quad 1)\,jp(e_j;x)j^2:$$

So that $\ ^{;x}(h\exp(iY)) \quad ^{;x}(\mathrm{Id}) \quad 0$ with equality if and only if $\exp(iY):x = x$. Hence when $\ ^{;x}$ has a critical point at the identity, it is necessarily a minimum.

□

Proposition 2.2.12. *Let* $\ \ 2\ X\ (G)^{\mathrm{R}}$ *then* $\ ^1(\ )\quad X^{\ \text{-ss}}$*. Moreover, a point* x_0 *is -stable if and only if the orbit* $G:x_0$ *intersects* $\ ^1(\ )$ *exactly in a* H*-orbit.*

Proof. First statement follows from definition of stability 2.2.4 and Lemma 2.2.11.

Assume x_0 is -stable, then its orbit is closed in $X^{\ \text{-ss}}$ and $G:x_0 \setminus\ \ ^1(\ )$ is not empty. Let x lies in this intersection, then $\ ^{;x}$ has a minimum at the identity. For all $g;g^0\ 2\ G$

$$^{;g:x}(g^0) = \ ^{;x}(g^0g) + \log\left(\left|\ \ (g)\right|^2\right)$$

Hence $\ ^{;g:x}(g^0)$ is minimum for $g^0 = g\ ^1$. Now if $g\ 2\ G$ verifies $g:x\ 2\ \ ^1(\ )$ by Lemma 2.2.11, $\ ^{;g:x}(g^0)$ has a minimum not only at $g^0 = g\ ^1$ but also at the identity. By the second statement of previous lemma, $g\ ^1 = h\exp(iY)$ with $h\ 2\ H$ and $\exp(iY):x = x$. As x is stable, its stabilizer is finite so that $\exp(iY) = \mathrm{Id}$ and $g\ ^1\ 2\ H$. Moreover for any $h\ 2\ H$, the map $\ ^{;h:x}$ has a minimum at identity hence $h:x\ 2\ \ ^1(\ )$ so that $G:x_0 \setminus\ \ ^1(\ ) = H:x$.

Conversely suppose $G:x_0 \setminus\ \ ^1(\ ) = H:x$. First x_0 is -semistable. By Migliorini [Mig96, Proposition A.9], the orbit $G:x_0$ is closed in $X^{\ \text{-ss}}$. It remains to prove that the stabilizer of x_0 is finite. By Lemma 2.2.11 the map $\ ^{;x}$ is minimum at the identity. Let $Y\ 2$ h such that $\exp(iY)$ is in the stabilizer of x. Then $\left|\ \ (\exp(iY))\right| = 1$, otherwise either $\ ^{;x}(\exp(iY)) < \ ^{;x}(\mathrm{Id})$ or $\ ^{;x}(\exp(\ \ iY)) < \ ^{;x}(\mathrm{Id})$. Hence $\ ^{;x}(\exp(iY)) = \ ^{;x}(\mathrm{Id})$ and $\exp(iY)\ 2\ H$ so that $Y = 0$ and the stabilizer of x is finite.

□

Remark 2.2.13. *For* $\ \ 2\ X\ (G)$ *such that -stability and -semistability coincide. Last proposition implies that the inclusion* $\ ^1(\ )\quad X^{\ \text{-ss}}$ *goes down to a continuous bijective map*

$$^1(\ )=H\ \ !\ \ X^{\ \text{-ss}}==G:$$

This result is a particular instance of Kempf-Ness theory, it gives a natural bijection between a symplectic quotient and a GIT quotient. Hoskins [Hos13] proved that this map is actually an homeomorphism.

2.2.4 Hilbert-Mumford criterion for stability

Next theorem is a variant of the usual Hilbert-Mumford criterion for stability. It applies to real parameters $\ \ 2\ X\ (G)^{\mathrm{R}}$ not only to to linear characters. Instead of algebraic one-parameter subgroups it relies on one-parameter real Lie groups defined for $Y\ 2$ h by

$$\begin{array}{ccc} \mathrm{R} & ! & G \\ t & 7! & \exp(itY) \end{array}$$

Many variants of Hilbert-Mumford criterion for one-parameter real Lie groups are given in [GRS13]. Before proving the criterion, two classical technical lemmas are necessary.

Lemma 2.2.14. *Let* $2\ X\ (G)^{\mathbb{R}}$ *and* $Y\ 2\ \mathfrak{h}$*, for* $t\ 2\ \mathbb{R}$

$$\log|\ \ (\exp(itY))|^2 = 2h\ ;Y\,i\ t:$$

Proof. We prove it for $2\ X\ (G)$ and deduce for elements in $X\ (G)^{\mathbb{R}}$ by $\mathbb{R}$-linearity.

$$\begin{aligned} \frac{d}{dt}\Big|_{t=s} \log|\ \ (\exp(itY))|^2 &= \frac{1}{j\ \ (\exp(isY))j^2}\frac{d}{dt}\Big|_{t=s}|\ \ (\exp(itY))|^2 \\ &= \frac{d}{dt}\Big|_{t=s}|\ \ (\exp(i(t\ \ s)Y))|^2 \\ &= \frac{d}{dt}\Big|_{t=0}|\ \ (\exp(itY))|^2 \\ &= 2d\ \ _{\mathrm{Id}}(iY) \end{aligned}$$

By the construction of the element $2\ Z(\mathfrak{h})$ from 2.2.2 we conclude that

$$\frac{d}{dt}\Big|_{t=s} \log|\ \ (\exp(itY))|^2 = 2h\ ;Y\,i$$

and

$$\log|\ \ (\exp(itY))|^2 = 2h\ ;Y\,i\ t:$$

□

Lemma 2.2.15. *Let* $x_0\ 2\ X^{\ \text{-s}}$ *such that* $\ ^{;x_0}$ *is minimum at the identity. Let* $Z\ 2\ \mathfrak{h}$ *and decompose* x_0 *in a basis of eigenvectors of the hermitian endomorphism* iZ

$$x_0 = \sum x^0$$

with

$$\exp(iZ)x^0 = \exp(\ \ )x^0:$$

Then either $h\ ;Z\,i < 0$ *or there exists* $\ > 0$ *with* $x^0 \not= 0$*.*

Proof. By Lemma 2.2.11 and Proposition 2.2.12, as x_0 is -stable, the Kempf-Ness map $\ ^{;x_0}$ reaches its minimum exactly on H. For $Z\ 2\ \mathfrak{h}$ consider the map f_Z defined for t real by

$$f_Z(t) = \ \ ^{;x_0}(\exp(iZt)):$$

f_Z reaches its minimum only at $t = 0$. We can compute $f_Z(t)$ using the decomposition of x_0 in eigenvectors of iZ and Lemma 2.2.14

$$f_Z(t) = \sum \exp(2t\ \ )\,||x^0||^2\ \ \ 2h\ ;Z\,i\ t: \tag{2.6}$$

Its second derivative is

$$f_Z^{00}(t) = \sum 4\ \ ^2\exp(2t\ \ )\,||x^0||^2:$$

Then f_Z is convex, moreover it reaches its minimum only at $t = 0$ so that

$$\lim_{t\ !\ +1} f_Z(t) = +1:$$

Looking at equation (2.6) this implies either $h\ ;Z\,i < 0$ or there exists $\ > 0$ with $x^0 \not= 0$. □

Theorem 2.2.16 (Hilbert-Mumford criterion for stability)**.** *Let $\alpha \in X(G)^{\mathbb{R}}$ and $x \in X$. The following statements are equivalent*

(i) x is α-stable.

(ii) For all $Y \in \mathfrak{h}$, different from zero, such that $\lim_{t \to +\infty} \exp(itY).x$ exists then $\langle \alpha, Y \rangle < 0$.

Proof. not (i) implies not (ii)

Let $x \in X \setminus X^{\alpha\text{-s}}$. Then if $\Phi^{\alpha,x}$ admits a minimum, the stabilizer of x is not finite and this minimum is reached on an unbounded subset of G. Thus there exists an unbounded minimizing sequence for $\Phi^{\alpha,x}$. By polar decomposition and H invariance we can assume it has the following form $(\exp iY_n)_{n\in\mathbb{N}}$ with $(Y_n)_{n\in\mathbb{N}} \in \mathfrak{h}^{\mathbb{N}}$ unbounded. The hermitian space W admits an orthonormal basis $B^n = (e_1^n, \ldots, e_d^n)$ made of eigenvectors of iY_n with associated eigenvalues $\lambda_1^n, \ldots, \lambda_d^n$.

$$\exp(iY_n).e_k^n = \exp(\lambda_k^n)e_k^n.$$

This basis allows to compute:

$$\Phi^{\alpha,x}(\exp iY_n) = \sum_{k=1}^{d} \exp(2\lambda_k^n)\|x_k^n\|^2 - 2\langle \alpha, Y_n \rangle$$

with $x_k^n = p(x, e_k^n)e_k^n$ the components of x in the basis B^n. By compactness of the set of orthonormal frames, we can assume the sequence of basis $(B^n)_{n\in\mathbb{N}}$ converges to an orthonormal basis $B = (e_1, \ldots, e_k)$. Let $x_k = p(x, e_k)e_k$ the components of x in the basis B. Then $\lim_{n\to+\infty} x_k^n = x_k$. Let

$$\Lambda_n = \sum_{k=1}^{d} |\lambda_k^n|$$

As $(Y_n)_{n\in\mathbb{N}}$ is unbounded, up to an extraction of a subsequence, we can assume that $\lim_{n\to+\infty} \Lambda_n = +\infty$ and that the following limit exist and are finite:

$$Y := \lim_{n\to+\infty} \frac{Y_n}{\Lambda_n}$$

and

$$\mu_k := \lim_{n\to+\infty} \frac{\lambda_k^n}{\Lambda_n}.$$

Now one can bound from bellow the values $\Phi^{\alpha,x}(\exp iY_n)$ of the minimizing sequence

$$\begin{aligned}\Phi^{\alpha,x}(\exp iY_n) &\geq \sum_{\{k \mid x_k \neq 0\}} \exp(2\lambda_k^n)\|x_k^n\|^2 - 2\langle \alpha, Y_n \rangle. \\ &\geq \sum_{\{k \mid x_k \neq 0\}} \exp(2(\mu_k + o(1))\Lambda_n)\left(\|x_k\|^2 + o(1)\right) \\ &\quad - 2(\langle \alpha, Y \rangle + o(1))\Lambda_n\end{aligned}$$

with $o(1)$ some sequences going to zero when n goes to infinity. As the left-hand side is the value of a minimizing sequence, it cannot go to plus infinity. Hence $\langle \alpha, Y \rangle \leq 0$, moreover if $x_k \neq 0$ Then $_k$ 0. We conclude as Y satisfies $\lim_{t \to +\infty} \exp(itY).x$ exists and $\langle\ ; Y\rangle$ 0.

<u>(i) implies (ii)</u>

Let $x \in X^{-s}$, by Lemma 2.2.11 and Proposition 2.2.12 there exists $g_0 \in G$ such that for $x_0 = g_0.x$, the Kempf-Ness map $^{;x_0}$ reaches its minimum exactly on H. Now let $Y \in \mathfrak{h}$ such that $\lim_{t \to +\infty} \exp(itY).x$ exists then $\lim_{n \to +\infty} \exp(inY).x$ exists. For all $n \in \mathbb{N}$ polar decomposition provides unique $h_n \in H$ and $Z_n \in \mathfrak{h}$ such that

$$\exp(inY) = h_n \exp(iZ_n) g_0.$$

Then Z_n is unbounded. Proceed as in the first part of the proof, iZ_n is an hermitian endomorphism denote by $^n_1, \dots, ^n_d$ its eigenvalues and let

$$_n = \sum_{k=1}^{d} | ^n_k |.$$

We can assume that $\lim_{n \to +\infty} \ _n = +\infty$ and that the following limits exist and are finite:

$$Z := \lim_{n \to +\infty} \frac{Z_n}{_n}$$

and

$$_k := \lim_{n \to +\infty} \frac{^n_k}{_n}.$$

Then denoting by x^0_k the components of x_0 in an orthonormal basis of eigenvectors of iZ

$$^{;x}(\exp(iZ_n)g_0) \quad \sum_{\{k | x_k \neq 0\}} \exp\left(2\left(\ _k + o(1)\right)\ _n\right)\left(\|x_k\|^2 + o(1)\right)$$

$$2\left(\langle\ ; Z\rangle + o(1)\right)\ _n + \log\left|\ (g_0)\right|^2$$

By Lemma 2.2.15 either $\langle\ ; Z\rangle < 0$ or there exists $_k > 0$ with $x^0_k \neq 0$. In any case

$$\lim_{n \to +\infty} \ ^{;x}(\exp(iZ_n)g_0) = +\infty.$$

Then the relation (2.2.4) defining Z_n implies

$$\lim_{n \to +\infty} \ ^{;x}(\exp(inY)) = +\infty. \tag{2.7}$$

Decompose x in a basis of eigenvectors of the hermitian endomorphism iY

$$x = \sum x$$

then

$$^{;x}(\exp(inY)) = \sum \exp(2n\ ) \|x\|^2 \quad 2\langle\ ; Y\rangle n.$$

As the limit $\lim_{n \to +\infty} \exp(inY).x$ is assumed to exist, 0 if $x \neq 0$. Then the condition (2.7) implies $\langle\ ; Y\rangle < 0$.

□

2.2.5 Regular locus

In this subsection the closed subvariety X is not relevant, the action of G and H on the ambient hermitian vector space W is studied. First note that the moment map can be defined not only on X but on the whole space W. Let $T \subset H$ a maximal torus. As in Example 2.2.9 the ambient space W decomposes as an orthogonal direct sum $W = \bigoplus W_\chi$ with χ characters of T and

$$W_\chi = \{x \in W \mid t.x = \chi(t)x \text{ for all } t \in T\}.$$

Denote by A the finite subset of elements $\alpha \in \mathfrak{t}$ such that for the character χ_α the space W_{χ_α} is not zero then

$$W = \bigoplus_{\alpha \in A} W_\alpha.$$

As before the link between linear characters and elements in $\mathfrak{t}$ is through the invariant pairing $\langle \ldots, \ldots \rangle$

$$id\,\chi_{\alpha\,\mathrm{Id}}(\xi) = \langle \alpha, \xi \rangle.$$

Hence if ξ is orthogonal to the $\mathbb{R}$ vector space spanned by A

$$\chi_\alpha(\exp t\xi) = 1$$

for all $\alpha \in A$ so that $\exp t\xi$ is in the kernel of the action of H on W. From the beginning this kernel is assumed to be trivial, hence the vector space spanned by A is $\mathfrak{t}$. As in Example 2.2.9, the image of μ_T, the moment map relative to the T-action, is the cone spanned by positive combinations of A. For any A' finite subset of $\mathfrak{t}$ the cone spanned by positive combinations of A' is:

$$C(A') := \left\{ \sum_{\alpha \in A'} a_\alpha \alpha \mid a_\alpha \geq 0 \text{ for all } \alpha \in A' \right\}.$$

For any $\xi \in \mathfrak{t}$

$$\begin{aligned} \langle \mu(x), \xi \rangle &= \frac{d}{dt} \|\exp(it\xi).x\|^2 \Big|_{t=0} \\ &= \langle \mu_T(x), \xi \rangle. \end{aligned}$$

Hence, as noted by Kirwan [Kir84], if $\mu(x) \in \mathfrak{t}$ then $\mu(x) = \mu_T(x)$. For A' a finite subset of $\mathfrak{t}$ we denote by $\dim A'$ the dimension of the vector space spanned by A'.

Lemma 2.2.17. *Let $x \in W$ such that for all $A' \subset A$ with $\dim A' < \dim \mathfrak{t}$, the value of the moment map $\mu_T(x)$ does not lie in $C(A')$. Then the stabilizer of x is finite.*

Proof. Decompose x according to its weight $x = \sum_{\alpha \in A} x_\alpha$ then

$$\mu_T(x) = \sum \|x_\alpha\|^2 \alpha.$$

Denote by A_x the set of elements α such that $x_\alpha \neq 0$. The hypothesis about $\mu_T(x)$ implies that $\dim A_x = \dim \mathfrak{t}$. Now for $\xi \in \mathfrak{t}$

$$\exp(\xi t).x = \sum_{\alpha \in A_x} \chi_\alpha(\exp \xi t) x_\alpha.$$

Hence if $\exp \xi t$ is in the stabilizer of x, for all $\alpha \in A_x$ the pairing with ξ vanishes $\langle \alpha, \xi \rangle = 0$. As A_x spans $\mathfrak{t}$ this implies that $\xi = 0$ and the stabilizer of x in T is finite. ☐

Previous lemma justifies the introduction of the following nonempty open subset of $\mathfrak{t}$

$$C(A)^{\mathrm{reg}} := C(A) \setminus \bigcap_{\substack{A' \subset A \\ \dim A' < \dim \mathfrak{t}}} C(A').$$

As all maximal torus of H are conjugated, the set $C(A)^{\mathrm{reg}} \cap Z(\mathfrak{h})$ is independent of a choice of maximal torus T.

Proposition 2.2.18. *For $\theta \in C(A)^{\mathrm{reg}} \cap Z(\mathfrak{h})$, every θ-semistable points are θ-stable, $W^{\theta\text{-ss}} = W^{\theta\text{-s}}$ and in particular $X^{\theta\text{-ss}} = X^{\theta\text{-s}}$.*

Proof. Let $x \in W^{\theta\text{-ss}}$, then $\overline{G.x}$ meets $\mu^{-1}(\theta)$. But $\overline{G.x} \setminus G.x$ is a union of G-orbits of dimension strictly smaller than $G.x$, points in those orbits has stabilizer with dimension greater than one. By previous lemma every point in $\mu^{-1}(\theta)$ has a finite stabilizer. Thus $G.x \cap \mu^{-1}(\theta) \neq \emptyset$ and the stabilizer of x is finite so that x is θ-stable. □

Kirwan [Kir84], Ness-Mumford [NM84], Sjamaar [Sja98], Harada-Wilkin [HW08] and Hoskins [Hos13] studied a stratification of W. It relies on the Morse theory of the following map. For $\theta \in Z(\mathfrak{h})$

$$\begin{aligned} h_\theta : W &\to \mathbb{R} \\ x &\mapsto |\mu(x) - \theta|^2 \end{aligned}$$

with $|\ldots|$ the norm defined by the invariant pairing $\langle \ldots, \ldots \rangle$ on $\mathfrak{h}$. A critical point of a smooth map f is a point x where the differential vanishes $d_x f = 0$. A critical value of f is the image $f(x)$ of a critical point x. The gradient of h_θ is the vector field defined thanks to the hermitian pairing $p(\ldots, \ldots)$ for $x \in W$ and $v \in T_x W$ by

$$p(\mathrm{grad}_x h_\theta, v) = d_x h_\theta . v$$

For $x \in W$ the negative gradient flow relative to h_θ is the map

$$\begin{aligned} \gamma_x : \mathbb{R}_{\geq 0} &\to W \\ t &\mapsto \gamma_x(t) \end{aligned}$$

uniquely determined by the condition

$$\left.\frac{d\gamma_x(s)}{ds}\right|_{s=t} = -\mathrm{grad}_{\gamma_x(t)} h_\theta .$$

and $\gamma_x(0) = x$. By [Sja98] and [HW08] it is well defined and for any x the limit $\lim_{t\to+\infty} \gamma_x(t)$ exists and is a critical point of h_θ. S_θ is the set of point $x \in W$ with negative gradient flow for h_θ converging to a point where h_θ reaches its minimal value 0:

$$S_\theta := \left\{ x \in W \,\middle|\, \lim_{t\to+\infty} \gamma_x(t) \in \mu^{-1}(\theta) \right\}.$$

This is the open strata of the stratification, Sjamaar called it the set of analitically semistable points. When the stability parameter is a true character i.e. $\theta \in X(G)$, Hoskins [Hos13] proved that this strata coincides with the θ-semistable locus. Here we want to consider any $\theta \in X(G)^{\mathbb{R}}$, the proof of the inclusion $S_\theta \subset W^{\theta\text{-ss}}$ is the same and is enough for our purpose.

Proposition 2.2.19. *S is a subset of $W^{\text{-ss}}$.*

Proof. The flow $_x(t)$ belongs to the orbit $G{:}x$ hence $\lim_{t \to +\infty} {}_x(t) \ \overline{G{:}x}$. Therefore if $x \in S$ then $\overline{G{:}x} \cap {}^{-1}(\ ) \neq \emptyset$. □

An important feature of the map h is that its critical points lie in a finite union $\bigcup_{A' \ A} {}^{-1}(H{:}\ (A';\ ))$ indexed by the subsets of the finite set A. With $(A';\ )$ the projection of to the closed convex $C(A')$ and $H{:}\ (A';\ )$ the adjoint orbit of $(A';\ )$.

Lemma 2.2.20. *By definition of the projection to a closed convex in an euclidian space $j\ (A';\ )\ \ j$ is the distance between and the cone $C(A')$, define*

$$d = \inf_{\substack{A' \ A \\ (A';\ ) \neq}} j\ (A';\ )\ \ j^2 \qquad (2.8)$$

then $d > 0$ and $h^{-1}[0; d[\ \ S$.

Proof. For any $h \in H$ by invariance of the scalar product under the adjoint action and as $\in Z(\mathrm{h})$

$$jh{:}\ (A';\ )\ \ j^2 = j\ (\ ; A')\ \ j^2 :$$

Hence if x is a critical point of h not in ${}^{-1}(\ )$, then $x \in {}^{-1}(H{:}\ (A';\ ))$ for some $(A';\ )$ different from and

$$j\ (x)\ \ j^2 = j\ (\ ; A')\ \ j^2 > d :$$

So that the only critical value of h_0 in the intervalle $[0; d[$ is 0.

Now for any $x \in W$, the map $t \mapsto h\ (\ _x(t))$ can only decrease, and it converges to a critical value. Therefore if $x \in h^{-1}[0; d[$ the negative gradient flow converges necessarily to a point $\lim_{t \to +\infty} {}_x(t)$ which belongs to $h^{-1}(0) = {}^{-1}(\ )$ so that $x \in S$. □

Theorem 2.2.21. *Let $_0 \in C(A)^{\text{reg}} \cap Z(\mathrm{h})$, there is an open neighborhood V_{0} of $_0$ in $C(A)^{\text{reg}} \cap Z(\mathrm{h})$ such that for all $\in V_0$, -stability and $_0$-stability coincide $W^{_0\text{-ss}} = W^{\text{-ss}}$.*

Proof. Let > 0 such that $B(\ _0;\ )$ the ball of center $_0$ and radius in t is included in $C(A)^{\text{reg}}$. Then when varies in $B(\ _0;\ )$ it does not meet any frontier of a cone $C(A')$ with $A'\ \ A$. So that for $\in B(\ _0;\ )$, for all $A'\ \ A$, $(\ ; A') \neq 0$ if and only if $(\ _0; A') \neq 0$. Thus the subset indexing the infima defining d and d_{0} in (2.8) are identical. As the projection to closed convex is a continuous map, the map $\mapsto d$ is continuous on $B(\ _0;\ )$. Therefore one can chose $' > 0$ such that

- $d > \frac{d_0}{2}$ for all $\in B(\ _0;\ ')$.

Moreover $'$ can be chosen to satisfy the following conditions

- $B(\ _0;\ ')\ \ C(A)^{\text{reg}}$
- $'^2 < \frac{d_0}{2}$

Let in $B(\ _0;\ ^{0}) \cap Z(\mathfrak{h})$, we shall see that $W^{\ \text{-ss}} = W^{\ 0\text{-ss}}$. First note that $\ \in C(A)^{\text{reg}} \cap Z(\mathfrak{h})$ and Proposition 2.2.18 implies $W^{\ \text{-ss}} = W^{\ \text{-s}}$.

For $x \in W^{\ \text{-ss}} = W^{\ \text{-s}}$, by Proposition 2.2.12 there exists $g \in G$ such that $g.x \in \ ^{-1}(\ )$. Then $j\ (g.x) \quad _0 j^{<} \frac{d\ _0}{2}$ and $g.x \in h\ _0^{-1}[0; d\ _0[$. By Lemma 2.2.20, $g.x \in S\ ^{0}$ and by Proposition 2.2.19 $g.x$ is $\ _0$-semistable so that $x \in W^{\ 0\text{-ss}}$.

Similarly for $x \in W^{\ 0\text{-ss}}$, there exists $g \in G$ such that $g.x \in \ ^{-1}(\ _0)$. Then $j\ (g.x) \quad j^2 < \frac{d\ _0}{2}$ and as $\frac{d\ _0}{2} < d\ $, the point $g.x$ lies in $h\ ^{-1}[0; d\ [$ therefore x is -stable. □

Considering again the closed subvariety $X \quad W$ one defines the regular locus:

Definition 2.2.22 (Regular locus). *The regular locus B^{reg} is the set of elements $\ \in C(A)^{\text{reg}} \cap Z(\mathfrak{h})$ such that for all $x \in X^{\ \text{-ss}}$ the stabilizer of x in G is trivial and $X^{\ \text{-ss}} \neq \emptyset$.*

Proposition 2.2.23. *The regular locus B^{reg} is the union of some connected components of $C(A)^{\text{reg}} \cap Z(\mathfrak{h})$.*

Proof. By Theorem 2.2.21, if and $\ ^{0}$ are in the same connected component of $C(A)^{\text{reg}} \cap Z(\mathfrak{h})$ then $W^{\ \text{-ss}} = W^{\ 0\text{-ss}}$. Hence if $\ \in C(A)^{\text{reg}} \cap Z(\mathfrak{h})$ is such that for all $x \in X^{\ \text{-ss}}$ the stabilizer of x in G is trivial and $X^{\ \text{-ss}} \neq \emptyset$, the same holds for $\ ^{0}$ in the same connected component of $C(A)^{\text{reg}} \cap Z(\mathfrak{h})$. □

Remark 2.2.24. *Note that the regular locus B^{reg} can be empty, for instance if the center $Z(\mathfrak{h})$ is a subset of a cone $C(A^0)$ with $\dim A^0 < \dim \mathfrak{t}$. Fortunately it is non-empty for the application to Nakajima's quiver varieties of next sections.*

In next subsection we prove that the real moment map is a locally trivial fibration over the regular locus B^{reg}.

2.2.6 Trivialization of the real moment map over the regular locus

Next construction follows ideas from Hitchin-Karlhede-Lindström-Roček and is illustrated in [Hit+87, Figure 3 p.348].

Proposition 2.2.25. *For $\ \in X\ (G)^{\mathbb{R}}$ and x a -stable point with trivial stabilizer, there exists a unique $Y^{\ ;x} \in \mathfrak{h}$ such that $\exp\left(iY^{\ ;x}\right).x \in \ ^{-1}(\ )$. Moreover for $h \in H$ the adjoint action of h on $Y^{\ ;x}$ satisfies*

$$h.Y^{\ ;x} = Y^{\ ;h.x}. \tag{2.9}$$

Let $\ ^{0} = \ (x)$ and $x^0 = \exp\left(iY^{\ ;x}\right).x$, then

$$Y^{\ ^0;x^0} = \ Y^{\ ;x}. \tag{2.10}$$

.

Proof. As x is -stable, by Proposition 2.2.12 the orbit $G.x$ intersects $\ ^{-1}(\ )$ exactly on a H-orbit. There exists $g \in G$ such that $g.x \in \ ^{-1}(\ )$. Apply polar decomposition to this element $g = h_0 \exp\left(iY^{\ ;x}\right)$ with $h_0 \in H$ and $Y^{\ ;x} \in \mathfrak{h}$. Then

$$\ ^{-1}(\ ) \cap G.x = H.\exp\left(iY^{\ ;x}\right).x$$

Take Y' such that $\exp(iY').x \in {}^{1}(\ )$ then

$$\exp(iY').x = h \exp(iY^{\ ;x}).x$$

for some h in H. By triviality of the stabilizer of x and uniqueness of polar decomposition $Y' = Y^{\ ;x}$ hence $Y^{\ ;x}$ is uniquely determined. Let us check H-equivariance, for $h \in H$

$$^{1}(\ ) \ni h \exp(iY^{\ ;x}).x = \exp\left(ih.Y^{\ ;x}\right).h.x$$

by uniqueness $Y^{\ ;h.x} = h.Y^{\ ;x}$. Equation (2.10) is clear. □

Remark 2.2.26. *The assumption that x has a trivial stabilizer can be relaxed. Then there exists $Y^{x;} \in$ h such that*

$$\left\{Y \in \mathfrak{h} \,|\, \exp(iY).x \in {}^{1}(\ )\right\} = (\mathrm{Stab}_H\, x).Y^{\ ;x}$$

The right-hand side is the orbit of $Y^{\ ;x}$ under the adjoint action of the stabilizer of x in H. For applications to quiver varieties we only need to consider the case of a trivial stabilizer.

Lemma 2.2.27. *Let $\ \in Z(\mathfrak{h})$ and x_0 a -stable point with trivial stabilizer. There exists an open neighborhood $U_{\ ;x_0}$ of $(\ ;x_0)$ in* h $\ X$ *and a smooth map*

$$\begin{array}{rccc} Y : & U_{\ ;x_0} & \to & \mathfrak{h} \\ & (\ ';x') & \mapsto & Y(\ ';x') \end{array}$$

such that $(\exp(iY(\ ';x')).x') = \ '$.

Proof. Note that when $\ \in Z(\mathfrak{h})$ necessarily $Y(\ ;x)$ is equal to the $Y^{\ ;x}$ introduced in previous proposition. Let $Y^{\ ;x_0}$ such that $x := \exp\left(iY^{\ ;x_0}\right).x_0$ is in the intersection $G.x_0 \cap {}^{1}(\ )$. Consider the map

$$\begin{array}{rccc} f : & \mathfrak{h} \quad \mathfrak{h} \quad X & \to & \mathfrak{h} \\ & (Y';\ ';x') & \mapsto & (\exp(iY').x') \quad ' \end{array}$$

in order to use the implicit function theorem on a neighborhood of $\left(Y^{\ ;x_0};\ ;x_0\right)$ we first prove that the differential of f with respect to Y' at $(Y^{\ ;x_0};\ ;x_0)$ is invertible. As x has a finite stabilizer, the embedding of tangent spaces $T_xH.x \hookrightarrow T_xG.x$ identifies with the embedding

$$\mathfrak{h} = T_{\mathrm{Id}}H \hookrightarrow T_{\mathrm{Id}}G = \mathfrak{h} \quad i\mathfrak{h}. \tag{2.11}$$

By Proposition 2.2.10, d is surjective so that ${}^{1}(\ )$ is a smooth manifold and $\ker d\ _x = T_x\ {}^{1}(\ )$. Proposition 2.2.12 implies ${}^{1}(\ ) \cap G.x = H.x$ Restricting d_x to the tangent space of the G-orbit we obtain the following short exact sequence

$$0 \hookrightarrow T_xH.x \hookrightarrow T_xG.x \xrightarrow{d_x\ |_{T_xG.x}} \mathfrak{h} \to 0.$$

the surjectivity follows from dimension counting and the identification of the tangent spaces with (2.11). Thus we obtain the expected invertibility of the differential with respect to Y' of f at $(Y^{\ ;x_0};\ ;x_0)$, the map $d_{Y'}f_{(Y^{\ ;x_0};\ ;x_0)}$, identifies with an invertible map $i\mathfrak{h} \to \mathfrak{h}$. The implicit function theorem applies and gives the existence of $U_{\ ;x_0}$ h $\ X$ an open neighborhood of $(\ ;x_0)$ and the expected smooth map $Y(\ldots;\ldots)$.

□

Next theorem is a first result concerning local triviality of the moment map, over the regular locus B^{reg} the real moment map is a locally trivial fibration.

Theorem 2.2.28. *Let $_0$ in B^{reg}, and $U_{\ 0}$ the connected component of B^{reg} containing $_0$. There is a di eomorphism f such that the following diagram commutes*

$$\begin{array}{ccc} U_{\ 0} \quad {}^{1}(\ _0) & \xrightarrow[f]{} & {}^{1}(U_{\ 0}) \\ & \searrow & \downarrow \\ & & U_{\ 0} \end{array}$$

Moreover f is H equivariant so that the diagram goes down to quotient

$$\begin{array}{ccc} U_{\ 0} \quad {}^{1}(\ _0)=H & \longrightarrow & {}^{1}(U_{\ 0})=H \\ & \searrow & \downarrow \\ & & U_{\ 0} \end{array}$$

Proof. For $\ 2\ U_{\ 0}$ we know from 2.2.5 that $X^{\ \text{-s}} = X^{\ 0\text{-s}} \neq ;$. Define f by

$$f(\ ;x) := \exp(iY(\ ;x)) \,.x$$

It follows from Proposition 2.2.25 that it is invertible with inverse

$$f^{\ 1}(x^0) = (\ (x^0); \exp(iY(\ _0;x^0)) \,.x^0) \,.$$

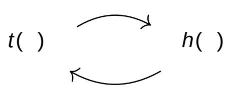
Lemma 2.2.27 implies that f is a di eomorphism. Equivariance follows from equation (2.9) so that $f(\ ;h.x) = h.f(\ ;x)$ and f goes down to a di eomorphism between quotients. □

In next sections Nakajima's quiver varieties are considered, they admit an additional hyperkähler structure. A similar trivialization is established in this hyperkähler context.

2.3 Quiver varieties and stability

2.3.1 Generalities about quiver varieties

The quiver varieties considered in this thesis were introduced by Nakajima [Nak94]. Let be a quiver with vertices $_0$ and edges $_1$. For an edge $\ 2\ _1$ we denote $t(\ ) \ 2\ _0$ its tail and $h(\ ) \ 2\ _0$ its head, we define the reverse edge $\bar{}$ such that $t(\bar{}) = h(\ )$ and $h(\bar{}) = t(\ )$.



Let $\bar{}_1 := f\bar{}\,j\ 2\ _1g$ and $\tilde{} := \ _1\ t\ \bar{}_1$. For $\bar{}\ 2\ \bar{}_1$ we set $\bar{\bar{}} :=$ to obtain an involution on $\tilde{}$. The extended quiver $\tilde{}$ is obtained by adding an inverse to all edges in Γ_1, its set of vertices is Γ_0 and its set of edges is $\tilde{\Gamma}$. Let $\varepsilon : \tilde{\Gamma} \to \{-1, 1\}$ be the map

$$\begin{cases} \varepsilon(\gamma) = 1 & \text{if } \gamma \in \Gamma_1 \\ \varepsilon(\bar{\gamma}) = -1 & \text{if } \bar{\gamma} \in \bar{\Gamma}_1 \end{cases}$$

We fix a dimension vector $v \in \mathbb{N}^{\Gamma_0}$. A representation of the quiver Γ with dimension vector v is a pair (V, x) with $V = \bigoplus_{j \in \Gamma_0} V_j$ a graded vector space with $\dim V_j = v_j$ and $x = (x_\gamma)_{\gamma \in \Gamma_1}$ a collection of linear maps $x_\gamma : V_{t(\gamma)} \to V_{h(\gamma)}$. A subrepresentation is a subspace $W \subset V$ with a compatible Γ_0-grading and preserved by x. The set of quiver representations with dimension vector v is identified with

$$\operatorname{Rep}(\Gamma, v) := \bigoplus_{\gamma \in \Gamma_1} \operatorname{Mat}_{\mathbb{C}}(v_{h(\gamma)}, v_{t(\gamma)}).$$

For construction of quiver varieties it is interesting to consider representations of the extended quiver $\tilde{\Gamma}$

$$\operatorname{Rep}\left(\tilde{\Gamma}, v\right) := \bigoplus_{\gamma \in \tilde{\Gamma}} \operatorname{Mat}_{\mathbb{C}}(v_{h(\gamma)}, v_{t(\gamma)}).$$

It is a complex vector space, the complex structure considered in this section is

$$I.(x_\gamma)_{\gamma \in \tilde{\Gamma}} = (i x_\gamma)_{\gamma \in \tilde{\Gamma}}$$

The group $\mathrm{GL}_v := \prod_{i \in \Gamma_0} \mathrm{GL}_{v_i}(\mathbb{C})$ acts linearly on $\operatorname{Rep}(\tilde{\Gamma}, v)$

$$g.(x_\gamma)_{\gamma \in \tilde{\Gamma}} := \left(g_{h(\gamma)} x_\gamma g_{t(\gamma)}^{-1}\right)_{\gamma \in \tilde{\Gamma}}.$$

The diagonal embedding of $\mathbb{C}^*$ in GL_v acts trivially so that the action goes down to an action of the group

$$G_v := \mathrm{GL}_v / \mathbb{C}^*,$$

which identifies with

$$G_v = \left\{ (g_j)_{j \in \Gamma_0} \in \mathrm{GL}_v \,\Big|\, \prod_{j \in \Gamma_0} \det(g_j) = 1 \right\}.$$

Note that G_v is isomorphic to a product of a special linear group and a finite number of general linear groups so that it is a reductive group. The Lie algebra of GL_v, respectively G_v is $\mathfrak{gl}_v = \bigoplus_{j \in \Gamma_0} \mathfrak{gl}_{v_j}(\mathbb{C})$ respectively.

$$\mathfrak{g}_v = \left\{ (x_j)_{j \in \Gamma_0} \in \mathfrak{gl}_v \,\Big|\, \sum_{j \in \Gamma_0} \operatorname{tr} x_j = 0 \right\}$$

The center of $\mathfrak{g}_v$ is

$$Z(\mathfrak{g}_v) = \left\{ (\lambda_j \operatorname{Id}_{v_j})_{j \in \Gamma_0} \,\Big|\, (\lambda_j)_{j \in \Gamma_0} \in (\mathbb{C})^{\Gamma_0} \text{ with } \sum_{j \in \Gamma_0} v_j \lambda_j = 0 \right\}.$$

Let $\theta \in \mathbb{Z}^{\Gamma_0}$ such that $\sum_{j \in \Gamma_0} v_j \theta_j = 0$, define χ_θ a character of G_v by

$$\chi_\theta((g_j)_{j \in \Gamma_0}) = \prod_{j \in \Gamma_0} \det(g_j)^{\theta_j}. \tag{2.12}$$

The θ-semistable locus, respectively θ-stable locus in the sense of Mumford's Geometric Invariant Theory [MF82], are denoted by $\operatorname{Rep}(\tilde{\Gamma}, v)^{\theta\text{-ss}}$, respectively $\operatorname{Rep}(\tilde{\Gamma}, v)^{\theta\text{-s}}$.

Definition 2.3.1 (Complex moment map). *The complex moment map is defined by*

$$\begin{array}{rccc} \mu_{\mathbb{C}} : & \operatorname{Rep}(\tilde{Q}, v) & \longrightarrow & \mathfrak{g}_v \\ & (x_\alpha)_{\alpha \in \tilde{Q}_1} & \longmapsto & \sum_{\alpha \in \tilde{Q}_1} \epsilon(\alpha) x_\alpha x_{\bar{\alpha}} \end{array}$$

it is G_v-equivariant for the adjoint action on $\mathfrak{g}_v$.

This complex moment map will be related to the real moment map of Definition 2.2.8 in next section.

Definition 2.3.2 (Nakajima's quiver variety). *For $\lambda \in Z(\mathfrak{g}_v)$, the set $\mu_{\mathbb{C}}^{-1}(\lambda)$ is an affine variety in $\operatorname{Rep}(\tilde{Q}, v)$, it inherits a G_v action. Nakajima's quiver varieties are defined as GIT quotients:*

$$\mathcal{M}_v(\lambda, \theta) := \mu_{\mathbb{C}}^{-1}(\lambda) \cap \operatorname{Rep}(\tilde{Q}, v)^{\theta\text{-ss}} /\!/ G_v.$$

Those varieties are interesting from the differential geometry point of view and have an hyperkähler structure. We are interested in the family formed by those varieties when the parameters λ and θ are varying. Before studying those family, we introduce another kind of variety: Nakajima's framed quiver variety.

Fix another dimension vector $w \in \mathbb{N}^{Q_0}$ and denote

$$\operatorname{Rep}(v, w) := \bigoplus_{j \in Q_0} \operatorname{Mat}_{\mathbb{K}}(v_i, w_i)$$

$$\operatorname{Rep}(w, v) := \bigoplus_{j \in Q_0} \operatorname{Mat}_{\mathbb{K}}(w_i, v_i)$$

An element $g \in \mathrm{GL}_v$ acts on $a = (a_j)_{j \in Q_0} \in \operatorname{Rep}(v, w)$ by

$$g.a := (a_j g_j^{-1})_{j \in Q_0}$$

and on $b = (b_j)_{j \in Q_0} \in \operatorname{Rep}(v, w)$ by

$$g.b := (g_j b_j)_{j \in Q_0}$$

Introduce framed quiver representations

$$\operatorname{Rep}\left(\tilde{Q}, v, w\right) := \operatorname{Rep}(v, w) \oplus \operatorname{Rep}(w, v) \oplus \operatorname{Rep}\left(\tilde{Q}, v\right)$$

and extend the moment map

$$\begin{array}{rccc} \mu^0 : & \operatorname{Rep}(\tilde{Q}, v, w) & \longrightarrow & \mathfrak{gl}_v \\ & (a, b, x) & \longmapsto & (\mu(x)_j - b_j a_j)_{j \in Q_0} \end{array}.$$

Definition 2.3.1. *Let $\theta \in \mathbb{R}^{Q_0}$, a representation $(a, b, x) \in \operatorname{Rep}(\tilde{Q}, v, w)$ is θ-semistable if for any x-invariant subspace $S \subset V$ such that $S_j \subset \ker a_j$ the following inequality holds*

$$\sum_{j \in Q_0} \theta_j \dim S_j \leq 0$$

and for any x-invariant subspace $T \subset V$ such that $\operatorname{Im} b_j \subset T_j$

$$\sum_{j \in Q_0} \theta_j \dim T_j \leq \sum_{j \in Q_0} \theta_j v_j.$$

It is stable if the inequality are strict unless $S = 0$ and $T = V$.

The result of King extends to the framed case.

Theorem 2.3.2. *Let $\theta \in \mathbb{Z}^I$, a point of the affine variety* $\operatorname{Rep}\left(\bar{Q}; v; w\right)$ *is stable (respectivly semistable) with respect to the linearization*

$$\begin{array}{rccc} \chi_\theta : & \mathrm{GL}_v & \to & \mathbb{C} \\ & (g_i)_{i \in I} & \mapsto & \prod_{i \in I} \det(g_i)^{-\theta_i} \end{array}$$

in the sense of GIT, if and only if it is θ-stable, respectivly θ-semistable in the sense of definition 2.3.1.

Proof. A result of Crawley-Boevey identifying framed quiver varieties to unframed ones (remark at the end of Section 1 in [Cra01]) and the discussion following Definition 4.2.1 in [Let11] bring this theorem back to the unframed case. □

Definition 2.3.3 (Nakajima's framed quiver varieties). *For λ in the center of* $\mathfrak{gl}_v$ *and $\theta \in \mathbb{Z}^{Q_0}$ the Nakajima's framed quiver variety is defined as a GIT quotient*

$$\mathcal{M}_{v,w}(\lambda, \theta) := \mu^{-1}(\lambda) \cap \operatorname{Rep}\left(\tilde{Q}; v; w\right)^{\theta\text{-ss}} // \mathrm{GL}_v$$

2.3.2 King's characterization of stability of quiver representations

As in Section 2.2 the geometric invariant theory has a symplectic counterpart. $\operatorname{Rep}\left(\tilde{Q}; v\right)$ is an hermitian vector space with norm

$$\left|\left|(\varphi_\alpha)_{\alpha \in \tilde{\Omega}}\right|\right|^2 = \sum_{\alpha \in \tilde{\Omega}} \operatorname{tr}(\varphi_\alpha \varphi_\alpha^\dagger).$$

The G_v-action restricts to a unitary action of the maximal compact subgroup

$$U_v = \left\{ (g_j)_{j \in Q_0} \in \prod_{j \in Q_0} U_{v_j} \;\middle|\; \prod_{j \in Q_0} \det(g_{v_j}) = 1 \right\}$$

The Lie algebra of U_v is

$$\mathfrak{u}_v = \left\{ (x_j)_{j \in Q_0} \in \bigoplus_{j \in Q_0} \mathfrak{u}_{v_j} \;\middle|\; \sum_{j \in Q_0} \operatorname{tr} x_j = 0 \right\}$$

with U_{v_j}, respectively $\mathfrak{u}_{v_j}$, the group of unitary matrices, respectively the space of skew-hermitian matrices of size v_j. The real moment map μ_I for the U_v action satisfies

$$\langle \mu_I(x), Y \rangle = \frac{1}{2} \frac{d}{dt} ||\exp(it.Y).x||^2 \Big|_{t=0}$$

for $Y \in \mathfrak{u}_v$. The pairing is defined for $Y = (Y_j)_{j \in Q_0}$ and $Z = (Z_j)_{j \in Q_0}$ by

$$\langle Y, Z \rangle = \sum_{j \in Q_0} \operatorname{tr}(Y_j Z_j). \tag{2.13}$$

As in 2.2.2, to the character χ_θ defined by (2.12) is associated the following element of the Lie algebra $\mathfrak{u}_v$

$$\theta = (i\theta_j \mathrm{Id}_{V_j})_{j\in Q_0} \in \mathfrak{u}_v. \tag{2.14}$$

Indeed for $Y = (Y_j)_{j\in Q_0}$ in the Lie algebra $\mathfrak{u}_v$, by the usual differentiation of the determinant map at identity

$$d\chi_{\theta\,\mathrm{Id}}(iY_j) = \sum_{j\in Q_0} i\theta_j \operatorname{tr}(Y_j) = \langle \theta, Y \rangle.$$

We recall here an important result from King giving a characterization of θ-stability for quiver representations.

Theorem 2.3.3 (King [Kin94] Proposition 3.1). *Let $\theta \in \mathbb{Z}^{Q_0}$ such that $\sum \theta_j v_j = 0$ and χ_θ the associated character defined by* (2.12) *.*

1. *A quiver representation $(V, \phi) \in \mathrm{Rep}\left(\tilde{Q}, v\right)$ is θ-semistable if and only if for all subrepresentation $W \subset V$*

$$\sum_{j\in Q_0} \theta_j \dim W_j \leq 0.$$

2. *A quiver representation (V, ϕ) is a θ-stable if and only if for all subrepresentation W different from 0 and (V, ϕ)*

$$\sum_{j\in Q_0} \theta_j \dim W_j < 0.$$

The symplectic point of view allows to consider real parameters $\theta \in \mathbb{R}^{Q_0}$ such that $\sum_{j\in Q_0} v_j \theta_j = 0$. They are associated to elements $\chi_\theta \in X(G_v)^{\mathbb{R}}$ with well-defined modulus:

$$\left|\chi_\theta((g_j)_{j\in Q_0})\right| = \prod_{j\in Q_0} |\det(g_j)|^{\theta_j}.$$

The set of θ-stable points in $\mathrm{Rep}\left(\tilde{Q}, v\right)$ is defined by Definition 2.2.4. The end of this section is devoted to a generalization of the second point of King's theorem for real parameters $\theta \in \mathbb{R}^{Q_0}$ such that $\sum \theta_j v_j = 0$.

Let $Y = (Y_j)_{j\in Q_0} \in \mathfrak{u}_v$, the iY_j are hermitian endomorphisms of V^j. For $\lambda \in \mathbb{R}$ denote by V^j_λ the subspace of V^j spanned by eigenvectors of iY_j with eigenvalues smaller than λ then define

$$V_\lambda := \bigoplus_{j\in Q_0} V^j_\lambda.$$

Lemma 2.3.4. *Let $x = (V, \phi)$ in $\mathrm{Rep}\left(\tilde{Q}, v\right)$ and $Y \in \mathfrak{u}_v$. The limit*

$$\lim_{t\to+\infty} \exp(itY).x$$

exists if and only if for every λ real, V_λ defines a subrepresentation of (V, ϕ)

Proof. For all $j \in {}_0$ take a basis of V^j formed by eigenvectors of iY_j and assume the eigenvalues repeated according to multiplicities are ordered

$$ {}^j_1 \quad {}^j_2 \quad \quad {}^j_{v_j}. $$

In those basis of eigenvectors, for $\in \tilde{}$ one can write the matrix of and compute the action of $\exp(itY)$

$$ (\exp(itY). \;) = \begin{pmatrix} & e^{t({}^{h(\;)}_1 \; {}^{t(\;)}_2)} \; {}_{1;2} & \dots & \\ {}_{1;1} & & & \\ e^{t({}^{h(\;)}_2 \; {}^{t(\;)}_1)} \; {}_{2;1} & & & \\ \vdots & \dots & e^{t({}^{h(\;)}_a \; {}^{t(\;)}_b)} \; {}_{a;b} & \dots \\ e^{t({}^{h(\;)}_{v_{h(\;)}} \; {}^{t(\;)}_1)} \; {}_{v_{h(\;)};1} & & \dots & \end{pmatrix} $$

the limit exists if and only if the matrix is upper triangular i.e. $(V \;) \quad V$ and V defines a subrepresentation of $(V; \;)$. □

Next result is the generalization of King's theorem relative to -stability of quiver representations for a real parameter . Its proof relies on previous lemma and the Hilbert-Mumford criterion for real one-parameter Lie groups 2.2.16.

Theorem 2.3.4. *Let $\in \mathbb{R}^{\,0}$ such that $\sum_{j \in {}_0} {}_j v_j = 0$ and the associated element in $X \; (G_v)^{\mathbb{R}}$. A quiver representation $(V; \;)$ is -stable if and only if for all subrepresentation $W \quad V$ different from 0 and $(V; \;)$*

$$ \sum_{j \in {}_0} {}_j \dim W_j < 0. $$

Proof. Let $x = (V; \;)$ in $\mathrm{Rep}\left(\tilde{}; v\right)^{\text{-s}}$ a -stable point. By Hilber-Mumford criterion (Theorem 2.2.16), for all $Y \in \mathfrak{u}_v$ such that $\lim_{t \to +\infty} \exp(itY).x$ exists then $\langle \; ; Y \rangle < 0$.

Let W be a subrepresentation of $(V; \;)$ different from 0 and $(V; \;)$. For all $j \in {}_0$ define Y_j in $\mathfrak{u}_{v_j}$ such that W_j is an eigenspace of iY_j with eigenvalue ${}_1$ and $W_j^{\perp}$ the orthogonal complement of W_j is an eigenspace of iY_j with eigenvalue ${}_2$ and ${}_2 > {}_1$. By previous lemma $\lim_{t \to +\infty} \exp(itY).x$ exists.

$$ \begin{aligned} \langle \; ; Y \rangle &= \sum_{j \in {}_0} {}_j ({}_1 \dim W_j + {}_2 (\dim V_j \quad \dim W_j)) \\ &= \sum_{j \in {}_0} ({}_1 \quad {}_2) \, {}_j \dim W_j \end{aligned} $$

because $\sum {}_j v_j = 0$. Then Hilbert-Mumford criterion implies $\langle \; ; Y \rangle < 0$, hence $\sum_{j \in {}_0} {}_j \dim W_j < 0$.

Conversely let $x = (V; \;)$ a quiver representation such that for all subrepresentation $W \quad V$ different from 0

$$ \sum_{j \in {}_0} {}_j \dim W_j < 0. $$

Let $Y = (Y_j)_{j \in {}_0} \in \mathfrak{u}_v$ different from zero. The set of eigenvalues of iY_j is ordered ${}^j_1 < \quad < {}^j_{d_j}$. The set of all eigenvalues for all $j \in {}_0$ is also ordered

$$ \left\{ {}^j_k \right\}_{\substack{j \in {}_0 \\ 1 \;\; k \;\; d_j}} = \{ {}_1; \; {}_2; \dots; \; {}_m \} $$

with $_k < \ _{k+1}$. For convenience add an element $_0 < \ _1$. If $\lim_{t \to \infty} \exp(itY).x$ exists, by previous lemma V is a subrepresentation of $(V; \ )$. Moreover

$$
\begin{aligned}
h \ ; Y i &= \sum_{j \in \ _0} \ _j \sum_{k=1}^{d_j} \ ^j_k \left(\dim V^j_{\ ^j_k} - \dim V^j_{\ ^j_{k-1}} \right) \\
&= \sum_{j \in \ _0} \ _j \sum_{k=1}^{m} \ _k \left(\dim V^j_{\ _k} - \dim V^j_{\ _{k-1}} \right) \\
&= \sum_{j \in \ _0} \ _j \sum_{k=1}^{m-1} (\ _k - \ _{k+1}) \dim V^j_{\ _k} \\
&\quad \ _m \sum_{j \in \ _0} \ _j \dim V^j_{\ _m}.
\end{aligned}
$$

The last summand vanishes as $\sum \ _j v_j = 0$,

$$
h \ ; Y i = \sum_{k=1}^{m} (\ _k - \ _{k+1}) \sum_{j \in \ _0} \ _j \dim V^j_{\ _k}
$$

As $Y \neq 0$, it has at least two distinct eigenvalues. Then $V_{\ _1}$ is a subrepresentation different from zero and V and

$$
(\ _0 - \ _1) \sum_{j \in \ _0} \ _j \dim V^j_{\ _1} < 0
$$

so that $h \ ; Y i < 0$. □

This result is useful in next section to characterize a regular locus for the hyperkähler moment map.

2.4 Nakajima's quiver varieties as hyperkähler quotients and trivialization of the hyperkähler moment map

After some reminder about the hyperkähler structure of Nakajima's quiver varieties, trivializations of the hyperkähler moment map are discussed.

2.4.1 Hyperkähler structure on the space of representations of an extended quiver

The space $\mathrm{Rep}\left(\tilde{\ }; v\right)$ is endowed with three complex structures

$$
\begin{aligned}
I.(\ ; \ \_) &= (i \ ; i \ \_) \\
J.(\ ; \ \_) &= (\ \_^y ; \ ^y) \\
K.(\ ; \ \_) &= (\ i \ \_^y ; i \ ^y)
\end{aligned}
$$

satisfying quaternionic relations

$$I^2 = J^2 = K^2 = IJK = -1 \tag{2.15}$$

and a norm

$$\left\|(\ )_{\in e}\right\|^2 = \sum_{\in e} \operatorname{tr}\left(\ \ ^{\dagger}\right).$$

For each complex structure, polarisation identity defines an hermitian pairing compatible with $\|\ldots\|$. For example the hermitian pairing compatible with the complex structure I used in previous section is

$$p_I(u, v) = \frac{1}{4}\left(\|u+v\|^2 - \|u-v\|^2 + i\|u+I.v\|^2 - i\|u-I.v\|^2\right)$$

$p_J(\ldots,\ldots)$ and $p_K(\ldots,\ldots)$ are similarly defined. One expression is particularly simple

$$p_I\left((\ )_{\in e}, (\ )_{\in e}\right) = \sum_{\in e} \operatorname{tr}(\ \ ^{\dagger}).$$

Remark 2.4.1. *Even if the hermitian metric relies on the choice of complex structure, by the polarisation identity the real part remains the same, it is the hyperkähler metric*

$$g(\ldots,\ldots) := \operatorname{Re} p_I(\ldots,\ldots) = \operatorname{Re} p_J(\ldots,\ldots) = \operatorname{Re} p_K(\ldots,\ldots).$$

Definition 2.4.2 (Real symplectic forms). *As in equation (2.4) we define a real symplectic form for each complex structure*

$$\begin{aligned} \omega_I(\ldots,\ldots) &:= g(I\ldots,\ldots) \\ \omega_J(\ldots,\ldots) &:= g(J\ldots,\ldots) \\ \omega_K(\ldots,\ldots) &:= g(K\ldots,\ldots) \end{aligned}$$

Notations 2.4.3. *I-linear means* $\mathbb{C}$*-linear with respect to the complex structure I and similarly for J-linear and K-linear.*

Proposition 2.4.4 (Permutation of complex structures). *Consider the map*

$$\begin{aligned} : \operatorname{Rep}\left(\tilde{\ }, v\right) &\to \operatorname{Rep}\left(\tilde{\ }, v\right) \\ x &\mapsto \tfrac{1}{2}(1 + I + J + K).x \end{aligned}$$

It is an isomorphism from the hermitian vector space $\operatorname{Rep}\left(\tilde{\ }, v\right)$ *with the complex structure I and hermitian pairing p_I to the hermitian vector space* $\operatorname{Rep}\left(\tilde{\ }, v\right)$ *with the complex structure J and pairing p_J.*

More generally it cyclically permutes the three complex structure I, J, K

$$\begin{aligned} (I.x) &= J.\ (x) \\ (J.x) &= K.\ (x) \\ (K.x) &= I.\ (x). \end{aligned} \tag{2.16}$$

Such a map is sometimes called an hyperkähler rotation.

Proof. Relations (2.16) follow from a computation with the quaternionic relations (2.15). To prove the compatibility with the hermitian structures it is enough to check that $||\ (x)|| = ||x||$.

$$||(1 + I + J + K).x||^2 = g((1 + I + J + K).x, (1 + I + J + K).x).$$

The expected result is obtain after cancellations from the identity $g(I.u, u) = 0$, similar relations for the other complex structures and quaternionic relations (2.15). □

In 2.3.1 an I-linear action of G_v is described. The hyperkähler rotation provides the following construction for J-linear and K-linear actions. This three actions coincide when restricted to the compact subgroup U_v.

Definition 2.4.5 (Complexification of the action). *Thanks to polar decomposition, to define a linear action of G_v compatible with the complex structure J it is enough to define the action of* $\exp(i.Y)$ *for $Y \in \mathfrak{u}_v$. To highlight the complex structure used, this action is written* $\exp(J.Y)\ldots$ *and defined by*

$$\exp(J.Y).x := \ \left(\exp(i.Y).\ \ ^{-1}(x)\right)$$

with the element $\exp(i.Y)$ *of G_v acting by the natural I-linear action previously described. Similarly*

$$\exp(K.Y).x := \ ^{-1}(\exp(i.Y).\ (x)).$$

Remark 2.4.6. *A point x is -(semi)stable with respect to the I-linear action if and only if (x) is -(semi)stable with respect to the J-linear action.*

2.4.2 Hyperkähler structure and moment maps

By Proposition 2.4.4 the various G_v-actions previously described are compatible with the hermitian metrics so that the constructions of section 2.2 apply. They provide a moment map for each complex structure.

$$\begin{aligned}
\langle\ _I(x), Y\rangle &= \frac{1}{2}\frac{d}{dt}||\exp(t.I.Y).x||^2\Big|_{t=0}\\
\langle\ _J(x), Y\rangle &= \frac{1}{2}\frac{d}{dt}||\exp(t.J.Y).x||^2\Big|_{t=0}\\
\langle\ _K(x), Y\rangle &= \frac{1}{2}\frac{d}{dt}||\exp(t.K.Y).x||^2\Big|_{t=0}.
\end{aligned}$$

The pairing is defined by (2.13).

Definition 2.4.7 (Hyperkähler moment map). *Those three real moment maps fit together in an hyperkähler moment map* $\ _H : \mathrm{Rep}\left(\tilde{\ }, v\right) \to \mathfrak{u}_v \ \ \mathfrak{u}_v \ \ \mathfrak{u}_v$ *defined by* $\ _H = (\ _I, \ _J, \ _K)$.

The moment map $\ _C$ defined in 2.3.1 by

$$\ _C\left((\ \ )_{\ \in e}\right) := \sum_{\ \in e} (\ ) \ \ \ \ \ . \tag{2.17}$$

can be expressed from the real moment maps

$$\mu_{\mathbb{C}} := \mu_J + i\mu_K ;$$

it is a polynomial map with respect to the complex structure I.

Remark 2.4.8. *By cyclic permutation of the complex structure,* $\mu_K + i\mu_I$ *is polynomial with respect to the complex structure* J *and* $\mu_I + i\mu_J$ *is polynomial with respect to the complex structure* K.

Take $(\lambda_{J,j})_{j\in Q_0}$ and $(\lambda_{K,j})_{j\in Q_0}$ in $\mathbb{R}^{Q_0}$ such that $\sum_j v_j\lambda_{J,j} = \sum_j v_j\lambda_{K,j} = 0$. Associate to each of them an element in the center of the Lie algebra $\mathfrak{u}_v$

$$\begin{aligned} \zeta_J &:= \left(i\lambda_{J,j}\,\mathrm{Id}_{V_j}\right)_{j\in Q_0} \\ \zeta_K &:= \left(i\lambda_{K,j}\,\mathrm{Id}_{V_j}\right)_{j\in Q_0} . \end{aligned}$$

Then $\zeta_J + i\zeta_K$ defines an element in the center of $\mathfrak{g}_v = \mathfrak{u}_v \oplus i\mathfrak{u}_v$. Hence $\mu_J^{-1}(\zeta_J) \cap \mu_K^{-1}(\zeta_K) = \mu_{\mathbb{C}}^{-1}(\zeta_J + i\zeta_K)$ is an affine variety embedded in the vector space $\mathrm{Rep}\left(\tilde{Q}, v\right)$ endowed with the complex structure I and stable under the G_v-action. Section 2.2 does not apply directly to this situation as $\mu_{\mathbb{C}}^{-1}(\zeta_J + i\zeta_K)$ might be singular. However it applies to the action of G_v on the ambiant space $\mathrm{Rep}\left(\tilde{Q}, v\right)$. For $\lambda_I \in \mathbb{R}^{Q_0}$ such that $\sum_{j\in Q_0} v_j\lambda_{I,j} = 0$ consider the associated element $\zeta_I \in X^*(G_v)^{\mathbb{R}}$.

Definition 2.4.9 (Hyperkähler regular locus). *For* $w \in \mathbb{N}^{Q_0}$ *a dimension vector*

$$H_w := \left\{ (\lambda_I, \lambda_J, \lambda_K) \in \left(\mathbb{R}^{Q_0}\right)^3 \middle| \sum_j w_j\lambda_{I,j} = \sum_j w_j\lambda_{J,j} = \sum_j w_j\lambda_{K,j} = 0 \right\} .$$

The regular locus is

$$H_v^{\mathrm{reg}} = H_v \setminus \bigcup_{w<v} H_w \tag{2.18}$$

the union is over dimension vector $w \neq v$ *such that* $0 \leq w_i \leq v_i$.

Remark 2.4.10. *This regular locus is empty unless the dimension vector* v *is indivisible, then* H_v^{reg} *is the complementary of a finite union of codimension* 3 *real vector space.*

Thanks to Kempf-Ness theory, Nakajima's quiver varieties can be constructed as hyperkähler quotients. The underlying manifold of the variety $\mathcal{M}_{v}^{\lambda_I}(\zeta_J + i\zeta_K)$ (see definition 2.3.2) is :

$$\mathfrak{m}_v(\lambda_I, \lambda_J, \lambda_K) = \mu_{\mathbb{H}}^{-1}(\lambda_I, \lambda_J, \lambda_K)/U_v$$

2.4.3 Trivialization of the hyperkähler moment map

We study the family of Nakajima's quiver varieties when the parameters $(\lambda_I, \lambda_J, \lambda_K)$ are varying. Nakajima proved by consecutive uses of different complex structures that for λ and λ' in H_v^{reg} the manifolds $\mathfrak{m}_v(\lambda_I, \lambda_J, \lambda_K)$ and $\mathfrak{m}_v(\lambda'_I, \lambda'_J, \lambda'_K)$ are diffeomorphic [Nak94, Corollary 4.2]. We use this idea of consecutive uses of different complex structures to prove that those manifolds fit in a locally trivial family over the regular locus H_v^{reg}. First let us highlight relevant facts about the regular locus.

Lemma 2.4.11. *Let $(\zeta_I, \zeta_J, \zeta_K) \in H_v^{\mathrm{reg}}$ and $x \in \mu_J^{-1}(\zeta_J) \cap \mu_K^{-1}(\zeta_K)$. Then x is ζ_I-stable if and only if it is ζ_I-semistable.*

Proof. If $x_0 \in \mu_H^{-1}(\zeta_I, \zeta_J, \zeta_K)$ its stabilizer in G_v is trivial. Indeed Maffei proved that the differential of the moment map at x_0 is surjective [Maf02, Lemma 48], then Proposition 2.2.10 implies the triviality of the stabilizer of x_0.

Let $x \in \mu_J^{-1}(\zeta_J) \cap \mu_K^{-1}(\zeta_K)$ a ζ_I-semistable point. Then $\overline{G_v.x} \cap \mu_I^{-1}(\zeta_I)$ is not empty. As $\mu_J^{-1}(\zeta_J) \cap \mu_K^{-1}(\zeta_K) = \mu_C^{-1}(\zeta_J + i\zeta_K)$ is G_v stable, the closure of the orbit $\overline{G_v.x}$ meets $\mu_H^{-1}(\zeta_I, \zeta_J, \zeta_K)$ at a point x_0. This point necessarily has a trivial stabilizer, hence $x_0 \in G_v.x$ and x is ζ_I-stable. □

Let $(\zeta_I, \zeta_J, \zeta_K) \in H_v^{\mathrm{reg}}$ and consider first the complex structure I. By previous lemma and King's characterisation of stability (Theorem 2.3.4), for ζ_I' in an open neighborhood of ζ_I, stability with respect to ζ_I' is the same as stability with respect to ζ_I.

Now consider the complex structure J. Thanks to Remark 2.4.6 on the affine variety $\mu_K^{-1}(\zeta_K) \cap \mu_I^{-1}(\zeta_I)$ all ζ_J-semistable points are ζ_J-stable. Moreover for ζ_J' in an open neighborhood of ζ_J, stability with respect to ζ_J' is the same as stability with respect to ζ_J. Similarly for the complex structure K.

Assume that the dimension vector v is a root of the quiver so that the moment map is surjective, see [Cra06, Theorem 2]. Consider the diagram

$$\begin{array}{ccc} \mu_H^{-1}(H_v^{\mathrm{reg}}) & \longrightarrow & \mathrm{Rep}\left(\tilde{Q}, v\right) \\ \downarrow & & \downarrow \mu_H \\ H_v^{\mathrm{reg}} & \longrightarrow & \mathfrak{u}_v \quad \mathfrak{u}_v \quad \mathfrak{u}_v \end{array}.$$

.

Theorem 2.4.12 (Local triviality of the hyperkähler moment map)**.** *Over the regular locus H_v^{reg}, the hyperkähler moment map μ_H is a locally trivial fibration compatible with the U_v-action:*

Any $(\zeta_I, \zeta_J, \zeta_K) \in H_v^{\mathrm{reg}}$ admits an open neighborhood V, and a diffeomorphism f such that the following diagram commutes

$$\begin{array}{ccc} V \times \mu_H^{-1}(\zeta_I, \zeta_J, \zeta_K) & \xrightarrow[f]{} & \mu_H^{-1}(V) \\ & \searrow & \downarrow \mu_H \\ & & V \end{array}$$

Moreover f is compatible with the U_v-action so that the diagram goes down to quotient

$$\begin{array}{ccc} V \times \mathfrak{m}_v(\zeta_I, \zeta_J, \zeta_K) & \longrightarrow & \mu_H^{-1}(V)/U_v \\ & \searrow & \downarrow \\ & & V \end{array}$$

Proof. The method is similar to the proof of Theorem 2.2.28 applied consecutively to the three complex structures. The idea of using different complex structures comes from [Nak94] and [Kro89]. Take $(\zeta_I, \zeta_J, \zeta_K) \in H_v^{\mathrm{reg}}$ and a connected open neighborhood $U_I \times U_J \times U_K$ such that for $\xi'_I \in U_I$, any $x \in \mu_J^{-1}(U_J) \cap \mu_K^{-1}(U_K)$ is ξ'_I-semistable if and only if it is ξ_I-stable. Similarly for U_J and U_K. For any x with $\mu_{\mathbb{H}}(x) = (\xi'_I, \xi'_J, \xi'_K) \in U_I \times U_J \times U_K$, by Proposition 2.2.25 applied to the I-linear action of G_v on $\operatorname{Rep}\left(\tilde{Q}, v\right)$, there exists a unique $Y_I(\xi_I, x) \in \mathfrak{u}_v$ such that

$$\exp\left(I.Y_I(\xi_I, x)\right).x \in \mu_{\mathbb{H}}^{-1}(\xi_I, \xi'_J, \xi'_K).$$

Then by exchanging the three complex structures with hyperkähler rotations, there exists unique $Y_J(\xi_J, x)$ and $Y_K(\xi_K, x)$ such that

$$\exp\left(J.Y_J(\xi_J, x)\right)\exp\left(I.Y_I(\xi_I, x)\right).x \in \mu_{\mathbb{H}}^{-1}(\xi_I, \xi_J, \xi'_K)$$

and

$$\exp\left(K.Y_K(\xi_K, x)\right)\exp\left(J.Y_J(\xi_J, x)\right)\exp\left(I.Y_I(\xi_I, x)\right).x \in \mu_{\mathbb{H}}^{-1}(\xi_I, \xi_J, \xi_K).$$

This defines the map f^{-1}

$$f^{-1}(x) := \left((\xi'_I, \xi'_J, \xi'_K), \exp\left(K.Y_K(\xi_J, x)\right)\exp\left(J.Y_J(\xi_J, x)\right)\exp\left(I.Y_I(\xi_I, x)\right).x\right).$$

Lemma 2.2.27 implies the smoothness of f^{-1}. This map induces a diffeomorphism, indeed exchanging ξ and ξ' in previous construction produces the expected inverse

$$f\left(x, (\xi'_I, \xi'_J, \xi'_K)\right) := \exp\left(I.Y_I(\xi'_I, x)\right)\exp\left(J.Y_J(\xi'_J, x)\right)\exp\left(K.Y_K(\xi'_K, x)\right).x$$

It follows from equation (2.10) that the maps are inverse of each others. The exchange in the order of appearance of the complex structures I, J and K in the definition of f and f^{-1} are necessary as the exponentials do not necessarily commute. The U_v-equivariance follows from equation (2.9). □

Similarly one can consider the complex moment map $\mu_{\mathbb{C}} = \mu_J + i\mu_K$ instead of $\mu_{\mathbb{H}}$. The complex regular locus is $C_v^{\mathrm{reg}} := C_v \setminus \bigcup_{w<v} C_w$ with

$$C_w = \left\{ \xi \in \mathbb{C}^{Q_0} \,\middle|\, \sum_{j \in Q_0} w_j \xi_j = 0 \right\}$$

Theorem 2.4.13. *The complex moment map is a locally trivial fibration over C_v^{reg}. Any $\xi \in C_v^{\mathrm{reg}}$ admits an open neighborhood V, and a diffeomorphism f such that the following diagram commutes*

$$\begin{array}{ccc} V \times \mu_{\mathbb{C}}^{-1}(\xi) & \xrightarrow{\ f\ } & \mu_{\mathbb{C}}^{-1}(V) \\ & \searrow & \downarrow \mu_{\mathbb{C}} \\ & & V \end{array}$$

Proof. The proof is similar to the hyperkähler situation. □

Denote $\pi : \mu_{\mathbb{H}}^{-1}(H_v^{\mathrm{reg}})/U_v \to H_v^{\mathrm{reg}}$ the map obtained taking the quotient of $\mu_{\mathbb{H}}$. Consider the cohomology sheaves $\mathcal{H}^i \pi_* \overline{\mathbb{Q}}_l$ of the derived pushforward of the constant sheaf and the cohomology sheaves $\mathcal{H}^i \pi_! \overline{\mathbb{Q}}_l$ of the derived compactly supported pushforward of the constant sheaf.

Corollary 2.4.14. *The sheaves $H^i \quad \overline{\mathbb{Q}}_l$ and $H^i \;{}_! \overline{\mathbb{Q}}_l$ are constant sheaves over H_v^{reg}.*

Proof. By Theorem 2.4.12 those sheaves are locally constant. H_v^{reg} is a complementary of a finite union of codimension 3 real vector spaces, hence it is simply connected so that the locally constant sheaves are constant. □

Nakajima explained to us that this corollary can also be obtained by generalizing Slodowy's construction [Slo80] to quiver varieties.

Finally we extend the trivialization of the hyperkähler moment map over lines constructed by Crawley-Boevey and Van den Bergh [CV04] using twistor spaces as told to us by Nakajima.

Denote by $\mathbb{H}$, respectively $\mathbb{H}_0$, the set of quaternions, respectively the set of purely imaginary quaternions and $\mathbb{H}_0 = \mathbb{H}_0 \setminus \{0\}$. The space $\mathfrak{u}_v{}^3$ is identified with $\mathbb{H}_0 \quad_{\mathbb{R}} \mathfrak{u}_v$. Then the hyperkähler moment map reads

$$ {}_{\mathbb{H}} = I \quad {}_I + J \quad {}_J + K \quad {}_K. $$

Once an orthonormal basis of $\mathbb{R}^3$ is fixed, the triple of complex structures I, J and K is fixed and we write ${}_{\mathbb{R}} = \;{}_I$, ${}_{\mathbb{C}} = \;{}_J + i \;{}_K$. The hyperkähler moment map is assumed to be surjective and the dimension vector indivisible. Then H_v^{reg} is the open subset of generic parameters in $\mathbb{H}_0 \quad_{\mathbb{R}} Z(\mathfrak{u}_v)$. For $\quad \in H_v^{\text{reg}}$ a generic parameter and S a contractible subset of $\mathbb{H}_0$, Crawley-Boevey and Van den Bergh constructed a trivialization of the hyperkähler moment map over $S \quad$, see [CV04] proof of Lemma 2.3.3 (in the statement of this lemma S is chosen to be a complex line). The assumption contractible is relaxed in next theorem. It relies on the theory of twistor spaces developped by Penrose [Pen76], Atiyah-Hitchin-Singer [AHS78] and Salamon [Sal82][Sal86]. The main point is the compatibility between hyperkähler quotients and twistor spaces from Hitchin-Karlhede-Lindström-Roček [Hit+87] p.560, see also Hitchin [Hit92]. The following Theorem as well as its proof was told to us by Nakajima.

Theorem 2.4.15. *For $\quad$ generic in $\mathbb{H}_0 \quad_{\mathbb{R}} Z(\mathfrak{u}_v)$ define*

$$ \mathbb{H}_0 \; := \{ h \quad | h \in \mathbb{H}_0 \}. $$

There exists a di eomorphism f such that the following diagram commutes

$$ \begin{array}{ccc} {}_{\mathbb{H}}^{-1}(\mathbb{H}_0 \;)/U_v & \xrightarrow{f} & {}_{\mathbb{H}}^{-1}(\quad)/U_v \quad \mathbb{H}_0 \\ & \searrow^{{}_{\mathbb{H}}} & \downarrow \\ & & \mathbb{H}_0 \end{array} $$

the vertical arrow is the projection to $\mathbb{H}_0$.

Proof. Consider the quaternionic vector space $\mathrm{Rep}\left(\tilde{\quad}, v\right)$ and the projection

$$ \mathrm{Rep}\left(\tilde{\quad}, v\right) \quad S^2 \to S^2. $$

With S^2 the 2-sphere of imaginary quaternions with unit norm

$$ S^2 = \left\{ aI + bJ + cK \,\middle|\, a^2 + b^2 + c^2 = 1 \right\}. $$

S^2 is given the usual complex structure of the projective line. The twistor space associated to Rep $\left(\tilde{\ };v\right)$ is the manifold Rep $\left(\tilde{\ };v\right)$ S^2 endowed with a complex structure such that the fiber over I_u 2 S^2 is Rep $\left(\tilde{\ };v\right)$ seen as a vector space with complex structure I_u.

As detailed in [CV04], the group of quaternion of unit norm, identified with SU(2), acts on H^0 $Z(\mathrm{u}_v)$ by

$$h.(h^0 \quad) = hh^0\overline{h} \quad :$$

with $\overline{aI + bJ + cK + d} = \quad aI \quad bJ \quad cK + d$. Let a generic parameter, up to the choice of orthonormal basis of R^3 we can assume $= I \qquad _I$. The SU(2) orbit of thus identifies with S^2 as

$$\mathrm{SU}(2). \ = \left\{ I_u \quad \middle| I_u \ 2 \ S^2 \right\}: \tag{2.19}$$

The twistor space of the hyperkähler manifold $_{\mathrm{H}}{}^1(\ )/U_v$ is a complex manifold T with an holomorphic map p to S^2

$$T \xrightarrow{p} S^2:$$

The underlying di erential manifold of the twistor space is just a product and p the projection to the second factor

$$_{\mathrm{H}}{}^1(\ )/U_v \quad S^2 \longrightarrow S^2:$$

The twistor spaces construction is compatible with hyperkähler quotients as explained in [Hit+87] p.560. Thus the fiber of p over I_u is $_{\mathrm{H}}{}^1(\ )/U_v$ endowed with the complex structure inherited from the complex structure I_u on Rep $\left(\bar{\ };v\right)$. Namely if $I_u \qquad = (\ ^0_I;\ ^0_J;\ ^0_K)$ then the fiber of the twistor space over I_u is the complex manifold

$$p^{\ 1}(I_u) = \ _{\mathrm{C}}{}^1(\ ^0_J + i\ ^0_K) \setminus \ _{\mathrm{R}}{}^1(\ ^0_I)/U_v$$

Thus fibers of p are exactly fibers of $_{\mathrm{H}}$ and the twistor space provides trivialization of the hyperkähler moment map over the orbit SU(2). :

$$\begin{array}{ccccc} _{\mathrm{H}}{}^1(\mathrm{SU}(2). \ )/U_v & \to T & \longrightarrow & _{\mathrm{H}}{}^1(\ )/U_v \quad S^2 \\ \downarrow_{\mathrm{H}} & & \searrow^{p} & \swarrow \\ \mathrm{SU}(2). & \longrightarrow & S^2 & \end{array}$$

is defined thanks to (2.19), the map is the identity on the fibers and forgets the complex structure. This diagram traduces the equivalence between, on the right, varying complex structure on a fixed fiber $_{\mathrm{H}}{}^1(\ )/U_v$ and on the left varying the fiber for a fixed complex structure I.

The construction is similar to Crawley-Boevey and Van den Bergh's construction except that the twistor space formalism allows to obtain a trivialization over the non-contractible space SU(2). .

As in [CV04], the trivialization can be extended thanks to the $R_{>0}$ action. Note that for t a positive real number $\ _H(tx) = t^2\ _H(x)$. Then identifying $S^2 \quad R_{>0}$ with H_0 we obtain the trivialization

$$\begin{array}{ccc} {}_H{}^1(H_0; \ )/U_v & \longrightarrow & {}_H{}^1(\ )/U_v \quad H_0 \\ \downarrow & & \downarrow \\ H_0; & \longrightarrow & H_0 \end{array}$$

The SU(2)-action on the base of this trivialization traduces the variation of complex structure on the hyperkähler manifold ${}_H{}^1(\ )/U_v$ whereas the $R_{>0}$ action traduces the rescaling of the metric. □

Chapter 3

Geometric and combinatoric background

This chapter recalls the geometric and combinatoric tools necessary to study character varieties and their cohomology. The base field $\mathbb{K}$ is either $\mathbb{C}$ or an algebraic closure $\overline{\mathbb{F}}_q$ of a finite field $\mathbb{F}_q$. Section 3.1 introduces the notations for perverse sheaves and intersection cohomology.

In Section 3.2 some properties of symmetric functions are recalled. They are used to define Hausel-Letellier-Villegas kernel $\mathbb{H}_n^{HLV}$. This kernel is fundamental in the description of cohomology of character varieties. Moreover symmetric functions formalism is useful to study representation of Weyl groups. They are also necessary to define the algebra spanned by Kostka polynomials mentioned in the introduction 1.3.2. In order to relate this algebra with cohomology of quiver varieties, an important result of Garsia-Haiman [GH96] is recalled in 3.2.4.

Section 3.3 contains various notations for conjugacy classes and their Jordan type, they will be used throughout the thesis.

In 3.4, Springer theory [Spr76; BM83], Lusztig parabolic induction [Lus84; Lus85; Lus86] and the associated resolutions of closure of conjugacy classes are recalled.

Construction and basic properties of character varieties are given in 3.5. Moreover construction of resolutions from 3.4 are extended to character varieties, following Letellier [Let11; Let13].

Finally some conjecture and theorems relating the cohomology of character varieties with the kernel $\mathbb{H}_n^{HLV}$ are stated in 3.6.

3.1 Perverse sheaves and intersection cohomology

3.1.1 Perverse sheaves

In this section classical results about perverse sheaves and intersection cohomology are stated. The constructions come from Beilinson, Bernstein, Deligne and Gabber [Bei+18].

$\mathbb{K}$ is either $\mathbb{C}$ or an algebraic closure $\overline{\mathbb{F}}_q$ of a finite field $\mathbb{F}_q$ with q elements. X is an algebraic variety over $\mathbb{K}$. Let l be a prime different from the characteristic and denote by $_X$ the constant l-adic sheaf on X with coefficients in $\overline{\mathbb{Q}}_l$.

Notations 3.1.1. *The category of -constructible sheaves on X is denoted by $D_c^b(X)$. Its objects are represented by complexes of sheaves K such that the cohomology* *sheaves H^iK are -constructible sheaves on X and finitely many of them are non-zero. For Y a variety over* K *and $f : X \to Y$ a morphism one has the usual functors*

$$
\begin{aligned}
f \ ; f^! &: D^b_c(Y) \to D^b_c(X) \\
f \ ; f_! &: D^b_c(X) \to D^b_c(Y)
\end{aligned}
$$

For m an integer $K[m]$ is the shifted complex such that $H^iK[m] = H^{i+m}K$. For x a point in X, the stalk at x of the i-th cohomology sheave of the complex K is denoted by H^i_xK . The structural morphism of X is $p : X \to$ Spec K*. The k-th cohomology space of X with coe cients in is*

$$H^k(X; \ ) := H^k p \ \ _X$$

and the k-th compactly supported intersection cohomology space of X is

$$H^k_c(X; \ ) := H^k p_! \ \ _X.$$

The Verdier dual operator is denoted by $D_X : D^b_c(X) \to D^b_c(X)$.

Theorem 3.1.2 (Base change). *Consider $K \in D^b_c(Y')$ and a cartesian square*

$$
\begin{array}{ccc}
X' & \xrightarrow{g} & Y' \\
{\scriptstyle b}\downarrow & & \downarrow{\scriptstyle a} \\
X & \xrightarrow[f]{} & Y
\end{array}
\tag{3.1}
$$

then the natural morphism $f \ a_!K \to b_!g \ K$ is an isomorphism.

Remark 3.1.3. *Let $\ , \to X$ a geometric point of X and its image by f. Consider the fibers of the vertical arrows:*

$$X \ := X' \ \ _X \ ; \quad Y \ := Y' \ \ _Y$$

In the following diagram h is an isomorphism

$$
\begin{array}{ccc}
X & \xrightarrow{h} & Y \\
\downarrow & & \downarrow \\
 & \longrightarrow &
\end{array}
$$

The base change isomorphism for this diagram identifies with the stalk at of the base change isomorphism of Diagram (3.1):

$$f \ a_!K \ \to b_!g \ K$$

which is nothing but the morphism obtained by functoriality of the compactly supported cohomology

$$H_c(Y \ ; K) \xrightarrow{h} H_c(X \ ; h \ K).$$

Definition 3.1.4. *Let W a finite group acting from the left on a variety X. For all $w \in W$ there is a morphism $w : X \to X$. An action of W on an element $K \in D^b_c(X)$ is the data of morphisms $_w : w K = K$ satisfying the following relation for all $w, w' \in W$*

$$_{w'w} = {}_w w (_{w'}) \tag{3.2}$$

and such that $_1 = \mathrm{Id}$. Then we say that the complex K is W-equivariant.

Remark 3.1.5. *When the action of W on X is trivial, an action of W on $K \in D^b_c(X)$ is just a group morphism from the opposite group W^{op} to the group of automorphism $\mathrm{Aut}(K)$.*

Proposition 3.1.6. *Let $f : X \to Y$ a W-equivariant morphism between varieties with left W-action. Let W act on K by morphisms $_w : w K = K$. Then W acts on $f_!K$.*

Proof. The action is defined for $w \in W$ the following way. Base change formula provides an isomorphism $w f_!K \to f_!w K$. Compose this isomorphism with $f_! {}_w$ to obtain an isomorphism $\tilde{}_w : w f_!K \to f_!K$. The compatibility (3.2) follows from functoriality of base change. □

Definition 3.1.7 (Perverse sheave). *A perverse sheave is an object K in $D^b_c(X)$ such that for all $i \in \mathbb{N}$*

$$\begin{aligned} \dim\left(\mathrm{Supp}\, H^i K\right) & \quad i \\ \dim\left(\mathrm{Supp}\, H^i D_X K\right) & \quad i. \end{aligned}$$

The category of perverse sheaves on X is denoted by $M(X)$, it is an abelian category.

3.1.2 Intersection cohomology

Definition 3.1.8 (Intersection complex). *Let $Y \hookrightarrow X$ a closed embedding and $j : U \hookrightarrow Y$ an open embedding. Assume U is smooth, irreducible and $\overline{U} = Y$. Let be a local system on U. $\underline{IC}_{Y;}$ is the unique perverse sheave K on Y characterized by*

$$H^i K = 0 \text{ if } i < \dim Y \tag{3.3}$$

$$H^{\dim Y} K_{|U} = \tag{3.4}$$

$$\dim\left(\mathrm{Supp}\, H^i K\right) < i \text{ if } i > \dim Y \tag{3.5}$$

$$\dim\left(\mathrm{Supp}\, H^i D_Y K\right) < i \text{ if } i > \dim Y. \tag{3.6}$$

We also denote $\underline{IC}_{Y;}$ its extension $j \underline{IC}_{Y;}$. The intersection complex defined by Goresky-MacPherson [GM83] and Deligne is obtained by shifting this perverse sheaf

$$IC_{Y;} := \underline{IC}_{Y;}[\ \dim Y].$$

Remark 3.1.9 (Continuation principle). *The intersection complex of can also be defined as the intermediate extension $\underline{IC}_{Y;} = j_{!}$. Moreover the functor $j_{!}$ is fully faithful (see Kiehl-Weissauer [KW01, III - Corollary 5.11]).*

Remark 3.1.10. *The intersection complex does not depend on the choice of smooth open subset in Y. When the local system is not specified, it is chosen to be the constant sheaf $_X$ and $\underline{IC}_X := \underline{IC}_{X;\ _U}$.*

Definition 3.1.11 (Intersection cohomology). *Let $p : X \to \operatorname{Spec} \mathbb{K}$ the structural morphism and k an integer. The k-th intersection cohomology space of X is*

$$IH^k(X;\kappa) := H^k p_* IC_X$$

and the k-th compactly supported intersection cohomology space of X is

$$IH^k_c(X;\kappa) := H^k p_! IC_X.$$

For $\mathbb{K} = \mathbb{C}$, Saito [Sai86] proved that the intersection cohomology spaces carry a mixed-Hodge structure. Thus there exists on $IH^k_c(X;\mathbb{Q})$ an increasing finite filtration called the weight filtration and denoted by W^k such that the complexified quotient $\mathbb{C} \otimes_{\mathbb{Q}} W^k_r/W^k_{r-1}$ carries a pure Hodge structure of weight r. The Hodge numbers of this structure are denoted $h_c^{i,j;k}(X)$ and satisfy $i + j = r$.

Definition 3.1.12. *The mixed-Hodge structure is encoded in the mixed-Hodge polynomial:*

$$IH_c(X;x,y,v) := \sum_{i,j,k} h_c^{i,j;k}(X) x^i y^j v^k. \tag{3.7}$$

This polynomial has two important specialisations, the Poincaré polynomial

$$P_c(X;t) := IH_c(X;1,1,v) = \sum_k \dim IH^k_c(X;\kappa) v^k \tag{3.8}$$

and the E-polynomial

$$E_c(X;x,y) := IH_c(X;x,y,-1). \tag{3.9}$$

Remark 3.1.13. *For X a smooth variety the intersection cohomology is the usual l-adic cohomology*

$$\begin{aligned} IH^i(X;\kappa) &= H^i(X;\kappa) \\ IH^i_c(X;\kappa) &= H^i_c(X;\kappa). \end{aligned}$$

3.2 Symmetric functions

3.2.1 Lambda ring and symmetric functions

In this section the combinatorics involved in the cohomology of character varieties is recalled.

Notations 3.2.1. *A partition of an integer $n \in \mathbb{N}$ is a decreasing sequence of non-negative integers*

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{l(\lambda)}) \text{ with } |\lambda| := \lambda_1 + \lambda_2 + \cdots + \lambda_{l(\lambda)} = n.$$

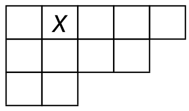
The length of λ is the number $l(\lambda)$ of non-zero terms. The set of partitions of n is denoted by $\mathcal{P}_n$ and

$$\mathcal{P}^* := \bigcup_{n \in \mathbb{N}_{>0}} \mathcal{P}_n \text{ and } \mathcal{P} = \bigcup_{n \in \mathbb{N}} \mathcal{P}_n$$

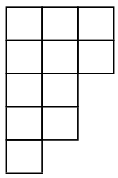
with P_0 a set with a unique element 0 *called the empty partition. The Young diagram of a partition is the set*

$$f(i;j)j1 \quad i \quad l(\ ) \text{ and } 1 \quad j \quad {}_i g:$$

A partition is often identified with its Young diagram so that $(i;j)$ *2 means that* $(i;j)$ *belongs to the Young diagram of . The transpose of a Young diagram is obtained by permuting i and j. The transpose* ^{0} *of a partition is the partition with Young diagram the transpose of the Young diagram of . The Young diagram of the partition* $= (5;4;2)$ *has the following form*



with x the box $(i;j) = (1;2)$*. The arm length of x is number of box right of x, here* $a(x) = 3$*. The leg length is the number of box under x, here* $l(x) = 2$*. The transpose of* $= (5;4;2)$ *is the partition* $^0 = (3;3;2;2;1)$ *with Young diagram*



For $= (\ _1; : : : ;\ _l)$ *a partition then*

$$P \ := P_{\ 1} \qquad P_{\ l}:$$

Definition 3.2.2 (Dominance orderin). *The dominance ordering on P is defined by if and only if j j = j j and*

$$\sum_{i=1}^{k} \ _i \quad \sum_{i=1}^{k} \ _i \text{ for all } k \ 2 \ \mathrm{N}$$

Let $X = (x_1; x_2; : : :)$ be an infinite set of variable and $\mathrm{Sym}[X]$ be the ring of symmetric functions in $(x_1; x_2; : : :)$ over Q. This ring is graded by the degree and $\mathrm{Sym}_n[X] \quad \mathrm{Sym}[X]$ are the symmetric functions homogeneous of degree n. We use the usual notations from Macdonald's book [Mac15]. A basis of $\mathrm{Sym}[X]$ is given by monomial symmetric functions $(m\ )_{\ 2P}$. If is a partition of length l, m is obtained by summing all distinct monomials of the form $x_{i_1}^{\ 1} x_{i_2}^{\ 2} : : : x_{i_l}^{\ l}$ with distinct indices i_k. Elementary symmetric functions $(e_n)_{n2\mathrm{N}}$, complete symmetric functions $(h_n)_{n2\mathrm{N}}$ and power sums $(p_n)_{n2\mathrm{N}}$ are defined for $n \ 2 \ \mathrm{N}_{>0}$ by

$$\begin{aligned} e_n[X] &:= \sum_{1 \quad i_1 < \quad < i_n} x_{i_1} x_{i_2} \quad x_{i_n} \\ h_n[X] &:= \sum_{1 \quad i_1 \quad \quad i_n} x_{i_1} x_{i_2} \quad x_{i_n} \\ p_n[X] &:= x_1^n + x_2^n + : : : \end{aligned}$$

and $e_0 := h_0 := p_0 := 1$. Each one of this family freely generates the ring $\mathrm{Sym}[X]$. We introduce the corresponding basis labelled by partitions $\lambda \in \mathcal{P}$

$$
\begin{aligned}
e_\lambda &:= e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_l} \\
h_\lambda &:= h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_l} \\
p_\lambda &:= p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_l}
\end{aligned}
$$

When talking about symmetric functions, if we do not need to specify the set of variable we write just F instead of $F[X]$, we can think of F as an element in the $\mathbb{Q}$-algebra freely generated by $(e_n)_{n \in \mathbb{N}}$. A convenient formalism to study symmetric functions is provided by lambda rings. The following reminder on this topic comes from Mellit [Mel17a; Mel18].

Definition 3.2.3 (Lambda ring). *A lambda ring over* $\mathbb{Q}$ *is a commutative and unitary* $\mathbb{Q}$*-algebra* Λ *endowed, for* $n \in \mathbb{N}_{>0}$*, with ring morphisms*

$$
\begin{aligned}
p_n \ : \ \Lambda &\to \Lambda \\
a &\mapsto p_n[a]
\end{aligned}
$$

such that $p_n \circ p_m = p_{nm}$ *for* $n, m \in \mathbb{N}_{>0}$*. The* p_n *are called the Adams operators. We use square brackets instead of parenthesis for evaluation of Adams operators.*

Example 3.2.4 (Symmetric functions over $\mathbb{Q}$). *The ring of symmetric functions* $\mathrm{Sym}[X]$ *is freely generated, as a* $\mathbb{Q}$*-algebra by the power sums* $p_n[X]$*. Adams operators on* $\mathrm{Sym}[X]$ *can be defined by their values on the power sums*

$$
p_m[p_n[X]] := p_{mn}[X] \quad \text{for } m \in \mathbb{N}_{>0} \text{ and } n \in \mathbb{N}.
$$

This gives $\mathrm{Sym}[X]$ *a lambda ring structure.*

Remark 3.2.5. *Note that for all* $n \in \mathbb{N}_{>0}$*,* $p_n[X] = p_n[p_1[X]]$ *then let* $X := p_1[X]$*. All the power sums* $p_n[X]$ *with* $n > 0$ *are obtained applying Adams operator to* X*. The notations used for the power sums agree with the one resulting of applications of Adams operators to* X*.*

Example 3.2.6. $\mathbb{Q}(q,t)$ *is endowed with the Adams operator defined by* $p_n[f(q,t)] = f(q^n, t^n)$ *for any* $f(q,t) \in \mathbb{Q}(q,t)$*.*

Example 3.2.7 (Symmetric functions over $\mathbb{Q}(q,t)$). *The ring of symmetric functions over* $\mathbb{Q}(q,t)$ *is still denoted* $\mathrm{Sym}[X]$*. It is a lambda ring, the Adams operators act by* $p_n[f(q,t)F[X]] = f(q^n, t^n) p_n[F[X]]$*.*

Example 3.2.8 (Multivariate symmetric functions).

$$
\mathrm{Sym}[X_1, \dots, X_k] := \mathrm{Sym}[X_1] \otimes \mathrm{Sym}[X_2] \otimes \cdots \otimes \mathrm{Sym}[X_k]
$$

is the ring of functions in k *sequences of variables* $X_i = (x_{i,1}, x_{i,2}, \dots)$ *symmetric in each sequence. The Adams operators are defined by*

$$
p_n[F_1[X_1] \otimes \cdots \otimes F_k[X_k]] = p_n[F_1[X_1]] \otimes \cdots \otimes p_n[F_k[X_k]]
$$

Notations 3.2.9 (Conventions for variable in ring of symmetric functions). *When considering symmetric functions, uppercase characters such as $X; Y; Z; X_i$ will be infinite set of variable so that $(p_n[X])_{n\in\mathbb{N}}$ are algebraically free. Lowercase characters such as $q; r; s; t; u; v; w; z$ will be single variables and Adams operator act on them as $p_n[u] = u^n$.*

Definition 3.2.10 (Plethystic action). *Let $\varepsilon \in \Lambda$ be an element in a lambda ring and $F[X] \in \mathrm{Sym}[X]$. As the power sums freely generate the ring of symmetric functions there exists a unique polynomial f such that $F[X] = f(p_1[X]; p_2[X]; \ldots)$. To compute the plethystic action of F on ε we evaluate the polynomial f on the Adams operator $F[\varepsilon] := f(p_1; p_2; \ldots)[\varepsilon]$. It defines a ring morphism from $\mathrm{Sym}[X]$ to Λ. It is also called plethystic substitution of X by ε.*

Remark 3.2.11. *Once again, denoting by X the element $p_1[X] \in \mathrm{Sym}[X]$, notations are compatible. $p_n[X]$ is both the evaluation of the Adams operator p_n on the element X and the plethystic action of the symmetric function p_n on the element X. Plethystic action can be thought of as evaluation of a symmetric function F on an element of a lambda ring.*

For Λ a lambda ring, $\Lambda[[s]]$ is the ring of power series in s with coefficients in Λ. It is endowed with a lambda ring structure such that $p_n[s] = s^n$. Elements of $\mathrm{Sym}[X][[s]]$ such that the symmetric function in front of s^n is of degree n can be thought of as elements in $\mathrm{Sym}[[X]]$ the completion of $\mathrm{Sym}[X]$ with respect to the ideal $\mathrm{Sym}[X]^+$ of symmetric functions without constant terms.

Definition 3.2.12 (Plethystic exponential and logarithm). *Let Λ be a lambda ring and $s\Lambda[[s]]$ the formal series in s with coefficient in Λ without terms in s^0. For $G \in s\Lambda[[s]]$ the plethystic exponential is defined by*

$$\mathrm{Exp}[G] := \sum_{n=0}^{+\infty} h_n[G] = \exp\left(\sum_{n=1}^{+\infty} \frac{p_n[G]}{n}\right)$$

and the plethystic logarithm

$$\mathrm{Log}[1+G] := \sum_{n=1}^{+\infty} \frac{\mu(n)}{n} p_n\left[\log(1+G)\right].$$

with μ the usual Mobius function. Contrarily to the ordinary ones, the plethystic exponential and logarithm start with an uppercase character.

Remark 3.2.13. *As the Adams operators are ring morphism*

$$\begin{aligned} \mathrm{Exp}[F+G] &= \mathrm{Exp}[F]\,\mathrm{Exp}[G] \\ \mathrm{Log}[(1+F)(1+G)] &= \mathrm{Log}[1+F] + \mathrm{Log}[1+G] \end{aligned}$$

Remark 3.2.14. *As expected, the plethystic logarithm is the inverse of the plethystic exponential*

$$\begin{aligned} \mathrm{Log}[\mathrm{Exp}[G]] &= \sum_{n,m\geq 1} \frac{\mu(n)}{nm} p_{nm}[G] \\ &= \sum_{n\geq 1}\sum_{d|n} \frac{\mu(d)}{n} p_n[G] \\ &= G \end{aligned}$$

This computation follows from the characterisation of Mobius function and the fact that p_1 acts as identity.

Proposition 3.2.15. *Let F 2 [[s]], write the expansion of its logarithm and plethystic logarithm*

$$\log(F) = \sum_n \frac{U_n}{n} s^n$$
$$\mathrm{Log}[F] = \sum_n V_n s^n :$$

Then the coe cients of those expansions are related by

$$V_n = \frac{1}{n} \sum_{d|n} (d) p_d \left[U_{\frac{n}{d}} \right] :$$

Proof.

$$\mathrm{Log}[F] = \sum_{d;n} \frac{(d)}{d} p_d \left[\frac{U_m}{m} s^m \right]$$
$$= \sum_{d;n} \frac{(d)}{d} \frac{p_d[U_m]}{m} s^{md}$$

As p_d is a ring morphism and $p_d[s^m] = s^{md}$. Conclusion follows by taking the coe□cient in front of s^n. □

Remark 3.2.16. *There is natural way to embed* Sym$[X]$ *in* Sym$[X]$[[s]], *we can add a variable s to keep track of the degree. For F 2* Sym$[X]$ *written in the basis of power sums as*

$$F[X] = \sum_{2P} c\ p\ [X]$$

we denote also by F the element in Sym$[X]$[[s]]

$$F = \sum_{2P} c\ p\ [X] s^{j\ j}$$

with c the coordinate of F in the basis $(p\ )_{2P}$.

Proposition 3.2.17 ([HLR13] proof of proposition 3.1). *Plethystic logarithm and plethystic substitution commute. Namely for any 2 and F symmetric function without constant term*

$$\mathrm{Log}[1 + F[\ ]] = \mathrm{Log}[1 + F][\ ]$$

where $F[\ ]$ 2 [[s]] and Log$[1 + F][\ ]$ *means plethystic substitution in each coe - cients of the power series in s.*

Proof. Use notations from Proposition 3.2.15 and Remark 3.2.16. First the $U_n[X]$ are obtained from the $c\ p\ [X]$ by additions and multiplications. Then the $V_n[X]$ are obtained from the $U_n[X]$ by additions, multiplications and Adams operator. Conclusion follows as plethystic action is a ring morphism commuting with Adams operator. □

This section ends with the introduction of Hall pairing. Other related pairings will be discussed in 3.2.3.

Definition 3.2.18 (Hall pairing). *The Hall pairing is a symmetric bilinear pairing on* $\mathrm{Sym}[X]$ *such that the power sums form an orthogonal basis*

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda,\mu} z_\lambda \tag{3.10}$$

$\delta_{\lambda,\mu}$ *is* 1 *if* $\lambda = \mu$ *and* 0 *otherwise.* z_λ *is the order of a the stabilizer of a partition of cycle type* λ*. Namely*

$$z_\lambda = \prod_{l=1}^{k} i_l^{m_l} m_l!$$

for a partition $\lambda = (\underbrace{i_1, \dots, i_1}_{m_1}, \dots, \underbrace{i_k, \dots, i_k}_{m_k})$.

3.2.2 Characters of the symmetric group and symmetric functions

Well-known results relating symmetric functions and representation theory of the symmetric group are recalled, see [Mac15] for more details. A class function on a finite group W is a $\mathbb{Q}$-valued function constant over conjugacy classes. Important examples of class functions are given by characters of finite dimensional representations of the group W. The space of class functions is actually spanned by irreducible characters. It is endowed with a scalar product defined by

$$\langle f, g \rangle_W = \frac{1}{|W|} \sum_{\sigma \in W} f(\sigma) g(\sigma^{-1}).$$

Irreducible characters then form an orthonormal basis of the space of class functions.

Remark 3.2.19. *For* χ *and* ψ *two characters of a finite group* W *and* V_χ, V_ψ *the associated representations*

$$\dim \mathrm{Hom}_W(V_\chi, V_\psi) = \langle \chi, \psi \rangle_W$$

Definition 3.2.20. *Let* $R = \bigoplus_{n \in \mathbb{N}} R_n$ *with* R_n, $n > 0$ *the space of class function on the symmetric group* $\mathfrak{S}_n$ *and* $R_0 := \mathbb{Q}$*. It is endowed with a non-degenerate pairing* $\langle \dots, \dots \rangle$ *and a product* $\cdot$*:*

- $\langle f, g \rangle = \langle f, g \rangle_{\mathfrak{S}_n}$ *for* $f, g \in R_n$ *and* R_n *is orthogonal to* R_m *if* $n \neq m$.
- *Let* $f \in R_n$ *and* $g \in R_m$ *then* $f \otimes g$ *defines a class function on* $\mathfrak{S}_n \times \mathfrak{S}_m$*. Fix an embedding* $\mathfrak{S}_n \times \mathfrak{S}_m \to \mathfrak{S}_{n+m}$ *so that class functions can be induced from* $\mathfrak{S}_n \times \mathfrak{S}_m$ *to* $\mathfrak{S}_{n+m}$ *and define the product* $f.g := \mathrm{Ind}_{\mathfrak{S}_n \times \mathfrak{S}_m}^{\mathfrak{S}_{m+n}} f \otimes g$*. This product is sometime called external tensor product.*

Definition 3.2.21 (Characteristic map). *Conjugacy classes in* $\mathfrak{S}_n$ *are indexed by partitions of* n *specifying the cycle type. For a class function* $f \in R_n$ *define*

$$\mathrm{ch}(f) = \langle f, \psi_n \rangle_{\mathfrak{S}_n}$$

where

$$\psi_n : \mathfrak{S}_n \to \mathrm{Sym}[X]$$
$$\sigma \mapsto p_{\mathrm{cyc}(\sigma)}$$

with $\mathrm{cyc}(\sigma)$ *the partition giving the cycle type of the permutation* σ*. The map* ch *extends by linearity to give the characteristic map* $\mathrm{ch} : R \to \mathrm{Sym}[X]$*.* R_0 *is sent to constants by* ch*.*

Remark 3.2.22. *It is convenient to express* ch *with partitions rather than permutations. For a class function* $f \in R_n$ *with value* f_λ *on the conjugacy class of cycle type* λ

$$\mathrm{ch}(f) = \frac{1}{|\mathfrak{S}_n|}\sum_{\sigma \in \mathfrak{S}_n} f(\sigma) p_{\mathrm{cyc}(\sigma)} \tag{3.11}$$

$$= \sum_{|\lambda| = n} z_\lambda^{-1} f_\lambda p_\lambda \tag{3.12}$$

In last line the sum over elements of the symmetric group is turned into a sum over partitions indexing conjugacy classes of the symmetric group. We used that $\frac{|C_\lambda|}{|\mathfrak{S}_n|} = z_\lambda^{-1}$ *where* C_λ *is the conjugacy class of cycle type* λ*.*

Proposition 3.2.23. *The characteristic map* ch *is an isomorphism between* R *and* $\mathrm{Sym}[X]$ *compatible with the products and the pairings (*$\mathrm{Sym}[X]$ *being endowed the Hall pairing 3.2.18).*

Proof. First let us check that $\langle \mathrm{ch}(f), \mathrm{ch}(g)\rangle = \langle f, g\rangle$, by linearity we just have to check it for $f, g \in R^n$. Orthogonality properties of the power sums (Definition 3.2.18) and (3.12) give

$$\langle \mathrm{ch}(f), \mathrm{ch}(g)\rangle = \sum_{|\lambda| = n} z_\lambda^{-1} f_\lambda g_\lambda = \langle f, g\rangle_{\mathfrak{S}_n}$$

last equality comes from the previous trick used to go from a sum over the symmetric group to a sum over partitions in Remark 3.2.22.

To check that it is a ring morphism take $f \in R^n$ and $g \in R^m$. By adjunction between induction and restriction of representations:

$$\mathrm{ch}(f.g) = \langle f.g, \psi\rangle_{\mathfrak{S}_{m+n}} = \langle f \otimes g, \psi\rangle_{\mathfrak{S}_m \times \mathfrak{S}_n}$$

last term splits into a product of sum overs $\mathfrak{S}_m$ and $\mathfrak{S}_n$

$$\frac{1}{|\mathfrak{S}_m \times \mathfrak{S}_n|}\sum_{(\sigma,\tau) \in \mathfrak{S}_m \times \mathfrak{S}_n} f(\sigma) g(\tau) p_\sigma p_\tau = \langle f, \psi\rangle_{\mathfrak{S}_m} \langle g, \psi\rangle_{\mathfrak{S}_n}$$

so that

$$\mathrm{ch}(f.g) = \mathrm{ch}(f)\,\mathrm{ch}(g).$$

Let $\mathbb{1}_\lambda$ the map $\mathbb{1}_\lambda(\sigma) = 1$ if σ is of cycle type λ and $\mathbb{1}_\lambda(\sigma) = 0$ otherwise. $(\mathbb{1}_\lambda)_{\lambda \in \mathcal{P}}$ is a basis of R. It is sent to $\left(z_\lambda^{-1} p_\lambda\right)_{\lambda \in \mathcal{P}}$ by ch. Hence the characteristic map send a basis to a basis, it is an isomorphism.

□

Remark 3.2.24. *Under the characteristic map* ch*, the symmetric function h_n is sent to the constant class function with value* 1*. The symmetric function e_n is sent to the sign.*

Irreducible characters of S_n are indexed by partitions of n as in [Mac15] in such a way that $^{(n)}$ corresponds to the trivial representation and $^{(1^n)}$ to the sign representation.

Definition 3.2.25. *The Schur functions are the images of the irreducible characters of the symmetric group under the characteristic map. For $\ \ 2\ P_n$*

$$s\ [X] := \mathrm{ch}(\ \ ) = \sum_{j\ j=j\ j} \frac{p\ [X]}{z} \tag{3.13}$$

where is the value of the character of type evaluated on a conjugacy class of cycle type .

Proposition 3.2.26. *Schur functions $(s\ )_{\ 2P}$ form an orthonormal basis with respect to the Hall pairing and equation* (3.13) *might be inverted to express the power sums from the Schur functions*

$$p\ [X] = \sum_{j\ j=j\ j} \ \ s\ [X]: \tag{3.14}$$

Proof. The family of Schur functions is the image under the characteristic map of an orthonormal basis of R. Equation (3.14) follows from (3.13) and orthogonality of characters of the symmetric group. □

Remark 3.2.27. *Let $_V\ 2\ R_n$ the class function defined as the character of a representation V of S_n. The Schur functions and the power sums have the following representation theoretic interpretation:*

- *$hs\ ;\mathrm{ch}(\ _V)i$ is the multiplicity of the irreducible representation V in the representation V.*
- *$hp\ ;\mathrm{ch}(\ _V)i$ is the trace of an element in S_n with cycle type on the representation V.*

Lemma 3.2.28. *For a partition of n let the sign representation of $S\ =\ S_{\ 1}\ \ \ S_{\ l}$. A choice of inclusion $S\ \ \ S_n$ allows to induce . Then for $\ 2\ P_n$*

$$\dim \mathrm{Hom}_{S_n}\left(\mathrm{Ind}_{S}^{S_n}\ \ ; V\ \right) = he\ ; s\ \ i = hh\ ; s\ _0 i:$$

Proof. $\dim \mathrm{Hom}_{S_n}\left(\mathrm{Ind}_{S}^{S_n}\ \ ; V\ \right)$ is the multiplicity of the irreducible representation V in $\mathrm{Ind}_{S}^{S_n}$. For $m\ 2\ \mathbb{N}_{>0}$ the symmetric function e_m is the characteristic of the sign representation of S_m. Thanks to the compatibility between induction and product, e is the characteristic of $\mathrm{Ind}_{S}^{S_n}$. First equality now follows from Remark 3.2.27. To obtain the second equality, notice that $V_{\ 0}$ is the representation V twisted by the sign. □

Definition 3.2.29 (Frobenius characteristic). *We extend the characteristic map* ch *to bigraded representations of* $\mathfrak{S}_n$ *by adding variable* q *and* t *to keep track of the degree. To a bigraded representation of the symmetric group* $V = \bigoplus_{(i,j)\in\mathbb{N}^2} V_{i,j}$ *is associated a symmetric function over* $\mathbb{Z}(q,t)$ *given by*

$$\mathrm{ch}(V) = \sum_{\in P_n} \sum_{(i,j)\in\mathbb{N}^2} \langle V_{i,j}, \quad \rangle q^i t^j s \tag{3.15}$$

where the representation $V_{i,j}$ *is identified with its character so that* $\langle V_{i,j}, \quad \rangle$ *is the multiplicity of the irreducible representation of type* in $V_{i,j}$. *The symmetric function* $\mathrm{ch}(V)$ *is called the Frobenius characteristic of the bigraded representation* V.

Example 3.2.30. *For any* $\in P_n$ *the Macdonald polynomial* $\tilde{H}\,[X;q,t]$ *is obtained in this way from a bigraded representations of the symmetric group. This is the famous* $n!$*-conjecture of Garsia-Haiman [GH93], proved by Haiman [Hai01].*

3.2.3 Orthogonality and Macdonald polynomials

In this section Mellit [Mel17a; Mel18] characterisation of modified Macdonald polynomials is recalled.

Generalities about scalar products on Sym[X]

A scalar product on Sym[X] is a $\mathbb{Q}(q,t)$-bilinear form

$$\begin{array}{rccl} (\ldots,\ldots)^S : & \mathrm{Sym}[X] \quad \mathrm{Sym}[X] & \to & \mathbb{Q}(q,t) \\ & F, G & \mapsto & (F[X], G[X])^S \end{array}$$

which is non-degenerate. It can be extended to multivariate symmetric functions by specifying the variable acted upon in index

$$(\ldots,\ldots)^S_X : \mathrm{Sym}[X,Y_1, \quad ,Y_k] \quad \mathrm{Sym}[X,Z_1, \quad ,Z_l] \to \mathrm{Sym}[Y_1, \quad ,Y_k,Z_1, \quad ,Z_l]$$

on pure tensors it reads

$$(F[X] \quad F'[Y_1, \quad ,Y_k], G[X] \quad G'[Z_1, \quad ,Z_l])^S_X := (F[X], G[X])^S G'[Z_1, \quad ,Z_l] F'[Y_1, \quad ,Y_k]$$

and it extends by linearity.

Assumption 3.2.31 (Homogeneity). *When considering families of symmetric functions indexed by partitions such as* $(u\ )_{\in P}$, *the symmetric function* u *is always assumed to be homogeneous of degree* $|\ |$.

Definition 3.2.32. *Let* $(u\ )_{\in P}$, $(v\ )_{\in P}$ *two basis dual with respect to a scalar product* $(\ldots,\ldots)^S$. *Then the element* $K_S[X,Y] \in \mathrm{Sym}[X,Y]$ *defined by*

$$K_S[X,Y] := \sum_{\in P} u\ [X] v\ [Y]$$

is called the reproducing kernel of the scalar product $(\ldots,\ldots)^S$. *It depends only on the scalar product but not on the choice of dual basis as detailed in next proposition.*

Proposition 3.2.33. *With the notations of previous definition, two families of symmetric functions* $(a\ )_{2P}$, $(b\ )_{2P}$ *are dual basis with respect to* $(:::;:::)^S$ *if and only if*

$$K_S[X;Y] = \sum_{2P} a\ [X] b\ [Y] \tag{3.16}$$

Proof. Express a and b in the basis $(u\ )_{2P}$ and $(v\ )_{2P}$

$$a\ [X] = \sum_{j\ j=j\ j} c\ u\ [X] \tag{3.17}$$

$$b\ [Y] = \sum_{j\ j=j\ j} d\ v\ [Y]: \tag{3.18}$$

Equation (3.16) now reads

$$\sum_{;} c\ d\ u\ [X] v\ [Y] = \sum_{2P} u\ [X] v\ [Y]:$$

As the family $(u\ [X] v\ [Y])_{;\ 2P}$ is free in $\mathrm{Sym}[X;Y]$ this last equation is equivalent to

$$\sum_{j\ j=j\ j} c\ d\ =\ _{;}: \tag{3.19}$$

Now $(a\ )_{2P}$ and $(b\ )_{2P}$ are dual with respect to $(:::;:::)^S$ if and only if

$$(a\ ; b\ )^S =\ _{;}:$$

Using expansions (3.17), (3.18) and duality of $(u\ )_{2P}$, $(v\ )_{2P}$ this is equivalent to

$$\sum_{j\ j=j\ j} c\ d\ =\ _{;}: \tag{3.20}$$

Two last equations can be written with matrices with columns and rows indexed by partitions of a given integer then Equation (3.19) reads $C^tD = \mathrm{Id}$ which is clearly equivalent to (3.20) : $CD^t = \mathrm{Id}$. □

Remark 3.2.34. *The name reproducing kernel comes from the notion of kernel of an operator see [Mel18].* K_S *is the kernel of the identity operator with respect to the pairing* $(:::;:::)^S$*, indeed for any* $F[X]\ 2\ \mathrm{Sym}[X]$

$$(K_S[X;Y]; F[X])^S_X = F[Y]:$$

Hall pairing and $(q;t)$-deformations

The Hall pairing was defined in 3.2.18, it satisfies

$$hp\ ; p\ i =\ _{;}\ z$$

Remark 3.2.35. $(p\ )_{2\overline{P}}$ *and* $\left(z\ ^{1}p\ \right)_{2\overline{P}}$ *form dual basis with respect to the Hall pairing so that the kernel of Hall pairing is*

$$\mathrm{Exp}[XY] = \sum_{2P} p\ [X] \frac{p\ [Y]}{z} = \sum_{n} h_n[XY]:$$

Before introducing deformations of the Hall pairing we need the following lemma.

Lemma 3.2.36. *For $F, G \in \mathrm{Sym}[X]$ and $S \in \mathbb{Q}(q,t)$*

$$\langle F[X], G[SX] \rangle = \langle F[SX], G[X] \rangle \tag{3.21}$$

Proof. It follows from successive applications of remark 3.2.34

$$\begin{aligned} \langle F[X], G[SX] \rangle_X &= \langle F[X], \langle \mathrm{Exp}[XSY], G[Y] \rangle_Y \rangle_X \\ &= \langle \langle F[X], \mathrm{Exp}[XSY] \rangle_X, G[Y] \rangle_Y \\ &= \langle F[SY], G[Y] \rangle_Y . \end{aligned}$$

□

Definition 3.2.37 (deformations of Hall pairing). *The (q,t)-deformation of the Hall pairing is defined by*

$$(F[X], G[X])^{q,t} := \langle F[X], G[(q-1)(1-t)X] \rangle .$$

Previous lemma implies that $(\ldots,\ldots)^{q,t}$ defines scalar products on $\mathrm{Sym}[X]$.

Remark 3.2.38. *The reproducing kernel of the (q,t) Hall pairing is*

$$\mathrm{Exp}\left[\frac{XY}{(q-1)(1-t)}\right].$$

Definition 3.2.39 (Modified Macdonald polynomials). *M_λ is the subspace of* $\mathrm{Sym}[X]$ *spanned by monomials symmetric functions $m_\mu[X]$ with $\mu \leq \lambda$. Macdonald polynomials $\left(\tilde{H}_\lambda[X;q,t]\right)_{\lambda \in P}$ are uniquely determined by*

- $\tilde{H}_\lambda[X(t-1);q,t] \in M_\lambda$
- $\tilde{H}_\lambda[X(q-1);q,t] \in M_{\lambda'}$
- *normalization* $\tilde{H}_\lambda[1;q,t] = 1$.

Proposition 3.2.40. *An equivalent characterization of MacDonald polynomials is*

- *Orthogonality* $(\tilde{H}_\lambda[X;q,t], \tilde{H}_\mu[X;q,t])^{q,t} = 0$ *if* $\lambda \neq \mu$
- *One of the triangularity condition* $\tilde{H}_\lambda[X(t-1)] \in M_\lambda$ *or* $\tilde{H}_\lambda[X(q-1)] \in M_{\lambda'}$
- *Normalization* $\tilde{H}_\lambda[1;q,t] = 1$

Moreover

$$a_\lambda(q,t) := \left(\tilde{H}_\lambda[X;q,t], \tilde{H}_\lambda[X;q,t]\right)^{q,t} = \prod_{x \in \lambda} (q^{a(x)+1} - t^{l(x)})(q^{a(x)} - t^{l(x)+1}). \tag{3.22}$$

the product is over the Young diagram of λ and $a(x)$ is the arm length and $l(x)$ the leg length (see Notations 3.2.1).

Proof. [Mel17a] corollary 2.8. □

Definition 3.2.41 (Modified Kostka polynomials). *The modified Kostka polynomials* $\left(\widetilde{K}_{;}(q;t)\right)_{;\,2P_n}$ *are defined as the coe cients of the transition matrix between the basis of Schur functions and the basis of modified Macdonald polynomials:*

$$\tilde{H}[X;q;t] = \sum_{2P_n} \widetilde{K}_{;}(q;t)s :$$

Notations 3.2.42. *The variables* $(q;t)$ *will often be omitted and the modified Kostka polynomial denoted by* $\widetilde{K}_{;}$ *and the modified Macdonald polynomial by* $\tilde{H}[X]$.

The Macdonald polynomials $\tilde{H}[X;q;t]$ where first introduced by Garsia-Haiman [GH96] as a deformation of polynomials defined by Macdonald [Mac15]. The definition recalled here comes from [Mel17a].

The remaining of this combinatoric background section is devoted to the presentation of a result of Garsia-Haiman [GH96, Theorem 3.4]. This result will be used in 4.4 when discussing a combinatoric interpretation of traces of Weyl group actions on cohomology of quiver and character varieties.

3.2.4 A result of Garsia-Haiman

Proposition 3.2.43. *Define an operator* $_1$ *by*

$$_1F[X] := F[X] \quad F\left[X + \frac{(1 \quad q)(1 \quad t)}{z}\right] \mathrm{Exp}[\ zX]|_{z^0}$$

Where $|_{z^0}$ *means take the coe cient in front of* z^0. *Then*

$$_1\tilde{H}[X;q;t] = (1 \quad t)(1 \quad q) \sum_{(i;j)2} q^{j\ 1}t^{i\ 1}\tilde{H}[X;q;t]$$

Moreover

$$\tilde{H}[1 \quad u;q;t] = \prod_{(i;j)2} \left(1 \quad uq^{j\ 1}t^{i\ 1}\right) \tag{3.23}$$

Proof. [GH96] Corollary 3.1 and theorem 3.2 □

Lemma 3.2.44. *At first order in* u

$$\tilde{H}[1+u;q;t] = 1 + u \sum_{(i;j)2} q^{j\ 1}t^{i\ 1} + O(u^2) \tag{3.24}$$

Proof. One should be careful with plethystic substitution, to compute left hand side of (3.24) one cannot just substitute u for u in (3.23). Indeed $p_n[1 \quad u] = 1 \quad u^n$ and $p_n[1+u] = 1+u^n$ so that substituting u for u in the latter gives back the former only when n is odd. Denote by $d_{;}$ the coe□cient of p in the power sum expansion of $\tilde{H}$ then

$$\begin{aligned} \tilde{H}[1 \quad u;q;t] &= \sum_{j\ j=j\ j} d_{;} \prod_i (1 \quad u^{\ i}) \\ \tilde{H}[1+u;q;t] &= \sum_{j\ j=j\ j} d_{;} \prod_i (1+u^{\ i}): \end{aligned}$$

We conclude by comparing the coe□cient in front of u and using (3.23). □

Lemma 3.2.45. *Let F 2 $\mathrm{Sym}_n[X]$ be a symmetric function of degree n 2. Then the coe cient in front of u in $F[1+u]$ is given by the Hall pairing with a complete symmetric function*

$$F[1+u]|_u = \langle h_{(n\ 1;1)}[X]; F[X] \rangle$$

Proof. The coe□cient of m in the monomial expansion of F is denoted by c . The plethystic substitution $F[1+u]$ corresponds to the evaluation of the symmetric function F on the set of variables $(1; u; 0; \ldots)$.

$$F[1+u] = \sum_{j\ j=n} c\ m\ [1+u]$$

Hence the only m contributing are the one with of length at most two, and the coe□cient in front of u is $c_{(n\ 1;1)}$. Conclusion follow as complete symmetric functions and monomial symmetric functions are dual with respect to the Hall pairing. □

Lemma 3.2.46. *Let F 2 $\mathrm{Sym}_n[X]$ be a symmetric function of degree n then*

$$\left.\frac{F[1\ \ u]}{1\ \ u}\right|_{u=1} = hF[X]; p_n[X]i:$$

Proof. Let d be the coe□cient in front of p in the power sum expansion of F.

$$\begin{aligned} F[1\ \ u] &= \sum_{j\ j=n} d\ p\ [1\ \ u] \\ &= \sum_{j\ j=n} d\ \prod_i (1\ \ u^{\ i}): \end{aligned}$$

When dividing by $(1\ \ u)$ and setting $u = 1$ all terms coming from partitions of length at least two will vanish as $(1\ \ u)^2$ divides them

$$\left.\frac{F[1\ \ u]}{1\ \ u}\right|_{u=1} = d_{(n)} \left.\frac{1\ \ u^n}{1\ \ u}\right|_{u=1} = nd_{(n)}:$$

The size of the centralizer of an n-cycle in S_n is $z_{(n)} = n$, conclusion follows by orthogonality of power sums (3.10). □

Let us recall an important combinatorics theorem that will be related later to cohomology of character and quiver varieties.

Theorem 3.2.47 (Garsia, Haiman [GH96] theorem 3.4)**.** *We denote by $\prod'_{(i;j)2}$ a product over the young diagram of a partition omitting the top left corner with $(i; j) = (1; 1)$.*

$$(\ \ 1)^{n\ \ 1} s_{(1^n)}[X] = (q\ \ 1)(1\ \ t) \sum_{j\ j=n} \frac{\sum_{(i;j)2} q^{j\ \ 1} t^{i\ \ 1} \prod'_{(i;j)2} (1\ \ q^{j\ \ 1} t^{i\ \ 1}) \tilde{H}\ [X]}{a\ (q; t)} \tag{3.25}$$

Proof. The reproducing kernel of the $(q; t)$-Hall pairing was given in Remark 3.2.38. The n-degree term of $\mathrm{Exp}[Z]$ is $h_n[Z]$. The basis $\left(\tilde{H}\ [X]\right)_{2P}$ and $\left(\frac{\tilde{H}\ [X]}{a}\right)_{2P}$ are dual with respect to this scalar product. Following Proposition 3.2.33, the degree n term of the reproducing kernel of the (q,t)-Hall pairing is

$$h_n\left[\frac{XY}{(q-1)(1-t)}\right] = \sum_{|\mu|=n}\frac{\tilde H_\mu[X]\tilde H_\mu[Y]}{a_\mu}.$$

Now expand h_n in the basis of power sums, proceed to plethystic substitution $Y = 1-u$ and apply (3.23)

$$\sum_{|\mu|=n} z_\mu^{-1}p_\mu\left[\frac{X(1-u)}{(q-1)(1-t)}\right] = \sum_{|\mu|=n}\frac{\tilde H_\mu[X]\prod_{(i,j)\in\mu}(1-uq^{j-1}t^{i-1})}{a_\mu}.$$

Now divide by $(1-u)$ and set $u=1$. Apply lemma 3.2.46 to left hand side and compute explicitly the right hand side

$$\sum_{|\mu|=n} z_\mu^{-1}\left(p_\mu\left[\frac{XY}{(q-1)(1-t)}\right], p_{(n)}[Y]\right)_Y = \sum_{|\mu|=n}\frac{\tilde H_\mu[X]\prod'_{(i,j)\in\mu}(1-q^{j-1}t^{i-1})}{a_\mu}$$

as Adams operator are ring morphisms

$$p_\mu\left[\frac{XY}{(q-1)(1-t)}\right] = p_\mu\left[\frac{X}{(q-1)(1-t)}\right]p_\mu[Y]$$

and using orthogonality of power sums (3.10)

$$p_{(n)}\left[\frac{X}{(q-1)(1-t)}\right] = \sum_{|\mu|=n}\frac{\tilde H_\mu[X]\prod'_{(i,j)\in\mu}(1-q^{j-1}t^{i-1})}{a_\mu}. \tag{3.26}$$

. We apply the operator Δ_{-1} to (3.26). According to Proposition 3.2.43, Δ_{-1} is diagonal in the basis of Macdonal polynomials and we obtain, up to a sign, the right hand side of (3.25). Let us compute the left hand side

$$\begin{aligned}\Delta_{-1}p_{(n)}\left[\tfrac{X}{(q-1)(1-t)}\right] &= p_{(n)}\left[\tfrac{X}{(q-1)(1-t)}\right] - p_{(n)}\left[\tfrac{X}{(q-1)(1-t)} - \tfrac{1}{z}\right]\mathrm{Exp}[-zX]|_{z^0}\\ &= p_{(n)}\left[\tfrac{X}{(q-1)(1-t)}\right] - p_{(n)}\left[\tfrac{X}{(q-1)(1-t)}\right]\mathrm{Exp}[-zX]|_{z^0} + p_{(n)}\left[\tfrac{1}{z}\right]\mathrm{Exp}[-zX]|_{z^0}\\ &= \tfrac{1}{z^n}\mathrm{Exp}[-zX]|_{z^0}.\end{aligned}$$

In second line we used that Adam operator p_n is a ring morphism and in the last line that it acts on z as raising to power n. Now $\mathrm{Exp}[-zX]$ is the inverse of $\mathrm{Exp}[zX]$ so that if $X = (x_1 + x_2 + \dots)$

$$\mathrm{Exp}[-zX] = \prod_i (1-zx_i)$$

the coefficient in front of z^n is $(-1)^n e_n[X]$ so that

$$(-1)^n e_n[X] = -(q-1)(1-t)\sum_{|\mu|=n}\frac{\sum_{(i,j)\in\mu} q^{j-1}t^{i-1}\prod'_{(i,j)\in\mu}(1-q^{j-1}t^{i-1})\tilde H_\mu[X]}{a_\mu}.$$

Conclusion follows as $e_n = s_{(1^n)}$. □

3.3 Conjugacy classes and adjoint orbits for general linear group

$\mathbb{K}$ is either $\mathbb{C}$ or an algebraic closure $\overline{\mathbb{F}}_q$ of the finite field with q elements $\mathbb{F}_q$.

3.3.1 Notations for adjoint orbits and conjugacy classes

For r an integer and $z \in \mathbb{K}$, denote by $J_r(z)$ the Jordan block of size r with eigenvalue z

$$J_r(z) := \begin{pmatrix} z & 1 & & & \\ & z & \ddots & & \\ & & \ddots & 1 & \\ & & & z & 1 \\ & & & & z \end{pmatrix} \in \mathfrak{gl}_r .$$

Let $\mu = (\mu_1, \mu_2, \dots, \mu_s)$ a partition of an integer m and let $z \in \mathbb{C}$. Denote by $J_\mu(z)$ the matrix with eigenvalue z and Jordan blocks of size μ_j.

$$J_\mu(z) := \begin{pmatrix} J_{\mu_1}(z) & & & \\ & J_{\mu_2}(z) & & \\ & & \ddots & \\ & & & J_{\mu_s}(z) \end{pmatrix} \in \mathfrak{gl}_m .$$

Let $\alpha = (\alpha_1, \dots, \alpha_l) \in \mathcal{P}_n$ a partition of n, introduce the following notation

$$\mathcal{P}_\alpha := \mathcal{P}_{\alpha_1} \times \mathcal{P}_{\alpha_2} \times \dots \times \mathcal{P}_{\alpha_l}.$$

Consider a diagonal matrix

$$\sigma = \begin{pmatrix} \sigma_1 \mathrm{Id}_{\alpha_1} & & & \\ & \sigma_2 \mathrm{Id}_{\alpha_2} & & \\ & & \ddots & \\ & & & \sigma_l \mathrm{Id}_{\alpha_l} \end{pmatrix} \tag{3.27}$$

with $\sigma_i \neq \sigma_j$ for $i \neq j$, so that α_i is the multiplicity of the eigenvalue σ_i. Let $\underline{\mu} = (\mu^1, \dots, \mu^l) \in \mathcal{P}_\alpha$.

Notations 3.3.1. *Denote by $\mathcal{O}_{\underline{\mu},\sigma}$ the adjoint orbit of the matrix:*

$$J_{\underline{\mu},\sigma} := \begin{pmatrix} J_{\mu^1}(\sigma_1) & & & \\ & J_{\mu^2}(\sigma_2) & & \\ & & \ddots & \\ & & & J_{\mu^l}(\sigma_l) \end{pmatrix}.$$

If all the eigenvalue are non-zero, this adjoint orbit is also a conjugacy class in GL_n, *it is then denoted by $\mathcal{C}_{\underline{\mu},\sigma}$.*

We recall a well-known proposition.

Proposition 3.3.2. *The Zariski closure of the adjoint orbit* $O_{\_;}$ *is*

$$\overline{O}_{\_;} = \bigcup_{\_\ \_} O_{\_;}$$

the union is over l*-uple* $\_ = ({}^1; : : : ; {}^l)$ *with* ${}^j \quad {}^j$ *for all* $1 \quad j \quad l$*. The dominance order on partition was recalled in Definition 3.2.2 .*

Proof.

$$O_{\_;} = \bigcap_{1 \ j \ l} \left\{ X \ 2 \ \mathrm{gl}_n \middle| \dim \ker(X \quad {}_j)^k = \sum_{1 \ i \ k} {}^{j0}_i \text{ for all } k \ 2 \ \mathrm{N} \right\}$$

with ^{j0} the transpose of the partition ^{j} so that ${}^{j0}_i = \mathrm{card} \, fr \ 2 \ \mathrm{N} j \ {}^j_r \quad ig$. The Zariski closure is

$$\overline{O}_{\_;} = \bigcap_{1 \ j \ l} \left\{ X \ 2 \ \mathrm{gl}_n \middle| \dim \ker(X \quad {}_j)^k \quad \sum_{1 \ i \ k} {}^{j0}_i \text{ for all } k \ 2 \ \mathrm{N} \right\} :$$

Indeed the inequality on the dimension of the kernel is a close condition, it corresponds to the vanishing of all minors of $(X \quad {}_j)^k$ of size ${}_j + 1 \quad \sum_{1 \ i \ k} {}^{j0}_i$. Then $O_{\_;} \quad \overline{O}_{\_;}$ if and only if ${}^{j0} \quad {}^{j0}$ which is equivalent to ${}^j \quad {}^j$. □

3.3.2 Types and conjugacy classes over finite fields

ix a total order on $\mathrm{N}_{>0} \quad P$.

Definition 3.3.3 (Type). *A type is a non-increasing sequence* $! = (d_1; !^1) : : : (d_l; !^l)$ *with* $(d_i; !^i) \ 2 \ \mathrm{N}_{>0} \quad P$ *. Denote by* T_n *the set of type* $!$ *with* $\sum d_i j!^i j = n$ *and* $T = \bigcup_{n 2 \mathrm{N}_{>0}} T_n$*.*

Definition 3.3.4 (Type of a $\mathrm{GL}_n(\mathrm{F}_q)$ conjugacy class or of a $\mathrm{gl}_n(\mathrm{F}_q)$ adjoint orbit). *Let* C *be a conjugacy class in* $\mathrm{GL}_n(\mathrm{F}_q)$*, its characteristic polynomial has its coe -cients in* F_q *so that its eigenvalues, which live in* $\overline{\mathrm{F}}_q$*, are permuted by the Frobenius. The spectrum of* C*, with multiplicity, reads*

$$\left(\underbrace{\left({}_1; : : : ; {}^{q^{d_1} \ 1}_1 \right) ; : : : ; \left({}_1; : : : ; {}^{q^{d_1} \ 1}_1 \right)}_{m_1} ; : : : ; \underbrace{\left({}_l; : : : ; {}^{q^{d_l} \ 1}_l \right) ; : : : ; \left({}_l; : : : ; {}^{q^{d_l} \ 1}_l \right)}_{m_l} \right)$$

with ${}_i \ 2 \ \overline{\mathrm{F}}_q$ *such that* ${}^{q^{d_i} \ 1}_i \ \not\in \ {}_i$*,* ${}^{q^{d_i}}_i = {}_i$ *and* ${}_i \not\in {}_j$ *for* $i \not\in j$*. Then the conjugacy class* C *determines partitions* $!^i \ 2 \ P_{m_i}$ *giving the size of the Jordan blocks of the Frobenius orbit of eigenvalues* $\left({}_i; : : : ; {}^{q^{d_i} \ 1}_i \right)$*. After reordering it defines a type* $! \ 2 \ T_n$ *given by* $! = (d_1; !_1) : : : (d_l; !_l)$*. The same description holds for adjoint orbits instead of conjugacy classes.*

Notations 3.3.5. *For any family of symmetric functions* $(u \ )_{\ 2P}$ *indexed by partitions and any type* $! = (d_1; !^1) : : : (d_l; !^l)$ *introduce the following notation*

$$u_! := \prod_{i=1}^{l} p_{d_i}[u_{!^i}] = \prod_{i=1}^{l} u_{!^i}[X^{d_i}]$$

3.3.3 Resolutions of Zariski closure of conjugacy classes and adjoint orbits

Consider a conjugacy class $C_{\underline{\ };\ }$. Notations are introduced in previous section, in GL_n is a diagonal matrix like in (3.27), denote by M its centralizer in GL_n.

$$M = \begin{pmatrix} \mathrm{GL}_{\ _1} & 0 & \\ 0 & \mathrm{GL}_{\ _2} & \\ \vdots & 0 & \ddots \end{pmatrix}$$

$\underline{\ } = (\ ^1; : : : ;\ ^l)$ with $\ ^i$ a partition of $\ _i$. The transposed partition is denoted by $\overline{\ }^{i0} = (\ ^{i0}_1;\ ^{i0}_2; : : :)$. Let L the subgroup of GL_n formed by block diagonal matrices with blocks of size $\ ^{i0}_r$, it is a subgroup of M with the following form

$$L = \begin{pmatrix} \overbrace{\begin{matrix} \mathrm{GL}_{\ ^{10}_1} & 0 & \\ 0 & \mathrm{GL}_{\ ^{10}_2} & \\ \vdots & 0 & \ddots \end{matrix}}^{1} & & \\ & \overbrace{\begin{matrix} \mathrm{GL}_{\ ^{20}_1} & 0 & \\ 0 & \mathrm{GL}_{\ ^{20}_2} & \\ \vdots & 0 & \ddots \end{matrix}}^{2} & \\ & & \ddots \end{pmatrix} :$$

Notations 3.3.6. *For $\ = (\ _1; : : : ;\ _l)$ a partition let*

$$\mathrm{S}_{\ } = \mathrm{S}_{\ _1} \quad \mathrm{S}_{\ _l} \text{ and } \mathrm{GL}_{\ } := \mathrm{GL}_{\ _1} \quad \mathrm{GL}_{\ _l} :$$

For $\underline{\ } = (\ ^1; : : : ;\ ^l) \in P$

$$\mathrm{GL}_{\underline{\ }} := \mathrm{GL}_{\ ^1} \ : : : \mathrm{GL}_{\ ^l} = \prod_{r;s} \mathrm{GL}_{\ ^r_s}$$

and

$$\mathrm{S}_{\underline{\ }} := \mathrm{S}_{\ ^1} \ : : : \mathrm{S}_{\ ^l} = \prod_{r;s} \mathrm{S}_{\ ^r_s}$$

Then the previously introduced Levi subgroups satisfy $M = \mathrm{GL}_{\ }$ and $L = \mathrm{GL}_{\underline{\ }^0}$.

Denote by P the parabolic subgroup of blocks upper triangular matrices having L as a Levi factor, $P = LU_P$ with

$$U_P = \begin{pmatrix} \overbrace{\begin{matrix} \mathrm{Id}_{\ ^{10}_1} & & \\ 0 & \mathrm{Id}_{\ ^{10}_2} & \\ \vdots & 0 & \ddots \end{matrix}}^{1} & * & \\ & \overbrace{\begin{matrix} \mathrm{Id}_{\ ^{20}_1} & & \\ 0 & \mathrm{Id}_{\ ^{20}_2} & \\ \vdots & 0 & \ddots \end{matrix}}^{2} & \\ & & \ddots \end{pmatrix} :$$

Now we can construct a resolution of singularities of $\overline{C}_{\lambda,\mu}$:

$$\widetilde{X}_{L,P,\lambda} := \left\{ (X, gP) \in GL_n \times GL_n/P \,\middle|\, g^{-1}Xg \in \lambda U_P \right\}$$

Proposition 3.3.7 (Resolution of Zariski closure of conjugacy classes). *The image of the projection to the first factor $\widetilde{X}_{L,P,\lambda} \to GL_n$ is the Zariski closure of the conjugacy class $C_{\lambda,\mu}$. Moreover the following map is a resolution of singularities*

$$p_\lambda : \begin{array}{ccc} \widetilde{X}_{L,P,\lambda} & \to & \overline{C}_{\lambda,\mu} \\ (X, gP) & \mapsto & X \end{array}.$$

There is a similar result for adjoint orbits. For λ a diagonal matrix in $\mathfrak{gl}_n$ as in (3.27), let $\mathfrak{l}$, $\mathfrak{p}$, respectively $\mathfrak{u}_P$ the Lie algebras of L, P, respectively U_P, then $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}_P$.

$$\widetilde{Y}_{L,P,\lambda} := \left\{ (X, gP) \in \mathfrak{gl}_n \times GL_n/P \,\middle|\, g^{-1}Xg \in \lambda + \mathfrak{u}_P \right\}.$$

Proposition 3.3.8 (Resolution of Zariski closure of adjoint orbits). *The image of the projection to the first factor $\widetilde{Y}_{L,P,\lambda} \to \mathfrak{gl}_n$ is the Zariski closure of the adjoint orbit $O_{\lambda,\mu}$. Moreover the following map is a resolution of singularities*

$$p_\lambda : \begin{array}{ccc} \widetilde{Y}_{L,P,\lambda} & \to & \overline{O}_{\lambda,\mu} \\ (X, gP) & \mapsto & X \end{array}.$$

3.4 Resolution of conjugacy classes and Weyl group actions

3.4.1 Borho-MacPherson approach to Springer theory

The approach of Borho-MacPherson [BM83] to Springer theory is recalled, it relies on perverse sheaves. It follows work of Lusztig [Lus81] for the general linear group. G is a reductive group over $\mathbb{K}$ and B a Borel subgroup of G. There is a decomposition $B = TU$ with T a maximal torus and U the unipotent radical of B. Consider the Grothendieck-Springer resolution

$$\widetilde{G} = \left\{ (X, gB) \in G \times G/B \,\middle|\, g^{-1}Xg \in B \right\}.$$

Denote by $G^{\mathrm{reg}} \subset G$ the subset of regular semi-simple elements and

$$\widetilde{G}^{\mathrm{reg}} = \left\{ (X, gB) \in G^{\mathrm{reg}} \times G/B \,\middle|\, g^{-1}Xg \in B \right\}.$$

Let $T^{\mathrm{reg}} := G^{\mathrm{reg}} \cap T$, one has the following isomorphism

$$\begin{array}{ccc} T^{\mathrm{reg}} \times G/T & \to & \widetilde{G}^{\mathrm{reg}} \\ (t, gT) & \mapsto & (gtg^{-1}, gB) \end{array}$$

The Weyl group $W = N_G(T)/T$ acts on $T^{\mathrm{reg}} \times G/T$, for $w \in W$ and $\underline{w} \in G$ a representative

$$w.(t, gT) := (\underline{w}t\underline{w}^{-1}, g\underline{w}^{-1}T).$$

Thus W acts on $\widetilde{G}^{\mathrm{reg}}$ by

$$w.(X, gB) = (X, g\dot{w}^{-1}B).$$

Consider the following map

$$p^G : \begin{array}{ccc} \widetilde{G} & \to & G \\ (X, gB) & \mapsto & X \end{array}.$$

Denote by p^{reg} its restriction to $\widetilde{G}^{\mathrm{reg}}$. Then p^{reg} is a Galois cover with group W. Denote by $U \subset G$ the subset of unipotent elements and

$$\widetilde{U} = \left\{(X, gB) \in U \times G/B \,\middle|\, g^{-1}Xg \in U\right\}.$$

Consider the following diagram, both squares are cartesian

$$\begin{array}{ccccc} \widetilde{U} & \hookrightarrow & \widetilde{G} & \overset{j}{\hookleftarrow} & \widetilde{G}^{\mathrm{reg}} \\ \downarrow{\scriptstyle p^U} & & \downarrow{\scriptstyle p^G} & & \downarrow{\scriptstyle p^{\mathrm{reg}}} \\ U & \hookrightarrow & G & \underset{i}{\hookleftarrow} & G^{\mathrm{reg}}. \end{array}$$

Proposition 3.4.1 (Borho-MacPherson [BM83], 2.6). *The Weyl group W acts on $p^G_!\overline{\mathbb{Q}}_\ell$ and on $p^U_!\overline{\mathbb{Q}}_\ell$.*

Proof. Let $\overline{\mathbb{Q}}_{\ell,\widetilde{G}^{\mathrm{reg}}} \in D^b_c\left(\widetilde{G}^{\mathrm{reg}}\right)$ be the constant sheaf concentrated in degree 0. It is W-equivariant with $\varphi_w : w^*\overline{\mathbb{Q}}_{\ell,\widetilde{G}^{\mathrm{reg}}} \to \overline{\mathbb{Q}}_{\ell,\widetilde{G}^{\mathrm{reg}}}$ a morphism which is the identity on stalks. p^{reg} is equivariant for the trivial action of W on G^{reg} so that by Proposition 3.1.6, W acts on $p^{\mathrm{reg}}_!\overline{\mathbb{Q}}_{\ell,\widetilde{G}^{\mathrm{reg}}}$ and there is a group morphism $W^{\mathrm{op}} \to \mathrm{Aut}(p_!\overline{\mathbb{Q}}_{\ell,\widetilde{G}^{\mathrm{reg}}})$. This morphism is composed with inversion in order to obtain a left action.

This rather formal construction will be relevant later to compare various actions. In the present situation the action can be easily described without the formalism of W-equivariant complexes. The complex $p^{\mathrm{reg}}_!\overline{\mathbb{Q}}_{\ell,\widetilde{G}^{\mathrm{reg}}}$ is concentrated in degree 0, its stalk is isomorphic to the group algebra of W, the group W acts by right multiplication.

Springer theory extends this action to the derived pushforward $p^G_!\overline{\mathbb{Q}}_\ell$. First p^G is small, and by base change $i^*p^G_!\overline{\mathbb{Q}}_\ell = p^{\mathrm{reg}}_!\iota^*\overline{\mathbb{Q}}_{\ell,\widetilde{G}^{\mathrm{reg}}}$. Therefore $p^G_!\overline{\mathbb{Q}}_\ell[\dim\widetilde{G}] = \underline{IC}_{p^{\mathrm{reg}}_!\overline{\mathbb{Q}}_{\ell,\widetilde{G}^{\mathrm{reg}}}}$. Then $\mathrm{Aut}(p^{\mathrm{reg}}_!\overline{\mathbb{Q}}_{\ell,\widetilde{G}^{\mathrm{reg}}}) = \mathrm{Aut}\left(IC_{p^{\mathrm{reg}}_!\overline{\mathbb{Q}}_{\ell,\widetilde{G}^{\mathrm{reg}}}}\right)$ so that W acts on $p^G_!\overline{\mathbb{Q}}_\ell$. To conclude, by base change $p^U_!\overline{\mathbb{Q}}_\ell$ is isomorphic to the restriction of $p^G_!\overline{\mathbb{Q}}_\ell$ to U.

□

To study characters varieties, this construction is used when G is either GL_n or a Levi subgroup of a parabolic subgroup of GL_n.

Example 3.4.2. *When $G = \mathrm{GL}_n$, the Weyl group is isomorphic to a symmetric group $\mathfrak{S}_n$. The irreducible representations of the symmetric group $\mathfrak{S}_n$ are indexed by partitions of n. For $\lambda \in \mathcal{P}_n$ the associated irreducible representation is V_λ. The trivial representation is $V_{(n)}$ and $V_{(1^n)}$ is the signature. Then there is a nice description of the left W-action on $p^U_!\overline{\mathbb{Q}}_\ell$*

$$p^U_!\overline{\mathbb{Q}}_\ell\left[\dim\widetilde{U}\right] = \bigoplus_{\lambda\in\mathcal{P}_n} V_\lambda \otimes \underline{IC}_{\overline{C_\lambda}}.$$

With C_λ the unipotent class with Jordan type λ. With notations from previous section $C_\lambda = C_{\lambda,1}$.

Example 3.4.3. *For a Levi subgroup M of a parabolic subgroup of GL_n with*

$$M = \mathrm{GL}$$

the Weyl group $W_M = N_M(T)/T$ is isomorphic to S (Notations 3.3.6 are used). Let U_M M the subset of unipotent element in M and $\widetilde{U}_M$ its Springer resolution. The result for GL_n easily generalizes to

$$p_!^{U_M} \quad \mathbb{Q}_M \left[\dim \widetilde{U}_M\right] = \bigoplus_{\_ \in P} V_\_ \quad \underline{IC_{C^M_\_}} \tag{3.28}$$

with $C^M_\_$ the unipotent conjugacy class in M defined for $\_ = (^1, \dots, ^l)$ by

$$C^M_\_ := C_{1} \quad C_{l} \quad \mathrm{GL}_{1} \quad \mathrm{GL}_{l}.$$

Remark 3.4.4. *The same construction exists for adjoint orbits. Denote by $\mathfrak{g}$, $\mathfrak{b}$, respectively $\mathfrak{u}$ the Lie algebras of G, B respectively U. Denote by $\mathfrak{n}$ the subset of nilpotents elements in $\mathfrak{g}$.*

$$\widetilde{\mathfrak{g}} := \left\{(X, gB) \in \mathfrak{g} \quad G/B \,\middle|\, g^{-1}Xg \in \mathfrak{b}\right\}$$

and

$$\widetilde{\mathfrak{n}} := \left\{(X, gB) \in \mathfrak{n} \quad G/B \,\middle|\, g^{-1}Xg \in \mathfrak{u}\right\}.$$

They fit in a diagram

$$\begin{array}{ccc} \widetilde{\mathfrak{n}} & \hookrightarrow & \widetilde{\mathfrak{g}} \\ {\scriptstyle p^{\mathfrak{n}}}\downarrow & & \downarrow{\scriptstyle p^{\mathfrak{g}}} \\ \mathfrak{n} & \hookrightarrow & \mathfrak{g}. \end{array}$$

The Weyl group W acts on $p_!^{\mathfrak{g}} \; \mathbb{Q}_{\mathfrak{g}}$ and on $p_!^{\mathfrak{n}} \; \mathbb{Q}_{\mathfrak{n}}$. Moreover

$$p_!^{\mathfrak{n}} \; \mathbb{Q}_{\mathfrak{n}}[\dim \widetilde{\mathfrak{n}}] = \bigoplus_{\in P_n} V \quad \underline{IC_{O}}.$$

With O the nilpotent adjoint orbit of Jordan type .

3.4.2 Parabolic induction

In this section Lusztig parabolic induction is recalled [Lus84; Lus85; Lus86]. Most results hold for any reductive algebraic group G, for our purpose we assume G is either GL_n or a Levi factor of a parabolic subgroup of GL_n. Let P be a parabolic subgroup of G with Levi decomposition $P = LU_P$. The projection to L with respect to this decomposition is $_P : LU_P \to L$. Consider the diagram

$$L \longleftarrow V_1 \xrightarrow{\;\;'\;\;} V_2 \xrightarrow{\;\;''\;\;} G \tag{3.29}$$

with

$$\begin{aligned} V_1 &= \left\{(x, g) \in G \quad G \,\middle|\, g^{-1}xg \in LU_P\right\} \\ V_2 &= \left\{(x, gP) \in G \quad G/P \,\middle|\, g^{-1}xg \in LU_P\right\} \end{aligned}$$

$$\begin{aligned} (x; g) &= \ _P(g^{\ 1}xg) \\ {}^0(x; g) &= (x; gP) \\ {}^{00}(x; gP) &= x \end{aligned}$$

Parabolic induction is a functor $\mathrm{Ind}^G_{L \ P}$ from the category of L-equivariant perverse sheaves on L to the derived category of G-equivariant complexes of sheaves on G. Take K an L-equivariant perverse sheaf on L. The morphism is smooth with connected fibers of dimension $m = \dim G + \dim U_P$. Therefore the shifted pull-back $K[m]$ is an L-equivariant perverse sheaf on V_1. Hence there exists a perverse sheaf $\widetilde{K}$ on V_2, unique up to isomorphism, such that ${}^0\ \widetilde{K}[\dim P] = \ K[m]$. Then the parabolic induction of K is defined by $\mathrm{Ind}^G_{L \ P} K := {}^{00}_{!}\widetilde{K}$.

Example 3.4.5. *The Springer complex* $p^G_{!}\ \ _{\mathbb{C}}$ *is nothing but* $\mathrm{Ind}^G_{T \ B \ \ T}$ *and the* W*-action on this complex is a particular cass of a more general situation studied by Lusztig [Lus86].*

Example 3.4.6. *Parabolic induction also relates to the resolution of closure of conjugacy classes from 3.3.3. Consider the following diagram with the first line being the diagram of parabolic induction*

$$\begin{array}{ccccccc} L & \longleftarrow & V_1 & \longrightarrow & V_2 & \longrightarrow & \mathrm{GL}_n \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ f \ \ g & \longleftarrow & \widehat{X}_{L;P;} & \longrightarrow & \widetilde{X}_{L;P;} & \longrightarrow & \overline{C}_{\_;} \end{array}$$

then

$$p_{!} \ \ _{\widetilde{X}_{L;P;}} \left[\dim \widetilde{X}_{L;P;}\ \right] = \mathrm{Ind}^{\mathrm{GL}_n}_{L \ P} \ \ _{f \ \ g}.$$

with $\ _{f \ )}$ *the constant sheaf with support* $f \ \ g$*.*

Proposition 3.4.7 (Lusztig [Lus85] I-4.2). *Let* $P; Q$ *be parabolic subgroups of* G *with Levi decomposition* $P = LU_P$*,* $Q = MU_Q$ *such that* $P \ \ Q$ *and* $L \ \ M$*, then* $P \cap M$ *is a parabolic sugroup of* M *with* L *as a Levi subgroup. Let* K *a* L*-equivariant perverse sheaf on* L *such that* $\mathrm{Ind}^M_{L \ P \cap M} K$ *is a perverse sheaf on* M*. Then*

$$\mathrm{Ind}^G_{L \ P} K = \mathrm{Ind}^G_{M \ G}\left(\mathrm{Ind}^M_{L \ P \cap M} K\right).$$

Let us detail the implication of this proposition for Springer complexes. As in previous section, $G = \mathrm{GL}_n$, B is a Borel subgroup of G and T a maximal torus in B. M is a Levi factor of P a parabolic subgroup of G containing B, it has the following form for some $\ 2 P_n$

$$M = \mathrm{GL} \ \ .$$

By transitivity of the parabolic induction from previous proposition

$$\mathrm{Ind}^G_{T \ B \ \ T} = \mathrm{Ind}^G_{M \ P} \mathrm{Ind}^M_{T \ B \cap M \ \ T}. \tag{3.30}$$

The left hand side is the Springer complex for G so that it carries a W-action, this action restricts to a W_M-action as $W_M \ \ W$. Similarly $\mathrm{Ind}^M_{T \ B \cap M \ \ T}$ carries a W_M-action as it is isomorphic to the Springer complex for M. Under the parabolic induction functor $\mathrm{Ind}^G_{M \ P}$, this W_M-action on $\mathrm{Ind}^M_{T \ B \cap M \ \ T}$ induces a W_M-action on $\mathrm{Ind}^G_{M \ P} \mathrm{Ind}^M_{T \ B \cap M \ \ T}$. Lusztig [Lus86, 2.5] proved that both W_M-action coincide under the isomorphism (3.30), this implies in particular the next theorem:

Theorem 3.4.8. *Let 2 $Z(L)$ and $_{f\ g}$ the constant sheaf with support $f\ g$. Let $M = Z_{\mathrm{GL}_n}(\ )$, assume M = GL*

$$\mathrm{Ind}_{L\ P}^{\mathrm{GL}_n}\ {}_{f\ g} = \bigoplus_{\_2P} \mathrm{Hom}_{W_M}\left(\mathrm{Ind}_{W_L}^{W_M}\ ; V\_\right)\ \underline{IC}_{\overline{C}\_;}$$

with the sign representation of W_L.

Remark 3.4.9. *The same constructions exist for Lie algebras, see for instance [Let05]. Previous theorem then becomes:*

$$\mathrm{Ind}_{\mathrm{l}\ \mathrm{p}}^{\mathrm{gl}_n}\ {}_{f\ g} = \bigoplus_{\_2P} \mathrm{Hom}_{W_M}\left(\mathrm{Ind}_{W_L}^{W_M}\ ; V\_\right)\ \underline{IC}_{\overline{O}\_;}$$

3.4.3 Relative Weyl group actions on multiplicity spaces

An interesting feature of the multiplicity spaces $\mathrm{Hom}_{W_M}\left(\mathrm{Ind}_{W_L}^{W_M}\ ; V\_\right)$ is that they carry a relative Weyl group action. Before describing this action, we recall a general result about symmetric group, see Letellier [Let11, 6.1, 6.2].

Consider a type $! = (d_1; !^1) ::: (d_l; !^l)$ 2 T_n (the set of types was defined in 3.3.3). The associated Schur function is

$$s_! = s_{!^1}\left[X^{d_1}\right] ::: s_{!^l}\left[X^{d_l}\right]$$

and

$$r(!) := \sum_{i=1}^{l} (d_i \quad 1) j!^i j: \tag{3.31}$$

Definition 3.4.10 (Twisted Littlewood-Richardson coe□cients). *As the usual Schur funtions $(s\ )\ _{2P_n}$ form a basis of $\mathrm{Sym}_n[X]$, there exist coe cients $c_!$ such that*

$$s_! = \sum_{2P_n} c_! s\ :$$

Coe cients $c_!$ are called the twisted Littlewood-Richardson coe cients.

Lemma 3.4.11. *Let $!^0$ the transpose of $!$, i.e $!^0 = (d_1; !^{1^0}) ::: \left(d_l; !^{l^0}\right)$. Then*

$$c_{!^0}^{\ 0} = (\ \ 1)^{r(!)} c_!:$$

with $r(!)$ defined in (3.31).

Proof. This follows from a computation in the ring of symmetric functions using the basis of power sums, see Letellier [Let11, 6.2.4]. □

Let us recall their interpretation in terms of representations of symmetric group. The type $!$ defines an irreducible representation $V_!$ of the group $S_! := \prod_{i=1}^{l} S_{j!^i j}^{d_i}$.

$$V_! := \ {}_{i=1}^{l} V_{!^i}^{\ d_i}$$

with $V_{!^i}$ the representation of $\mathfrak{S}_{j!^i j}$ indexed by the partition $!^i$. Denote by $f_!$ the morphism $\mathfrak{S}_! \; ! \; \mathrm{GL}(V_!)$ induced by the representation $V_!$. Introduce the relative Weyl group

$$W_{\mathfrak{S}_n}(\mathfrak{S}_!; V_!) = \left\{ n \; 2 \; N_{\mathfrak{S}_n}(\mathfrak{S}_!) \,\middle|\, f_!(n \;\; ^1 ::: n) = f_!(:::) \right\} = \mathfrak{S}_!$$

This is the group of permutations of the blocks of $\mathfrak{S}_!$ corresponding to the same representation $V_{!^i}$.

Proposition 3.4.12 (Letellier [Let11] Proposition 6.2.5). *For $\;\; 2 \; P_n$ and $V \;$ the associated representation of $\mathfrak{S}_n$. For $! \; 2 \; T_n$ a type. The relative Weyl group $W_{\mathfrak{S}_n}(\mathfrak{S}_!; V_!)$ acts on*

$$\mathrm{Hom}_{\mathfrak{S}_n}\left(\mathrm{Ind}_{\mathfrak{S}_!}^{\mathfrak{S}_n} V_!; V \;\right):$$

Let $w \; 2 \; W_{\mathfrak{S}_n}(\mathfrak{S}_!; V_!)$ acting by cyclic permutation of the d_i blocks with representation $V_{!^i}$ for $1 \;\; i \;\; l$. Then

$$\mathrm{tr}\left(w; \mathrm{Hom}_{\mathfrak{S}_n}\left(\mathrm{Ind}_{\mathfrak{S}_!}^{\mathfrak{S}_n} V_!; V \;\right)\right) = c_! :$$

Remark 3.4.13. *Assume the type $!$ has the following form*

$$! = (\;_1; (1)) ::: (\;_l; (1)) \text{ with } \;\; = (\;_1; :::; \;_l) \; 2 \; P_n:$$

Then $s_! = p \;$ and by (3.14), for $\;\; 2 \; P_n$

$$c_! =$$

Notice that $W_{\mathfrak{S}_n}(\mathfrak{S}_n; V_!) = \mathfrak{S}_n$ and the element w associated to $!$ has cycle type $\;$. Therefore the proposition implies that as a $W_{\mathfrak{S}_n}(\mathfrak{S}_n; V_!)$ representation

$$\mathrm{Hom}_{\mathfrak{S}_n}\left(\mathrm{Ind}_{\mathfrak{S}_!}^{\mathfrak{S}_n} V_!; V \;\right) = V \;:$$

With this general result about symmetric group, we go back to the Weyl groups relative to resolution of conjugacy classes.

Definition 3.4.14 (Relative Weyl group). *For L a Levi subgroup of M, The relative Weyl group is*

$$W_M(L) := N_M(L)/L:$$

Take L and M similarly to Section 3.3.3. Denote by $(m_1^i; :::; m_{k_i}^i)$ the multiplicity of the parts of $\;^{i0}$ so that it has the following form

$$^{i0} = \left(\underbrace{a_1^i; :::; a_1^i}_{m_1^i}; \underbrace{a_2^i; :::; a_2^i}_{m_2^i}; :::; \underbrace{a_{k_i}^i; :::; a_{k_i}^i}_{m_{k_i}^i} \right):$$

Then with notations 3.3.6 $L = \mathrm{GL}_{\underline{\;\;}^0}$ and the relative Weyl group is

$$W_M(L) = \prod_{\substack{1 \; i \; l \\ 1 \; r \; k_i}} \mathfrak{S}_{m_r^i}:$$

When $M = \mathrm{GL}_n$ then the relative Weyl group is the group of permutations of same-sized blocks of L.

Notations 3.4.15. *Conjugacy classes in $W_M(L)$ are indexed by elements*

$$= (^{i,r})_{\substack{1 \leq i \leq l \\ 1 \leq r \leq k_i}} \in \prod_{\substack{1 \leq i \leq l \\ 1 \leq r \leq k_i}} \mathcal{P}_{m^i_r}.$$

A conjugacy class then determined l distinct types ω_i with parts $\left(^{i,r}_s, (1^{a^i_r}) \right)_{\substack{1 \leq r \leq k_i \\ 1 \leq s \leq l(^{i,r})}}$.

Note that

$$s_{\omega'_i} = \prod_{r=1}^{k_i} \prod_{s=1}^{l(^{i,r})} h_{a^i_r}\left[X^{ ^{i,r}_s}\right]$$

Following notations will be convenient to compute Weyl group actions on the cohomology of character varieties.

$$\widetilde{h} := \prod_{i=1}^{l} s_{\omega'_i}$$

and

$$r( ) := \sum_{i=1}^{l} r(\omega_i).$$

with $r(\omega_i)$ defined by (3.31).

Those data describe the $W_M(L)$ action on the multiplicity spaces, Proposition 3.4.12 implies:

Theorem 3.4.16. *Let $_0$ the sign representation of W_L and $\in \mathcal{P}$. The relative Weyl group $W_M(L)$ acts on $\mathrm{Hom}_{W_M}\left(\mathrm{Ind}_{W_L}^{W_M} \ _0, V\right)$. The trace of the action of an element with conjugacy class indexed by $\in \prod_{\substack{1 \leq i \leq l_j \\ 1 \leq r \leq k_i}} \mathcal{P}_{m^i_r}$ is*

$$\mathrm{tr}\left(; \mathrm{Hom}_{W_M}\left(\mathrm{Ind}_{W_L}^{W_M} \ _0, V\right)\right) = \prod_{i=1}^{l} c_{w_i}^{ i}.$$

3.4.4 Relative Weyl group actions and Springer theory

There is another construction of relative Weyl group action using another variant of Springer theory. It will be useful to construct relative Weyl group actions when considering family of comet-shaped quiver varieties.

Let P be a parabolic subgroup of GL_n and L a Levi factor of P. L is isomorphic to a group of blocks diagonal matrices $\mathrm{GL}_{c_1} \times \dots \times \mathrm{GL}_{c_r}$. The Lie algebra of L, respectivly U_P are denoted $\mathfrak{l}$ respectivly $\mathfrak{u}_P$. At the level of the Lie algebras the Levi decomposition becomes $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}_P$. The center of this Lie algebra $\mathfrak{l}$ is denoted $Z(\mathfrak{l})$ and its regular locus is

$$Z(\mathfrak{l})^{\mathrm{reg}} = \{x \in Z(\mathfrak{l}) \mid Z_G(x) = L\}.$$

Define

$$\widetilde{\mathbb{Y}}^{\mathrm{reg}}_{L,P} = \left\{(x, gL) \in \mathfrak{gl}_n \times \mathrm{GL}_n/L \,\middle|\, g^{-1}xg \in Z(\mathfrak{l})^{\mathrm{reg}}\right\}$$

Consider the projection on the first factor $p^{\mathrm{reg}} : \widetilde{\mathbb{Y}}^{\mathrm{reg}}_{L,P} \to \mathfrak{gl}_n$, denote $\mathbb{Y}^{\mathrm{reg}}_{L,P}$ its image. This image consists of semisimple elements with r distinct eigenvalues with multiplicities $c_1, \dots, c_r$. Consider the relative Weyl group $W_{\mathrm{GL}_n}(L) = N_{\mathrm{GL}_n}(L)/L$, and for each $w \in W_{\mathrm{GL}_n}(L)$ chose a representative $\dot{w} \in N_{\mathrm{GL}_n}(L)$. This relative Weyl group acts on $Z(\mathfrak{l})$ by

$$w.\xi := \dot{w}\,\xi\,\dot{w}^{-1}.$$

Consider the fiber product

$$\begin{array}{ccc} Z(\mathfrak{l})^{\mathrm{reg}} & \longleftarrow & \mathbb{Y}^{\mathrm{reg}}_{L,P} \times_{Z(\mathfrak{l})^{\mathrm{reg}}/W_{\mathrm{GL}_n}(L)} Z(\mathfrak{l})^{\mathrm{reg}} \\ \downarrow & & \downarrow \\ Z(\mathfrak{l})^{\mathrm{reg}}/W_{\mathrm{GL}_n}(L) & \xleftarrow{\chi} & \mathbb{Y}^{\mathrm{reg}}_{L,P} \end{array}$$

with χ the characteristic polynomial. Note that the following map is an isomorphism

$$\begin{array}{ccc} \widetilde{\mathbb{Y}}^{\mathrm{reg}}_{L,P} & \to & \mathbb{Y}^{\mathrm{reg}}_{L,P} \times_{Z(\mathfrak{l})^{\mathrm{reg}}/W_{\mathrm{GL}_n}(L)} Z(\mathfrak{l})^{\mathrm{reg}} \\ (x, gL) & \mapsto & (x, g^{-1}xg) \end{array} \tag{3.32}$$

Therefore the $W_{\mathrm{GL}_n}(L)$ action on $Z(\mathfrak{l})^{\mathrm{reg}}$ induces an action on $\widetilde{\mathbb{Y}}^{\mathrm{reg}}_{L,P}$. It is given explicitly by

$$w.(x, gL) = (x, g\dot{w}^{-1}L).$$

Then

$$\widetilde{\mathbb{Y}}^{\mathrm{reg}}_{L,P} \xrightarrow{p^{\mathrm{reg}}} \mathbb{Y}^{\mathrm{reg}}_{L,P}$$

is a Galois cover with group $W_{\mathrm{GL}_n}(L)$. This relative Weyl group acts on the push forward of the constant sheaf $p^{\mathrm{reg}}_* \overline{\mathbb{Q}}_\ell$. Define

$$\widetilde{\mathbb{Y}}_{L,P} = \left\{ (x, gP) \in \mathfrak{gl}_n \times \mathrm{GL}_n/P \;\middle|\; g^{-1}xg \in Z(\mathfrak{l}) + \mathfrak{u}_P \right\}$$

Remark 3.4.17. *An element $gP \in \mathrm{GL}_n/P$ identifies with a partial flag*

$$0 = E_r \subset E_{r-1} \subset \cdots \subset E_1 \subset \mathbb{K}^n$$

such that $\dim E_{i-1}/E_i = c_i$ for all $1 \leq i \leq r$. Indeed GL_n acts transitively on such flags and the stabilizer is P. Then a point (x, gP) in $\widetilde{\mathbb{Y}}_{L,P}$ consists of an endomorphism $x \in \mathfrak{gl}_n$ and a partial flag gP preserved by x such that x acts as a scalar on E_{i-1}/E_i for all $1 \leq i \leq r$.

Denote $\mathbb{Y}_{L,P}$ the image of the projection to the first factor $p : \widetilde{\mathbb{Y}}_{L,P} \to \mathfrak{gl}_n$. Note that the map p is proper. The following theorem is a particular case of [Lus84, Lemma 4.3 and Proposition 4.5]. It can be seen as a generalization of Borho-MacPherson result.

Theorem 3.4.18. *$\mathbb{Y}^{\mathrm{reg}}_{L,P}$ is an open, dense, smooth subset of $\mathbb{Y}_{L,P}$ and the following square is cartesian*

$$\begin{array}{ccc} \widetilde{\mathbb{Y}}^{\mathrm{reg}}_{L,P} & \xrightarrow{i} & \widetilde{\mathbb{Y}}_{L,P} \\ {\scriptstyle p^{\mathrm{reg}}}\downarrow & & \downarrow{\scriptstyle p} \\ \mathbb{Y}^{\mathrm{reg}}_{L,P} & \hookrightarrow & \mathbb{Y}_{L,P} \end{array} \tag{3.33}$$

with i the map $(x, gL) \to (x, gP)$. Moreover $p_! \overline{\mathbb{Q}}_\ell = IC^\bullet(\mathbb{Y}_{L,P}, p^{\mathrm{reg}}_! \overline{\mathbb{Q}}_\ell)$ so that $W_{\mathrm{GL}_n}(L)$ acts on $p_* \overline{\mathbb{Q}}_\ell$.

Remark 3.4.19. *p^{reg} is a Galois cover and i an open embedding so that the dimensions can be easily computed:*

$$\dim \mathrm{Y}_{L;P} = \dim \widetilde{\mathrm{Y}}_{L;P} = \dim \widetilde{\mathrm{Y}}^{\mathrm{reg}}_{L;P} = \dim \mathrm{GL}_n \quad \dim L + \dim Z(L): \qquad (3.34)$$

Let us describe the relation with the resolution of closure of adjoint orbits introduced in 3.3.8. Let $\quad 2\ Z(\mathrm{l})$ and $M := Z_{\mathrm{GL}_n}(\ )$. Then use the same notations as in 3.3.3 so that $M = \mathrm{GL}\ $ for $\quad$ a partition of n. Moreover $L \quad M$ and the integers $(c_1; c_2; :::; c_r)$ are relabelled $(\ ^{1\,0}_{1};\ ^{1\,0}_{2}; :::)$ so that $^{i\,0}$ is a partition of $\ _i$. The inclusion $L \quad M$ comes from inclusions

$$\mathrm{GL}_{\ ^{i\,0}_{1}} \qquad \mathrm{GL}_{\ ^{i\,0}_{l_i}} \quad \mathrm{GL}_{\ _i}:$$

The resolution of the closure of $\overline{O}_{\_;}$ fits in the following diagram

$$\begin{array}{ccc} \widetilde{\mathrm{Y}}_{L;P} & \longleftarrow & \widetilde{\mathrm{Y}}_{L;P;} \\ \downarrow p & & \downarrow p \\ \mathrm{Y}_{L;P} & \longleftarrow \overline{O}_{\_;} = \bigsqcup_{\_\ \_} & O_{\_;} \end{array} \qquad (3.35)$$

The decomposition $\overline{O}_{\_;} = \bigsqcup_{\_\ \_} O_{\_;}$ actualy comes from a decomposition of $\mathrm{Y}_{L;P}$. Define

$$\mathrm{Y}^{M;\_}_{L;P} := \bigsqcup_{0 2 Z(\mathrm{m})^{\mathrm{reg}}} O_{\_;\ 0}:$$

This decomposition is similar to the one introduced by Shoji [Sho88].

Proposition 3.4.20. *$\mathrm{Y}^{M;\_}_{L;P}$ is smooth of dimension*

$$\dim \mathrm{Y}^{M;\_}_{L;P} = \dim O_{\_;} + \dim Z(\mathrm{m}):$$

Then $\mathrm{Y}_{L;P}$ admits the following decomposition

$$\mathrm{Y}_{L;P} = \bigsqcup_{M} \bigsqcup_{\_\ \_} \mathrm{Y}^{M;\_}_{L;P}:$$

The first union is over the set of centralizer of elements $\quad 2\ Z(\mathrm{l})$. In the second union, $\_$ depends on M as previously described. The unique part indexed by $M = L$ is $\mathrm{Y}^{\mathrm{reg}}_{L;P}$.

Proof. Denote by $Z_{\_}$ the centralizer in GL_n of the element $J_{\_;}$ in $O_{\_;}$ (see Notations 3.3.1). Then there is a natural finite cover

$$\begin{array}{ccc} Z(\mathrm{m})^{\mathrm{reg}} \quad \mathrm{GL}_n = Z_{\_} & ! & \mathrm{Y}^{M;\_}_{L;P} \\ (\ 0; gZ_{\_}) & 7! & gJ_{\_;\ 0}\, g^{\ 1} \end{array}$$

Therefore $\mathrm{Y}^{M;\_}_{L;P}$ is smooth and

$$\dim \mathrm{Y}^{M;\_}_{L;P} = \dim O_{\_;} + \dim Z(\mathrm{m}):$$

□

3.5 Character varieties and their additive counterpart

In this section the main objects studied in this thesis are introduced.

3.5.1 Character varieties

Let $\overline{\Sigma}$ be a compact Riemann surface of genus g. Consider the punctured Riemann surface $\Sigma = \overline{\Sigma} \setminus \{p_1, \dots, p_k\}$ where p_j are distinct points on $\overline{\Sigma}$ called punctures. The field $\mathbb{K}$ is either $\mathbb{C}$ or an algebraic closure $\overline{\mathbb{F}}_q$ of a finite field $\mathbb{F}_q$ with q elements. Fix a non negative integer n. We are concerned by n-dimensional $\mathbb{K}$-representations of the fundamental group of Σ with prescribed monodromy around the punctures.

For each puncture, specify a conjugacy class C_{σ^j, μ^j}. The notations are the same as in previous section, with the addition of an upper index $1 \leq j \leq k$ labelling the punctures. σ^j is a diagonal matrix with diagonal coefficients

$$(\underbrace{\sigma^j_1, \dots, \sigma^j_1}_{|\mu^j_1|}, \dots, \underbrace{\sigma^j_{l_j}, \dots, \sigma^j_{l_j}}_{|\mu^j_{l_j}|})$$

and $\sigma^j_r \neq \sigma^j_s$ for $r \neq s$. Moreover, $\mu^j = \left(\mu^{j,1}, \dots, \mu^{j,l_j}\right)$ with $\mu^{j,r} \in \mathcal{P}_{|\mu^j_r|}$ the partition giving the size of the Jordan blocks of the eigenvalue σ^j_r.

A bold symbol is used to represent k-uple:

$$\begin{aligned} \boldsymbol{\mu} &:= \left(\mu^1, \dots, \mu^k\right) \\ \boldsymbol{\sigma} &:= \left(\sigma^1, \dots, \sigma^k\right) \\ \mathbf{C}_{\boldsymbol{\sigma},\boldsymbol{\mu}} &:= \left(C_{\sigma^1,\mu^1}, \dots, C_{\sigma^k,\mu^k}\right) \end{aligned} \tag{3.36}$$

The representations of the fundamental group of Σ with monodromy around p_j in the closure $\overline{C}_{\sigma^j,\mu^j}$ form the following affine variety

$$R_{\overline{\mathbf{C}}_{\boldsymbol{\sigma},\boldsymbol{\mu}}} := \Big\{ (A_1, B_1, \dots, A_g, B_g, X_1, \dots, X_k) \in \mathrm{GL}_n^{2g} \times \overline{C}_{\sigma^1,\mu^1} \times \cdots \times \overline{C}_{\sigma^k,\mu^k} \Big| \; A_1 B_1 A_1^{-1} B_1^{-1} \dots A_g B_g A_g^{-1} B_g^{-1} X_1 \dots X_k = \mathrm{Id} \Big\}.$$

The group GL_n acts by simultaneous conjugation on $R_{\overline{\mathbf{C}}_{\boldsymbol{\sigma},\boldsymbol{\mu}}}$

$$g.(A_1, \dots, B_g, X_1, \dots, X_k) = \left(gA_1g^{-1}, \dots, gB_gg^{-1}, gX_1g^{-1}, \dots, gX_kg^{-1}\right).$$

The center of GL_n acts trivially so this action factors through an action of PGL_n.

Definition 3.5.1 (Character variety). *The character variety we are interested in is the following GIT quotient*

$$\mathcal{M}_{\overline{\mathbf{C}}_{\boldsymbol{\sigma},\boldsymbol{\mu}}} := R_{\overline{\mathbf{C}}_{\boldsymbol{\sigma},\boldsymbol{\mu}}} /\!/ \mathrm{PGL}_n := \mathrm{Spec}\, \mathbb{K}\left[R_{\overline{\mathbf{C}}_{\boldsymbol{\sigma},\boldsymbol{\mu}}}\right]^{\mathrm{PGL}_n}.$$

It is an affine variety with regular functions the PGL_n*-invariants functions on* $R_{\overline{\mathbf{C}}_{\boldsymbol{\sigma},\boldsymbol{\mu}}}$.

Under some generecity assumptions, the PGL_n action is free.

Definition 3.5.2 (Generic conjugacy classes). *Denote $(^{j})$ the multiset of eigenvalues of ^{j} repeated according to multiplicities. $^{j}_{r}$ appears exactly $^{j}_{r}$ times in the multiset $(^{j})$. The k-uple of conjugacy classes $C_{;}$ is generic if and only if it satisfy the two following conditions*

1. $$\prod_{j=1}^{k} \prod_{\in (^{j})} = 1$$

2. *For any $r \quad n \quad 1$, for all $(R_1;\dots;R_k)$ with $R_j \quad (^{j})$ of size r*

$$\prod_{j=1}^{k} \prod_{\in R_j} \neq 1$$

Throughout the thesis, every character varieties considered are assumed to have generic conjugacy classes at the punctures.

Remark 3.5.3. *If the k-uple of conjugacy classes $\overline{C}_{;}$ is generic and V is a non-zero subspace of $\mathbb{K}^n$ stable by some elements $X_j \in C_{\_j; j}$ such that*

$$\prod_{j=1}^{k} \det(X_{j|V}) = 1$$

then $V = \mathbb{K}^n$.

Definition 3.5.4. *Let $R_{C;} := R_{\overline{C};} \cap \left(\mathrm{GL}_n(\mathbb{K})^{2g} \quad \prod_{j=1}^{k} C_{\_j; j}\right)$ and $M_{C;}$ the image of $R_{C;}$ in $R_{\overline{C};}$.*

Proposition 3.5.5. *If $C_{;}$ is generic then points of $R_{C;}$ correspond to irreducible representations of the fundamental group of the punctured Riemann surface .*

Proof. Let V be a subrepresentation of $(A_1;B_1;\dots;A_g;B_g;X_1;\dots;X_k) \in R_{C;}$. Then V is stable by thoses matrices and the equation defining $R_{C;}$ restricts to

$$(A_{1|V};B_{1|V})\dots(A_{g|V};B_{g|V})X_{1|V}\dots X_{k|V} = \mathrm{Id}_V:$$

Taking determinant, the genericity implies $V = 0$ or $V = \mathbb{K}^n$. □

We recall a proposition from [Let13], and [HLR11] for the semisimple case.

Proposition 3.5.6. *If $C_{;}$ is generic then $R_{C;}$ is non-singular, when non-empty its dimension is*

$$\dim R_{C;} = 2gn^2 \quad n^2 + 1 + \sum_{j=1}^{k} \dim C_{\_j; j}:$$

Proof. The proof combines the one of theorem 2.2.5 in [HR08] and proposition 5.2.8 in [EOR04].
$R_{\mathcal{C}} = \quad ^{1}(I_n)$ where is the map

$$\Phi: \quad \mathrm{GL}_n(\mathbb{K})^{2g} \times \prod_{i=1}^{k} \overline{C}_i \to \mathrm{SL}_n(\mathbb{K})$$
$$(A_1, B_1, \dots, A_g, B_g, X_1, \dots, X_k) \mapsto (A_1, B_1) \dots (A_g, B_g) X_1 \dots X_k.$$

It is enough to check that the differential $d\Phi_z$ of this map at a point

$$z = (A_1, B_1, \dots, A_g, B_g, X_1, \dots, X_k) \in R_{\mathcal{C}}$$

is a surjective map between tangent space. The tangent space of $\mathrm{SL}_n(\mathbb{K})$ is the Lie algebra $\mathrm{sl}_n(\mathbb{K})$. The tangent space of C_i at X_i is made of Lie brackets $[r_i, X_i]$ for $r_i \in \mathrm{gl}_n(\mathbb{K})$. Hence an element of the tangent space at z reads

$$v = (h_1, l_1, \dots, h_g, l_g, [r_1, X_1], \dots, [r_k, X_k])$$

for $r_i, h_j, l_j \in \mathrm{gl}_n(\mathbb{K})$. First we compute the differential with respect to A_j of the commutator $A_j \mapsto A_j B_j A_j^{-1} B_j^{-1}$

$$d_j(h_j) := h_j B_j A_j^{-1} B_j^{-1} - A_j B_j A_j^{-1} h_j A_j^{-1} B_j^{-1}$$

similarly with respect to B_j

$$d'_j(l_j) := A_j l_j A_j^{-1} B_j^{-1} - A_j B_j A_j^{-1} B_j^{-1} l_j B_j^{-1}.$$

We use the usual rule to differentiate a product of matrix

$$\begin{aligned} d\Phi_z(v) &= \sum_{i=1}^{k} (A_1, B_1) \dots (A_g, B_g) X_1 \dots X_{i-1} [r_i, X_i] X_{i+1} \dots X_k \\ &+ \sum_{j=1}^{g} (A_1, B_1) \dots (A_{j-1}, B_{j-1}) d_j(h_j) (A_{j+1}, B_{j+1}) \dots (A_g, B_g) X_1 \dots X_k \\ &+ \sum_{j=1}^{g} (A_1, B_1) \dots (A_{j-1}, B_{j-1}) d'_j(l_j) (A_{j+1}, B_{j+1}) \dots (A_g, B_g) X_1 \dots X_k \end{aligned}$$

the first, respectively second and third lines correspond to differenciation with respect to X_i respectively A_j and B_j. Now we use that z satisfies the equation defining $R_{\mathcal{C}}$

$$\begin{aligned} d\Phi_z(v) &= \sum_{i=1}^{k} (X_i \dots X_k)^{-1} [r_i, X_i] X_{i+1} \dots X_k \\ &+ \sum_{j=1}^{g} (A_1, B_1) \dots (A_{j-1}, B_{j-1}) d_j(h_j) \left((A_1, B_1) \dots (A_{j-1}, B_{j-1})(A_j, B_j)\right)^{-1} \\ &+ \sum_{j=1}^{g} (A_1, B_1) \dots (A_{j-1}, B_{j-1}) d'_j(l_j) \left((A_1, B_1) \dots (A_{j-1}, B_{j-1})(A_j, B_j)\right)^{-1} \end{aligned}$$

We rewrite to exhibit some conjugation

$$\begin{aligned} d\Phi_z(v) &= \sum_{i=1}^{k} (X_{i+1} \dots X_k)^{-1} X_i^{-1} [r_i, X_i] X_{i+1} \dots X_k \\ &+ \sum_{j=1}^{g} (A_1, B_1) \dots (A_{j-1}, B_{j-1}) d_j(h_j) (A_j, B_j)^{-1} \left((A_1, B_1) \dots (A_{j-1}, B_{j-1})\right)^{-1} \\ &+ \sum_{j=1}^{g} (A_1, B_1) \dots (A_{j-1}, B_{j-1}) d'_j(l_j) (A_j, B_j)^{-1} \left((A_1, B_1) \dots (A_{j-1}, B_{j-1})\right)^{-1} \end{aligned}$$

. To prove that this differential is surjective we take $u \in \mathrm{sl}_n(\mathbb{K})$ such that for any v tangent to z we have $\mathrm{tr}(d\ _z(v)u) = 0$ and show that $u = 0$. For any $r_i \in \mathrm{gl}_n(\mathbb{K})$ we must have

$$\mathrm{tr}((X_{i+1} \dots X_k)^{-1} X_i^{-1} [r_i, X_i] X_{i+1} \dots X_k u) = 0 \tag{3.37}$$

Let us prove by recursion that u commutes with X_i for all $1 \leq i \leq k$.
When $i = k$

$$0 = \mathrm{tr}(X_k^{-1}[r_k, X_k]u) = \mathrm{tr}(X_k^{-1} r_k X_k u) - \mathrm{tr}(r_k u) = \mathrm{tr}(r_k X_k u X_k^{-1} - r_k u)$$

in last equality we use the cyclicity of the trace. This must be true for any r_k so that $u = X_k u X_k^{-1}$. Now let us assume that u commutes with X_m for any $i < m \leq k$, (3.37) implies

$$\mathrm{tr}(X_i^{-1}[r_i, X_i]u) = 0$$

so that u commutes with X_i. Similarly u commutes with A_j and B_j. By genericity and Schur lemma this implies that u is a scalar matrix, as it is in $\mathrm{sl}_n(\mathbb{K})$ it must be zero which achieves the proof. □

Proposition 3.5.7 (Stratification of $\mathcal{M}_{\overline{C}}$, [Let13] Corollary 3.6). *We assume C is generic. The stratification of Zariski closure of conjugacy classes induces a stratification of the character variety:*

$$\mathcal{M}_{\overline{C}} = \bigsqcup \mathcal{M}_{C} .$$

The union is over $= (\ ^1, \dots, \ ^k)$ *with* $\ ^j = (\ ^{j,1}, \dots, \ ^{j,l_j})$ *such that*

$$\ ^{j,i} \ \ ^{j,i}, \text{ for all } 1 \leq j \leq k, \ 1 \leq i \leq l_j$$

with the dominance order on P_i^j.

Moreover if $\mathcal{M}_{\overline{C}}$ is non empty, then $\mathcal{M}_{C}$ is also non empty. Therefore when $\mathcal{M}_{\overline{C}}$ is non empty, its dimension is

$$\dim \mathcal{M}_{\overline{C}} = d := n^2(2g - 2) + 2 + \sum_{j=1}^{k} \dim C_{\ ^j} . \tag{3.38}$$

3.5.2 Additive analogous of Character varieties

Instead of the multiplicative equation in GL_n defining $\mathcal{R}_{\overline{C}}$, one can consider additive equation in gl_n. This is called the additive Deligne-Simpson problem. It was studied by Crawley-Boevey [Cra03b], [Cra06] in the case $g = 0$, by Hausel, Letellier and Rodriguez-Villegas [HLR11] for semisimple adjoint orbits and by Letellier [Let11] in general.

As before, notation from 3.3.1 are used, and a k-uple of adjoint orbits in gl_n is introduced:

$$\mathcal{O} := \left(\mathcal{O}_{\ ^1}, \dots, \mathcal{O}_{\ ^k}\right).$$

Consider the affine variety

$$\mathcal{V}_{\overline{\mathcal{O}}} := \left\{ (A_1, B_1, \dots, A_g, B_g, X_1, \dots, X_k) \in \mathrm{gl}_n^{2g} \times \overline{\mathcal{O}}_{\ ^1} \times \dots \times \overline{\mathcal{O}}_{\ ^k} \ \middle| \ \sum_{i=1}^{g} [A_i, B_i] + \sum_{j=1}^{k} X_j = 0 \right\}.$$

This is an a□ne variety acted upon by GL_n by coordinate-wise adjoint action. The center of GL_n acts trivialy so that the action factors through a PGL_n action. Consider the GIT quotient

$$Q_{\overline{O}\ ;} := V_{\overline{O}\ ;} \,//\, \mathrm{PGL}_n = \mathrm{Spec}\left(\mathrm{K}\left[V_{\overline{O}\ ;}\right]^{\mathrm{GL}_n}\right): \tag{3.39}$$

Definition 3.5.8 (Generic adjoint orbits). *Denote (^{j}) the multiset of eigenvalues of ^{j} repeated according to multiplicities. ^j_r appears exactly ^j_r times in the multiset (^{j}). The k-uple of adjoint orbits $O_{\ ;}$ is generic if and only if it satisfy the two following conditions*

1.
$$\sum_{j=1}^{k} \sum_{2\ (\ ^j)} = 0$$

2. *For any r n 1, for all $(R_1; : : : ; R_k)$ with R_j (^{j}) of size r*
$$\sum_{j=1}^{k} \sum_{2 R_j} \neq 0$$

.

Remark 3.5.9. *Contrarily to the multiplicative case, generic k-uple of adjoint orbits do not exist for every multiplicities ($^1; : : : ; ^k$). In particular if an integer $d > 1$ divides all ^{j} for 1 j k.*

Character varieties and their additive analogous share many properties in common. They have the same dimension and similar stratifications. Let

$$U_{O\ ;} := U_{\overline{O}\ ;} \cap \mathfrak{gl}_n^{2g} \quad O_{\_1;\ 1} \qquad O_{\_k;\ k}$$

and $Q_{O\ ;}$ the image of $U_{O\ ;}$ in $Q_{\overline{O}\ ;}$.

Proposition 3.5.10 (Stratification of $Q_{\overline{O}\ ;}$). *Assume $\overline{O}_{\ ;}$ is generic, then*

$$Q_{\overline{O}\ ;} = \bigsqcup Q_{O\ ;}$$

is a stratification of $Q_{\overline{O}\ ;}$. Moreover

$$\dim Q_{\overline{O}\ ;} = d \quad = n^2(2g \quad 2) + 2 + \sum_{j=1}^{k} \dim O_{\_j;\ j}:$$

3.5.3 Resolutions of character varieties

The resolutions of conjugacy classes introduced in 3.3.3 induce resolutions of character variety. As before we consider a generic k-uple of conjugacy classes

$$C_{\ ;} = \left(C_{\_1;\ 1}; : : : ; C_{\_k;\ k}\right)$$

and upper indices $1 \quad j \quad k$ label the puncture. As usual ^{j} is a diagonal matrix with diagonal coe□cients

$$(\underbrace{{}^{j}_{1}, \ldots, {}^{j}_{1}}_{{}^{j}_{1}}, \ldots, \underbrace{{}^{j}_{l_j}, \ldots, {}^{j}_{l_j}}_{{}^{j}_{l_j}}).$$

Let $M^{j} := Z_{\mathrm{GL}_n}(\ ^{j})$ then with Notation 3.3.6

$$M^{j} = \mathrm{GL}_{\ j}$$

As usual $^{j,i} \in P_{\ ^{j}_{i}}$ is the Jordan type of $^{j}_{i}$. Denote by $^{j,i\,0} = \left(^{j,i\,0}_{1}, ^{j,i\,0}_{2}, \ldots \right)$ the transposed partition. Let $L^{j} \quad M^{j}$ the subgroup of diagonal matrices as in 3.3.3

$$L^{j} = \underbrace{\mathrm{GL}_{\ ^{j,1\,0}_{1}} \quad \mathrm{GL}_{\ ^{j,1\,0}_{2}} \quad \ldots}_{\mathrm{GL}_{\ ^{j}_{1}}} \qquad \underbrace{\mathrm{GL}_{\ ^{j,l_j\,0}_{1}} \quad \mathrm{GL}_{\ ^{j,l_j\,0}_{2}} \quad \ldots}_{\mathrm{GL}_{\ ^{j}_{l_j}}}$$

Let $\widetilde{\mathrm{X}}_{L^{j},P^{j},\ ^{j}}$ a resolution of $\overline{C}_{\ ^{j},\ ^{j}}$ as constructed in 3.3.3. Let

$$\widetilde{\mathrm{X}}_{\mathbf{L},\mathbf{P},} := \prod_{1 \quad j \quad k} \widetilde{\mathrm{X}}_{L^{j},P^{j},\ ^{j}}.$$

Letellier [Let13] constructed resolutions of singularities for character varieties.

Definition 3.5.11 (Resolutions of character variety). *Define*

$$\widetilde{M}_{\mathbf{L},\mathbf{P},} := \Big\{(A_i, B_i)_{1 \quad i \quad g}, (X_j, g_j P^{j})_{1 \quad j \quad k} \in \mathrm{GL}_n^{2g} \quad \widetilde{\mathrm{X}}_{\mathbf{L},\mathbf{P},}$$
$$\Big| A_1 B_1 A_1^{\ 1} B_1^{\ 1} \ldots B_g^{\ 1} X_1 \ldots X_k = \mathrm{Id}\Big\} // \mathrm{PGL}_n. \quad (3.40)$$

The maps $p^{\ j} : \widetilde{\mathrm{X}}_{L^{j},P^{j},\ ^{j}} \to \overline{C}_{\ ^{j},\ ^{j}}$ *induce a map*

$$p \ : \widetilde{M}_{\mathbf{L},\mathbf{P},} \to M_{\overline{C}\ ,}$$

this map is a resolution of singularity.

Next theorem is a particular case of a result of Letellier [Let13, Theorem 5.4]

Theorem 3.5.12.

$$p_! \quad [d\ ] = \bigoplus_{\preceq} A_{\ ',} \quad \underline{IC}_{M_{\overline{C}\ ,}}$$

and in terms of cohomology:

$$H_c^{i+d}\left(\widetilde{M}_{\mathbf{L},\mathbf{P},}\ ;\ \right) = \bigoplus_{\preceq} A_{\ ',} \quad IH_c^{i+d}\left(M_{\overline{C}\ ,}\ ;\ \right). \quad (3.41)$$

The multiplicity space $A_{\ ',}$ *will be described in the remaining of the section.*

Like the resolutions of closure of conjugacy classes, the resolution of character varieties come with a Weyl group action *à la* Springer. First we present the Weyl groups involved. The Weyl group of M^j is $W_{M^j} = N_{M^j}(T)/T$ then

$$W_{M^j} = S_{\mu^j}.$$

For $\underline{\mu}^j = (\mu^{j,1}, \dots, \mu^{j,l_j}) \in \mathcal{P}_{\mu^j}$ and $V_{\mu^{j,i}}$ the irreducible representation of $S_{\mu^j_i}$ indexed by $\mu^{j,i}$ let

$$V_{\underline{\mu}^j} := \bigotimes_{i=1}^{l_j} V_{\mu^{j,i}}$$

it is an irreducible representation of W_{M^j}.

The Weyl group of L^j is $W_{L^j} = N_{L^j}(T)/T$, it is a subgroup of W_{M^j}

$$W_{L^j} = \underbrace{S_{\mu^{j,1'}_1} \times S_{\mu^{j,1'}_2} \times \dots}_{S_{\mu^j_1}} \times \dots \times \underbrace{S_{\mu^{j,l_j'}_1} \times S_{\mu^{j,l_j'}_2} \times \dots}_{S_{\mu^j_{l_j}}}$$

The sign representation for this Weyl group is

$$\varepsilon_{\underline{\mu}^{j'}} := \bigotimes_{i=1}^{l_j} \bigotimes_{r} \varepsilon_{\mu^{j,i'}_r}.$$

It was previously denoted only by ε, the index is now added to remind the form of the Weyl group $W_{L^j} = S_{\underline{\mu}^{j'}}$.

Definition 3.5.13. *The multiplicity space relative to the j-th puncture is*

$$A_{\underline{\mu}^{j'}, \underline{\mu}^j} = \mathrm{Hom}_{W_{M^j}}\left(\mathrm{Ind}_{W_{L^j}}^{W_{M^j}} \varepsilon_{\underline{\mu}^{j'}}, V_{\underline{\mu}^j}\right).$$

Remark 3.5.14. *The expression is particularly simple when L^j is a torus T. Then the multiplicity space is just $V_{\underline{\mu}^j}$.*

Define $W_{\boldsymbol{M}} := \prod_{j=1}^k W_{M^j}$ and similarly $W_{\boldsymbol{L}} := \prod_{j=1}^k W_{L^j}$. The parameter $\omega = (\underline{\mu}^1, \dots, \underline{\mu}^k) \in \mathcal{P}_{\mu^1} \times \dots \times \mathcal{P}_{\mu^k}$ indexes irreducible representations of

$$W_{\boldsymbol{M}} = \prod_{j=1}^k \prod_{i=1}^{l_j} S_{\mu^j_i}.$$

V_ω is the following irreducible representation of $\prod_{j=1}^k \prod_{i=1}^{l_j} S_{\mu^j_i}$

$$V_\omega = \bigotimes_{j=1}^k \bigotimes_{i=1}^{l_j} V_{\mu^{j,i}}. \tag{3.42}$$

Now $\varepsilon_{\omega'}$ is the sign representation of $W_{\boldsymbol{L}}$, namely

$$\varepsilon_{\omega'} := \bigotimes_{j=1}^k \varepsilon_{\underline{\mu}^{j'}}$$

The description of the multiplicity space for resolutions of closure of conjugacy classes (Theorem 3.4.8) extends to $A_{\omega',\omega}$:

Notations 3.5.15. *The multiplicity space $A_{\ \prime;}$ is*

$$A_{\ \prime;} = \mathrm{Hom}_{W_{\boldsymbol{M}}}\left(\mathrm{Ind}_{W_{\boldsymbol{L}}}^{W_{\boldsymbol{M}}} \ \ \prime; V\ \right) = \bigotimes_{j=1}^{k} A_{\underline{\ }j^0;\underline{\ }j}$$

Everything in this section also apply to the additive case.

Definition 3.5.16. *For $O_{\ ;}$ a generic k-uple of adjoint orbits, define*

$$\widetilde{Q}_{\boldsymbol{L};\boldsymbol{P};} := \Big\{(A_i;B_i)_{1\ i\ g};(X_j;g_jP^j)_{1\ j\ k}\ 2\ \mathrm{GL}_n^{2g}\ \ \widetilde{\mathrm{Y}}_{\boldsymbol{L};\boldsymbol{P};} \ \Big|\ \sum_{i=1}^{g}[A_i;B_i] + \sum_{j=1}^{k} X_j = 0\Big\} == \mathrm{PGL}_n: \quad (3.43)$$

The maps $p^{\ j}: \widetilde{\mathrm{Y}}_{L^j;P^j;\ j}\ !\ \overline{O}_{\underline{\ }j;\ j}$ induce a resolution of singularities:

$$p\ : \widetilde{Q}_{\boldsymbol{L};\boldsymbol{P};}\ \ !\ Q_{\overline{O}\ ;}:$$

Theorem 3.5.17.

$$p_!\ \ [d\ ] = \bigoplus_{\preceq} A_{\ \prime;}\ \ \ \underline{IC}_{Q_{\overline{O}\ ;}}$$

and in terms of cohomology:

$$H_c^{i+d}\left(\widetilde{Q}_{\boldsymbol{L};\boldsymbol{P};}\ ;\ \right) = \bigoplus_{\preceq} A_{\ \prime;}\ \ \ IH_c^{i+d}\left(M_{\overline{O}\ ;}\ ;\ \right): \quad (3.44)$$

3.5.4 Relative Weyl group actions

An interesting feature of the multiplicity spaces $A_{\ \prime;}$ is that they carry a relative Weyl group action. It is constructed by Letellier [Let11, 6.1, 6.2]. The relative Weyl group is

$$W_{\boldsymbol{M}}(L) := \prod_{j=1}^{k} W_{M^j}(L^j)$$

with $W_{M^j}(L^j)$ the relative Weyl groups described in 3.4.3. Their action on the multiplicity spaces provide a $W_{\boldsymbol{M}}(L)$-action on $A_{\ \prime;}$. As usual an index $1\ \ j\ \ k$ is added to label the puncture. Conjugacy classes in $W_{\boldsymbol{M}}(L)$ are labelled by elements

$$= (\ ^j)_{1\ j\ k}$$

with $\ ^j\ 2 \prod_{\substack{1\ i\ l_j \\ 1\ r\ k_{j;i}}} P_{m_r^{j;i}}$ as in 3.4.3 with an additional index j for the puncture.

$$^j = (\ ^{j;i;r})_{\substack{1\ i\ l_j \\ 1\ r\ k_{j;i}}}\ 2 \prod_{\substack{1\ i\ l_j \\ 1\ r\ k_{j;i}}} P_{m_r^{j;i}}:$$

Notations 3.4.15 extend to k-uple:

Notations 3.5.18.

$$\widetilde{h} \;\; := \prod_{j=1}^{k} \prod_{i=1}^{l_j} s_{! \;\; j;i^0} [X_j]$$

and

$$r(\;\;) := \sum_{j=1}^{k} \sum_{i=1}^{l_j} r(! \;\; {}_{j;i}):$$

Proposition 3.5.19. *The relative Weyl group $W_{\boldsymbol{M}}(L)$ acts on $A \;\; {}_{\prime;}$ and the trace of an element in the conjugacy class indexed by is*

$$\mathrm{tr}(\;\;; A \;\; {}_{\prime;}\;) = \prod_{j=1}^{k} \prod_{i=1}^{l_j} c_{\;\; j;i}^{\;\; j;i}:$$

This proposition will be useful together with the decomposition of the cohomology of resolutions of character varieties (3.44).

Theorem 3.5.20. *Let $C \;\; {}_{;}$ a generic k-uple of conjugacy classes and $\widetilde{M}_{\boldsymbol{L};\boldsymbol{P};}$ the resolution of $M_{\overline{C} \;\; ;}$. The relative Weyl group $W_{\boldsymbol{M}}(L)$ acts on the cohomology of $\widetilde{M}_{\boldsymbol{L};\boldsymbol{P};}$. The trace of an element in the conjugacy class indexed by is*

$$\mathrm{tr}\left(\;\;; H_c^{i+d}\left(\widetilde{M}_{\boldsymbol{L};\boldsymbol{P};} \;\;;\;\;\right)\right) = \sum_{\preceq} \mathrm{tr}(\;\;; A \;\; {}_{\prime;}\;) H_c^{i+d}\left(M_{\overline{\boldsymbol{C}} \;\; ;} \;\;;\;\;\right):$$

3.6 Cohomology of character varieties: some results and conjectures

3.6.1 Conjectural formula for the mixed-Hodge polynomial

Hausel, Letellier and Rodriguez-Villegas [HLR11] introduced a generating function conjecturally encoding mixed-Hodge structure on the cohomology of character varieties. Let g be a non-negative integer, the genus, and k a positive integer, the number of punctures.

Definition 3.6.1 (Generating function and Hausel-Letellier-Villegas kernel). *The k-points, genus g Cauchy function is defined by*

$$\;{}_k^g(z;w) := \sum_{2P} H \;\; (z;w) \prod_{i=1}^{k} \tilde{H} \;\; \left[X_i; z^2; w^2\right] s^{j \;\; j} \tag{3.45}$$

with

$$H \;\; (z;w) := \prod \frac{\left(z^{2a+1} \quad w^{2l+1}\right)^{2g}}{\left(z^{2a+2} \quad w^{2l}\right)\left(z^{2a} \quad w^{2l+2}\right)}: \tag{3.46}$$

The degree n Hausel-Letellier-Villegas kernel is defined by

$$\mathbb{H}_n^{HLV}(z;w) := (z^2 \quad 1)(1 \quad w^2)\,\mathrm{Log} \;\; {}_k^g(z;w)\Big|_{s^n}:$$

The generating function $^{g}_{k}(z;w)$ belongs to the lambda ring $\mathrm{Sym}[X_1;:::;X_k][[s]]$. This Cauchy function is known to encode cohomological information about character varieties and quiver varieties, let us recall these various conjectures and theorems.

When the conjugacy classes are semisimple Hausel, Letellier, Rodriguez-Villegas stated a conjecture for the mixed-Hodge polynomial of the character variety [HLR11]. They proved the specialisation corresponding to the E-polynomial. Letellier generalized this conjecture to arbitrary types and intersection cohomology.

Let $C_{\ ;}$ a k-uple of generic conjugacy classes. Then $= (\_^1;:::;\_^k)$ with $\_^j = (\ ^{j;1};:::;\ ^{j;l_j})$. The transposition of the partition $^{j;i} \in P_{\ ^j_i}$ is denoted by $^{j;i^0}$ and

$$s_{\ '} := \prod_{j=1}^{k}\prod_{i=1}^{l_j} s_{\ ^{j;i^0}}[X_j] \tag{3.47}$$

Conjecture 3.6.2 (Letellier [Let13], Conjecture 1.5). *For $C_{\ ;}$ a generic k-uple of conjugacy classes, the mixed-Hodge polynomial of the character variety $M_{\overline{C}\ ;}$ is*

$$IH_c(M_{\overline{C}\ ;};q;v) = (v\sqrt{q})^{d}\ \left\langle s_{\ '};\mathbb{H}_n^{HLV}\left(\frac{1}{\sqrt{q}};v\sqrt{q}\right)\right\rangle$$

with $q = xy$. In particular after specializing to the Poincaré polynomial

$$P_c(M_{\overline{C}\ ;};v) = v^{d}\ \langle s_{\ '};\mathbb{H}_n^{HLV}(\ 1;v)\rangle: \tag{3.48}$$

Some specializations of this conjecture are already proved. The formula obtained after specialization to the E-polynomial is proved by Hausel, Letellier and Rodriguez-Villegas [HLR11] for semisimple conjugacy classes and by Letellier [Let13] for any type of conjugacy classes. The proof relies on counting points of character varieties over finite fields and representation theory of $\mathrm{GL}_n(\mathbb{F}_q)$. The formula obtained after specialization to the Poincaré polynomial is proved by Schiffmann [Sch16] for one central conjugacy class and by Mellit [Mel17a] for any k-uple of semisimple conjugacy classes. The proof relies on counting point of moduli space of stable parabolic Higgs bundles over finite field.

For the additive case the Poincaré polynomial is known, the cohomology is pure so that it is obtained by counting points over finite fields. It was computed in the semisimple case by Hausel, Letellier and Rodriguez-Villegas [HLR11], for any types of adjoint orbits by Letellier [Let11].

Theorem 3.6.3. *Let $O_{\ ;}$ a generic k-uple of adjoint orbits. The Poincaré polynomial for compactly supported intersection cohomology of $Q_{\overline{O}\ ;}$ is*

$$P_c\left(Q_{\overline{O}\ ;};v\right) = v^{d}\ \langle s_{\ '};\mathbb{H}_n^{HLV}(0;v)\rangle:$$

3.6.2 Poincaré polynomial of character varieties with semisimple conjugacy classes at punctures

Let us recall Mellit's result and check that it is a particular case of the conjecture. Let $\mathcal{S} = (S_1;:::;S_k)$ a generic k-uple of semisimple conjugacy classes. Then S_j has the form $C_{\_j;\ j}$ with $\_\ ^j = (1\ ^j_1;:::;1\ ^j_{l_j})$ and

$$s\ ,\ = \prod_{j=1}^{k}\prod_{i=1}^{l_j} s_{(\ ^j_i)}[X_j] = \prod_{j=1}^{k} h\ _j[X_j] = h\ :$$

Lemma 3.6.4. *If $M_{\mathcal{S}}$ is non-empty, its dimension is*

$$d_{\mathcal{S}} = n^2(2g + k \quad 2) + 2 \quad \sum_{i;j}(\ ^i_j)^2 \tag{3.49}$$

which is even.

Proof. First note that the centralizer in GL_n of an element in S_i is isomorphic to $\prod_j \mathrm{GL}\ _{^i_j}$ so that

$$\begin{aligned} \dim S_i &= \dim \mathrm{GL}_n \quad \sum_j \dim \mathrm{GL}\ _{^i_j} \\ &= n^2 \quad \sum_j (\ ^i_j)^2 : \end{aligned}$$

Equation (3.49) then follows from the general formula (3.38). Reducing modulo 2

$$\begin{aligned} d_{\mathcal{S}} \quad & n^2k \quad \sum_{i;j}(\ ^i_j)^2 \quad \bmod 2 \\ & nk \quad \sum_{i;j}\ ^i_j \quad \bmod 2 \\ & 0 \quad \bmod 2 \end{aligned}$$

□

The conjecture from Hausel, Letellier, Rodriguez-Villegas [HLR11] for the mixed-Hodge structure of the character varieties with monodromies specified by $\mathcal{S}$ reads

$$IH_c(M_{\mathcal{S}};q;v) = (v\sqrt{q})^{d_{\mathcal{S}}}\left\langle h\ ;\mathbb{H}_n^{HLV}\left(\frac{1}{\sqrt{q}};v\sqrt{q}\right)\right\rangle :$$

Note that as the conjugacy classes are generic semisimple, the character variety is smooth and the intersection cohomology coincides with the usual cohomology. Then the specialization to compactly supported Poincaré polynomial of the conjecture is

$$P_c(M_{\mathcal{S}};v) = \sum_i v^i \dim H_c^i(M_{\mathcal{S}};\ \ ) = v^{d_{\mathcal{S}}}\left\langle h\ ;\mathbb{H}_n^{HLV}(\ \ 1;v)\right\rangle : \tag{3.50}$$

In order to compare this formula with Mellit's result we perform a change of variable $v = \frac{1}{\sqrt{u}}$

$$\sum_i(\ \ 1)^i u^{\frac{i}{2}} \dim H_c^i(M_{\mathcal{S}};\ \ ) = \left(\frac{1}{\sqrt{u}}\right)^{d_{\mathcal{S}}}\left\langle h\ ;\mathbb{H}_n^{HLV}\left(\ \ 1;\frac{1}{\sqrt{u}}\right)\right\rangle :$$

Note that $(z; w) = (w; z) = (w; z)$, moreover the dimension is even by Lemma 3.6.4, therefore the conjecture is equivalent to

$$\sum_i (1)^{2d_{\mathcal{S}}} {}^i u^{\frac{2d_{\mathcal{S}} \ i}{2}} \dim H_c^i(M_{\mathcal{S}}; \ ) = (^p \overline{u})^{d_{\mathcal{S}}} \left\langle h \ ; \mathrm{H}_n^{HLV} \left(\frac{1}{p \overline{u}}; 1 \right) \right\rangle :$$

By Poincaré duality this formula becomes

$$\sum_i (1)^i u^{\frac{i}{2}} \dim H^i(M_{\mathcal{S}}; \ ) = (^p \overline{u})^{d_{\mathcal{S}}} \left\langle h \ ; \mathrm{H}_n^{HLV} \left(\frac{1}{p \overline{u}}; 1 \right) \right\rangle :$$

Thus the Poincaré polynomial specialization of the conjecture is equivalent to the formula proved by Mellit [Mel17a, Theorem 7.12] and we have the following theorem.

Theorem 3.6.5. *For $S = (S_1; : : : ; S_k)$ a generic k-uple of semisimple conjugacy classes. If the multiplicities of the eigenvalues of S_j are given by a partition $^j \ 2 \ P_n$ for $1 \quad j \quad k$. Then the Poincaré polynomial of the character variety $M_{\mathcal{S}}$ is*

$$P_c(M_{\mathcal{S}}; v) = v^{d_{\mathcal{S}}} \langle h \ ; \mathrm{H}_n^{HLV}(\ 1; v) \rangle : \tag{3.51}$$

3.6.3 Weyl group actions on the cohomology

In 3.5.4 a Weyl group action on the cohomology of resolutions of character varieties was introduced. The conjecture about the mixed-Hodge structure also concerns this Weyl group action. We present the implications in terms of Poincaré polynomial using Notations 3.4.15 and 3.5.18.

Definition 3.6.6 (-twisted Poincaré polynomial). *$C \ ;$ is a generic k-uple of conjugacy classes and $\widetilde{M}_{\boldsymbol{L};\boldsymbol{P};}$ is the resolution of $M_{\overline{C} \ ;}$. For indexing a conjugacy class in $W_{\boldsymbol{M}}(L)$, the -twisted Poincaré polynomial of $\widetilde{M}_{\boldsymbol{L};\boldsymbol{P};}$ is*

$$P_c \left(\widetilde{M}_{\boldsymbol{L};\boldsymbol{P};} \ ; v \right) := \sum_i \mathrm{tr} \left(\ ; H_c^i \left(\widetilde{M}_{\boldsymbol{L};\boldsymbol{P};} \ ; \ \right) \right) v^i :$$

In the additive case, -twisted Poincaré polynomial were computed by Letellier [Let11, Corollary 7.4.3]. It is a consequence of Theorem 3.5.17 and Theorem 3.6.3.

Theorem 3.6.7. *Let $O \ ;$ a generic k-uple of adjoint orbits and $\widetilde{Q}_{\boldsymbol{L};\boldsymbol{P};}$ the resolution of $Q_{\overline{O} \ ;}$. Let representing a conjugacy class in the $W_{\boldsymbol{M}}(L)$ the -twisted Poincaré polynomial is*

$$\sum_i \mathrm{tr} \left(\ ; H_c^i(\widetilde{Q}_{\boldsymbol{L};\boldsymbol{P};} \ ; \ ) \right) v^i = (\ 1)^{r(\ )} v^{d} \left\langle \widetilde{h} \ ; \mathrm{H}_n^{HLV}(0; v) \right\rangle :$$

Remark 3.6.8. *The description of the Weyl group action is particularly simple when all the L^j are maximal torus. The notations $L = T$ and $P = B$ are used. Then $W_{\boldsymbol{M}}(T) = \prod_{j=1}^{k} \prod_{i=1}^{l_j} \mathrm{S}_{\ i}^{\ j}$. The irreducible representation V are indexed by $2 \ P_{\ 1} \qquad P_{\ k}$ as in (3.42). Then from 3.44 and the description of the action on the multiplicity spaces 3.5.15, 3.5.14, the isotypical component of type V is*

$$\mathrm{Hom}_{W_{\boldsymbol{M}}(\boldsymbol{T})} \left(V \ ; H_c^{i+d} \ \left(\widetilde{Q}_{\boldsymbol{T};\boldsymbol{B};} \ ; \ \right) \right) = H_c^{i+d} \ \left(Q_{\overline{O} \ ;} \ ; \ \right) :$$

In terms of Poincaré polynomial

$$v^{\ d} \sum_i v^i \dim \mathrm{Hom}_{W_{\boldsymbol{M}}(\boldsymbol{T})} \left(V \ ; H_c^{i+d} \ \left(\widetilde{Q}_{\boldsymbol{T};\boldsymbol{B};} \ ; \ \right) \right) = \langle s \ ; \mathrm{H}_n^{HLV}(0; v) \rangle$$

Remark 3.6.9. *It is also interesting to study the action of a Weyl group relative to a particuar puncture, for instance the first puncture. This will be used in 4.4.2 to describe some structure coe cients of an algebra spanned by Kostka polynomial. A particularly interesting case is when* L^1 *is a maximal torus and* $M^1 = \mathrm{GL}_n$*. Then the component of the Weyl group relative to the first puncture is* $W_{M^1}(L^1) = \mathrm{S}_n$ *and*

$$W_{\boldsymbol{M}}(L) = \mathrm{S}_n \quad \prod_{j=2}^{k} W_{M^j}(L^j):$$

According to this decomposition consider an element $(w;1;:::;1)$ 2 $W_{\boldsymbol{M}}(L)$ *with* w 2 S_n *an element of cycle type* 2 P_n*. Then*

$$\widetilde{h} \quad = p \ [X_1]h_{\_0 2}[X_2]:::h_{\_0 k}[X_k]$$

and $(\ \ 1)^{r(\ )} = \ (\ )$ *the sign of the permutation* w *with cycle type* . *Previous theorem reads*

$$P_c\left(\widetilde{Q}_{\boldsymbol{L};\boldsymbol{P};}\ ;v\right) = v^{d}\ \ (\ )\left\langle p\ [X_1]h_{\_0 2}[X_2]:::h_{\_0 k}[X_k];\mathrm{H}_n^{HLV}(0;v)\right\rangle:$$

This can be understood in terms of Frobenius characteristic, see Definition 3.2.29. Consider the representation of S_n *on the cohomology of* $\widetilde{Q}_{\boldsymbol{L};\boldsymbol{P};}$ *twisted by the sign:* $H\ (\widetilde{Q}_{\boldsymbol{L};\boldsymbol{P};}\ ;\ )$. *Its graded Frobenius characteristic is given by the following symmetric function in* X_1

$$v^{d}\ \left\langle h_{\_0 2}[X_2]:::h_{\_0 k}[X_k];\mathrm{H}_n^{HLV}(0;v)\right\rangle_{X_2;:::;X_k}:$$

Notice that $V \quad = V_{\ 0}$*, by Remark 3.2.27, the multiplicity of the irreducible component* V *in* $H\ (\widetilde{Q}_{\boldsymbol{L};\boldsymbol{P};}\ ;\ )$ *is given by*

$$v^{d}\ \left\langle s_{\ 0}[X_1]h_{\_0 2}[X_2]:::h_{\_0 k}[X_k];\mathrm{H}_n^{HLV}(0;v)\right\rangle:$$

Letellier proved that the Weyl group action on the cohomology of the resolution $\widetilde{M}_{\boldsymbol{L};\boldsymbol{P};}$ preserves the weight filtration. Therefore similarly to the -twisted Poincaré polynomial one can defined the -twisted mixed-Hodge polynomial $IH_c\left(\widetilde{M}_{\boldsymbol{L};\boldsymbol{P};}\ ;q;v\right)$.

Conjecture 3.6.10 (Letellier [Let13] Conjecture 1.8). *Let* $C_{\ ;}$ *a generic k-uple of conjugacy classes. For* $\widetilde{M}_{\boldsymbol{L};\boldsymbol{P};}$ *the resolution of a character variety* $M_{\overline{C}\ ;}$ *and* *a conjugacy class in* $W_{\boldsymbol{M}}(L)$*, the* *-twisted Poincaré polynomial is*

$$IH_c\left(\widetilde{M}_{\boldsymbol{L};\boldsymbol{P};}\ ;q;v\right) = (\ \ 1)^{r(\ )}(v\sqrt{q})^{d}\ \left\langle\widetilde{h}\ ;\mathrm{H}_n^{HLV}\left(\frac{1}{\sqrt{q}};v\sqrt{q}\right)\right\rangle:$$

Chapter 4

Weyl group actions on the cohomology of comet-shaped quiver varieties and combinatorics

4.1 Introduction

In this chapter the construction of the varieties $Q_{\overline{O}\ ;}$ and their resolutions $\widetilde{Q}_{\boldsymbol{L};\boldsymbol{P};}$ as comet-shaped quiver varieties is recalled. The base field K is either C or an algebraic closure $\overline{\mathrm{F}}_q$ of a finite field F_q. We consider $\widetilde{Q}_{\boldsymbol{L};\boldsymbol{P}}$ the family formed by resolutions $\widetilde{Q}_{\boldsymbol{L};\boldsymbol{P};}$ when is varying. In terms of comet-shaped quiver varieties this family is induced by the moment map. Weyl group actions on the cohomology of quiver varieties have been studied by Nakajima [Nak94; Nak98], Lusztig [Lus00] and Maei [Maf02]. With those methods we construct a monodromic Weyl group action on the cohomology of fibers of the family $\widetilde{Q}_{\boldsymbol{L};\boldsymbol{P}}$. The construction of this action relies on the moment map being locally trivial. The local triviality of such moment was recalled in Chapter 2. Similar Weyl group action were used by Hausel, Letellier and Rodriguez-Villegas [HLR13] to prove Kac conjecture. Moreover they computed traces of those actions thanks to Grothendieck trace formula. The same method is applied in this chapter.

Notice that Theorem 3.5.17 also provides a relative Weyl group action, *à la* Springer, on the cohomology of resolutions $\widetilde{Q}_{\boldsymbol{L};\boldsymbol{P};}$. Letellier [Let11] computed the trace of the action by counting points over finite fields. In this chapter we check that the monodromic and the Springer action are isomorphic.

Some combinatoric interpretation are given for those Weyl group actions. Surprisingly, some traces of those actions are related to some structure coecients of an algebra spanned by modified Kostka polynomials $\left(\widetilde{K}_{\ ;}\right)_{\ ;\ 2P_n}$. The structure coecents $\left(c_{\ ;}\right)_{\ ;\ ;\ 2P_n}$ were introduces by Rodriguez-Villegas in unpublished notes, they are defined by

$$\widetilde{K}_{\ ;}\widetilde{K}_{\ ;} = \sum_{2P_n} c_{\ ;}\widetilde{K}_{\ ;}:$$

We prove that the specialization $c^{1^n}_{\ ;}(0;t)$ of the coecients has an interpretation in terms of Weyl group action on the cohomology of comet-shaped quiver varieties.

Theorem 4.1.1. *Consider a generic* 4*-uple of adjoint orbits of the following type:*

- *O_1 has one eigenvalue with Jordan type $^0 \; 2 \; P_n$*
- *O_2 has one eigenvalue with Jordan type $^0 \; 2 \; P_n$.*
- *O_3 is semisimple regular, it has n distinct eigenvalues.*
- *O_4 is semisimple with one eigenvalue of multiplicity $n \quad 1$ and the other of multiplicity 1.*

Then the Weyl group with respect to O_3 is the symmetric group S_n and it acts on the cohomology of $Q_{\overline{\mathcal{O}}}$. Let w a n-cycle in this Weyl group then

$$c^{1^n}_{;} (0; t) = t^{-\frac{d_{\mathcal{O}}}{2}} \sum_r \operatorname{tr}\left(w; IH^{2r}_c (Q_{\overline{\mathcal{O}}}; \mathbb{C})\right) t^r :$$

4.2 Nakajima's quiver varieties

4.2.1 Resolution of Zariski closure of adjoint orbits as Nakajima's framed quiver varieties

In this section we recall the construction of resolutions of closure of adjoint orbits as Nakajima's framed quiver varieties, see Definition 2.3.3. Those results come from Kraft-Procesi [KP81], Nakajima [Nak98; Nak01], Crawley-Boevey [Cra03a; Cra03b], Shmelkin [Shm09] and Letellier [Let11].

Let $O_{\_;}$ an adjoint orbit with semisimple part and Jordan type $\_ \; 2 \; P$ as in 3.3.1. Consider the resolution $\widetilde{Y}_{L;P;} \quad ! \; \overline{O}_{\_;}$ as in 3.3.8. There is a Nakajima's framed quiver variety realizing this resolution. Let $d := \sum_{i=1}^{l} \; {}^{i}_{1}$ and recall that

$$L = \prod_{i=1}^{l} \prod_{r=1}^{\;^{i}_{1}} \mathrm{GL}_{\;^{i\,0}_{r}} :$$

The indices $(\;^{i\,0}_{r})_{1 \; i \; l, \; 1 \; r \; ^{i}_{1}}$ are relabelled $(c_s)_{1 \; s \; d}$ so that

$$L = \prod_{s=1}^{d} \mathrm{GL}_{c_s}$$

and introduce the parameter $= (\; _s)_{1 \; s \; d}$ such that $_s = \; _i$ if c_s corresponds to $\;^{i\,0}_{r}$ for some r. Consider the quiver $_{O_{\_;}}$ of type $A_{d \; 1}$ with summit indexed by integers between 1 and $d \quad 1$ and arrows going in the decreasing direction. Introduce the dimension vector $v_{O_{\_;}} := (v_1; :::; v_{d \; 1})$ with

$$v_1 := n \quad c_1; \;\; v_i := v_{i \; 1} \quad c_i \;\; \text{for } i > 1$$

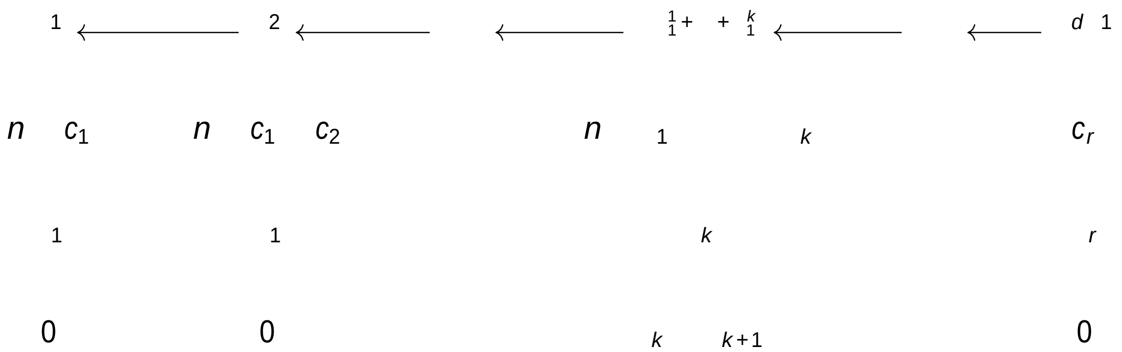
and $w := (n; 0; : : : ; 0)$.

Define the parameter $_{O_{\_;}} = (\; _1; :::; \; _{d \; 1})$ by

$$_i := \begin{cases} \;_k \quad \;_{k+1} & \text{if} \quad i = \;^1_1 + \quad + \;^k_1 \\ 0 & \text{otherwise} \end{cases} \tag{4.1}$$

$O_{\_;}$ is identified with the element $(_j \, \mathrm{Id}_{V_j})_{1 \quad j \quad d \quad 1}$.

We summarize everything in the following diagram showing the quiver, the dimension vector, the parameter and the parameter .



Remark 4.2.1. *When writing the dimension vector under the quiver, we used that* $j \,^i j = \, _i$.

Consider a second dimension vector $w = (n; 0; : : : ; 0)$ and an extended representation $(a; b; \ ) \ 2 \ \mathrm{Rep}\left(\tilde{\ }_{O_{\_;}} ; v_{O_{\_;}} ; w\right)$. As $w_i = 0$ unless $i = 1$, a is just a linear map $a : V_1 \ ! \ W_1$ and $b : W_1 \ ! \ V_1$ with $W_1 = \mathrm{K}^n$. For $1 \quad i \quad d \quad 2$, denote by $_{i+1;i}$ the linear map associated to the edge from $i + 1$ to i and by $_{i;i+1}$ the map associated to the reverse edge from i to $i + 1$. Such a representation belongs to $^{0 \ 1}(\ _{O_{\_;}})$ if and only if

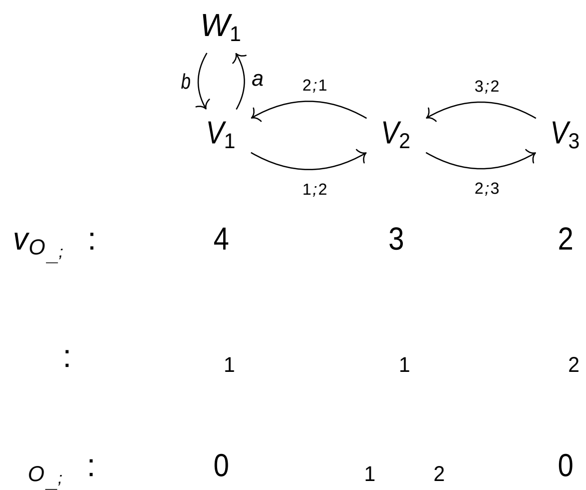
$$\begin{cases} \quad _{2;1} \quad _{1;2} \quad ba & = \quad (\ _1 \quad \ _2) \, \mathrm{Id}_{V_1} \\ _{i+1;i} \quad _{i;i+1} \quad \ _{i \ 1;i} \quad _{i;i \ 1} & = \quad (\ _i \quad \ _{i+1}) \, \mathrm{Id}_{V_i} \qquad \text{for } 2 \quad i \quad d \quad 2 \\ \quad _{d \ 1;d \ 2} \quad _{d \ 1;d \ 2} & = \quad (\ _{d \ 1} \quad \ _d) \, \mathrm{Id}_{V_{d \ 1}} \end{cases} \tag{4.2}$$

those equations are called the preprojective relations.

Example 4.2.2. *For the adjoint orbit of*

$$\begin{pmatrix} _1 & 1 & 0 & 0 & 0 & 0 \\ 0 & _1 & 1 & 0 & 0 & 0 \\ 0 & 0 & _1 & 0 & 0 & 0 \\ 0 & 0 & 0 & _1 & 0 & 0 \\ 0 & 0 & 0 & 0 & _2 & 0 \\ 0 & 0 & 0 & 0 & 0 & _2 \end{pmatrix}$$

the Jordan type is $\_ = ((3; 1); (1; 1)) \ 2 \ P_4 \quad P_2$ *and we obtain*



Theorem 4.2.3. *First consider the Nakajima's framed quiver variety* $M^0_{v_O;w}(\ \ _{O\_;}\ )$ *obtained from previous data and stability parameter* $= 0$*. The following map is well defined and is a bijection (it is an isomorphism when* $\mathbb{K} = \mathbb{C}$*)*

$$
\begin{array}{rcl}
{}_0 : M^0_{v_{O\_;}\ ;w}\left({}_{O\_;} \right) & ! & \overline{O}_{\_;} \\
(a;b;\ ) & 7! & ab \quad {}_1 \mathrm{Id}_n
\end{array}
$$

Now take a stability parameter $2\ \mathbb{Z}^{d\ 1}_{>0}$*, the following map is a bijection (an isomorphism when* $\mathbb{K} = \mathbb{C}$*).*

$$
\begin{array}{rcl}
: M_{v_{O\_;}\ ;w}\left({}_{O\_;} \right) & ! & \widetilde{\mathrm{Y}}_{L;P;} \\
(a;b;\ ) & 7! & (ab + {}_1 \mathrm{Id}_n; f_{a;b;}\ )
\end{array}
$$

with $f_{a;b;}$ *the flag* $0 \quad E_{d\ 1} \qquad E_1 \quad \mathbb{C}^n$ *defined by*

$$
\begin{array}{rcl}
E_1 & := & \mathrm{Im}(a) \\
E_i & := & \mathrm{Im}(a \quad {}_{2;1} \quad {}_{3;2} \qquad {}_{i;i\ 1}) \quad \text{for } 2 \quad i \quad d \quad 1.
\end{array}
$$

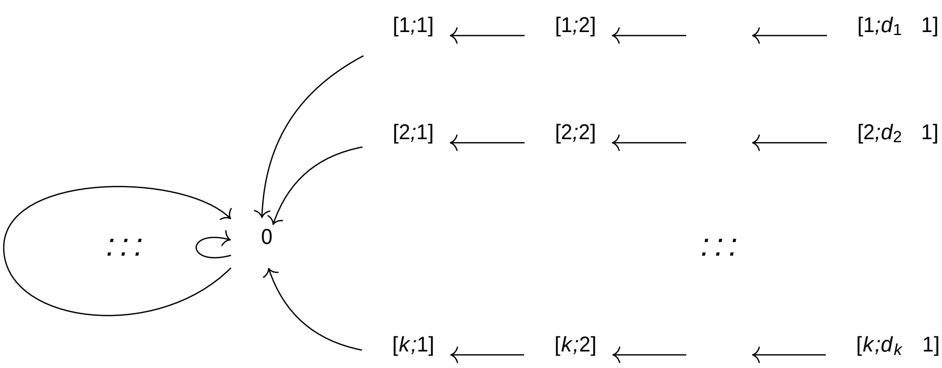
Moreover, the following diagram commutes

$$
\begin{array}{ccc}
M_{v_{O\_;}\ ;w}\left({}_{O\_;} \right) & \longrightarrow & \widetilde{\mathrm{Y}}_{L;P;} \\
\downarrow & & \downarrow p \\
M^0_{v_{O\_;}\ ;w}\left({}_{O\_;} \right) & \xrightarrow[0]{} & \overline{O}_{\_;}
\end{array}
$$

with p *the resolution of* $\overline{O}_{\_;}$ *from Proposition 3.3.8 and the natural map from GIT theory.*

4.2.2 Comet-shaped quiver varieties

Let $O_{\ ;} = \left(O_{\_1;\ 1}; \dots ; O_{\_k;\ k}\right)$ be a genric k-uple of adjoint orbits in $\mathfrak{gl}_n$. We recall Crawley-Boevey's result relating the variety $Q_{\overline{O}\ ;}$ defined in the introduction to a quiver variety. The idea is to glue together k quivers of type A corresponding to each adjoint orbit $O_{\ j;\ j}$ to a central vertex 0 and add g loops to this central vertex, we obtain the following comet-shaped quiver ${}_{O\ ;}$



The j-th leg is a quiver of type A with vertices labelled from $[j;1]$ to $[j;d_j\quad 1]$. The dimension vector $v_{O\ ;}$ is defined such that its coordinate at the central vertex is n and its coordinates on the j-th leg coincide with the dimension vector $v_{\mathcal{O}_{j};\,j}$ described in previous section. Similarly the parameter $_{\mathcal{O}\,;}$ is defined such that its coordinates on the j-th leg coincide with the parameter $_{\mathcal{O}_{j};\,j}$. The component at the central vertex $_{\mathcal{O}\,;\,;0}$ is defined such that $v_{\mathcal{O}\,;} \cdot {}_{\mathcal{O}\,;} = 0$ hence

$$n_{\mathcal{O}\,;\,;0} = \sum_{j=1}^{k} \sum_{i=1}^{d_j\ 1} v_{\mathcal{O}\,;\,;[j;i]}\ {}_{\mathcal{O}\,;\,;[j;i]}.$$

Consider a representation of the extended quiver $\ 2 \operatorname{Rep}\left(\tilde{}_{\mathcal{O}\,;}\,; v_{\mathcal{O}\,;}\right)$.

- Denote by $_{[j;i]}$ the linear map associated to the arrow with tail $[j;i]$ and $\overline{_{[j;i]}}$ the linear map associated to the reversed arrow with head $[j;i]$.
- For $1 \quad i \quad g$ the map associated to the i-th loop is denoted $_i$ and the one associated to the reverse loop is denoted $_{\bar{i}}$.

As usual is the moment map and $_{\mathcal{O}\,;}$ is identified with an element in the center of the Lie algebra $\mathfrak{g}_{v_{\mathcal{O}\,;}}$. Let

$$X_j := {}_{[j;1]}\ \overline{_{[j;1]}}\ \ {}_{[j;1]}$$

If belongs to $^{1}(\ {}_{\mathcal{O}\,;}\ )$ then $X_j\ 2\ \overline{\mathcal{O}}_{j;\,j}$. Indeed it follows from previous description of closure of adjoint orbits as framed quiver varieties and identification, for each legs, of the vector space at the central vertex with the framing vector space W_1 from previous section.

Now if A_i is the linear map associated to the i-th loop of the quiver and B_i the map associated to the reversed loop, the preprojective relation at the central vertex is exactly the equation defining $V_{\overline{\mathcal{O}}}$. Hence the following map is well defined

$$\begin{aligned} {}_{\mathcal{O}\,;} \ : \quad {}^{1}(\ {}_{\mathcal{O}\,;}\ ) \ &! \qquad\qquad V_{\overline{\mathcal{O}}\,;} \\ &7! \quad (A_1; B_1; \ldots; A_g; B_g; X_1; \ldots; X_k) \end{aligned}$$

Theorem 4.2.4. *In the following diagram where the vertical arrows are quotient maps, the application* $_{\mathcal{O}\,;}$ *goes down to the quotient to a bijective morphism* $_{\mathcal{O}\,;}$ *(when* $\mathbb{K} = \mathbb{C}$ *it is an isomorphism).*

$$\begin{array}{ccc} {}^{1}(\ {}_{\mathcal{O}\,;}\ ) & \xrightarrow{\ {}_{\mathcal{O}\,;}\ } & V_{\overline{\mathcal{O}}\,;} \\ \downarrow & & \downarrow \\ M^{0}_{v_{\mathcal{O}\,;}}(\ {}_{\mathcal{O}\,;}\ ) & \xrightarrow[\ {}_{\mathcal{O}\,;}\ ]{} & Q_{\overline{\mathcal{O}}\,;} \end{array}$$

Proof. It is proved by Crawley-Boevey [Cra01; Cra03b], see also Letellier [Let11, Proposition 5.2.2] for any genus. □

$\widetilde{Q}_{\mathbf{L};\mathbf{P};}$, the resolution of $Q_{\overline{\mathcal{O}}\,;}$ introduced in 3.5.16, is also interpreted as Nakajima's quiver variety for the quiver $_{\mathcal{O}\,;}$.

Theorem 4.2.5. *Consider a stability parameter associated to the quiver $Q_{\overline{O}\,;}$ such that $_{[j;i]} > 0$ for all vertices $[j;i]$. There is a bijective morphism $^{O\,;\,;}$: $M_{v_{O\,;}}(\ _{O\,;}\ )\ !\ \widetilde{Q}_{\mathbf{L};\mathbf{P};}$ which is an isomorphism when $\mathbb{K} = \mathbb{C}$ and the following diagram commutes*

$$\begin{array}{ccc} M_{v_{O\,;}}(\ _{O\,;}\ ) & \xrightarrow{\ O\,;\,;\ } & \widetilde{Q}_{\mathbf{L};\mathbf{P};} \\ \downarrow & & \downarrow p \\ M^{0}_{v_{O\,;}}(\ _{O\,;}\ ) & \xrightarrow[\ O\,;\ ]{} & Q_{\overline{O}\,;} \end{array}$$

with the natural projection from GIT theory.

Proof. It follows from Letellier's article [Let11], where the construction of the map $^{O\,;\,;}$ is given in Section 5.3. This map is induced by the map of Theorem 4.2.3. Contrarily to Letellier's article, we do not consider partial resolution so that our parameter has non-zero components. Therefore the dimension vector for the quiver variety $M_{v_{O\,;}}(\ _{O\,;}\ )$ describing the resolution $\widetilde{Q}_{\mathbf{L};\mathbf{P};}$ is the same as the dimension vector of the quiver variety describing $Q_{\overline{O}\,;}$. □

The quiver variety point of view gives a criteria for non-emptiness. The question of emptiness of $Q_{\overline{O}\,;}$ and $M_{\overline{C}\,;}$ is known as the Deligne-Simpson problem. Kostov [Kos04] gave a survey about this problem. For a di erent approach see Soibelman [Soi16]. The additive version was answered by Crawley-Boevey in terms of roots of the quiver [Cra03b]. The multiplicative case (for generic conjugacy classes and genus $g = 0$) is solved by Crawley-Boevey [Cra03a, Theorem 8.3]. For any genus, the result follows from Hausel, Letellier, Rodriguez-Villegas [HLR11, 5.2] and Letellier [Let11, Corollary 3.15]. Those results are summarized in the following theorem:

Theorem 4.2.6. *Let $O_{\,;}$ a generic k-uple of adjoint orbit. The variety $Q_{\overline{O}\,;}$ is not empty if and only $Q_{O\,;}$ is not empty. This happens if and only if the dimension vector $v_{O\,;}$ is a root of the quiver $_{O\,;}$. This is always the case for $g > 0$.*

Let $C_{\,;\,,}$ a generic k-uple of conjugacy classes. The variety $M_{\overline{C}\,;\,,}$ is not empty if and only if $M_{C\,;\,,}$ is not empty. This happens if and only if the the dimension vector $v_{O\,;\,,}$ is a root of the quiver $_{O\,;\,,}$. This is always the case for $g > 0$.

4.2.3 Family of comet-shaped quiver varieties

When the eigenvalues are varying, one obtains a family of varieties.

Notations 4.2.7. *From now on the pair $\mathbf{L};\mathbf{P}$ is fixed. For short, let*

$$Z(l) := Z(\mathfrak{l}_1) \qquad Z(\mathfrak{l}_k):$$

Denote by B the subset of elements $2\ Z(l)$ such that the k-uple of adjoint orbits $O_{\,;}$ is generic. Note that the genericity condition depends only the semisimple part and not on the type . The set B is a Zariski open subset of a codimension one subspace of $Z(l)$ given by the vanishing of the sum of the traces. Identifying $Z(l)$ with an a ne space, B is either empty or the complementary of a finite union of hyperplanes in the codimension one subspace.

Definition 4.2.8 (Family of varieties $\widetilde{Q}_{\mathbf{L};\mathbf{P};}$). *Define*

$$\widetilde{V}_{\mathbf{L};\mathbf{P}} := \{(\; ; (A_i; B_i)_{1 \quad i \quad g}; (X_j; g_j P^j)_{1 \quad j \quad k}) \mid$$
$$2\ B; \text{ and } (A_i; B_i)_{1 \quad i \quad g}; (X_j; g_j P^j)_{1 \quad j \quad k}\ 2\ V_{\mathbf{L};\mathbf{P};} \}$$
$$\widetilde{Q}_{\mathbf{L};\mathbf{P}} := \widetilde{V}_{\mathbf{L};\mathbf{P}} == \mathrm{GL}_n$$

and denote the map $: \widetilde{Q}_{\mathbf{L};\mathbf{P}} \; ! \; B$*. Thus the varieties* $\widetilde{Q}_{\mathbf{L};\mathbf{P};} \; = \; ^{1}(\;)$ *fit in a family* $\widetilde{Q}_{\mathbf{L};\mathbf{P}}$.

The choice of L determine a unique quiver $_{O\ ;}$ and a unique dimension vector $v_{O\ ;}$ independent of a choice of . Assume that the dimension vector is indivisible so that B is not empty. Then we can make the following assumption

Assumption 4.2.9 (Genericity of the stability parameter). *is a generic stability parameter, i.e. a stability parameter for the quiver* $_{O\ ;}$ *with dimension vector* $v_{O\ ;}$ *such that* $(\; ; 0; 0)\ 2\ H^{\mathrm{reg}}_{v_{O\ ;}}$ *with notations from 2.1.4.*

The construction of Theorem 3.5.17 extends to this family. It provides the following commutative diagram (the left vertical arrows is induced by the moment map)

$$\begin{array}{ccc} ^{1}(\mathfrak{z}^{\mathrm{gen}}_{v_{O\ ;}})^{\text{-ss}} == G_{v_{O\ ;}} & \longrightarrow & \widetilde{Q}_{\mathbf{L};\mathbf{P}} \\ \downarrow & & \downarrow \\ \mathfrak{z}^{\mathrm{gen}}_{v_{O\ ;}} & \longrightarrow & B \end{array} \tag{4.3}$$

is a fixed generic stability parameter. $\mathfrak{z}^{\mathrm{gen}}_{v_{O\ ;}}$ is the subset of the center of the Lie algebra $\mathfrak{g}_{v_{O\ ;}}$ corresponding to the subset B under the correspondence between parameters $_{O\ ;}$ and eigenvalues . Note that the correspondence between parameters of the quiver variety $_{O\ ;}\ 2\ Z(\mathfrak{g}_{v_{O\ ;}})$ and $Z(I)$ is not bijective. Thus the previous diagram relies on a choice of $k \quad 1$ eigenvalues. To $\ 2\ Z(I)$ associate the element $(\,_{O\ ;}\ ; \; ^{1}_{1}; \dots ; \; ^{k}_{1} \; ^{1})$ in $Z(\mathfrak{g}_{v_{O\ ;}}) \quad \mathbb{K}^{k \quad 1}$ this defines a bijective map

$$h : Z(I) \; ! \; \mathfrak{z}_{v_{O\ ;}} \qquad \mathbb{K}^{k \quad 1}: \tag{4.4}$$

Note that for a given parameter $_{O\ ;}$ the genericity conditions is independant of the choice of the $k \quad 1$ eigenvalues, namely $h^{\;1}(\,_{O\ ;}\ ; \; ^{1}_{1}; \dots ; \; ^{k}_{1} \; ^{1})$ is generic if and only if $h^{\;1}(\,_{O\ ;}\ ; 0; \dots ; 0)$ is generic. Therefore Diagram (4.3) can be modified to account for various choices of eigenvalues, then the horizontal arrows are bijections and isomorphism when $\mathbb{K} = \mathbb{C}$.

$$\begin{array}{cccc} \mathbb{K}^{k \quad 1} & ^{1}(\mathfrak{z}^{\mathrm{gen}}_{v_{O\ ;}})^{\text{-ss}} == G_{v_{O\ ;}} & \longrightarrow & \widetilde{Q}_{\mathbf{L};\mathbf{P}} \\ & \mathrm{Id} \downarrow & & \downarrow \\ & \mathbb{K}^{k \quad 1} \quad \mathfrak{z}^{\mathrm{gen}}_{v_{O\ ;}} & \longrightarrow & B \end{array} \tag{4.5}$$

Theorem 4.2.10. *If* $\mathbb{K} = \mathbb{C}$*, or if the characteristic is large enough, the cohomology sheaves* $H^i \; _!$ *are constant sheaves.*

Proof. When $\mathbb{K} = \mathbb{C}$, this is a consequence of Chapter 2 Corollary 2.4.14 and diagram (4.5). As is generic, To prove the result for $\mathbb{K} = \overline{\mathbb{F}}_q$ we can change characteristic as in [HLR13] proof of Theorem 2.3. This imply the result in large enough characteristic. ☐

4.3 Weyl group action

4.3.1 Decomposition of the family $Q_{L,P}$

Notations 4.3.1. *First we recall notations from 3.4.4 in this context. For* $1 \leq j \leq k$

$$\widetilde{\mathrm{Y}}_{L^j,P^j} := \left\{(X, g_j P^j) \in \mathfrak{gl}_n \times \mathrm{GL}_n/P^j \,\middle|\, g_j^{-1} X g_j \in Z(\mathfrak{l}^j) \oplus \mathfrak{u}_{P^j}\right\}$$

and define

$$\widetilde{\mathrm{Y}}_{\boldsymbol{L},\boldsymbol{P}} := \widetilde{\mathrm{Y}}_{L^1,P^1} \times \cdots \times \widetilde{\mathrm{Y}}_{L^k,P^k}.$$

Then $\mathrm{Y}_{\boldsymbol{L},\boldsymbol{P}}$ *is the image in* $\mathfrak{gl}_n^k$ *of the map forgetting the partial flags* $g_j P^j$*:*

$$\begin{array}{rccc} p: & \widetilde{\mathrm{Y}}_{\boldsymbol{L},\boldsymbol{P}} & \to & \mathfrak{gl}_n^k \\ & (X_j, g_j P^j)_{1 \leq j \leq k} & \mapsto & (X_j)_{1 \leq j \leq k}. \end{array}$$

Similarly $V_{\boldsymbol{L},\boldsymbol{P}}$*, respectively* $Q_{\boldsymbol{L},\boldsymbol{P}}$*, is obtained from* $\widetilde{V}_{\boldsymbol{L},\boldsymbol{P}}$*, respectively* $\widetilde{Q}_{\boldsymbol{L},\boldsymbol{P}}$*, by forgetting the partial flags.*

In this section a decomposition of the family $Q_{\boldsymbol{L},\boldsymbol{P}}$ is deduced from the decomposition $\overline{O}_{\lambda,\mu} = \bigsqcup O_{\underline{\lambda},\underline{\mu}}$ and the decomposition introduced in Proposition 3.4.20:

$$\mathrm{Y}_{L,P} = \bigsqcup_{M} \bigsqcup_{\underline{\lambda}\,\underline{\mu}} \mathrm{Y}_{L,P}^{M,\underline{\lambda}}.$$

The decomposition is used in next section (Lemma 4.3.4) in order to define a Weyl group action.

Let $\mathrm{Y}^B_{\boldsymbol{L},\boldsymbol{P}}$ the subset of elements in $\mathrm{Y}_{\boldsymbol{L},\boldsymbol{P}}$ with semisimple part generic, *i.e.* in B. The dimension of $\mathrm{Y}^B_{\boldsymbol{L},\boldsymbol{P}}$ is computed similarly to $\dim \mathrm{Y}_{L,P}$ in Remark 3.4.19:

$$\dim \mathrm{Y}^B_{\boldsymbol{L},\boldsymbol{P}} = kn^2 + \dim B - \sum_{j=0}^{k} \dim L^j.$$

The decomposition $\mathrm{Y}_{L,P} = \bigsqcup_M \bigsqcup_{\underline{\lambda}\,\underline{\mu}} \mathrm{Y}_{L,P}^{M,\underline{\lambda}}$ induces a similar decomposition for $\mathrm{Y}^B_{\boldsymbol{L},\boldsymbol{P}}$

$$\mathrm{Y}^B_{\boldsymbol{L},\boldsymbol{P}} = \bigsqcup_{\boldsymbol{M}} \bigsqcup \mathrm{Y}^{B,\boldsymbol{M},}_{\boldsymbol{L},\boldsymbol{P}}$$

With $\boldsymbol{M} = (M^1, \ldots, M^l)$ and $\mathrm{Y}^{B,\boldsymbol{M},}_{\boldsymbol{L},\boldsymbol{P}}$ the subset of elements in

$$\mathrm{Y}^{B,M^1,\underline{\lambda}^1}_{L^1,P^1} \times \cdots \times \mathrm{Y}^{B,M^k,\underline{\lambda}^k}_{L^k,P^k}$$

with generic semisimple parts. From the computation of the dimension of $\mathrm{Y}^{M,\underline{\lambda}}_{L,P}$ in Proposition 3.4.20, we deduce that when $Z(m) \cap B$ is not empty

$$\dim \mathrm{Y}^{B,\boldsymbol{M},}_{\boldsymbol{L},\boldsymbol{P}} = \sum_{j=1}^{n} \dim O_{\underline{\lambda}^j,\mu^j} + \dim Z(m) \cap B. \tag{4.6}$$

Now the decomposition of $\mathrm{Y}^B_{\boldsymbol{L},\boldsymbol{P}}$ induces a decomposition of the family of quiver varieties $Q_{\boldsymbol{L},\boldsymbol{P}}$. Let

$$Q^{\boldsymbol{M},}_{\boldsymbol{L},\boldsymbol{P}} := \left(V_{\boldsymbol{L},\boldsymbol{P}} \times_{\mathrm{Y}_{\boldsymbol{L},\boldsymbol{P}}} \mathrm{Y}^{B,\boldsymbol{M},}_{\boldsymbol{L},\boldsymbol{P}}\right) /\!/ \mathrm{PGL}_n.$$

We have the following proposition:

Proposition 4.3.2.

$$Q_{\boldsymbol{L},\boldsymbol{P}} = \bigsqcup_{\boldsymbol{M}} \bigsqcup Q^{\boldsymbol{M},}_{\boldsymbol{L},\boldsymbol{P}}.$$

When non-empty, the dimension of a part is

$$\dim Q^{\boldsymbol{M},}_{\boldsymbol{L},\boldsymbol{P}} = n^2(2g \quad 2) + 2 + \dim Z(m) \cap B + \sum_{j=1}^{k} \dim O_{\_j,\ j}. \tag{4.7}$$

Proof. The dimension of $Q^{\boldsymbol{M},}_{\boldsymbol{L},\boldsymbol{P}}$ can be computed just like the dimension of $Q_{O\ ,}$ (see Proposition 3.5.6 for the case of character varieties). The computation relies on the smoothness of $\mathrm{Y}^{B,\boldsymbol{M},}_{\boldsymbol{L},\boldsymbol{P}}$ which follows from the smoothness of $\mathrm{Y}^{B,M^j,\_j}_{L^j,P^j}$. Then from the dimension of $\mathrm{Y}^{B,\boldsymbol{M},}_{\boldsymbol{L},\boldsymbol{P}}$ given by (4.6) we obtain

$$\dim Q^{\boldsymbol{M},}_{\boldsymbol{L},\boldsymbol{P}} = n^2(2g \quad 2) + 2 + \dim Z(m) \cap B + \sum_{j=1}^{k} \dim O_{\_j,\ j}.$$

□

4.3.2 Construction of a Weyl group action on the cohomology of the quiver varieties in the family $Q_{L,P}$

The family $\widetilde{Q}_{\boldsymbol{L},\boldsymbol{P}} \to B$ is used to construct a Weyl group action on the cohomology of the varieties $\widetilde{Q}_{\boldsymbol{L},\boldsymbol{P},}$ for $\in B$. The Weyl group considered in this section is

$$W := W_{\mathrm{GL}_n}(L^1) \qquad W_{\mathrm{GL}_n}(L^k).$$

Each $W_{\mathrm{GL}_n}(L^j)$ is isomorphic to a symmetric group and acts on $Z(\mathrm{l}^j)$ by permuting the eigenvalues with same multiplicities. Therefore W acts on B, for $w = (w_1, \dots, w_k) \in W$ and $= (\ ^1, \dots, \ ^k) \in B$

$$w. \ := \left(\underline{w}_1 \ ^1\underline{w}_1^{\ 1}, \dots, \underline{w}_k \ ^k\underline{w}_k^{\ 1}\right)$$

with $\underline{w}_j$ a representative in GL_n of $w_j \in W_{\mathrm{GL}_n}(L^j)$. Consider the diagram:

$$\begin{array}{ccc} B & \longleftarrow & \widetilde{Q}_{\boldsymbol{L},\boldsymbol{P}} \\ {\scriptstyle 0}\downarrow & & \downarrow{\scriptstyle p} \\ B/W & \longleftarrow & Q_{\boldsymbol{L},\boldsymbol{P}} \end{array} \tag{4.8}$$

Thanks to the quiver variety point of view, the cohomology sheaves $H^i \ _!$ are constant (Theorem 4.2.10). In this section a W-equivariant structure on those cohomology sheaves is constructed. The method comes from Lusztig (see [Let05, Proof of Proposition 5.5.3]), it is also used by Laumon-Letellier [LL19, Section 5.2].

Before proving this result, let us define the regular locus. Denote by B^{reg} the subset of regular elements, i.e. elements $(\ ^1, \dots, \ ^k) \in B$ such that $Z_{\mathrm{GL}_n}(\ ^j) = L^j$.

It is the locus of B where the W-action is free. Diagram (4.8) is pulled back to the regular locus

$$\begin{array}{ccc} B^{\text{reg}} & \xleftarrow{\text{reg}} & \widetilde{Q}^{\text{reg}}_{\boldsymbol{L};\boldsymbol{P}} \\ {\scriptstyle \text{reg}}\downarrow & & \downarrow{\scriptstyle p^{\text{reg}}} \\ B^{\text{reg}}=W & \xleftarrow[\text{reg}]{} & Q^{\text{reg}}_{\boldsymbol{L};\boldsymbol{P}} \end{array} \tag{4.9}$$

Similarly to 3.32, notice that

$$Q^{\text{reg}}_{\boldsymbol{L};\boldsymbol{P}} \quad {}_{B^{\text{reg}}=W} B^{\text{reg}} = \widetilde{Q}^{\text{reg}}_{\boldsymbol{L};\boldsymbol{P}}: \tag{4.10}$$

Theorem 4.3.3. *The cohomology sheaves* $H^i \quad {}_!$ *admit a* W*-equivariant structure over* B*.*

Proof. Consider the diagram:

$$\begin{array}{ccccc} & & & & \widetilde{Q}_{\boldsymbol{L};\boldsymbol{P}} \\ & & & \swarrow{\scriptstyle c} & \\ B & \xleftarrow{a} & Q_{\boldsymbol{L};\boldsymbol{P}} \quad {}_{B=W} B & & \\ {\scriptstyle 0}\downarrow & & \downarrow{\scriptstyle b} & & \swarrow{\scriptstyle p} \\ B=W & \longleftarrow & Q_{\boldsymbol{L};\boldsymbol{P}} & & \end{array} \tag{4.11}$$

W acts on $Q_{\boldsymbol{L};\boldsymbol{P}} \quad {}_{B=W} B$ and the morphism a is W-equivariant. $Q^{\text{reg}}_{\boldsymbol{L};\boldsymbol{P}} \quad {}_{B^{\text{reg}}=W} B^{\text{reg}}$ is smooth, dense and open in $Q_{\boldsymbol{L};\boldsymbol{P}} \quad {}_{B=W} B$. The constant sheaf over $Q_{\boldsymbol{L};\boldsymbol{P}} \quad {}_{B=W} B$ is W-equivariant. Indeed for $w \; 2 \; W$ we can define a morphism

$$_w : w \quad !$$

which is the identity on the stalks. It satisfies the conditions of definitions 3.1.4. Applying the continuation principle from Remark 3.1.9, this W-equivariant structure extends to a W-equivariant structure on $IC_{Q_{\boldsymbol{L};\boldsymbol{P}} \quad {}_{B=W} B}$. Notice that $\;_! \quad = a_! c_! \;$. We shall see in Lemma 4.3.4 that

$$c_! \quad = IC_{Q_{\boldsymbol{L};\boldsymbol{P}} \quad {}_{B=W} B}:$$

Then the W-equivariant structure on $c_!$ induces a W-equivariant structure on $\;_!$. Up to the isomorphism $c_! \quad = IC_{Q_{\boldsymbol{L};\boldsymbol{P}} \quad {}_{B=W} B}$, the theorem is proved. □

It remains to prove the lemma:

Lemma 4.3.4. *There is an isomorphism* $c_! \quad = IC_{Q_{\boldsymbol{L};\boldsymbol{P}} \quad {}_{B=W} B}$.

Proof. Because of the isomorphism (4.10), the restriction of $c_!$ to the smooth locus $Q^{\text{reg}}_{\boldsymbol{L};\boldsymbol{P}} \quad {}_{B^{\text{reg}}=W} B^{\text{reg}}$ is the constant sheaf . In order to verify the hypothesis of Definition 3.1.8 it remains to prove that the map c is small, i.e. that it satisfies the following inequality

$$\dim\left\{x \; 2 \; Q_{\boldsymbol{L};\boldsymbol{P}} \quad {}_{B=W} B \,\middle|\, \dim c^{\;\;1}(x) \quad d\right\} \quad \dim Q_{\boldsymbol{L};\boldsymbol{P}} \quad {}_{B=W} B \quad 2d \text{ for all } d > 0:$$

It relies on dimension estimates from Lusztig [Lus84, 1.2], see also [Sho88, Theorem 1.4]. In the Lie algebra $\mathfrak{gl}_n$ the estimate becomes, for X in O an adjoint orbit

$$\dim\left\{gP \; 2 \; \mathrm{GL}_n=P \,\middle|\, g^{\;\;1}Xg \; 2 \quad + \mathfrak{u}_P\right\} \quad \frac{1}{2}\left(n^2 \quad \dim L \quad \dim O\right): \tag{4.12}$$

The proof is then standard in Springer theory. Let $d > 0$ and x such that

$$\dim c \;\; ^{1}(x) \quad d.$$

x belongs to some $Q_{O \;\; ;}$ for $\;\; 2\ B$ and some adjoint orbits $O_{\_ 1; \;\; 1}; \ldots ; O_{\_ k; \;\; k}$. The dimension estimate (4.12) implies

$$d \quad \frac{1}{2}\left(kn^2 \quad \sum_{j=1}^{k} \dim L^j \quad \dim O_{\_ j; \;\; j}\right)$$

so that

$$\sum_{j=1}^{k} \dim O_{\_ j; \;\; j} \quad kn^2 \quad \sum_{j=1}^{k} \dim L^j \quad 2d.$$

Using the decomposition from Proposition 4.3.2, $x\ 2\ Q^{B;\boldsymbol{M};}_{\boldsymbol{L};\boldsymbol{P}}$. Previous inequality and the expression (4.7) for the dimension of $Q^{B;\boldsymbol{M};}_{\boldsymbol{L};\boldsymbol{P}}$ give

$$\dim Q^{B;\boldsymbol{M};}_{\boldsymbol{L};\boldsymbol{P}} \quad n^2(2g \quad 2) + 2 + \dim Z(m) \setminus B + kn^2 \quad \sum_{j=1}^{k} \dim L^j \quad 2d. \quad (4.13)$$

Moreover

$$\dim Q^{B;\boldsymbol{M};}_{\boldsymbol{L};\boldsymbol{P}} \quad {}_{B=W}\ B = \dim Q^{B;\boldsymbol{M};}_{\boldsymbol{L};\boldsymbol{P}} \quad (4.14)$$

and

$$\dim Q_{\boldsymbol{L};\boldsymbol{P}} \quad {}_{B=W}\ B = \dim Q_{\boldsymbol{L};\boldsymbol{P}} \; = n^2(2g \quad 2) + 2 + \dim B + kn^2 \quad \sum_{j=1}^{k} \dim L^j. \quad (4.15)$$

Combining (4.13)(4.14) and (4.15):

$$\dim Q^{B;\boldsymbol{M};}_{\boldsymbol{L};\boldsymbol{P}} \quad {}_{B=W}\ B \quad \dim Q_{\boldsymbol{L};\boldsymbol{P}} \quad {}_{B=W}\ B + 2d + \dim Z(m) \setminus B \quad \dim B. \quad (4.16)$$

As d is assumed to be strictly positive, necessarily the inclusion $L \ (\ M$ is strict, hence

$$\dim Z(m) \setminus B < \dim B. \quad (4.17)$$

Now (4.16) and (4.17) provide the estimate

$$\dim Q^{B;\boldsymbol{M};}_{\boldsymbol{L};\boldsymbol{P}} \quad {}_{B=W}\ B < \dim Q_{\boldsymbol{L};\boldsymbol{P}} \quad {}_{B=W}\ B \quad 2d. \quad (4.18)$$

To conclude, the set $\left\{x\ 2\ Q_{\boldsymbol{L};\boldsymbol{P}} \quad {}_{B=W}\ B \,|\dim c \;\; ^{1}(x) \quad d\right\}$ is a finite union of varieties $Q^{B;\boldsymbol{M};}_{\boldsymbol{L};\boldsymbol{P}} \quad {}_{B=W}\ B$ with dimension satisfying previous estimate (4.18). □

Remark 4.3.5. *Let us study the restriction of the W-equivariant sheaves $H^i \;\; _!$ to the regular locus. Recall that $Q^{\mathrm{reg}}_{\boldsymbol{L};\boldsymbol{P}} \quad {}_{B^{\mathrm{reg}}=W}\ B^{\mathrm{reg}} = \widetilde{Q}^{\mathrm{reg}}_{\boldsymbol{L};\boldsymbol{P}}$, then for $\;\; 2\ B^{\mathrm{reg}}$*

$$H^i \;\; _! \;\; = H^i_c(\widetilde{Q}_{\boldsymbol{L};\boldsymbol{P};} \;\; ; \;\;).$$

For $w\ 2\ W$, the W-equivariant structure is given by the functoriality of the compactly supported cohomology (see Proposition 3.1.6 and Remark 3.1.3)

$$w \;\; : H^i_c\left(\widetilde{Q}_{\boldsymbol{L};\boldsymbol{P};w:} \;\; ; \;\; \right) \ ! \ H^i_c\left(\widetilde{Q}_{\boldsymbol{L};\boldsymbol{P};} \;\; ; \;\; \right).$$

This is called the monodromic action.

4.3.3 Frobenius morphism and monodromic action

The techniques in this section come from Hausel, Letellier and Rodriguez-Villegas [HLR13], though we do no consider regular semisimple values of the moment map. Instead each component of the moment map is central and each leg of the comet-shaped quiver corresponds to a particular adjoint orbit. Comet-shaped quiver varieties were also studied in this context by Letellier [Let12]. A slightly more general situation is considered here, as a leg can represents any adjoint orbit and not only a semisimple regular.

We proved in 4.2.10 that the cohomology sheaves $H^i \ _!$ are constant sheaves over B. Note that the fiber over of this constant sheaf is $H^i_c(\widetilde{Q}_{\mathbf{L};\mathbf{P};} \ ; \ )$. Thus for any $\ ; \ 2\ B$, there is an isomorphism

$$f_{\ ;} : H^i_c(\widetilde{Q}_{\mathbf{L};\mathbf{P};} \ ; \ ) \ ! \ H^i_c(\widetilde{Q}_{\mathbf{L};\mathbf{P};} \ ; \ ):$$

such that for any $\ ! \ 2\ B$

$$f_{\ ;} = f_{!;} \quad f_{;!}:$$

The W-equivariance of the local system $H^i \ _!$ implies the following theorem. It can also be proved directly, without referring to equivariance of the local system (see Ma□ei [Maf02, Section 5]).

Theorem 4.3.6. *Let $\ ; \ 2\ B$, the following diagram commutes*

$$\begin{array}{ccc}
H^i_c(\widetilde{Q}_{\mathbf{L};\mathbf{P};} \ ; \ ) & \xrightarrow{\ w\ } & H^i_c(\widetilde{Q}_{\mathbf{L};\mathbf{P};w \ 1:} \ ; \ ) \\
\downarrow f_{\ ;} & & \downarrow f_{w \ 1: \ ;w \ 1:} \\
H^i_c(\widetilde{Q}_{\mathbf{L};\mathbf{P};} \ ; \ ) & \xrightarrow{\ w\ } & H^i_c(\widetilde{Q}_{\mathbf{L};\mathbf{P};w \ 1:} \ ; \ )
\end{array}$$

Remark 4.3.7. *Note that if $\ 2\ B$ is not regular, then the map*

$$w \ : H^i_c\left(\widetilde{Q}_{\mathbf{L};\mathbf{P};} \ ; \ \right) \ ! \ H^i_c\left(\widetilde{Q}_{\mathbf{L};\mathbf{P};w \ 1:} \ \right)$$

is only the map coming from the W-equivariant structure of the constant sheaf $H^i \ _!$. It does not come by functoriality from a morphism a variety. At the level of variety, W only acts on $\widetilde{Q}^{\mathrm{reg}}_{\mathbf{L};\mathbf{P}}$.

This theorem allows to define a W-action on the compactly supported cohomology space $H^i_c(\widetilde{Q}_{\mathbf{L};\mathbf{P};} \ ; \ )$.

Proposition 4.3.8. *For $w\ 2\ W$ introduce the morphism*

$$_i(w) = f_{w: \ ;} \quad (w \ ^1)$$

This defines an action of W on $H^i_c(\widetilde{Q}_{\mathbf{L};\mathbf{P};} \ ; \ )$.

Proof. Let $w_1; w_2$ in W, the following diagram commutes by Theorem 4.3.6.

$$\begin{array}{ccccc}
H^i_c(\widetilde{Q}_{\mathbf{L};\mathbf{P};} \ ; \ ) & \xrightarrow{(w_2 \ ^1)} & H^i_c(\widetilde{Q}_{\mathbf{L};\mathbf{P};w_2:} \ ; \ ) & \xrightarrow{(w_1 \ ^1)} & H^i_c(\widetilde{Q}_{\mathbf{L};\mathbf{P};w_1 w_2:} \ ; \ ) \\
 & & \downarrow f_{w_2: \ ;} & & \downarrow f_{w_1 w_2: \ ;w_1:} \\
 & & H^i_c(\widetilde{Q}_{\mathbf{L};\mathbf{P};} \ ; \ ) & \xrightarrow{(w_1 \ ^1)} & H^i_c(\widetilde{Q}_{\mathbf{L};\mathbf{P};w_1:} \ ; \ ) \\
 & & & & \downarrow f_{w_1: \ ;} \\
 & & & & H^i_c(\widetilde{Q}_{\mathbf{L};\mathbf{P};} \ ; \ )
\end{array}$$

Going from top left corner to bottom right corner by top right corner is (w_1w_2). Going by the middle gives (w_1) (w_2). Therefore $(w_1w_2) = (w_1)$ (w_2).

□

The representation obtained when K = C is isomorphic to the representation obtained for K = $\overline{\mathbb{F}}_q$ and large enough characteristic. Indeed this can be proved by base change exactly like in [HLR13, Theorem 2.5]. Therefore from now on we assume:

Assumption 4.3.9. K = $\overline{\mathbb{F}}_q$ *and the characteristic is large enough.*

This assumption is very convenient as it allows to introduce Frobenius endomorphism and use Grothendiek's trace formula to compute the traces of the action obtained.

F is the Frobenius endomorphism on $\mathfrak{gl}_n$ raising coe□cients to the power q so that its set of fixed point if $\mathfrak{gl}_n(\mathbb{F}_q)$ and similarly for the group GL_n. Assume that the L^j are subgroups of bock diagonal matrices, and P^j subgroup of block upper triangular matrices, so that they are F-stable. F induces a Frobenius endomorphism on $\widetilde{Q}^{\mathrm{reg}}_{\boldsymbol{L},\boldsymbol{P}}$ and on B^{reg} also denoted by F

$$F\left(\ ;(A_i;B_i)_{1\ i\ g};(X_j;g_jL_j)_{1\ j\ k}\right) = \left(F(\ );(F(A_i);F(B_i))_{1\ i\ g};(F(X_j);F(g_j)L_j)_{1\ j\ k}\right)$$

This Frobenius can be twisted by an element $w = (w_1;\dots;w_k)$ in the Weyl group W. For $\ 2\ B^{\mathrm{reg}}$, define

$$wF(\ _1;\dots;\ _k) = (w_1.F(\ _1);\dots;w_k.F(\ _k)).$$

$(B^{\mathrm{reg}})^{wF}$ is the set of points fixed by wF. Similarly the w-twisted Frobenius on $\widetilde{Q}^{\mathrm{reg}}_{\boldsymbol{L},\boldsymbol{P}}$ is

$$wF := w\ \ F.$$

They are compatible $p^{\mathrm{reg}}\ \ wF = wF\ \ p^{\mathrm{reg}}$ so that for $\ ;\ \ 2\ B^{\mathrm{reg}}$ the following diagram commutes

$$\begin{array}{ccc} H^i_c(\widetilde{Q}_{\mathbf{L},\mathbf{P},\ }\ ;\ ) & \xrightarrow{F} & H^i_c(\widetilde{Q}_{\mathbf{L},\mathbf{P},F^{-1}(\ )};\ ) \\ \downarrow f_{\ ,\ } & & \downarrow f_{F^{-1}(\ ),F^{-1}(\ )} \\ H^i_c(\widetilde{Q}_{\mathbf{L},\mathbf{P},\ }\ ;\ ) & \xrightarrow{F} & H^i_c(\widetilde{Q}_{\mathbf{L},\mathbf{P},F^{-1}(\ )};\ ) \end{array}$$

Theorem 4.3.10. *Let* $\ 2\ (B^{\mathrm{reg}})^F$ *and* $\ 2\ (B^{\mathrm{reg}})^{wF}$. *The cardinal of the set of* wF *fixed points of* $\widetilde{Q}_{\boldsymbol{L},\boldsymbol{P},\ }$ *is*

$$\sharp\widetilde{Q}^{wF}_{\boldsymbol{L},\boldsymbol{P},\ } = \sum_i \mathrm{tr}\left(\ ^{2i}(w);H^{2i}_c(\widetilde{Q}_{\mathbf{L},\mathbf{P},\ }\ ;\ )\right)q^i$$

Proof. Consider the commutative diagram:

$$\begin{array}{ccccc} H^i_c(\widetilde{Q}_{\mathbf{L},\mathbf{P},\ }\ ;\ ) & \xrightarrow{w} & H^i_c(\widetilde{Q}_{\mathbf{L},\mathbf{P},w^{-1}.\ }\ ;\ ) & & \\ \uparrow f_{\ ,\ } & \overset{(w^{-1})}{\searrow} & \downarrow f_{w^{-1}.\ ,\ } & & \\ & & H^i_c(\widetilde{Q}_{\mathbf{L},\mathbf{P},\ }\ ;\ ) & \xrightarrow{F} & H^i_c(\widetilde{Q}_{\mathbf{L},\mathbf{P},\ }\ ;\ ) \\ & & \uparrow f_{F(\ ),\ } & & \uparrow f_{\ ,\ } \\ H^i_c(\widetilde{Q}_{\mathbf{L},\mathbf{P},\ }\ ;\ ) & \xrightarrow{w} & H^i_c(\widetilde{Q}_{\mathbf{L},\mathbf{P},F(\ )};\ ) & \xrightarrow{F} & H^i_c(\widetilde{Q}_{\mathbf{L},\mathbf{P},\ }\ ;\ ) \end{array}$$

Apply Grothendieck trace formula to wF

$$
\begin{aligned}
\sharp\widetilde{Q}^{wF}_{\boldsymbol{L},\boldsymbol{P},} &= \sum_i (\;1)^i \operatorname{tr}\left((wF)\;; H^i_c(\widetilde{Q}_{\boldsymbol{L},\boldsymbol{P},}\;;\;)\right) \\
&= \sum_i (\;1)^i \operatorname{tr}\left(F \quad {}^i(w\;\;^1); H^i_c(\widetilde{Q}_{\mathbf{L},\mathbf{P},}\;;\;)\right)
\end{aligned}
$$

The varieties $Q_{\mathbf{L},\mathbf{P},}$ are pure and polynomial count and $\;(w\;\;^1)$ commutes with F so that

$$
\begin{aligned}
\sharp\widetilde{Q}^{wF}_{\boldsymbol{L},\boldsymbol{P},} &= \sum_i \operatorname{tr}\left(F \quad {}^{2i}(w\;\;^1); H^{2i}_c(\widetilde{Q}_{\mathbf{L},\mathbf{P},}\;;\;)\right) \\
&= \sum_i \operatorname{tr}\left(\;{}^{2i}(w\;\;^1); H^{2i}_c(\widetilde{Q}_{\mathbf{L},\mathbf{P},}\;;\;)\right) q^i
\end{aligned}
$$

Now as W is isomorphic to a product of symmetric group, w is conjugated to its inverse $w\;\;^1$ and

$$
\sharp\widetilde{Q}^{wF}_{\boldsymbol{L},\boldsymbol{P},} = \sum_i \operatorname{tr}\left(\;{}^{2i}(w); H^{2i}_c(\widetilde{Q}_{\mathbf{L},\mathbf{P},}\;;\;)\right) q^i
$$

□

The Levi subgroup has the following form

$$
L^j = \underbrace{\mathrm{GL}_{c^j_1} \quad \mathrm{GL}_{c^j_1}}_{m^j_1} \qquad \underbrace{\mathrm{GL}_{c^j_{k_j}} \quad \mathrm{GL}_{c^j_{k_j}}}_{m^j_{k_j}}
$$

with $c^j_r \neq c^j_s$ for $r \neq s$. Then the relative Weyl group is

$$
W_{\mathrm{GL}_n}(L^j) = \mathrm{S}_{m^j_1} \qquad \mathrm{S}_{m^j_{k_j}}
$$

The symmetric group $\mathrm{S}_{m^j_r}$ acts by permuting the blocks of size c^j_r. Notations are similar to 3.5.4 except that the index i disappears as $M^j = \mathrm{GL}_n$. A conjugacy class in this Weyl group is determined by a k_j-uple $(\;^{j,1}, \ldots, \;^{j,k_j})$ with $\;^{j,r} \; 2 \; P_{m^j_r}$. Hence the conjugacy class of $w \; 2 \; W$ determines a k-uple of n-types $! = (!^1, \ldots, !^k)$ with

$$
!^j = \left(\;^{j,1}_1, 1^{c^j_1}\right) \ldots \left(\;^{j,1}_{l(\;^{j,1})}, 1^{c^j_1}\right) \ldots \left(\;^{j,k_j}_1, 1^{c^j_{k_j}}\right) \ldots \left(\;^{j,k_j}_{l(\;^{j,k_j})}, 1^{c^j_{k_j}}\right) \tag{4.19}
$$

Let $O_! = (O_{!_1}, \ldots, O_{!_k})$ be the k-uple of F-stable adjoint orbits such that the F-fixed points $(O^F_{!_1}, \ldots, O^F_{!_k})$ is of type $!$ (the type of adjoint orbit in $\mathfrak{gl}_n(\mathbb{F}_q)$ is defined in 3.3.4). Then the natural map $\widetilde{Q}_{\boldsymbol{L},\boldsymbol{P},} \; ! \; Q_{\overline{O}_!}$ is an isomorphism commuting with the Frobenius so that

$$
\sharp\widetilde{Q}^{wF}_{\boldsymbol{L},\boldsymbol{P},} = \sharp Q^F_{\overline{O}_!} \tag{4.20}
$$

Letellier [Let11] computed the number of points of $Q^F_{\overline{O}_!}$.

Theorem 4.3.11. *The cardinal of $Q^F_{\overline{O}_!}$ is given by*

$$
\sharp Q^F_{\overline{O}_!} = (\;1)^{r(\;)} q^{\frac{d}{2}} \left\langle \widetilde{h} \;; \mathrm{H}^{HLV}_n(0, q^{\frac{1}{2}}) \right\rangle
$$

Proof. As the orbits O_{ω_j} are semisimple, the variety $Q_{\overline{O}_\omega}$ is smooth so that the characteristic function of the intersection complex is constant with value 1. The result follows from Letellier [Let11, Theorem 6.9.1, Theorem 7.4.1 and Corollary 7.4.3]. □

Corollary 4.3.12. *For $\in B$ and representing a conjugacy class in the Weyl group as described in (4.19), the -twisted Poincaré polynomial of $\widetilde{Q}_{\mathbf{L};\mathbf{P};}$ is*

$$\sum_i \operatorname{tr}\left(; H_c^i(\widetilde{Q}_{\mathbf{L};\mathbf{P};} ;)\right) v^i = (-1)^{r()} v^d \left\langle \widetilde{h} ; \mathbb{H}_n^{HLV}(0;v)\right\rangle :$$

Proof. The action comes from the W-equivariant structure of the constant sheaves H^i. Therefore up to isomorphism the action does not depend on the choice of $\in B$ so that the twisted Poincaré polynomial can be computed for $\in (B^{\mathrm{reg}})^F$. Then from Theorem 4.3.10 and (4.20)

$$\sum_i \operatorname{tr}\left(^{2i}( ); H_c^{2i}\left(\widetilde{Q}_{\mathbf{L};\mathbf{P};} ; \right)\right) q^i = (-1)^{r()} q^{\frac{d}{2}} \left\langle \widetilde{h} ; \mathbb{H}_n^{HLV}(0;q^{\frac{1}{2}})\right\rangle :$$

This equality remains true after substituting q^n for q for $n > 0$. Thus it is an equality between two polynomials and the corollary is proved. □

It is interesting to notice that Letellier [Let11, Corollary 7.4.3] obtained exactly the same formula for twisted Poincaré polynomials with a different construction of the action. His construction is the one recalled in Theorem 3.5.20 for the character varieties setting. Notice that it does not necessarily involve the whole group W but only the subgroup of elements $w \in W$ such that $w. =$. Interestingly for such w the action from 4.3.8 is simply given by $_i(w) = (w^{-1})$. In the particular case where the Levi subgroup is a torus, in the character variety setting, we shall see in Chapter 5 that both action coincide. Except in that particular case, we do not have a direct prove that both action coincide. However as the twisted Poincaré polynomial coincide they are necessarily isomorphic.

It is also interesting to consider this action as an action on the cohomology of a quiver variety with semisimple adjoint orbits at punctures. Indeed notice that for $\in B^{\mathrm{reg}}$ the map $p : \widetilde{Q}_{\mathbf{L};\mathbf{P};} \to Q_{\mathcal{S}}$ from 3.5.16 is an isomorphism. Let $\mathcal{S} = (S_1; \dots; S_k)$ a generic k-uple of semisimple adjoint orbit, S_j is the adjoint orbit of ^{j}. The Weyl group $W_{\mathrm{GL}_n}(L^j)$ is the group of permutation of the eigenvalues of S_j with the same multiplicities. We have another formulation of previous corollary

Corollary 4.3.13. *For representing a conjugacy class in the Weyl group as described in (4.19), the -twisted Poincaré polynomial of $Q_{\mathcal{S}}$ is*

$$\sum_i \operatorname{tr}\left(; H_c^i(Q_{\mathcal{S}};)\right) v^i = (-1)^{r()} v^d \left\langle \widetilde{h} ; \mathbb{H}_n^{HLV}(0;v)\right\rangle :$$

4.4 Combinatorial interpretation in the algebra spanned by Kostka polynomials

4.4.1 Description of the algebra

In this section an algebra spanned by Kostka polynomials is studied and some structure coefficients are related to traces of Weyl group action on the cohomology of quiver varieties. Define a linear map $\Delta^{\#} : \mathrm{Sym}[X] \to \mathrm{Sym}[X;Y]$ such that on the basis of modified Macdonald polynomials

$$\Delta^{\#}\left(\tilde{H}_\lambda[X]\right) := \tilde{H}_\lambda[X]\tilde{H}_\lambda[Y] \text{ for } \lambda \in \mathcal{P}.$$

As in 3.2.42, the variable $(q;t)$ are implicit. Now as the Hall pairing is non-degenerate there is a uniquely determined bilinear map $\ldots\#\ldots$ such that for all F, G and H in $\mathrm{Sym}[X]$:

$$\langle F[X]\#G[X], H[X]\rangle = \left\langle F[X]G[Y], \Delta^{\#}(H[X])\right\rangle$$

The product $\#$ defines an associative and commutative algebra structure on $\mathrm{Sym}[X]$.

Definition 4.4.1. *For a k-uplet of partitions $\mu = (\mu^1, \ldots, \mu^k) \in \mathcal{P}_n^k$ and $\lambda \in \mathcal{P}_n$ we denote by c_μ^λ the structure coefficients of the product $\#$ in the basis of Schur functions*

$$s_{\mu^1}\#s_{\mu^2}\ldots\#s_{\mu^k} = \sum_{\lambda\,|\lambda|=n} c_\mu^\lambda s_\lambda. \tag{4.21}$$

Remark 4.4.2. *For $\mu = (\mu;\nu)$, the coefficient $c_{\mu,\nu}^\lambda$ coincides with the one introduced in the introduction, i.e. the following relation is satisfied*

$$\tilde{K}_{\mu,\lambda}\tilde{K}_{\nu,\lambda} = \sum c_{\mu,\nu}^{\rho}\tilde{K}_{\rho,\lambda}. \tag{4.22}$$

Proof. First let $\left(\tilde{L}_{\lambda,\mu}\right)_{\lambda,\mu\in\mathcal{P}_n}$ the inverse of the matrix of Kostka polynomials $\left(\tilde{K}_{\lambda,\mu}\right)_{\lambda,\mu\in\mathcal{P}_n}$ (see Definition 3.2.41)

$$s_\lambda = \sum_{\rho\in\mathcal{P}_n}\tilde{L}_{\lambda,\rho}\tilde{H}_\rho[X].$$

Now the coefficient $c_{\mu,\nu}^\lambda$ is defined by

$$\begin{aligned} c_{\mu,\nu}^\lambda &= \langle s_\mu\#s_\nu, s_\lambda\rangle \\ &= \left\langle s_\mu\#s_\nu, \sum_{\rho\in\mathcal{P}_n}\tilde{L}_{\lambda,\rho}\tilde{H}_\rho[X]\right\rangle. \end{aligned}$$

Then by definition of the product $\#$ and the coproduct $\Delta^{\#}$:

$$\begin{aligned} c_{\mu,\nu}^\lambda &= \sum_{\rho\in\mathcal{P}_n}\tilde{L}_{\lambda,\rho}\left\langle s_\mu[X]s_\nu[Y], \tilde{H}_\rho[X]\tilde{H}_\rho[Y]\right\rangle \\ c_{\mu,\nu}^\lambda &= \sum_{\rho\in\mathcal{P}_n}\tilde{L}_{\lambda,\rho}\tilde{K}_{\mu,\rho}\tilde{K}_{\nu,\rho} \end{aligned}$$

Multiply last equation by $\tilde{K}_{\lambda,\sigma}$ and sum over $\lambda \in \mathcal{P}_n$:

$$\tilde{K}_{\mu,\sigma}\tilde{K}_{\nu,\sigma} = \sum c_{\mu,\nu}^{\lambda}\tilde{K}_{\lambda,\sigma}.$$

Which is the relation used in introduction to define the coefficients $c_{\mu,\nu}^\lambda$. □

Example 4.4.3. *We computed some coefficients with Sage*

$$c^{(2,1,1)}_{(2,2),(2,1,1)} = q^3t - q^2t^2 - qt^3 - q^2t - t^2q + q^2 + qt + t^2$$
$$c^{(1,1,1,1)}_{(2,2),(2,1,1)} = q^3 + q^2t + qt^2 + t^3 + q^2 + 2qt + t^2 + q + t$$

Next conjecture comes from unpublished notes by Fernando Rodriguez Villegas.

Conjecture 4.4.4. *The structure coefficients $c^{\lambda}_{\mu,\nu}$ lie in $\mathbb{Z}[q,t]$.*

Some evidences supporting this conjecture will be provided. Following definition and remark were suggested by François Bergeron.

Definition 4.4.5. *Let F be a symmetric function, consider the operator*

$$F \# \ldots : \mathrm{Sym}[X] \to \mathrm{Sym}[X]$$
$$G \mapsto F \# G.$$

We denote Δ_F its adjoint with respect to the Hall pairing so that for any $G, H \in \mathrm{Sym}[X]$

$$\langle F \# G, H \rangle = \langle G, \Delta_F(H) \rangle \qquad (4.23)$$

Those operators are diagonal in the basis of modified Macdonald polynomials

$$\Delta_F(\tilde{H}_\mu[X;q,t]) = \left\langle F, \tilde{H}_\mu[X;q,t] \right\rangle \tilde{H}_\mu[X;q,t]. \qquad (4.24)$$

Remark 4.4.6. *Applying* (4.24) *with e_n*

$$\Delta_{e_n}\left(\tilde{H}_\mu[X;q,t]\right) = q^{n(\mu')}t^{n(\mu)}\tilde{H}_\mu[X;q,t]$$

we recognize the usual expression of the operator ∇ introduced by Bergeron-Garsia [BG98]. The higher (q,t)-Catalan sequence from Garsia-Haiman [GH96] (see also Haiman [Hai02, p.95]) is defined by

$$C_n^{(m)}(q,t) = \langle e_n, \nabla^m e_n \rangle$$

$\nabla = \Delta_{e_n}$ is the adjoint of $e_n \# \ldots$, moreover $s_{1^n} = e_n$ so that

$$C_n^{(m)}(q,t) = c^{1^n}_{\underbrace{1^n,\ldots,1^n}_{m+1}}.$$

The higher (q,t)-Catalan sequence are particular cases of the coefficients c^{1^n}.

We recall an important theorem which was first conjectured by Garsia-Haiman [GH96].

Theorem 4.4.7 ([Hai02] theorem 4.2.5). *The symmetric function $\nabla(e_n)$ is obtained as the Frobenius characteristic (see definition 3.2.29) of a bigraded representation of $\mathfrak{S}_n$, the so-called diagonal harmonics. In particular*

$$\langle \nabla(e_n), s_\lambda \rangle \in \mathbb{N}[q,t].$$

Corollary 4.4.8. *For any $\ 2 P_n$ the structure coe cients $c^{1^n}_{1^n;}$ gives the multiplicity of the irreducible representation of type in the bigraded representation of* S_n *on diagonal harmonics. In particular are $c^{1^n}_{1^n;}(q;t) \ 2\ \mathbb{N}[q;t]$ so that the conjecture 4.4.4 is true for those particular coe cients.*

Proof. According to remark 4.4.6 and adjonction relation (4.23)

$$hs\ ; r(e_n)i = he_n \# s\ ; e_n i \tag{4.25}$$

. By definition of the structure coe□cients $c_{\ ;}$ and as $e_n = s_{1^n}$

$$e_n \# s \ = \sum_{2P_n} c_{1^n} \ s$$

substituting in (4.25) we obtain

$$c^{1^n}_{1^n;}(q;t) = hs\ ; r(e_n)i$$

we conclude by the interpretation of $r(e_n)$ as a Frobenius characteristic from Theorem 4.4.7. □

Next theorem and corollary come from unpublished notes by Rodriguez-Villegas. The particular structure coe□cients c^{1^n} are related to the kernel $\mathbb{H}_n^{HLV}$.

Consider the generating function from Definition 3.6.1 for genus $g = 0$, $k+2$ punctures and with variable $z = q^{\frac{1}{2}}$, $w = t^{\frac{1}{2}}$. It is given by

$$^{0}_{k+2} := \sum_{2P} \frac{\prod_{i=1}^{k+2} \tilde{H}\ [X_i;q;t]}{a\ (q;t)} s^{j\ j}$$

with $a\ (q;t) = \left(\tilde{H}\ [X;q;t]; \tilde{H}\ [X;q;t]\right)^{q;t}$ as in 3.2.40.

Theorem 4.4.9. *We have the following relation:*

$$\left\langle p_{(n)}[X_{k+1}] h_{(n\ \ 1;1)}[X_{k+2}]; \mathrm{Log}\left[\ ^{g}_{k+2}\right]\right\rangle_{X_{k+1};X_{k+2}} = \sum_{j\ j=n} \frac{^{0}}{a} \prod_{i=1}^{k} \tilde{H}\ [X_i] s^{j\ j}$$

with

$$= \sum_{i;j\ 2} q^{j\ \ 1} t^{i\ \ 1}$$

$$^{0} = \prod_{i;j\ 2\ \ n(1;1)} (1 \ \ q^{j\ \ 1} t^{i\ \ 1})$$

Proof. According to Lemma 3.2.45, take the Hall pairing with $h_{(n\ \ 1;1)}[X_{k+2}]$ is equivalent to do plethystic substitution $X_{k+2} = 1+u$ and take the degree n coe□cient in front of u. As plethystic substitution and plethystic logarithm commute according to Proposition 3.2.17, we can perform this substitution inside the plethystic logarithm.

We consider terms of order 1 in u using (3.24)

$$\begin{aligned}\mathrm{Log}\left[\ {}^{0}_{k+2}\right] &= \mathrm{Log}\left[\ {}^{0}_{k+1} + u \sum_{2P} \frac{}{a} \prod_{i=1}^{k+1} H\ [X_i] s^{j\ j} + O(u^2)\right] \\ &= \mathrm{Log}\left[\ {}^{0}_{k+1}\left(1 + u \frac{1}{{}^{0}_{k+1}} \sum_{2P} \frac{}{a} \prod_{i=1}^{k+1} H\ [X_i] s^{j\ j} + O(u^2)\right)\right] \\ &= \mathrm{Log}\left[\ {}^{0}_{k+1}\right] + \mathrm{Log}\left[1 + u \frac{1}{{}^{0}_{k+1}} \sum_{2P} \frac{}{a} \prod_{i=1}^{k+1} H\ [X_i] s^{j\ j} + O(u^2)\right]\end{aligned}$$

We used that plethystic logarithm turns product into sum. From the definition of the plethystic logarithm, as $p_n[u] = u^n$, we easily see the coe□cient in front of u in previous expression

$$\mathrm{Log}\left[\ {}^{0}_{k+2}\right]\Big|_u = \frac{1}{{}^{0}_{k+1}} \sum_{2P} \frac{}{a} \prod_{i=1}^{k+1} H\ [X_i] s^{j\ j} :$$

Keeping the terms of degree n we obtain

$$\left\langle h_{(n\ \ 1;1)}[X_{k+2}]; \mathrm{Log}\left[\ {}^{0}_{k+2}\right]\right\rangle_{X_{k+2}} = \frac{1}{{}^{0}_{k+1}} \sum_{2P} \frac{}{a} \prod_{i=1}^{k+1} H\ [X_i] s^{j\ j}\Bigg|_{s^n} :$$

Inverting ${}^{0}_{k+1}$ is licit, it is defined by

$$\frac{1}{{}^{0}_{k+1}} = \frac{1}{1 + (\ {}^{0}_{k+1}\ \ 1)} = \sum_{k} (1\ \ \ \ {}^{0}_{k+1})^k :$$

Now we just have to take Hall pairing with the power sum $p_{(n)}\,[X_{k+1}]$. It is equivalent to take the coe□cient in front of $n\ \ ^1 p_{(n)}\,[X_{k+1}]$. But $p_{(n)}$ cannot be written as the product of two symmetric functions of degree strictly smaller than n so that the contribution of ${}^{0}_{k+1}$ in the denominator is irrelevant for the coe□cient in front of $n\ \ ^1 p_{(n)}\,[X_{k+1}]$ so that

$$\left(p_{(n)}[X_{k+1}] h_{(n\ \ 1;1)}[X_{k+2}]; \mathrm{Log}\left[\ {}^{0}_{k+2}\right]\right)_{X_{k+1};X_{k+2}} = \left(p_{(n)}[X_{k+1}]; \sum_{2P} \frac{}{a} \prod_{i=1}^{k+1} H\ [X_i] s^{j\ j}\right)_{X_{k+1}}$$

We conclude with Lemma 3.2.46 and (3.23). □

Corollary 4.4.10. *With the notations of previous theorem and definition 4.4.1*

$$(\ \ 1)^{n\ \ 1} c^{(1^n)} = (q\ \ \ 1)(1\ \ \ t)\left(\prod_{j=1}^{k} s\ _j[X_j] p_{(n)}[X_{k+1}] h_{(n\ \ 1;1)}[X_{k+2}]; \mathrm{Log}\left[\ {}^{0}_{k+2}\right]\right)_{X_1;:::;X_{k+2}} \tag{4.26}$$

Proof. We apply Theorem 4.4.9 to express the right hand side of (4.26) as

$$(q\ \ \ 1)(1\ \ \ t)\left(s\ _1[X_1] ::: s\ _k[X_k]; \sum_{j\ \ j=n} \frac{{}^{0}}{a} \prod_{i=1}^{k} H\ [X_i]\right)_{X_1;:::;X_k} :$$

By definition of the product $\#$:

$$(q \quad 1)(1 \quad t)\left(s_{\ 1}\# ::: \# s_{\ k}[X]; \sum_{j\ j=n} \frac{}{a}{}^{0} \tilde{H}\ [X]\right)_X :$$

Here we recognize the expression of Theorem 3.2.47

$$\left(s_{\ 1}\# ::: \# s_{\ k}[X]; (\quad 1)^{n \quad 1} s_{(1^n)}\right)_X$$

so that if we write

$$s_{\ 1}\# ::: \# s_{\ k}[X] = \sum c\ \ s\ [X]$$

the result follows from orthonormality of Schur functions. □

4.4.2 Interpretation of coefficients as traces of Weyl group action on the cohomology of quiver varieties

In this section a cohomological interpretation is given for the coefficients c . In order to lighten the notations the description is only given for the coefficient $c_{\ ;\ }$. The generalization to any   is straightforward.

First let us detail the data to describe the relevant variety $\tilde{Q}_{\boldsymbol{L};\boldsymbol{P};\ }$. The Levi subgroups are torus of diagonal matrices $L^j = T$ for $1 \quad j \quad 4$. The semisimple part $= (\ ^1; ::: ;\ ^4)$ is such that:

- $^1 = \ _1 \mathrm{Id}$ is central.
- $^2 = \ _2 \mathrm{Id}$ is central.
- $^3 = \begin{pmatrix} 1 & & & \\ & 2 & & \\ & & \ddots & \\ & & & n \end{pmatrix}$ with $_r \neq \ _s$ for $r \neq s$.
- $^4 = \begin{pmatrix} & & \\ & \ddots & \\ & & \end{pmatrix}$ has two eigenvalues $\neq$. The multiplicity of   is one and the multiplicity of   is $n \quad 1$.

Notice that such a choice can be made in the regular locus $\ \in B^{\mathrm{reg}}$.

First we consider Letellier's construction of the action in order to compute isotypical component. Let $M = M^1 \qquad M^4$ with M^j the centralizer in GL_n of $\ ^j$. Then $W_{\boldsymbol{M}}(L) = \mathrm{S}_n^2$. Letellier's construction provide an action of $W_{\boldsymbol{M}}(L)$ on the cohomology of $\tilde{Q}_{\boldsymbol{L};\boldsymbol{P};\ }$. Moreover

$$\mathrm{Hom}_{W_{\boldsymbol{M}}(\boldsymbol{L})}\left(V_{\ 0} \quad V_{\ 0}; H_c^{i+d_{\tilde{Q}_{\boldsymbol{L};\boldsymbol{P};}}}\left(\tilde{Q}_{\boldsymbol{L};\boldsymbol{P};\ }\ ;\ \right)\right) = H_c^{i+d_{Q_{\overline{\mathcal{O}}}}}(Q_{\overline{\mathcal{O}}};\ ) \tag{4.27}$$

With $\mathcal{O} = (O_1; ::: ; O_4)$ the 4-uple of generic adjoint orbits defined by:

- O_1 has Jordan type ^{0} and eigenvalue $_1$.
- O_2 has Jordan type ^{0} and eigenvalue $_2$.
- O_3 is the orbit of ^{3}.
- O_4 is the orbit of ^{4}.

Now with the construction from previous section, there is an action of the whole group $W = S_n^4$ on the cohomology of $\widetilde{Q}_{\mathbf{L};\mathbf{P};}$. The restriction of this W-action to $W_{\mathbf{M}}(L) = S_n^2$ is isomorphic to the Springer action. First take the $V_0 \quad V_0$ isotypical component with respect to the S_n^2-action. There remains an action of the Weyl group S_n^2 relative to the puncture 3 and 4 on the cohomology $IH_c^{i+d_{\overline{\mathcal{O}}}}(Q_{\overline{\mathcal{O}}}; \ )$.

Theorem 4.4.11. *Let w an n-cycle in the Weyl group relative to the third puncture. The coe cient $c^{1^n}_{;}$, after specialization $q = 0$, is given by the w-twisted Poincaré polynomial of $Q_{\overline{\mathcal{O}}}$, namely*

$$c^{1^n}_{;}(0;t) = t^{\frac{d_{\overline{\mathcal{O}}}}{2}} \sum_i \operatorname{tr}\left(w; IH_c^{2i}(Q_{\overline{\mathcal{O}}}; \ )\right) t^i$$

Proof. Combining (4.27), Theorem 3.6.7 and Remark 3.6.9

$$\sum_i \operatorname{tr}\left(w; IH_c^{i}(Q_{\overline{\mathcal{O}}}; \ )\right) v^i = (\ 1)^{n \ 1} v^{d_{\overline{\mathcal{O}}}} \left\langle s \ [X_1] s \ [X_2] p_{(n)}[X_3] h_{(n \ 1;1)}[X_4]; \mathbb{H}_n^{HLV}(0;v)\right\rangle :$$

The theorem follows from Corollary 4.4.10. □

4.4.3 Cohomological interpretation in the multiplicative case

Let us mention a conjectural similar interpretation in the multiplicative case. First introduce the relevant parameters. The Levi subgroups are tori of diagonal matrices $L^j = T$ for $1 \quad j \quad 4$. The semisimple par $\quad = (\ ^1; : : : ; \ ^4)$ is such that:

- $^1 = \ _1 \operatorname{Id}$ is central.
- $^2 = \ _2 \operatorname{Id}$ is central.
- $^3 = \ _3 \operatorname{Id}$ is central.
- $^4 = \begin{pmatrix} & & \\ & \ddots & \\ & & \end{pmatrix}$ has two eigenvalues $\quad 6= \quad$. The multiplicity of is one and the multiplicity of is $n \quad 1$.

Moreover this 4-uple can be chosen in the regular locus. Note that the same notations for the parameter are the same as in previous section, however objects are di erent as we now consider resolutions of character varieties. For instance the eigenvalues are now necessarily non zero and the genericity condition is the multiplicative one. The relative Weyl group is $W_{\mathbf{M}}(L) = S_n^3$. Now consider the following conjugacy classes

- C_1 has Jordan type ^{0} and eigenvalue $_1$.
- C_2 has Jordan type ^{0} and eigenvalue $_2$.
- C_3 has Jordan type (n) and eigenvalue $_3$.
- C_4 is the conjugacy class of $_4$.

Then $\widetilde{M}_{\mathbf{L};\mathbf{P};}$ is the resolution of $M_{\overline{\mathcal{C}}}$ with $\mathcal{C} = (C_1; : : : ; C_4)$. An intermediate between $\widetilde{M}_{\mathbf{L};\mathbf{P};}$ and $M_{\overline{\mathcal{C}}}$ is given by the variety

$$M_{\;0;\;0} = \Big\{(X_1; : : : ; X_4) \; 2 \; C_1 \qquad C_4; gB \; 2 \; \mathrm{GL}_n = B \, \big| g^{\;1} X_3 g \; 2 \;\; {}_3 U \qquad X_1 : : : X_4 = \mathrm{Id} g = = \mathrm{PGL}_n :$$

Then the resolution $\widetilde{M}_{\mathbf{L};\mathbf{P};}$! $M_{\overline{\mathcal{C}}}$ factors through $M_{\;;\;}$. This is a particular case of the *partial* resolutions of character varieties studied by Letellier [Let13]. As in the additive case, first take the $V_{\;0} \quad V_{\;0}$ isotypical component of the cohomology $H_c\left(\widetilde{M}_{\mathbf{L};\mathbf{P};} \; ;\right)$ then take the trace of an n-cycle with respect to the third puncture.

Just like Theorem 4.4.11 is derived from Theorem 3.6.7; next conjecture follows from Conjecture 3.6.10 for the twisted mixed-Hodge polynomial a resolution $\widetilde{M}_{\mathbf{L};\mathbf{P};}$.

Conjecture 4.4.12. *Let w an n-cycle in the Weyl group relative to the third puncture. The coe cient $c^{1^n}_{\;;}$ relates to the w-twisted mixed Hodge polynomial of $M_{\;0;\;0}$:*

$$c^{1^n}_{\;;}(q; t) = t^{\frac{\dim M_{\;0;\;0}}{2}} \, I H_c^{w}\left(M_{\;0;\;0}; \frac{1}{q}; \sqrt{qt}\right) :$$

In 6.2.2, the Poincaré polynomial specialization of this conjecture is proved.

Chapter 5

Intersection cohomology of character varieties with k 1 semisimple monodromies

5.1 Introduction

In this chapter the base field is C and we study character varieties with one monodromy of any type and the k 1 others semisimple. With this assumption, the Poincaré polynomial for intersection cohomology can be computed using only algebraic tools. In next chapter the hypothesis k 1 monodromies are semisimple is relaxed, then analytic tools are necessary.

Mellit computed the Poincaré polynomial of character varieties with semisimple conjugacy classes at each punctures [Mel17a]. He also constructed a family $\widetilde{M}$ of character varieties with their resolutions [Mel19]. This chapter relies on both results. In the family $\widetilde{M}$, the k 1 first conjugacy classes are fixed and are semisimple.

- The family's generic fiber is a character variety with a regular semisimple conjugacy class at the k-th puncture.
- Particular fibers are resolutions of character varieties with the closure of a regular conjugacy class at the k-th puncture.

This family comes with various Weyl group actions. There is a monodromic Weyl group action on the cohomology of the generic fibers (a character variety with regular semisimple conjugacy class at the k-th puncture). There is a Springer action on the cohomology of the particular fibers, it coincides with the action from 3.5.4. Mellit unified those actions on a local system equivariant for the action of the Weyl group W. This construction was a motivation for the construction of the Weyl group action on the cohomology of comet-shaped quiver varieties in 4.3.

Those constructions and the combinatoric relations between cohomology of resolutions and intersection cohomology of character varieties allow to compute the Poincaré polynomial. The idea Mellit suggested us, is to study the restriction of the W-action to subgroup W_M. Then the fiber of the W-equivariant local system over a point fixed by W_M carries a W_M-action. The isotypical component corresponding to the sign representation of W_M is the cohomology of a character variety with semisimple conjugacy classes at each punctures. The Poincaré polynomial of those character varieties is known. Considering various subgroup W_M, the relation can be inverted. This proves the Poincaré polynomial specialization of Letellier's conjecture [Let13] when k 1 conjugacy classes at punctures are semisimple.

Theorem 5.1.1. *$S_1; \dots; S_{k\ 1}$ are k 1 semisimple conjugacy classes, the multiplicities of their eigenvalues is determined by = ($^1; \dots;$ $^{k\ 1}$) 2 $P_n^{k\ 1}$. The conjugacy class $C_{\_;}$ is such that $(S_1; \dots; S_{k\ 1}; C_{\_;})$ is generic. Then the Poincaré polynomial for compactly supported intersection cohomology of the character variety $M_{S;\overline{C}_{\_;}}$ with k 1 monodromies in the semisimple conjugacy classes S_j and one monodromy in $\overline{C}_{\_;}$ is*

$$P_c\left(M_{S;\overline{C}_{\_;}}; t\right) = t^{d\ ;\_}\left\langle h\ \ s_{\_0}; \mathbb{H}_n^{HLV}(\ 1; t)\right\rangle :$$

Where

$$h\ \ s_{\_0} = h_{\ 1}[X_1] \dots h_{\ k\ 1}[X_{k\ 1}] s_{\_0}[X_k]$$

and $d_{\ ;\_} := \dim M_{S;\overline{C}_{\_;}}$

First in 5.2 we check compatibility between W-action on the restriction of the Springer complex and W_M-action from parabolic induction for regular conjugacy classes. Then it is applied to character varieties 5.3. In section 5.4, Mellit's construction of family of character varieties is detailed. Finally, in 5.5, combinatoric relations are inverted and the Poincaré polynomial for intersection cohomology is computed.

5.2 Resolutions of regular conjugacy classes and parabolic induction

In 3.3.3, resolutions of closure of conjugacy classes were discussed. Those resolutions come with Weyl group actions. In this section we focus on resolution of regular conjugacy classes. We check that the Weyl group action coming from such resolution is compatible with the action coming from restriction of the Springer complex.

Let 2 T and $M = Z_G(\ )$ the centralizer of in G. The Weyl group W_M is the stabilizer of in W. Let $\overline{C^{\mathrm{reg}}}$ the closure of the regular conjugacy class in G with semisimple part . Consider the Cartesian square

$$\begin{array}{ccc} \widetilde{G} & \longleftarrow & \bigsqcup_{w 2 W = W_M} \widetilde{\mathbb{X}}_{T;B;w:} \\ p^G \downarrow & & \downarrow \textstyle\bigsqcup_{w 2 W = W_M} p^{w:} \\ G & \xleftarrow{\ \ i\ \ } & \overline{C^{\mathrm{reg}}} \end{array}$$

with $w: := \underline{w}\ \ \underline{w}^{\ 1}$ for $\underline{w}$ a representative of w in G. Base change gives an isomorphism

$$i\ p_!^G = \bigoplus_{w 2 W = W_M} p_!^{w:} :$$

Springer theory recalled in 3.4.1 provides an action of the Weyl group W on $p_!^G$ therefore on $\bigoplus_{w 2 W = W_M} p_!^{w:}$. Next theorem is a direct application of Lusztig parabolic induction.

Theorem 5.2.1. *The W-action on $\bigoplus_{w 2 W = W_M} p_!^{w:}$ restricts to an action of W_M W on $p_!$ moreover*

$$p_! \; [\dim \widetilde{X}_{T;B;} \,] = \bigoplus_{\_ 2 P} V_{\_} \quad \underline{I C}_{\overline{C}_{\_;}} \; : \tag{5.1}$$

The sum is over l-uple $\_ \; 2 \; P \; = P_{\,1} \qquad P_{\,l}$ and $V_{\_}$ is the associated irreducible representation of W_M.

Proof. Note that (5.1) follows from Theorem 3.4.8, however it is detailed here in order to track the W-action from Springer theory.

Resolutions such as $\widetilde{X}_{T;B;}$ fit in the following diagram where the first line is the diagram of parabolic induction (3.29) from the torus T to G

$$\begin{array}{ccccccc}
T & \longleftarrow & \widehat{G} & \longrightarrow & \widetilde{G} & \longrightarrow & G \\
\uparrow & & \uparrow & & \uparrow & & \uparrow i \\
\bigsqcup_{w 2 W = W_M} fw: \; g & \longleftarrow & \bigsqcup_{w 2 W = W_M} \widehat{X}_{T;B;w:} & \longrightarrow & \bigsqcup_{w 2 W = W_M} \widetilde{X}_{T;B;w:} & \longrightarrow & \overline{C^{\mathrm{reg}}}
\end{array}$$

with

$$\widehat{X}_{L;P;} \; := \left\{ (x;g) \; 2 \; G \quad G \,\middle|\, g \;\, ^1 x g \; 2 \quad U_P \right\} :$$

From this diagram where squares are cartesian:

$$i \;\; \mathrm{Ind}_T^G \;\; _B \quad _T = i \quad \bigoplus_{w 2 W = W_M} \mathrm{Ind}_T^G \;\; _B \quad _{fw: \; g} = \bigoplus_{w 2 W = W_M} p_!^{w:}$$

With $_{fw: \; g}$ the constant sheaf supported on $fw: \; g$. A W-action is inherited from the action on $\mathrm{Ind}_T^G \;\; _B \quad _T$. This action restricts to a W_M-action.

Consider the same construction with M instead of G:

$$\begin{array}{ccccccc}
T & \longleftarrow & \widehat{M} & \longrightarrow & \widetilde{M} & \longrightarrow & M \\
\uparrow & & \uparrow & & \uparrow & & \uparrow j \\
f \;\; g & \longleftarrow & \widehat{X}_{T;B \setminus M;} & \longrightarrow & \widetilde{X}_{T;B \setminus M;} & \longrightarrow & \overline{C^M}
\end{array}$$

C^M is the regular conjugacy class in M with semisimple part and the squares are cartesian. One obtains

$$j \;\; \mathrm{Ind}_T^M \;\; _{B \setminus M} \quad _T = j \;\; \mathrm{Ind}_T^M \;\; _{B \setminus M} \quad _{f \;\; g}$$

and a W_M-action on this complex is inherited from the W_M-action on $\mathrm{Ind}_T^M \;\; _{B \setminus M} \quad _T$. It provides a W_M action on

$$i \;\; \mathrm{Ind}_T^G \;\; _B \quad _{f \;\; g} = i \;\; \mathrm{Ind}_M^G \;\; _P \; \mathrm{Ind}_T^M \;\; _{B \setminus M} \quad _{f \;\; g} :$$

Both W_M-actions coincide as detailed by Lusztig [Lus86]. Moreover Springer theory for M provides a description of this W_M action. Indeed

$$\overline{C^M} = \quad N_M$$

and left multiplication by provides an isomorphism between N_M and N_M. From the restriction of the Springer complex to N_M described in (3.28), we deduce

$$\mathrm{Ind}_{T\ B\cap M}^{M}\ f\ g = \bigoplus_{\_2P} V_{\_}\ \ \underline{IC_{\overline{C^M_{\_;}}}}$$

Finally for $\_ 2 P$ notice that

$$\mathrm{Ind}_{M\ P}^{G}\ \underline{IC_{\overline{C^M_{\_;}}}} = \underline{IC_{\overline{C}_{\_;}}}\ :$$

To conclude, the W-action on $\bigoplus_{w2W=W_M} p_!^{w:}$ restricts to a W_M-action such that $p_!$ is W_M-stable and

$$p_!\ \ [\dim \widetilde{X}_{T;B;}\ ] = \bigoplus_{\_2P\ 1\qquad P\ l} V_{\_}\ \ \underline{IC_{\overline{C}_{\_;}}}\ :$$

□

5.3 Resolution of the k-th conjugacy of character variety

In this section we detail how to apply previous resolution of regular conjugacy class to character varieties. Fix a $(k\ \ 1)$-uple of semisimple conjugacy classes $S = (S_1; : : : ; S_{k\ 1})$. Let $= (\ ^1; : : : ;\ ^{k\ 1})\ 2\ P_n^{k\ 1}$ with ^{i} the partition defined by the multiplicites of the eigenvalues of S_i. Let $2\ T$ such that the k-uple $\left(S_1; : : : ; S_{k\ 1}; \overline{C^{\mathrm{reg}}}\right)$ is generic.

Consider the resolution of the character variety with specified conjugacy classes at punctures $\left(S_1; : : : ; S_{k\ 1}; \overline{C^{\mathrm{reg}}}\right)$. This is a particular case of the situation described in 3.5.3, it is detailed here because a precise track of the Springer action is necessary.

$2\ P_n$ is the partition defined by the multiplicities of the eigenvalues of . The Levi subgroup $M = \mathrm{GL}$ is the centralizer of in GL_n. For any conjugacy class $C_{\_;}\ \ \overline{C^{\mathrm{reg}}}$ one can consider the character variety

$$M_{S;\overline{C}_{\_;}} := R_{S;\overline{C}_{\_;}} ==\mathrm{PGL}_n$$

with

$$R_{S;\overline{C}_{\_;}} := \Big\{(A_1; B_1; : : : ; A_g; B_g; X_1; : : : ; X_k)\ 2\ \mathrm{GL}_n^{2g}\ \ S_1 \qquad S_{k\ 1}\ \ \overline{C}_{\_;}\ \Big| \\ A_1B_1A_1^{\ 1}B_1^{\ 1} : : : A_gB_gA_g^{\ 1}B_g^{\ 1}X_1 : : : X_k = \mathrm{Id}\Big\} :$$

In previous section we considered $\widetilde{X}_{T;B;}$, the resolution of $\overline{C^{\mathrm{reg}}}$ the closure of the regular conjugacy class with semisimple part . This is used to construct a resolution of the character variety $M_{S;\overline{C^{\mathrm{reg}}}}$. Define

$$\widetilde{R}_{S;} := \Big\{(A_1; B_1; : : : ; B_g; X_1; : : : ; X_{k\ 1}; (X_k; gB))\ 2\ \mathrm{GL}_n^{2g}\ \ S_1 \qquad S_{k\ 1}\ \ \widetilde{X}_{T;B;}\ \Big| \\ A_1B_1A_1^{\ 1}B_1^{\ 1} : : : A_gB_gA_g^{\ 1}B_g^{\ 1}X_1 : : : X_k = \mathrm{Id}\Big\} :$$

The group PGL_n acts on this variety by

$$h.(A_1, \ldots, B_g, X_1, \ldots, (X_k, gB)) = \left(hA_1h^{-1}, \ldots, hX_1h^{-1}, \ldots, \left(hX_kh^{-1}, hgB\right)\right).$$

Consider the geometric quotient defined thanks to Mumford's geometric invariant theory

$$\widetilde{M}_{S,} = \widetilde{R}_{S,} /\!/ \mathrm{PGL}_n .$$

The map $p : \widetilde{\mathbb{X}}_{T,B,} \to \overline{C}^{\mathrm{reg}}$ induces a map

$$: \widetilde{M}_{S,} \to M_{S,\overline{C}^{\mathrm{reg}}}.$$

Those constructions fit in the following diagram where both squares are Cartesian

$$\begin{array}{ccccc}
\widetilde{M}_{S,} & \longleftarrow & \widetilde{R}_{S,} & \longrightarrow & \widetilde{\mathbb{X}}_{T,B,} \\
\downarrow & & \downarrow & & \downarrow p \\
M_{S,\overline{C}^{\mathrm{reg}}} & \longleftarrow & R_{S,\overline{C}^{\mathrm{reg}}} & \xrightarrow{\mathrm{pr}} & \overline{C}^{\mathrm{reg}}
\end{array}$$

This diagram is a particular case of Letellier's construction and we have the following theorem [Let13, Theorem 5.4].

Theorem 5.3.1. *The map* $: \widetilde{M}_{S,} \to M_{S,\overline{C}^{\mathrm{reg}}}$ *is a resolution of singularities. The Weyl group* W_M *acts on the derived pushforward of the constant sheaf* $_!$ *and*

$$_! [d] = \bigoplus_{2P} V \otimes \underline{IC}_{M_{S,\overline{C},}} .$$

The sum is over l-uple $\in P_1 \times \cdots \times P_l$*, the space* V *is the associated irreducible representation of* W_M *and* $d := \dim \widetilde{M}_{S,}$ *.*

Proof. It is a direct consequence of Theorem 5.2.1, base change, and the fact that $\mathrm{pr}\ \underline{IC}_{\overline{C},} = \underline{IC}_{M_{S,\overline{C},}}$, see [Let13, Theorem 4.10]. □

This theorem gives the compatibility between Springer action constructed from resolutions of closure of conjugacy classes and construction of character varieties. In particular it provides an action of W_M on the cohomology of the resolution $\widetilde{M}_{S,}$.

5.4 Family of character varieties

Mellit [Mel19] studied the family formed by the varieties $\widetilde{M}_{S,}$ when the parameter is varying. This construction is recalled and used to compute the intersection cohomology of the varieties $M_{S,\overline{C},}$. As in previous section, semisimple conjugacy classes are fixed $S = (S_1, \ldots, S_{k-1})$. For $\in T$ denote by S its conjugacy class.

Definition 5.4.1. *Let* $T_0 \subset T$ *the set of elements such that the k-uple* $(S_1, \ldots, S_{k-1}, S)$ *is generic.*

Denote by W the Weyl group $W = W_{\mathrm{GL}_n}(T)$. For $X \in \mathrm{GL}_n$ its characteristic polynomial is $(X) \in T/W$. The family of character varieties defined by Mellit is:

$$M = R /\!/ \mathrm{PGL}_n$$

with

$$R = \{A_1, \dots, B_g, X_1, \dots, X_k \in \mathrm{GL}_n \mid X_i \in S_i \text{ for } i < k \text{ and } (X_k) \in T_0/W$$
$$A_1 B_1 A_1^{-1} B_1^{-1} \dots A_g B_g A_g^{-1} B_g^{-1} X_1 \dots X_k = \mathrm{Id}\}.$$

(X_k) is the characteristic polynomial of X_k and we still denote by the induced map $: M \to T_0/W$. For $\in T_0$ denote by $[\ ]$ its class in T_0/W. Note that

$$^{-1}([\ ]) = M_{S, \overline{C^{\mathrm{reg}}}}.$$

The resolutions $\widetilde{M}_{S,}$ also fit in a family

$$\widetilde{M} = \widetilde{R} /\!/ \mathrm{PGL}_n$$

with

$$\widetilde{R} = \{A_1, \dots, B_g, X_1, \dots, X_k \in \mathrm{GL}_n, gB \in \mathrm{GL}_n/B \mid X_i \in S_i \text{ for } i < k$$
$$g^{-1} X_k g \in T_0 U \text{ and } A_1 B_1 A_1^{-1} B_1^{-1} \dots A_g B_g A_g^{-1} B_g^{-1} X_1 \dots X_k = \mathrm{Id}\}.$$

Denote by the map induced by the projection $T_0 U \to T_0$. Note that

$$^{-1}(\ ) = \widetilde{M}_{S,}$$

There is a natural map $: \widetilde{M} \to M$ forgetting gB those constructions fit in the following commutative diagram

$$\begin{array}{ccccccc} T_0 & \longleftarrow & \widetilde{M} & \longleftarrow & \widetilde{R} & \longrightarrow & \widetilde{\mathrm{GL}}_n \\ \downarrow{\scriptstyle 0} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle p^{\mathrm{GL}_n}} \\ T_0/W & \longleftarrow & M & \underset{q}{\longleftarrow} & R & \underset{\mathrm{pr}}{\longrightarrow} & \mathrm{GL}_n \end{array} \tag{5.2}$$

5.4.1 Springer action

Mellit uses the Springer action of W on $p_!^{\mathrm{GL}_n}$ to construct an action of W on $_!$. Let us recall This construction. The Springer action on $p_!^{\mathrm{GL}_n}$ gives a group morphism $W^{\mathrm{op}} \hookrightarrow \mathrm{Aut}\, p_!^{\mathrm{GL}_n}$. Base change applied to both squares at the right hand side of Diagram (5.2) provides an isomorphism

$$\mathrm{pr}\ p_!^{\mathrm{GL}_n} = q\ _!.$$

Consider the composition of group morphism s

$$W^{\mathrm{op}} \hookrightarrow \mathrm{Aut}\, p_!^{\mathrm{GL}_n} \xrightarrow{\mathrm{pr}} \mathrm{Aut}\, \mathrm{pr}\ p_!^{\mathrm{GL}_n} = \mathrm{Aut}\, q\ _!.$$

The quotient map q is smooth with connected fibers so that q is fully faithful and

$$\mathrm{Aut}\, q\ _! = \mathrm{Aut}\ _!.$$

Composition provides a group morphism

$$W^{\mathrm{op}} \to \mathrm{Aut}\ _!.$$

By Proposition 3.1.6, it induces an action on $(\ )_!$ also referred to as the Springer action.

5.4.2 Monodromic action

The Springer action on the complex $p_!^{\mathrm{GL}_n}$ comes from a W-equivariant structure on the constant sheaf over a regular locus $\widetilde{\mathrm{GL}}_n^{\mathrm{reg}}$. The same holds for $_!$. Let T_0^{reg} T_0 the subset of regular elements. An element $2\ T_0$ is regular if its centralizer in GL_n is $Z_{\mathrm{GL}_n}(\ ) = T$. Consider the pull back of Diagram (5.2) to the regular locus

$$\begin{array}{ccccccc} T_0^{\mathrm{reg}} & \xleftarrow{\ \mathrm{reg}\ } & \widetilde{M}^{\mathrm{reg}} & \longleftarrow & \widetilde{R}^{\mathrm{reg}} & \longrightarrow & \widetilde{\mathrm{GL}}_n^{\mathrm{reg}} \\ \downarrow {\scriptstyle 0;\mathrm{reg}} & & \downarrow & & \downarrow & & \downarrow \\ T_0^{\mathrm{reg}}=W & \xleftarrow[\ \mathrm{reg}\ ]{} & M^{\mathrm{reg}} & \longleftarrow & R^{\mathrm{reg}} & \longrightarrow & \mathrm{GL}_n^{\mathrm{reg}} \end{array} : \tag{5.3}$$

There is a W-action on $\widetilde{M}^{\mathrm{reg}}$ induced by the maps $gB \ 7! \ g\underline{w}\ ^{1}B$, for $w\ 2\ W$ and $\underline{w}\ 2\ \mathrm{GL}_n$ a representative. $2\ D_c^b\left(\widetilde{M}^{\mathrm{reg}}\right)$ is the constant sheaf concentrated in degree 0. Define a W-action on , for $w\ 2\ W$ let $_w : w \quad !$ be the morphism which is the identity on the stalks. Then by Proposition 3.1.6, W acts on $_!^{\mathrm{reg}}$. Let $2\ T_0^{\mathrm{reg}}$, for $w\ 2\ W$ the action on $\widetilde{M}^{\mathrm{reg}}$ induces an isomorphism

$$w : \widetilde{M}_{S;} \ ! \ \widetilde{M}_{S;w:} :$$

Note that $H^i \ _!^{\mathrm{reg}} \quad = H_c^i(M_{S;} \ ; \ )$. By Remark 3.1.3, on the stalks, the W-equivariant structure comes from the functoriality of the compactly supported cohomology

$$w \ : H_c^i(\widetilde{M}_{S;w:} \ ; \ ) \ ! \ H_c^i(\widetilde{M}_{S;} \ ; \ ):$$

Pushing forward to $T_0^{\mathrm{reg}}=W$ provides a W-action on $(\ ^{0;\mathrm{reg}} \ ^{\mathrm{reg}})_!$.

5.4.3 Comparison of monodromic action and Springer action

Mellit [Mel19] proved that the monodromic action and the Springer action coincide over the regular locus.

Theorem 5.4.2. *The monodromic action on* $(\ ^{0;\mathrm{reg}} \ ^{\mathrm{reg}})_!$ *coincides with the Springer action on* $(\ ^{\mathrm{reg}} \ )_!$ *under the isomorphism*

$$^0_! \ ^{\mathrm{reg}}_! \ = \ ^{\mathrm{reg}}_! \ _! \ :$$

Proof. Tracking the Springer action over the regular locus through Diagram (5.3), one sees that it comes from the W-equivariant structure of the constant sheaf over $\widetilde{M}^{\mathrm{reg}}$, just like the monodromic action. □

An important result is that the cohomology sheaves $H^i \ _!$ are local systems over T_0 [Mel19, Proposition 8.4.1]. This proposition together with Theorem 5.4.2 provide the following corollary

Corollary 5.4.3 (Corollary 8.4.3 [Mel19]). *There exists a W-equivariant structure on the local systems $H^i \ _!$ extending the W-equivariant structure on $H^i \ _!^{\mathrm{reg}}$ described in 5.4.2 and the pushforward of this W-equivariant structure on $(\ ^0 \ )_!$ over $T_0=W$ coincide with the Springer action on $(\ \ )_!$.*

Remark 5.4.4. *This result could also be obtained like in the additive case: Theorem 4.3.3. In the additive case, the cohomology sheaves H^i $_!$ are constant thanks to the quiver variety point of view. Here, in the multiplicative case, they are locally constant thanks to the cell decomposition from Mellit. In both case the W-equivariant structure can be obtained either with [Mel19, Corollary 8.4.3] or with Theorem 4.3.3. This last theorem also works when $\mathbb{K} = \overline{\mathbb{F}}_q$.*

As in 5.3, let $\ 2\ T_0$ and M the centraliser of $\ $ in G. The following notations are used

$$M = \mathrm{GL} \quad = \prod_i \mathrm{GL}_{\ i}$$
$$W_M = \mathrm{S} \quad = \prod_i \mathrm{S}_{\ i}$$
$$P \quad = \prod_i P_{\ i}:$$

Mellit suggested us to study restriction of the W action to the subgroup $W_M \quad W$.

Theorem 5.4.5. *W_M acts on $H^i_c\left(\widetilde{M}_{S; \ ;\ }\right)$ and there is an isomorphism of W_M-representations:*

$$H^i_c\left(\widetilde{M}_{S; \ ; \ }\right) = \bigoplus_{\_ 2P} V_{\_} \quad IH_c^{i+d \ ;\_ \ d}\left(M_{S;\overline{C}_{\_;}} \ ; \ \right)$$

with

$$d_{\ ;\_} := \dim M_{S;\overline{C}_{\_;}} \quad \text{and} \quad d \ = \dim \widetilde{M}_{S; \ }:$$

Proof. Consider the following pull back of Diagram (5.2):

$$\begin{array}{ccccccc}
W: & \longleftarrow & \bigsqcup_{w2W=W_M} \widetilde{M}_{S;w:} & \longleftarrow & \bigsqcup_{w2W=W_M} \widetilde{R}_{S;w:} & \longrightarrow & \bigsqcup_{w2W=W_M} \widetilde{X}_{T;B;w:} \\
\downarrow & & \downarrow F_{w2W=W_M} \ w: & & \downarrow & & \downarrow \\
[\ ] & \longleftarrow & M_{S;\overline{C}^{\mathrm{reg}}} & \longleftarrow & R_{S;\overline{C}^{\mathrm{reg}}} & \longrightarrow & \overline{C}^{\mathrm{reg}}
\end{array}$$

Previous corollary provides, for the stalk over $[\ ]\ 2\ T_0=W$, an isomorphism

$$\bigoplus_{w2W=W_M} H^i_c\left(\widetilde{M}_{S;w: \ ; \ }\right) = H^i_{[\ ]}(\ _0\ )_! \ = H^i_{[\ ]}(\ \ )_! \ = \bigoplus_{w2W=W_M} H^i_{[\ ]}(\ \ ^{w:}\ )_! \ :$$

This isomorphism is compatible with W-action and direct sum decomposition so that

$$H^i_c\left(\widetilde{M}_{S; \ ; \ }\right) = H^i_{[\ ]}(\ \ )_!$$

after restriction of the W action to $W_M \quad W$ this isomorphism holds as a W_M-representation isomorphism. A way to describe the action on the left hand side is that $H^i_c\left(\widetilde{M}_{S; \ ; \ }\right)$ is the stalk at $\ $ of a W-equivariant local system, W_M acts on this stalk as it fixes $\ $. The theorem then follows from the description of $(\ \ )_!$ from Theorem 5.3.1. $\square$

Let a central element in GL_n lying in T_0.

Theorem 5.4.6. *There is an isomorphism of W_M representations*

$$\bigoplus_{2P_n} \mathrm{Res}^W_{W_M} V \quad IH_c^{i+d\ ;} \left(M_{S;\overline{C}\ ;} ;\ \right) = \bigoplus_{\_2P} V\_ \quad IH_c^{i+d\ ;\_} \left(M_{S;\overline{C}\_;} ;\ \right):$$

with the notations from previous theorem and

$$d_{\ ;} = \dim M_{S;\overline{C}\ ;}:$$

Proof. Previous theorem applied with the central element instead of gives an isomorphism of W-representations

$$H_c^i\left(\widetilde{M}_{S;\ ;}\ \right) = \bigoplus_{2P_n} V \quad IH_c^{i+\dim M_{S;\overline{C}\ ;}\ \ \dim \tilde{M}_{S;}} \left(M_{S;\overline{C\ ;}};\ \right):$$

$H_c^i(M_{S;\ ;}\ ) = H^i\ _!$ and $H_c^i(M_{S;\ ;}\ ) = H^i\ _!$ are stalks of the same W-equivariant local system. Both and are fixed by W_M so that W_M acts on those stalks and the representations are isomorphic. The theorem then follows from Theorem 5.4.5. □

Corollary 5.4.7.

$$\bigoplus_{2P_n} \mathrm{Hom}_{W_M}\left(V\_; \mathrm{Res}^W_{W_M} V\right) \quad IH_c^{i+d\ ;} \left(M_{S;\overline{C}\ ;} ;\ \right) = IH_c^{i+d\ ;\_} \left(M_{S;\overline{C}\_;} ;\ \right) \tag{5.4}$$

in particular

$$\bigoplus_{2P_n} \mathrm{Hom}_{W_M}\left(\ ; \mathrm{Res}^W_{W_M} V\ \right) \quad IH_c^{i+d\ ;} \left(M_{S;\overline{C\ ;}};\ \right) = H_c^{i+d\ ;\_}(M_{S;S}\ ;\ ) \tag{5.5}$$

with $=\ _1 \quad _l$ *and* $_i$ *the signature representation of* $S\ _i$.

5.5 Poincaré polynomial for intersection cohomology of character varieties with k 1 semisimple monodromies

Notations 5.5.1. *As in previous section,* $(S_1; \ldots; S_{k\ 1})$ *is a fixed* $(k\ \ 1)$*-uple of semisimple conjugacy classes, their type is determined by* $= (\ ^1; \ldots;\ ^{k\ 1})\ 2\ P_n^{k\ 1}$. *For* $f\ 2\ \mathrm{Sym}[X]$ *a symmetric function*

$$h\ \ f := h\ _1[X_1] \ldots h\ _{k\ 1}[X_{k\ 1}] f[X_k]:$$

Let $2\ T_0$ with multiplicities of the eigenvalues given by a partition $2\ P_n$. As in previous subsection S is its conjugacy class in GL_n. Before generalizing to any conjugacy class, let us recall Mellit's result for semisimple monodromies at each punctures. From Equation (3.50) one obtains

$$P_c(M_{S;S}\ ; v) = v^{d\ ;} \left\langle h\ \ h\ ; \mathbb{H}_n^{HLV}(\ \ 1; v)\right\rangle: \tag{5.6}$$

with $d_{\ ;} = \dim M_{S;S}$:

Theorem 5.5.2. *Let $\ 2\ T_0$ central in GL_n and $\ 2\ P_n$ a partition. The conjugacy classe $C_{\ ;\ }$ has semisimple part and Jordan type . The Poincaré polynomial for intersection cohomology of $M_{S;\overline{C}_{\ ;\ }}$ is*

$$P_c\left(M_{S;\overline{C}_{\ ;\ }};v\right) = v^{d_{\ ;\ }}\left\langle h\ \ s_{\ 0};\mathrm{H}_n^{HLV}(\ 1;v)\right\rangle:$$

Proof. By adjunction

$$\dim \mathrm{Hom}_{W_M}\left(\ ;\mathrm{Res}^W_{W_M} V\ \right) = \dim \mathrm{Hom}_W\left(\mathrm{Ind}^W_{W_M}\ ;V\ \right):$$

Lemma 3.2.28 implies

$$\dim \mathrm{Hom}_W\left(\mathrm{Ind}^W_{W_M}\ ;V\ \right) = hh\ ;s_{\ 0}i:$$

Substituting (5.6) in (5.5) and taking the dimension

$$\sum_{2P_n} hh\ ;s_{\ 0}i\,t^{\ d_{\ ;\ }}P_c\left(M_{S;\overline{C_{\ ;\ }}};t\right) = \left\langle h\_h\ ;\mathrm{H}_n^{HLV}(\ 1;t)\right\rangle: \tag{5.7}$$

For $\ ;\ \ 2\ P_n$ let

$$M_{\ ;\ } = hh\ ;s\ \ i:$$

Schur functions form an orthonormal basis of $\mathrm{Sym}[X]$ for the Hall pairing, thus

$$h\ \ = \sum M_{\ ;\ }s\ :$$

As $(h\ )_{\ 2P_n}$ and $(s\ )_{\ 2P_n}$ are basis of $\mathrm{Sym}_n[X]$ the matrix $(M_{\ ;\ })_{\ ;\ 2P_n}$ is invertible its inverse is denoted by $(N_{\ ;\ })_{\ ;\ 2P_n}$. Such transition matrices are described by Macdonald [Mac15, I-6]. We conclude by multiplying (5.7) by $N_{\ ;\ }$ and summing over . □

For $\_ = (\ ^1;:::;\ ^l)\ 2\ P_{\ _1}\qquad P_{\ _l}$, we introduce the notation

$$s\_[X] := s_{\ _1}[X]:::s_{\ _l}[X]$$

so that

$$h\ \ s\_ = h_{\ _1}[X_1]:::h_{\ _k}[X_{k\ \ 1}]s_{\ _1}[X_k]:::s_{\ _l}[X_k]:$$

Corollary 5.5.3. *The Poincaré polynomial for intersection cohomology of the character variety $M_{S;\overline{C}_{\_;\ }}$ with $k\ \ 1$ monodromies in the semisimple conjugacy classes S and one monodromy in $\overline{C}_{\_;\ }$ is*

$$P_c\left(M_{S;\overline{C}_{\_;\ }};t\right) = t^{d_{\ ;\_}}\left\langle h\ \ s_{\_0};\mathrm{H}_n^{HLV}(\ 1;t)\right\rangle:$$

Proof. First note that after twisting both representations with the sign one has

$$\dim \mathrm{Hom}_{W_M}\left(V_{\_};\mathrm{Res}^W_{W_M} V\ \right) = \dim \mathrm{Hom}_{W_M}\left(V_{\_0};\mathrm{Res}^W_{W_M} V_{\ 0}\right):$$

Then as $\mathrm{Ind}^W_{W_M}$ is left adjoint to $\mathrm{Res}^W_{W_M}$

$$\dim \mathrm{Hom}_{W_M}\left(V_{\_0};\mathrm{Res}^W_{W_M} V_{\ 0}\right) = \dim \mathrm{Hom}_{W_M}\left(\mathrm{Ind}^W_{W_M} V_{\_0};V_{\ 0}\right) = \left\langle s_{\_0};s_{\ 0}\right\rangle:$$

Taking dimension in equation (5.4) and substituting result of previous theorem

$$t^{d\;;}{}_{-}P_c\left(M_{S;\overline{C}_{-};}\;;t\right) = \sum \left\langle s_{\underline{}0}; s_{0}\right\rangle \langle h\;\; s_{0}; \mathrm{H}_n^{HLV}(\;\;1;t)\rangle :$$

As Schur functions form an orthonormal basis of $\mathrm{Sym}_n[X]$,

$$s_{\underline{}0} = \sum_{2P_n} \left\langle s_{\underline{}0}; s_{0}\right\rangle s_{0}$$

so that

$$\sum_{2P_n} \left\langle s_{\underline{}0}; s_{0}\right\rangle t^{d}\;\; \langle h\;\; s\;; \mathrm{H}_n^{HLV}(\;\;1;t)\rangle = \left\langle h\;\; s_{\underline{}0}; \mathrm{H}_n^{HLV}(\;\;1;t)\right\rangle$$

□

Chapter 6

Intersection cohomology of character varieties through non-Abelian Hodge theory

6.1 Introduction

In this chapter, the base field is C, we compute the Poincaré polynomial for intersection cohomology of character varieties with the closure of conjugacy classes of any type at each puncture. This proves the Poincaré polynomial specialization of a conjecture from Letellier [Let13]. Mellit computed the Poincaré polynomial for character varieties with semi-simple monodromies [Mel17a]. In previous chapter we assumed $k \quad 1$ among k monodromies are semisimple. This assumption is now relaxed. As in previous chapter, the computation relies on the one hand on Mellit's result and on the other hand on resolutions of character varieties. Those constructions come with a combinatorial relation between the cohomology of the resolutions and the intersection cohomology of character varieties. The main technical di culty is to prove that the resolution is di eomorphic to a character variety with semisimple monodromies. Then the combinatorial relation can be inverted and gives a formula for the intersection cohomology of character varieties. Contrarily to previous chapter where everything was algebraic, analytic methods such as non-Abelian Hodge theory are now necessary to construct the di eomorphism.

6.1.1 Intersection cohomology of character varieties and Weyl group actions

Consider the resolution $\widetilde{M}_{\mathbf{L};\mathbf{P};}$ of a character variety $M_{\overline{C};}$ as introduced in 3.5.3. Springer theory provides a combinatoric relation between the cohomology of $\widetilde{M}_{\mathbf{L};\mathbf{P};}$ and intersection cohomology of character varieties $M_{\overline{C};}$:

$$H_c^{i+d}\left(\widetilde{M}_{\mathbf{L};\mathbf{P};}\ ;\ \right) = \bigoplus_{2P_1 \quad P_k} A_{\ ';} \quad IH_c^{i+d}\left(M_{\overline{C};}\ ;\ \right): \tag{6.1}$$

This relation is the main tool allowing to go from usual cohomology of smooth varieties to intersection cohomology of singular varieties. We shall see that the resolution $\widetilde{M}_{\mathbf{L};\mathbf{P};}$ is di eomorphic to a character variety $M_{\boldsymbol{\mathcal{S}}}$ with semisimple conjugacy classes at punctures. With $\boldsymbol{\mathcal{S}} = (S_1; : : : ; S_k)$ and S_j is the class of an element with centralizer in GL_n equal to $L^j = \mathrm{GL}_{\ j^0}$.

Mellit [Mel17a] computed the Poincaré polynomial of those character varieties. The Poincaré polynomial is invariant under di eomorphism so we deduce the Poincaré polynomial of the resolution. Then the combinatoric relation can be inverted using transition matrices between various basis of the space of symmetric funtions. This results in the following theorem:

Theorem 6.1.1. *For a generic k-uple of conjugacy classes $C_{\ ;}$, the Poincaré polynomial for compactly supported intersection cohomology of the character variety $M_{\overline{C}\ ;}$ is*

$$P_c\left(M_{\overline{C}\ ;}\ ; v\right) = v^{d}\ \left\langle s\ \prime; \mathrm{H}_n^{HLV}(\ 1; v)\right\rangle :$$

Moreover, as a by product of the di eomorphism between resolution $\widetilde{M}_{\mathbf{L};\mathbf{P};}$ and $M_{\boldsymbol{\mathcal{S}}}$, we obtain a Weyl group action on the cohomology of $M_{\boldsymbol{\mathcal{S}}}$ from the Springer action on the cohomology of $\widetilde{M}_{\mathbf{L};\mathbf{P};}$. Similarly to the additive case, the twisted Poincaré polynomial is computed in 6.2.2

Theorem 6.1.2. *$W_{\mathbf{M}}(L)$ acts on the cohomology of $M_{\boldsymbol{\mathcal{S}}}$ and the -twisted Poincaré polynomial is*

$$P_c\ (M_{\boldsymbol{\mathcal{S}}}; v) = (\ 1)^{r(\ )} v^{\dim M_{\boldsymbol{\mathcal{S}}}} \left\langle \widetilde{h}\ ; \mathrm{H}_n^{HLV}(\ 1; v)\right\rangle :$$

The symmetric functions $\widetilde{h}$ and $r(\ )$ are defined in 3.5.18.

6.1.2 Diffeomorphism between a resolution $\widetilde{M}_{L;P;}$ and a character variety with semisimple monodromies $M_{\boldsymbol{\mathcal{S}}}$

The technical part of the proof is to exhibit a di eomorphism between the resolution $\widetilde{M}_{\mathbf{L};\mathbf{P};}$ and $M_{\boldsymbol{\mathcal{S}}}$.

Theorem 6.1.3. *$C_{\ ;}$ is a generic k-uple of conjugacy classes and $\widetilde{M}_{\mathbf{L};\mathbf{P};}$ is the resolution of $M_{\overline{C}\ ;}$. Then $\widetilde{M}_{\mathbf{L};\mathbf{P};}$ is di eomorphic to a character variety $M_{\boldsymbol{\mathcal{S}}}$. With $\boldsymbol{\mathcal{S}} = (S_1; : : : ; S_k)$ and S_j is the class of an element with centralizer in GL_n equal to $L^j = \mathrm{GL}_{\ j^0}$.*

This theorem is proved in few steps in 6.6.1.

The first step is the Riemann-Hilbert correspondence, it gives a di eomorphism between the resolution $M_{\overline{C}\ ;}$ and a de Rham moduli space of parabolic connections. Riemann-Hilbert correspondence was developed by Deligne [Del70], and Simpson for the filtered case [Sim90]. Yamakawa proved that this correspondence induces a complex analytic isomorphism between moduli spaces [Yam08].

The second step is the non-Abelian Hodge theory, a di eomorphism between de Rham moduli space and Dolbeault moduli space. It was established by Hitchin [Hit87] and Donaldson [Don87] for compact curves. Corlette [Cor88] and Simpson [Sim88] generalized it for higher dimensions. The parabolic version over non-compact curves was proved by Simpson [Sim90]. This is the one needed here. It was generalized for higher dimension by Biquard [Biq97]. The relevant moduli spaces to obtain this correspondence as a diffeomorphism were introduced by Konno [Kon93] and Nakajima [Nak96]. Biquard-Boalch [BB04] generalized further to wild non-Abelian Hodge theory and constructed the associated hyperkähler moduli spaces. We use their construction of the moduli spaces. Biquard, García-Prada and Mundet i Riera [BGM15] established a parabolic non-Abelian Hodge correspondence for real groups, generalizing Simpson construction for GL_n.

After the diffeomorphism from non-Abelian Hodge theory we use the method from Nakajima [Nak96] for GL_2 and Biquard, García-Prada, Mundet i Riera [BGM15] for real groups. The weights defining the moduli space of parabolic Higgs bundles are changed. This is done before going back to another de Rham moduli space thanks to non-Abelian Hodge theory in the other direction. The change of stability on the Dolbeault side induces a change of eigenvalues of the residue on the de Rham side.

Finally Riemann-Hilbert correspondence is applied in the other direction. It gives a diffeomorphism to a character variety where the eigenvalues have been perturbed, the monodromies are now semisimple.

In Section 6.2, we compute the Poincaré polynomial for intersection of character varieties, assuming the resolution is diffeomorphic to a character variety with semisimple conjugacy classes at punctures.

In Section 6.3 the example of the sphere with four punctures and rank $n = 2$ is studied. There, we can obtain the expected diffeomorphism using only tools from algebraic geometry. This example has been studied for a long time by Vogt [Vog89] and Fricke-Klein [FK97]. The character varieties are affine cubic surfaces satisfying Fricke-Klein relation. Cubic surfaces and line over them have been extensively studied. They are classified for instance by Cayley [Cay69], see also Bruce-Wall [BW79], Manin [Man86] and Hunt [Hun96]. This rich theory proves that the minimal resolution is diffeomorphic to a character variety with semisimple monodromies. Both appear to be diffeomorphic to the projective plane blown up in six points minus three lines.

In Section 6.4 various filtered objects are introduced. First the filtered local system; the resolution $\widetilde{M}_{\mathbf{L};\mathbf{P};}$ appears to be the associated moduli space. Then the parabolic connections and finally the parabolic Higgs bundles.

In Sections 6.5 and 6.6 we recall Biquard-Boalch [BB04] analytic constructions of Hyperkähler moduli space. This provides the non-Abelian Hodge theory as a diffeomorphism. Then the stability parameters are perturbed following ideas of Nakajima [Nak96] for GL_2 and Biquard, García-Prada, Mundet i Riera [BGM15] for a larger family of groups. It finally provides the diffeomorphism between $\widetilde{M}_{\mathbf{L};\mathbf{P};}$ and $M_{\mathcal{S}}$.

6.2 Poincaré polynomial and twisted Poincaré polynomial

6.2.1 Computation of the Poincaré polynomial

Consider a generic k-uple of conjugacy classes $C_{;} = \left(C_{1;1}; \dots; C_{k;k}\right)$. As usual, the class $C_{j;j}$ is characterized by its eigenvalues

$$\underbrace{{}^{j}_{1}, \dots, {}^{j}_{1}}_{{}^{j}_{1}}, \dots, \underbrace{{}^{j}_{l_j}, \dots, {}^{j}_{l_j}}_{{}^{j}_{l_j}}$$

and by ${}^{j;i} \in P_{{}^{j}_{i}}$ the Jordan type of the eigenvalue ${}^{j}_{i}$. Denote by ${}^{j;i'}$ the transposed partition. For each of this conjugacy classes consider the resolution of the closure (see 3.3.3)

$$\widetilde{X}_{L^j;P^j;j} \to \overline{C}_{j;j}.$$

The group L^j used to construct the resolution is

$$L^j = \underbrace{GL_{{}^{j;1'}_{1}} \quad GL_{{}^{j;1'}_{2}} \quad \dots}_{GL_{{}^{j}_{1}}} \qquad \underbrace{GL_{{}^{j;l_j'}_{1}} \quad GL_{{}^{j;l_j'}_{2}} \quad \dots}_{GL_{{}^{j}_{l_j}}}.$$

As detailed in 3.5.3, resolution of closure of conjugacy classes fit together in $\widetilde{M}_{\mathbf{L};\mathbf{P};}$ a resolution of the character variety $M_{\overline{C};}$.

Definition 6.2.1 (Semisimple conjugacy classes of type $'$). *Consider a k-uple of conjugacy classes $\mathcal{S} = (S_1; \dots; S_k)$. We say that $\mathcal{S}$ is of type $'$ if one of the following equivalent condition is satisfied for all $1 \quad j \quad k$*

- *The multiplicities of the eigenvalues of S_j are given by the partition $\bigcup_{i=1}^{l_j} {}^{j;i'}$.*
- *The centralizer of an element in S^j is isomorphic to $L^j = GL_{j'}$.*

The proof of next theorem is postponed to the remaining sections of this chapter.

Theorem 6.2.2. *The resolution $\widetilde{M}_{\mathbf{L};\mathbf{P};}$ is di eomorphic to a character variety $M_{\mathcal{S}}$ with $\mathcal{S}$ a generic k-uple of semisimple conjugacy classes of type $'$.*

With this result we are ready to compute the Poincaré polynomial for intersection cohomology of character varieties $M_{\overline{C};}$. As the Poincaré polynomial is a topological invariant

$$P_c\left(\widetilde{M}_{\mathbf{L};\mathbf{P};}; t\right) = P_c(M_{\mathcal{S}}; t):$$

Let us translate (6.1) in terms of Poincaré polynomial.

$$t^{d} P_c(M_{\mathcal{S}}; t) = \sum_{\preceq} (\dim A_{';}) t^{d} P_c\left(M_{\overline{C};}; t\right): \tag{6.2}$$

The idea is now to invert this relation. First we compute the dimension of the multiplicity spaces $\dim A_{';}$.

Lemma 6.2.3. *The dimension of the multiplicity space is given by*

$$\dim A_{\prime;} = \prod_{\substack{1 \; j \; k \\ 1 \; i \; l_j}} \left\langle h_{\; j;i^0}; s_{\; j;i^0} \right\rangle$$

Proof. By definition

$$\begin{aligned} A_{\prime;} &= \mathrm{Hom}_{W_{\boldsymbol{M}}}\left(\mathrm{Ind}_{W_{\boldsymbol{L}}}^{W_{\boldsymbol{M}}} \; \prime; V\right) \\ &= \bigotimes_{1 \; j \; k} \left(\bigotimes_{1 \; i \; l_j} \mathrm{Hom}_{S_{\; i}^{\; j}}\left(\; _{j;i^0}; V_{\; j;i}\right)\right): \end{aligned}$$

We conclude with Lemma 3.2.28. □

Theorem 6.2.4. *For a generic k-uple of conjugacy classes* $C_{\; ;}$, *the Poincaré polynomial for compactly supported intersection cohomology of the character variety* $M_{\overline{C} \; ;}$ *is*

$$P_c\left(M_{\overline{C} \; ;}; v\right) = v^d \left\langle s_{\; \prime}; \mathrm{H}_n^{HLV}(\; 1; v)\right\rangle:$$

Proof. The complete symmetric functions $(h \;)_{\; 2P_m}$ and the Schur functions $(s \;)_{\; 2P_m}$ are two basis of the space of degree m symmetric functions. Let $(M_{\; ;})_{\; ; \; 2P_m}$ the transition matrix between between those basis then

$$h = \sum_{2P_m} M_{\; ;} s :$$

As the Schur functions form an orthonormal basis, the transition matrix is given explicitly by

$$M_{\; ;} = hh \; ; s \; i :$$

It is invertible and denote by $(N_{\; ;})_{\; ; \; 2P_m}$ its inverse. Combining Equation (6.2), Lemma 6.2.3 and the formula for Poincaré polynomial of character varieties with semisimple conjugacy classes:

$$\left\langle \prod_{j=1}^{k} \prod_{i=1}^{l_j} h_{\; j;i^0}[X_j]; \mathrm{H}_n^{HLV}(\; 1; v)\right\rangle = \sum_{\preceq} \prod_{j=1}^{k} \prod_{i=1}^{l_j} \left\langle h_{\; j;i^0}; s_{\; j;i^0}\right\rangle v^{\; d} P_c\left(M_{\overline{C} \; ;}; v\right):$$

This relation can now be inverted. Fix $\; 2 \; P_{\; 1} \qquad P_{\; k}$. Multiply previous equation by $N_{\; 1;1^0; \; 1;1^0}$ and sum over $^{1;1^0} \; 2 \; P_{\; 1}^{\; 1}$. Repeating this process gives the expected result:

$$\left\langle s_{\; 0}; \mathrm{H}_n^{HLV}(\; 1; v)\right\rangle = v^{\; d} P_c\left(M_{\overline{C} \; ;}; v\right):$$

□

6.2.2 Weyl group action and twisted Poincaré polynomial

As in [Let13, Proposition 1.9], twisted Poincaré polynomial can be computed thanks to previous theorem. Using notations from 3.5.4 and Definition 3.6.6 for -twisted Poincaré polynomial we have the following theorem

Theorem 6.2.5. *C ; is a generic k-uple of conjugacy classes and $\widetilde{M}_{\mathbf{L};\mathbf{P};}$ is the resolution of $M_{\overline{C};}$. For indexing a conjugacy class in $W_{\mathbf{M}}(L)$, the -twisted mixed-Hodge polynomial of $\widetilde{M}_{\mathbf{L};\mathbf{P};}$ is*

$$P_c\left(\widetilde{M}_{\mathbf{L};\mathbf{P};}\ ;v\right) = (\ \ 1)^{r(\ )}v^{d}\ \left\langle \widetilde{h}\ ;\mathbb{H}_n^{HLV}(\ \ 1;v)\right\rangle$$

Proof. Theorem 3.5.20 and Proposition 3.5.19 give

$$v^{\ d}P_c\left(\widetilde{M}_{\mathbf{L};\mathbf{P};}\ ;v\right) = \sum_{\preceq}\left(\prod_{j=1}^{k}\prod_{i=1}^{l_j} c_{\ j;i}^{\ j;i}\right) v^{\ d}P_c\left(M_{\overline{C}\ ;}\ ;t\right):$$

Apply Theorem 6.2.4:

$$v^{\ d}P_c\left(\widetilde{M}_{\mathbf{L};\mathbf{P};}\ ;v\right) = \sum_{\preceq}\left(\prod_{j=1}^{k}\prod_{i=1}^{l_j} c_{\ j;i}^{\ j;i}\right)\left\langle s\ \ ;\mathbb{H}_n^{HLV}(\ \ 1;v)\right\rangle:$$

Then using the relation $c_{!} = (\ \ 1)^{r(!)}c_{!0}^{\ 0}$ (see Lemma 3.4.11) and Notations 3.5.18

$$v^{\ d}P_c\left(\widetilde{M}_{\mathbf{L};\mathbf{P};}\ ;v\right) = (\ \ 1)^{r(\ )}\left\langle \widetilde{h}\ ;\mathbb{H}_n^{HLV}(\ \ 1;v)\right\rangle$$

□

Theorem 6.1.3 (which will be proved in Section 6.6) gives a diffeomorphism between $\widetilde{M}_{\mathbf{L};\mathbf{P};}$ and a character variety with semisimple monodromies $M_{\mathcal{S}}$. The diffeomorphism transports the action on the cohomology of $\widetilde{M}_{\mathbf{L};\mathbf{P};}$ to an action on the cohomology of $M_{\mathcal{S}}$ and we have the following corollary.

Corollary 6.2.6. *$W_{\mathbf{M}}(L)$ acts on the cohomology of $M_{\mathcal{S}}$ and the -twisted Poincaré polynomial is*

$$P_c\ (M_{\mathcal{S}};v) = (\ \ 1)^{r(\ )}v^{d}\ \left\langle \widetilde{h}\ ;\mathbb{H}_n^{HLV}(\ \ 1;v)\right\rangle:$$

One can proceed as in the additive case (see 4.4.2) to give a cohomological interpretation to another specialization of the coefficients $c_{\ ;}$.

Theorem 6.2.7. *For ; in P_n, there exists a generic 4-uple of conjugacy classes of the following type:*

- *C_1 has one eigenvalue with Jordan type $^0 \in P_n$*
- *C_2 has one eigenvalue with Jordan type $^0 \in P_n$.*
- *C_3 is semisimple regular, it has n distinct eigenvalues*
- *C_4 is semisimple with one eigenvalue of multiplicity $n\ \ 1$ and the other of multiplicity 1*

Then the Weyl group with respect to C_3 is the symmetric group $\mathfrak{S}_n$ and it acts on the cohomology of $M_{\overline{\mathcal{C}}}$. Let w a n-cycle in this Weyl group then

$$c_{\ ;}^{1^n}(1;t) = t^{-\frac{d_{\mathcal{C}}}{2}}\sum_{r}\mathrm{tr}\left(w;IH_c^r(M_{\overline{\mathcal{C}}};\ \ )\right)t^{\frac{r}{2}}:$$

6.3 Example of the sphere with four punctures and rank 2

We study the particular case $n = 2$, $k = 4$. Then the character varieties are affine cubic surfaces. The defining equation was known by Vogt [Vog89] and Fricke-Klein [FK97]. The theory of cubic surfaces allows to obtain the expected diffeomorphism. Cubic surfaces and lines over them have been extensively studied. They are classified for instance by Cayley [Cay69], see also Bruce-Wall [BW79], Manin [Man86] and Hunt [Hun96]. This particular example of character varieties also appear in the theory of Painlevé VI differential equation. In this context resolution of cubic surfaces were studied by Inaba-Iwasaki-Saito [IIS06a; IIS06b; IIS06c] with Riemann-Hilbert correspondence. It was also studied on the Dolbeault side by Hausel [Hau98].

6.3.1 Fricke relation

We consider representations of the fundamental group of the sphere with four punctures $\mathbb{P}^1 \setminus \{p_1, \ldots, p_4\}$. First we prescribe no particular condition on the monodromies around the puncture

$$R := \left\{(X_1, \ldots, X_4) \in \mathrm{GL}_2^4 \mid X_1 \ldots X_4 = \mathrm{Id}\right\}$$

The group GL_2 acts by conjugation on R, its center acts trivially, hence the action factors through an action of PGL_2. Points of the following GIT quotient represent closed orbits for this action.

$$M := R /\!/ \mathrm{PGL}_2 := \mathrm{Spec}\, \mathbb{C}[R]^{\mathrm{PGL}_2}$$

where $\mathbb{C}[R]^{\mathrm{PGL}_2}$ are the invariants under the GL_2 action in the algebra of functions of the affine variety R. There is an explicit description of this algebra. First note that $R = \mathrm{GL}_2^3$ as the fourth coordinate is determined by $X_4 = (X_1 X_2 X_3)^{-1}$. The algebra of functions on a k-uple of matrices invariant under conjugation was studied by Procesi.

Theorem 6.3.1 (Procesi [Pro76]). *Let $\mathbb{C}\left[\mathrm{GL}_n^k\right]^{\mathrm{PGL}_n}$ be the algebra of regular function $f : \mathrm{GL}_n^k \to \mathbb{C}$ invariant under simultaneous conjugation*

$$f(X_1, \ldots, X_k) = f(gX_1 g^{-1}, \ldots, gX_k g^{-1}).$$

This algebra is generated by

$$tr(X_{i_1} \ldots X_{i_l}) \tag{6.3}$$

where $0 \leq l \leq k$ and $i_1, \ldots, i_l \in \{1, \ldots, l\}$ not necessarily distinct. The relations between those functions are spanned by

$$\sum_{\sigma \in S_l} \epsilon(\sigma)\, \mathrm{tr}_\sigma(M_1, \ldots, M_l) = 0 \tag{6.4}$$

where M_i is any monomial in the coordinates $(X_j)_{1 \leq j \leq k}$ and tr_σ is defined by

$$\mathrm{tr}_\sigma(M_1, \ldots, M_l) := \mathrm{tr}(M_{a_{1,1}} \ldots M_{a_{1,l_1}}) \ldots \mathrm{tr}(M_{a_{r,1}} \ldots M_{a_{r,l_r}}). \tag{6.5}$$

for σ a product of r cycles with disjoint supports $\sigma = (a_{1,1} \ldots a_{1,l_1}) \ldots (a_{r,1} \ldots a_{r,l_r})$.

Moreover, to obtain a generating family we can restrict to function $tr(X_{i_1} \ldots X_{i_l})$ with $l \leq 2^n - 1$.

In particular $\mathbb{C}[R]^{\mathrm{PGL}_2}$ is generated by

$$\operatorname{tr}(X_i), \quad \operatorname{tr}(X_iX_j), \quad \operatorname{tr}(X_iX_jX_k) \tag{6.6}$$

for $i, j, k \in \{1, 2, 3\}$ not necessarily distincts. Our aim is to study character varieties with prescribed closure of conjugacy classes at punctures, we can continue with the assumption:

Assumption 6.3.2. *We assume that the* $(X_i)_{1 \leq i \leq 4}$ *have determinant* 1*.*

This assumption allows to get rid of some generators. Cayley-Hamilton theorem implies

$$X_i^2 - \operatorname{tr}(X_i)X_i + \mathrm{Id} = 0 \tag{6.7}$$

so that

$$\operatorname{tr}(X_i^2) = \operatorname{tr}(X_i)^2 - 2. \tag{6.8}$$

and multiplying (6.7) by X_j before taking trace

$$\operatorname{tr}(X_i^2X_j) = \operatorname{tr}(X_i)\operatorname{tr}(X_iX_j) - \operatorname{tr}(X_j). \tag{6.9}$$

Thus we can pick among (6.6) the following generators

$$\begin{array}{lllllllll} a & := & \operatorname{tr}(X_1), & b & := & \operatorname{tr}(X_2), & c & := & \operatorname{tr}(X_3), \\ x & := & \operatorname{tr}(X_2X_3), & y & := & \operatorname{tr}(X_1X_3), & z & := & \operatorname{tr}(X_1X_2), \\ d & := & \operatorname{tr}(X_1X_2X_3), & d' & := & \operatorname{tr}(X_1X_3X_2) & & & \end{array} \tag{6.10}$$

Moreover d' can be expressed with the other generators using relation (6.4) with the monomials $M_i = X_i$. The relations between those remaining generators are described in general by Procesi but it is convenient to obtain a finite description of the relations. Such a description was known by Vogt [Vog89] and Fricke-Klein [FK97], see also Goldman [Gol09] for a detailed discussion and Boalch-Paluba [BP16] for applications to G_2 character varieties. The relations boil down to a single equation known as the Fricke relation

$$xyz + x^2 + y^2 + z^2 + Ax + By + Cz + D = 0 \tag{6.11}$$

with

$$\begin{array}{lll} A & = & ad - bc \\ B & = & bd - ac \\ C & = & cd - ab \\ D & = & abcd + a^2 + b^2 + c^2 + d^2 - 4. \end{array}$$

The character varieties we are interested in are obtained by specifying the Zariski closure of the conjugacy class of each X_i. First we assume that they are all semi-simple regular. For $i = 1, \ldots, 4$, S_i is the conjugacy class of

$$\begin{pmatrix} \alpha_i & 0 \\ 0 & \alpha_i^{-1} \end{pmatrix}. \tag{6.12}$$

$\mathcal{S} = (S_1, \ldots, S_4)$ is assumed to be generic. In terms of invariant functions, $X_i \in S_i$ for all i, if and only if

$$\begin{array}{rcl} \operatorname{tr}(X_i) & = & \alpha_i + \alpha_i^{-1} \text{ for } 1 \leq i \leq 3 \\ \operatorname{tr}(X_1X_2X_3) & = & \alpha_4 + \alpha_4^{-1}. \end{array}$$

Then Fricke relation translates in next proproisition.

Proposition 6.3.3. *The character variety $M_{\mathcal{S}}$ is a smooth cubic surface in $\mathbb{A}^3$ given by Fricke relation (6.11) with coordinates x, y and z and constants A, B, C and D.*

Now consider non-semisimple conjugacy classes $\mathcal{C} = (C_1, C_2, C_3, C_4)$. With C_1 the conjugacy class of

$$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

and $C_2 = C_3 = C_4$ are the conjugacy classes of

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Note that this 4-uple of conjugacy classes is generic. The $(X_i)_{1 \leq i \leq 4}$ are already assumed to have determinant 1, then X_1 belongs to the closure $\overline{C}_1$ if and only if

$$\operatorname{tr} X_1 = -2.$$

Similarly the condition $(X_2, X_3, X_4) \in \overline{C}_2 \times \overline{C}_3 \times \overline{C}_4$ is equivalent to

$$\operatorname{tr} X_2 = \operatorname{tr} X_3 = \operatorname{tr}(X_1 X_2 X_3) = 2.$$

Substituting this parameters in Fricke relation, the character variety is again a cubic surface in $\mathbb{A}^3$ with equation:

$$xyz + x^2 + y^2 + z^2 - 4 = 0. \tag{6.13}$$

This cubic surface has exactly four singularities at $(-2, -2, -2)$, $(-2, 2, 2)$, $(2, -2, 2)$ and $(2, 2, -2)$. The classification of cubic surfaces (see for instance Bruce-Wall [BW79]) gives the following theorem:

Theorem 6.3.4. *After compactification in $\mathbb{P}^3$, the character variety $M_{\overline{\mathcal{C}}}$ is Cayley's nodal cubic, the only cubic surface with four singularities.*

This particular character variety was studied by Cantat-Loray [CL09] in the context of Painlevé VI.

In this example, using only elementary algebraic geometry, we can prove that the minimal resolution of $M_{\overline{\mathcal{C}}}$ is diffeomorphic to the character varieties with semisimple monodromies $M_{\mathcal{S}}$. We shall see that both varieties are obtained as the plane blown-up in six points minus three lines.

6.3.2 Projective cubic surfaces

Let us recall an important result in the classification of cubic surfaces. Smooth projective cubic surfaces in $\mathbb{P}^3$ can be constructed by a blow-up of $\mathbb{P}^2$ in six points.

Let $\mathbf{P} = (P_1, \ldots, P_6)$ be six distinct points in the projective plane $\mathbb{P}^2$. The blow-up of $\mathbb{P}^2$ with respect to those six points is denoted $Y_{\mathbf{P}} \to \mathbb{P}^2$.

Definition 6.3.5 (Generic configuration for six points in $\mathbb{P}^2$). *Such a configuration $\mathbf{P}$ of 6 points in $\mathbb{P}^2$ is called generic if no three of them lie on a line and no five of them lie on a conic.*

The two following theorems are well-known results about cubic surfaces, see for instance Manin [Man86] and Hunt [Hun96].

Theorem 6.3.6. *Up to isomorphism, smooth projective cubic surfaces in* $\mathbb{P}^3$ *are obtained as* $\mathbb{P}^2$ *blown-up in six points in generic position.*

Theorem 6.3.7. *If the six points* $\boldsymbol{P} = (P_1, \ldots, P_6)$ *are the intersection of four lines* $(L_1, \ldots, L_4)$ *in* $\mathbb{P}^2$*, then* $Y_{\boldsymbol{P}}$ *is isomorphic to a minimal resolution of singularities of Cayley's nodal cubic.*

The rest of this section is devoted to the proof of Theorem 6.3.7. Along the way, one direction of Theorem 6.3.6 is also proved: $\mathbb{P}^2$ blown-up in six points in generic position is isomorphic to a smooth cubic surface in $\mathbb{P}^3$.

Those results rely on the theory of linear systems we briefly recall. A detailed presentation can be found in Hartshorne [Har13, II-7].

Definition 6.3.8. *A divisor D on a smooth variety Y is a formal sum $D = \sum_V n_V V$ over subvarieties of codimension one with $n_V \in \mathbb{Z}$ and finitely many of them nonzero. D is effective if $n_V \geq 0$ for all V. A divisor D is principal if $D = (f)$ for f a non-zero global section of the sheaf of rational functions. Two divisors D and D' are linearly equivalent if $D - D'$ is principal.*

Definition 6.3.9. *Let D be a divisor on a projective space $\mathbb{P}^n$, the complete linear system denoted $|D|$ is the set of effective divisors linearly equivalent to D.*

Remark 6.3.10 (Hartshorne [Har13] II - 7.7, 7.8). *The complete linear system $|D|$ is identified with the projective space over the space of global sections of the invertible sheaf $\mathcal{L}(D)$ associated with D. Indeed the zero set $(s)_0$ of a section s is an effective divisor linearly equivalent to D*

$$\begin{array}{ccc} \mathbb{P}(H^0(Y, \mathcal{L}(D))) & \longrightarrow & |D| \\ [s] & \longmapsto & (s)_0 \end{array}.$$

Moreover if $\mathcal{L}(D)$ is generated by its global section, it provides a morphism

$$\begin{array}{cccc} \varphi : & Y & \longrightarrow & \mathbb{P}\left(H^0(Y, \mathcal{L}(D))^*\right) \\ & x & \longmapsto & [\varphi_x] \end{array} \tag{6.14}$$

Set theoretically, this morphism sends a point $x \in Y$ to $[\varphi_x]$ the line spanned by the linear form

$$\begin{array}{cccc} \varphi_x : & H^0(Y, \mathcal{L}(D)) & \longrightarrow & \mathbb{C} \\ & s & \longmapsto & s(x) \end{array}$$

Let $\boldsymbol{P} = (P_1, \ldots, P_6)$ be six points on $\mathbb{P}^2$, either in generic position or exactly the intersection points of four lines. Linear systems allow to construct a morphism from $Y_{\boldsymbol{P}}$ to $\mathbb{P}^3$.

Definition 6.3.11. *Let L a line in $\mathbb{P}^2$, the linear system $|3L - P_1 - \cdots - P_6|$ is a projective subspace of $|3L|$. It is defined under the identification $|3L| = \mathbb{P}\left(H^0(\mathbb{P}^2, \mathcal{L}(3L))\right)$ by $\mathbb{P}(V_{\boldsymbol{P}})$ with*

$$V_{\boldsymbol{P}} = \left\{ s \in H^0\left(\mathbb{P}^2, \mathcal{L}(3L)\right) | s(P_i) = 0, \ \text{for all } 1 \leq i \leq 6 \right\}.$$

It is the set of cubic curves in $\mathbb{P}^2$ containing all the $(P_i)_{1 \leq i \leq 6}$.

Now consider $Y_{\mathbf{P}}$ the blow up of $\mathbb{P}^2$ at $P_1 + \cdots + P_6$. Let E_i the exceptional divisor over P_i. There is a natural bijection from $|3L - P_1 - \cdots - P_6|$ to $|\pi^*(3L) - E_1 - \cdots - E_6|$. This bijection sends a cubic in $\mathbb{P}^2$ passing through all the P_i to its strict transform in $Y_{\mathbf{P}}$.

Lemma 6.3.12. *The line bundle $L(\pi^*(3L) - E_1 - \cdots - E_6)$ is generated by its global section and* $\dim H^0(Y_{\mathbf{P}}; L(\pi^*(3L) - E_1 - \cdots - E_6)) = 4$.

Proof. Under the identification between $|\pi^*(3L) - E_1 - \cdots - E_6|$ and $|3L - P_1 - \cdots - P_6|$, the space $H^0(Y_{\mathbf{P}}; L(\pi^*(3L) - E_1 - \cdots - E_6))$ corresponds to

$$\left\{s \in H^0(\mathbb{P}^2; L(3L)) \,|\, s(P_i) = 0 \ \text{ for } 1 \leq i \leq 6\right\}$$

which is a codimension 6 subspace of $H^0(\mathbb{P}^2; L(3L))$. The line bundle $L(3L)$ is nothing but $O(3)$. The statement about the dimension now follows from

$$\dim H^0(\mathbb{P}^2; O(3)) = 10.$$

To see that $L(\pi^*(3L) - E_1 - \cdots - E_6)$ is generated by its global section, we use that for any point P distinct from $P_1, \dots, P_6$ there exists a cubic containing the P_i but not containing P. This is detailed Hartshorne in [Har13, V - 4.3]. □

Thanks to previous lemma, the line bundle $L(\pi^*(3L) - E_1 - \cdots - E_6)$ provides a morphism $\varphi : Y_{\mathbf{P}} \to \mathbb{P}^3$ define as in (6.14).

Proposition 6.3.13. *The image of the morphism φ is a cubic surface in* $\mathbb{P}^3$.

Proof. We want to compute the number of intersection of the image of the morphism φ with a generic line L in $\mathbb{P}^3$. By construction the projective space of dimension three is naturally obtained as $\mathbb{P}(H^0(Y_{\mathbf{P}}; L(\pi^*(3L) - E_1 - \cdots - E_6))^*)$ the projective space of the space of sections of $L(\pi^*(3L) - E_1 - \cdots - E_6)$. Take two points P, Q distinct from the P_i. Then $[\varphi_P]$ and $[\varphi_Q]$ are two points in the image of φ. Now every cubic curve containing the eight points $P_1, \dots, P_6, P, Q$ also contains a ninth point R, see [Har13, V-4.5]. Thus the line in $\mathbb{P}(H^0(Y_{\mathbf{P}}; L(\pi^*(3L) - E_1 - \cdots - E_6))^*)$ containing $[\varphi_P]$ and $[\varphi_Q]$ also intersects the image of φ in a third point $[\varphi_R]$. Therefore the degree of the image of φ is three, it is a cubic surface in $\mathbb{P}^3$. □

Last proposition is true either if the points P are in generic position or if they are the intersection points of four lines. Next propositions present the difference between both situations.

Proposition 6.3.14. *If the points P are in generic position, then the map*

$$\varphi : Y_{\mathbf{P}} \to \mathbb{P}^3$$

is an embedding.

Proof. Let P and Q distinct points in $\mathbb{P}^2$. Among $(P_1, \dots, P_6, P)$, no four points are aligned. Then there exists a cubic in $\mathbb{P}^2$ containing $P_1, \dots, P_6, P$ but not containing Q, see [Har13, V-4.4]. Therefore $[\varphi_P] \neq [\varphi_Q]$. □

Remark 6.3.15. *Last two propositions prove one direction in the theorem of classification of smooth cubic surfaces 6.3.6. They prove that $\mathbb{P}^2$ blown-up in six points in generic position is a cubic surface in* $\mathbb{P}^3$.

Proposition 6.3.16. *If the points $(P_1, \dots, P_6)$ are exactly the intersection points of four lines $(L_1, \dots, L_4)$ in $\mathbb{P}^2$, then the map*

$$\varphi : Y_{\boldsymbol{P}} \to \mathbb{P}^3$$

is a blow-down along $(\widetilde{L}_1, \dots, \widetilde{L}_4)$ the strict transform of $(L_1, \dots, L_4)$. Therefore its image is a cubic surface with four singularities: the Cayley's nodal cubic.

Proof. Note that as each L_i contains three points blown-up, its strict transform $\widetilde{L}_i$ has self-intersection -2. Therefore $\widetilde{L}_i$ can be blown-down and its image is a singular point. Let us check that the morphism φ is indeed this blow-down. Let P a point in $\widetilde{L}_i$. If the strict transform of a conic passing through the $(P_j)_{1 \leq j \leq 6}$ also contains P, then this conic contains the line L_i. Indeed this conic either contains four points of the line L_i or it contains three points of L_i and is tangent to this line at one of this points. Therefore for all $P' \in \widetilde{L}_i$ one has $[\varphi_{P'}] = [\varphi_P]$. Therefore φ contracts the lines $(\widetilde{L}_i)_{1 \leq i \leq 4}$. As in the proof of Proposition 6.3.14, φ is an embedding away from the lines $(\widetilde{L}_i)_{1 \leq i \leq 4}$. □

Remark 6.3.17. *Last proposition proves Theorem 6.3.7: the projective plane blown-up at the six intersection points of four lines is a minimal resolution of singularities of Cayley's nodal cubic.*

Up to diffeomorphism, the manifold obtained by $\mathbb{P}^2$ blown-up in six distinct points, does not depend on the position of the points. This implies next proposition.

Proposition 6.3.18. *The minimal resolution of the projective Cayley's nodal cubic is diffeomorphic to a smooth projective cubic surface. Both are obtained as the projective plane $\mathbb{P}^2$ blown-up in six points.*

6.3.3 Lines on cubic surfaces

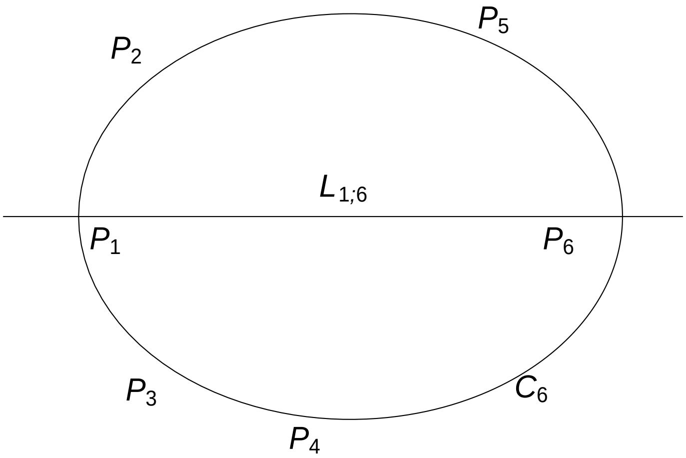
So far we saw that the minimal resolution of the projective Cayley's nodal cubic is diffeomorphic to a smooth projective cubic surface. However the variety we are interested in are not projective, they are affine. By Theorem 6.3.4 the variety $M_{\overline{\mathcal{C}}}$ is the projective Cayley's nodal cubic minus three lines at infinity. Those three lines are given by the equation $xyz = 0$, they form a triangle. Similarly the variety $M_{\mathcal{S}}$ is a smooth projective cubic surface minus the triangle at infinity $xyz = 0$. This triangle at infinity is a particular case of a general situation studied by Simpson [Sim16] for $n = 2$ and any number of punctures k.

The theory of lines on cubic surfaces has been thoroughly studied. See for instance Cayley [Cay69], Bruce-Wall [BW79], Manin [Man86] and Hunt [Hun96].

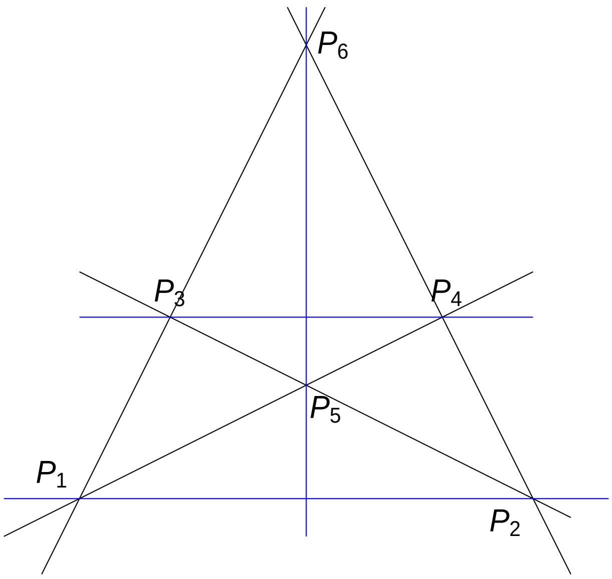
Proposition 6.3.19 (27 lines on smooth projective cubic surface)**.** *There are 27 lines on a smooth projective cubic surface. They all have a nice description in terms of $\mathbb{P}^2$ blown-up in six points $(P_1, \dots, P_6)$.*

- *Six of them are exceptional divisors E_i over P_i.*
- *Fifteen of them are the strict transform $\widetilde{L}_{i,j}$ of the line through P_i and P_j.*
- *Six of them are the strict transform $\widetilde{C}_j$ of the conic through all P_i except P_j.*

Following picture is an example of six generic points in the plan, the line $L_{1,6}$ as well as the conic C_6 are drawn.



Now consider six points not in generic position. Take four lines $(L_1, \dots, L_4)$ in $\mathbb{P}_2$ with exactly six intersection $(P_1, \dots, P_6)$, those lines are black in next figure. Consider the three lines $L_{1,2}, L_{3,4}$ and $L_{5,6}$ with $L_{i,j}$ containing P_i and P_j, those lines are blue in next figure. Up to relabelling we may assume $L_{i,j} \notin L_k$ for all i, j, k. Cayley's nodal cubic is obtained by blowing up the six points and then blowing down the strict transform of the four lines $(L_1, \dots, L_4)$. The four points image of this four lines under the blow-down are exactly the four singular points. See Hunt [Hun96, Chapter 4] for more pictures.



Proposition 6.3.20 (lines on Cayley's nodal cubic). *There are* 9 *lines on Cayley's nodal cubic.*

- *Six of them are the exceptional divisors E_i over P_i.*
- *Three of them are the strict transform of $L_{1,2}, L_{3,4}$ and $L_{5,6}$.*

Proposition 6.3.21. *The variety $M_{\overline{\mathcal{C}}}$ is Cayley's nodal cubic minus the images of $L_{1,2}, L_{3,4}$ and $L_{5,6}$.*

Proof. We saw that $M_{\overline{\mathcal{C}}}$ is Cayley's nodal cubic minus the three lines at infinity $xyz = 0$. Those three lines does not contains any of the four singularities. Therefore they are not the image of the exceptional divisors. Then they must be the three remaining lines, the blue lines on the picture. □

Theorem 6.3.22. *The character variety with generic semisimple conjugacy classes at punctures $M_{\mathcal{S}}$ is di eomorphic to the minimal resolution of singularities of the character variety $M_{\overline{\mathcal{C}}}$. Both are obtained as the projective plane $\mathbb{P}^2$ blown up in six points $(P_1, \dots, P_6)$ minus three lines $\widetilde{L}_{1,2}, \widetilde{L}_{3,4}, \widetilde{L}_{5,6}$.*

Proof. The statement about the minimal resolution of $M_{\overline{\mathcal{C}}}$ follows from previous proposition. $M_{\mathcal{S}}$ is a smooth projective cubic surface minus three lines forming a triangle. As those three lines intersect each other they cannot be any triple among the 27 lines over the surface, there are some restriction:

- Exceptional divisor E_i do not intersect each other.
- Strict transform $\widetilde{C}_j$ do not intersect each other.
- Strict transforms of two distinct line containing a same point P_i do not intersect.

Therefore the only possible triples of lines forming a triangle on a smooth cubic surface have the following form:

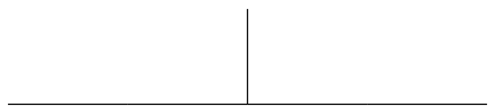
1. $(\widetilde{L}_{1,2}, \widetilde{L}_{3,4}, \widetilde{L}_{5,6})$
2. $(E_1, \widetilde{L}_{1,6}, \widetilde{C}_6)$.

The first case is exactly the expected result. To get an idea of the second case, consider the picture below Proposition 6.3.19, the conic C_6 and the line $L_{1,6}$ are drawn. To relate the second case to the first, proceed in two steps. First $\mathbb{P}^2$ is blown-up in the three points P_1, P_2 and P_3. The resulting variety is blown-down along $\widetilde{L}_{1,2}, \widetilde{L}_{1,3}$ and $\widetilde{L}_{2,3}$ (three lines with self-intersection -1). The variety obtained is again isomorphic to $\mathbb{P}_2$. We consider this copy of the projective plane as the starting point. This plane is blown up in six points $(P'_1, \dots, P'_6)$ with

- P'_1 the blow-down of $\widetilde{L}_{2,3}$
- P'_2 the blow-down of $\widetilde{L}_{1,3}$
- P'_3 the blow-down of $\widetilde{L}_{1,2}$
- P'_j the image of P_j for $j = 4, 5, 6$.

The construction obtained from the new copy of $\mathbb{P}^2$ and the points $(P'_1, \dots, P'_6)$ are labelled with a prime. Then the triple $\left(E_1, \widetilde{L}_{1,6}, \widetilde{C}_6\right)$ becomes $\left(\widetilde{L}'_{2,3}, \widetilde{L}'_{1,6}, \widetilde{L}'_{4,5}\right)$. In any cases the triangle of lined removed at infinity has the expected form. □

Remark 6.3.23. *There is an action of the Weyl group of* E_6 *on the configuration of the 27 lines on a smooth cubic surface. The Dynkin diagramm of* E_6 *is*



The generator of the upper vertex corresponds to the transformation previously described sending $\left(E_1, \widetilde{L}_{1,6}, \widetilde{C}_6\right)$ *to* $\left(\widetilde{L}'_{2,3}, \widetilde{L}'_{1,6}, \widetilde{L}'_{4,5}\right)$. *See Hartshorne [Har13, V-Exercise 4.11].*

6.4 Moduli spaces

6.4.1 de Rham moduli space

Parabolic holomorphic bundles were introduced by Mehta-Seshadri [MS80], they generalized Narasimhan-Seshadri [NS65] result to the parabolic case. Parabolic bundles appear in various area in mathematics and physics, for instance Pauly [Pau96] related those parabolic bundles with conformal field theory. In this section basic definitions are recalled.

Let a Riemann surface endowed with a complex structure. Let D the divisor $D = p_1 + \quad + p_k$.

Definition 6.4.1 (Filtered holomorphic bundles). *A filtered holomorphic bundle is the data of a holomorphic vector bundle* E *together with filtrations of* E^j *the fiber of* E *at* p_j *for* $j = 1, \dots, k$

$$\{0\} = E^j_0 \quad E^j_1 \quad \quad E^j_{m_j} = E^j.$$

The type of the filtration is defined by

$$^j_i = \dim E^j_i / E^j_{i-1}$$

for $j = 1, \dots, k$ *and* $i = 1, \dots, m_j$.

Definition 6.4.2 (parabolic degree). *Let* E *a filtered holomorphic bundle of type . Let* $= (^j_i)_{1 \ j \ k,\ 1 \ i \ m_j}$ *with* $^j_i \in \mathbb{R}$ *a stability parameter. The parabolic degree of* E *is*

$$\text{p-deg} \ E = \deg E + \sum_{i,j} {}^j_i \dim \left(E^j_i / E^j_{i-1}\right).$$

Let E a holomorphic vector bundle on . A logarithmic connection on E is a map of sheaves $D : E \to E \quad {}^1(\log D)$ satisfying the Leibniz rule

$$D(fs) = df \quad s + f D(s)$$

for all f holomorphic function and s section of E.

For z a coordinate vanishing at a point p_j, in a trivialization of E in a neighborhood of this point the connection reads

$$D = d + A(z)\frac{dz}{z}.$$

$A(0)$ is called the residue of D at p_j and denoted by $\mathrm{Res}_{p_j} D$

Fix some parabolic weights $\alpha_i^j \in [0,1[$ satisfying $\alpha_i^j > \alpha_{i-1}^j$. For $j = 1, \dots, k$ and $i = 2, \dots, m_j$ fix $A_i^j \in \mathbb{C}$ to specify a polar part. A logarithmic connection (E, D) is compatible with the parabolic structure if the endomorphism

$$\mathrm{Res}_{p_j} D : E^j \to E^j$$

satisfies $\left(\mathrm{Res}_{p_j} D\right) E_i^j \subseteq E_i^j$. A logarithmic connection compatible with the parabolic structure is called a parabolic connection.

It is compatible with the specified polar part if in addition the map induced by $\mathrm{Res}_{p_j} D$ on the graded spaces E_i^j / E_{i-1}^j is $A_i^j \,\mathrm{Id}$. A logarithmic connection compatible with the parabolic structure is α-semistable if and only if, for sub bundle $F \subset E$ preserved by D

$$\frac{\text{p-deg}_\alpha F}{\mathrm{rank}\, F} \leq \frac{\text{p-deg}_\alpha E}{\mathrm{rank}\, E}$$

it is stable if the inequality is strict unless $F = 0$. Two pairs of filtered holomorphic bundle and parabolic connections (E, D) and (E', D') are isomorphic if there is an isomorphism of holomorphic bundle $f : E \to E'$ compatible with the filtrations and such that $(f \otimes Id) \circ D = D' \circ f$.

Notations 6.4.3 (de Rham moduli space). *The de Rham moduli space $\mathcal{M}^{dR}_{A,\alpha}$ classifies isomorphism classes of α-stable parabolic connections with prescribed polar part A and parabolic degree 0.*

6.4.2 Filtered local systems and resolutions of character varieties

Definition 6.4.4 (Filtered local system). *A filtered local system is a local system L over $\Sigma \setminus \{p_1, \dots, p_k\}$ together with a filtration of the restrictions $L_{|U_j}$ to U_j some punctured neighborhood of p_j. Namely for all $j = 1, \dots, k$ there are local systems L_i^j such at*

$$0 = L_0^j \subset L_1^j \subset \cdots \subset L_{m_j}^j = L_{|U_j}.$$

The type μ of the filtered local system is defined by

$$\mu_i^j := \mathrm{rank}\, L_i^j / L_{i-1}^j.$$

Definition 6.4.5 (Parabolic degree of a filtered local system). *Let $\beta = \left(\beta_i^j\right)_{\substack{1 \leq j \leq k \\ 1 \leq i \leq m_j}}$ a stability parameter. The parabolic degree of the filtered local system is defined by*

$$\text{p-deg}_\beta L = \sum_{i,j} \beta_i^j \,\mathrm{rank}\, L_i^j / L_{i-1}^j$$

A filtered local system L is β-semistable if and only if for all sub local system $0 \subset L' \subset L$

$$\frac{\text{p-deg}_\beta L'}{\mathrm{rank}\, L'} \leq \frac{\text{p-deg}_\beta L}{\mathrm{rank}\, L}$$

it is β-stable if the inequality is strict.

Consider a character variety $\mathcal{M}_{\overline{\mathcal{C}}, \mu}$ with a resolution of singularities $\widetilde{\mathcal{M}}_{\boldsymbol{L}, \boldsymbol{P}, \beta}$. By the usual equivalence of category between local systems and representations of the fundamental group, the character variety $M_{\overline{C};}$ is the moduli space of local system with monodromy around p_j in $\overline{C}_{j;j}$. This correspondence extends to the resolution $\widetilde{M}_{\mathbf{L};\mathbf{P};}$ and the moduli space of filtered local system.

Proposition 6.4.6. *$\widetilde{M}_{\mathbf{L};\mathbf{P};}$ is the moduli space of filtered local system with filtration around p_j of type $\underline{\ }^{j\,0}$ and such that the endomorphism induced by the monodromy on $L_i^j = L_{i\ 1}^j$ is $\ _i^j\,\mathrm{Id}$.*

Proof. An element $g_j P^j\ 2\ \mathrm{GL}_n = P^j$ identifies with a partial flag of type $\underline{\ }^{j\,0}$ (see Remark 3.4.17). The condition $g_j^{\ 1} X_j g_j\ 2\ \ ^j U_{P^j}$ is exactly that the partial flag is preserved by X_j and that the induced endomorphism on the graded spaces are $\ _i^j\,\mathrm{Id}$. Note that we study only character varieties for generic choices of conjugacy classes at punctures. For such a generic choice, the stability parameter is irrelevant as the local system does not admit any sub local system. □

6.4.3 Dolbeault moduli space

A parabolic Higgs bundle is a pair $(E;\ )$ with E a filtered holomorphic vector bundle on X and a Higgs field $\ : E\ !\ E\quad ^1(\log D)$ such that $\mathrm{Res}_{p_j}\ (E_i^j)\quad E_i^j$. Let $\ = (\ _i^j)_{1\ j\ k,\ 1\ i\ n_j}$ a stability parameter. A parabolic Higgs bundle $(E;\ )$ is -semistable if and only if for all $0\ (\ F\ (\ E$ a sub bundle preserved by

$$\frac{\text{p-deg}\ \ F}{\text{rank}\, F}\qquad\frac{\text{p-deg}\ \ E}{\text{rank}\, E}:$$

it is -stable if the inequality is strict. As for the parabolic connections, it is interesting to specify the residue of the Higgs field. For all $i;j$ fix a semisimple adjoint orbit B_i^j in $\mathrm{gl}_{\ _i^j}$. The parabolic Higgs bundle has the prescribed residue if, in an holomorphic trivialization, the map induced on $E_i^j = E_{i\ 1}^j$ by the residue lies in the adjoint orbit B_i^j. Note that contrarily to the parabolic connections, the prescribed adjoint orbits on the graded spaces are not necessarily central. In fact much more general polar parts are considered by Biquard-Boalch, we restrict here to what is necessary for our purpose.

Notations 6.4.7 (Dolbeault moduli space). *The Dolbeault moduli space $M_{B;}^{Dol}$ classifies isomorphism classes of -stable parabolic Higgs bundles with prescribed residue B and parabolic degree* 0*.*

6.4.4 Various steps of the diffeomorphism

In the remaining of this chapter, analytic construction of the moduli spaces are recalled. Those spaces are endowed with a manifold structure. Those moduli spaces will be used to obtain a diffeomorphism from a resolution $\widetilde{M}_{\mathbf{L};\mathbf{P};}$ to a character variety $M_{\mathcal{S}}$ with semisimple conjugacy classes at punctures. The picture is is the following:

$$\begin{array}{ccccc} \widetilde{M}_{\mathbf{L};\mathbf{P};} & \xrightarrow{\text{R.H}} & M_{A;}^{dR} & \xrightarrow{\text{N.A.H}} & M_{B;}^{Dol} \\ & & & & \big\downarrow 7\ \ e \\ M_{\mathcal{S}} & \xleftarrow[\text{R.H}]{} & M_{A;e}^{dR} & \xleftarrow[\text{N.A.H}]{} & M_{B;e}^{Dol} \end{array} \qquad (6.15)$$

All the arrows are di□eomorphisms, R.H stands for Riemann-Hilbert correspondence and N.A.H for non-Abelian Hodge theory. The vertical arrow accounts for a change of stability parameter $\; ! \; \tilde{}$. This is the same idea as Biquard, García-Prada and Mundet i Riera [BGM15, Theorem 7.10]. It is detailed in the remaining of the chapter for this particular application.

6.5 Local model

In this section the local model used by Biquard-Boalch [BB04] to construct moduli spaces is recalled.

6.5.1 Local model for Riemann-Hilbert correspondence

Before constructing the moduli space, let us present what happens locally, near a puncture, and how the parameters of the moduli spaces are related. Consider L a rank n filtered local system on a punctured disk D^0 such that the monodromy induces a central endomorphism on the graded spaces. The monodromy X has eigenvalues $\;_i$ with multiplicity $\;_i$ for $1 \quad i \quad l$. We assume the filtration of the local system is finer than a filtration spanned by generalized eigenspaces of M. Then in a trivialization $(l_j)_{1 \; j \; n}$ compatible with the filtration, the monodromy reads

$$X = \begin{pmatrix} X_{\;1} & & \\ 0 & X_{\;2} & \\ \vdots & 0 & \ddots \end{pmatrix}$$

with $X_{\;i}$ a block of size $\;_i$ with further decomposition

$$X_{\;i} = \begin{pmatrix} {}_i \,\mathrm{Id}_{\;^{i\,0}_{1}} & & \\ 0 & {}_i \,\mathrm{Id}_{\;^{i\,0}_{2}} & \\ \vdots & 0 & \ddots \end{pmatrix}$$

The type of the filtration is $\underline{\;}^{0} = (\;^{1\,0}_{1};\;^{1\,0}_{2};:::;\;^{2\,0}_{1};\;^{2\,0}_{2};:::)$. Let $A_i \; 2 \; \mathbb{C}$ such that

$$\exp(\;2i\;A_i) = \;_i$$

and $0 \quad \mathrm{Re}\,A_i < 1$. Then A is the diagonal matrix with diagonal coe□cients

$$\Big(\underbrace{A_1;:::;A_1}_{\;1};:::;\underbrace{A_l;:::;A_l}_{\;l}\Big):$$

Let a a block strictly upper triangular matrix such that $\exp(\;2i\;(A+a)) = X$. Define E a rank n holomorphic bundle on the disk D spanned by $\;_j = e^{(A+a)\log z} l_j$ for $1 \quad j \quad n$. Let D the parabolic connection on E defined in the holomorphic trivialization $(\;_j)_{1 \; j \; n}$ by

$$\begin{aligned} D &= d + \frac{A+a}{z}dz \\ &= D_0 + \frac{a}{z}dz \end{aligned}$$

Then the parabolic local system L is nothing but the local system of flat sections of the parabolic connection (E, D). This describes locally the Riemann-Hilbert correspondence between a resolution of a character variety and a de Rham moduli space.

6.5.2 Metric and parabolic structure

The connection D_0 will be the local model for parabolic connections:

$$D_0 = d + \frac{A}{z}dz$$

with A diagonal. In order to continue the path presented in Diagram (6.15), we need to introduce an Hermitian metric. It will be related to a choice of stability parameter. Chose some stability parameters $\alpha_{r,s} \in [0,1[$ for each graded spaces of the filtration of type ξ. Introduce a diagonal matrix α with diagonal coefficients

$$(\alpha_1, \alpha_2, \dots, \alpha_n) := \Big(\overbrace{\underbrace{\alpha_{1,1}, \dots, \alpha_{1,1}}_{\xi^{1\prime}_1}, \underbrace{\alpha_{1,2}, \dots, \alpha_{1,2}}_{\xi^{1\prime}_2}, \dots}^{1}, \dots, \overbrace{\underbrace{\alpha_{l,1}, \dots, \alpha_{l,1}}_{\xi^{l\prime}_1}, \underbrace{\alpha_{l,2}, \dots, \alpha_{l,2}}_{\xi^{l\prime}_2}, \dots}^{l} \Big)$$

so that the α_i are the $\alpha_{r,s}$ repeated according to the multiplicities $\xi^{r\prime}_s$. Moreover assume that $\alpha_i \geq \alpha_{i+1}$ and $\alpha_{r,s} \neq \alpha_{u,v}$ if $(r,s) \neq (u,v)$.

Remark 6.5.1. *In this local model, there is a unique puncture p_1 so that the stability parameter intorduced in 6.4.1 are $(\alpha^1_i)_{1 \leq i \leq m_1}$. They are related to the stability parameters introduced in this section by*

$$(\alpha^1_1, \alpha^1_2, \dots, \alpha^1_{m_1}) = (\alpha_{1,1}, \alpha_{1,2}, \dots, \alpha_{2,1}, \alpha_{2,2}, \dots)$$

We apologize for the multiplication of similar notations. $(\alpha^1_i)_{1 \leq i \leq m_1}$ are adapted to the algebraic definition of stability whereas $(\alpha_{r,s})_{\substack{1 \leq r \leq l \\ 1 \leq s \leq \xi^r_1}}$ are adapted to the description of the connections and $(\alpha_1, \alpha_2, \dots, \alpha_n)$ to explicit construction of trivializations.

Define a Hermitian metric h on E such that $|e_j| = |z|^{\alpha_j}$. This metric determines the filtration of E:

$$E_i = \left\{ s \in E \,\middle|\, |s(z)|_h = O\left(|z|^{\alpha^1_i}\right) \right\}.$$

with $|\dots|_h$ the norm with respect to the metric h. We obtained an Hermitian vector bundle E on D with an orthonormal trivialization $\left(\frac{e_j}{|z|^{\alpha_j}}\right)_{1 \leq j \leq n}$.

Notations 6.5.2. *The symbol E represents a vector bundle in the sense of differential geometry, with smooth transition functions; whereas the symbol E represents a holomorphic bundle.*

The parabolic connection D_0 on the holomorphic bundle E induces a connection on E, in the orthonormal trivialization $\left(\frac{e_j}{|z|^{\alpha_j}}\right)_{1 \leq j \leq n}$ it reads

$$D_0 = d + \left(A - \frac{\alpha}{2}\right)\frac{dz}{z} - \frac{\alpha}{2}\frac{d\bar{z}}{\bar{z}}.$$

6.5.3 Local beahaviour for non-Abelian Hodge theory

D_0 decomposes as unitary connection plus a self-adjoint term

$$D_0 = D_0^h + {}_0.$$

In the orthonormal trivialization $\left(\frac{{}_j}{|z|^{{}_j}}\right)_{1 \quad j \quad n}$

$$D_0^h = d + \frac{A}{2}\frac{dz}{z} \qquad \frac{A^\dagger}{2}\frac{d\bar{z}}{\bar{z}}$$

and

$${}_0 = \frac{1}{2}\left(A\frac{dz}{z} + A^\dagger\frac{d\bar{z}}{\bar{z}} \qquad \frac{dz}{z} \qquad \frac{d\bar{z}}{\bar{z}}\right)$$

Consider the basis $(e_j)_{1 \quad j \quad n}$ defined by

$$e_j := \frac{{}_j}{|z|^{{}_j \quad i\,\mathrm{Im}\,A_j}}$$

with $\mathrm{Im}\,A_j$ the imaginary part of the j-th diagonal term of the matrix A.

Notations 6.5.3 (Canonical form). *The expression of D_0 in the orthonormal trivialization $(e_j)_{1 \quad j \quad n}$ is*

$$\begin{aligned} D_0 &= D_0^h + {}_0 \\ D_0^h &= d + \frac{1}{2}\mathrm{Re}(A)\left(\frac{dz}{z} \qquad \frac{d\bar{z}}{\bar{z}}\right) \\ {}_0 &= \frac{1}{2}\left(A\frac{dz}{z} + A^\dagger\frac{d\bar{z}}{\bar{z}} \qquad \frac{dz}{z} \qquad \frac{d\bar{z}}{\bar{z}}\right). \end{aligned}$$

Such expressions will be referred to as canonical forms.

Let $\bar{\partial}^F$ be the $(0,1)$-part of D_0^h and ${}_0$ the $(1,0)$-part of ${}_0$. In the basis $(e_j)_{1 \quad j \quad n}$ one has

$$\bar{\partial}^F = \bar{\partial} \qquad \frac{1}{2}\mathrm{Re}(A)\frac{d\bar{z}}{\bar{z}}.$$

This operator defines an holomorphic bundle over the punctured disk with holomorphic sections killed by $\bar{\partial}^F$. This holomorphic bundle can be extended over the puncture to an holomorphic bundle F, taking as a basis of holomorphic sections $(f_j)_{1 \quad j \quad n}$ defined by

$$f_j = |z|^{{}_j} e_j.$$

with ${}_j$ the real part of the j-th diagonal term of the matrix A. Then

$$|f_j|_h = |z|^{{}_j}$$

Similarly to the correspondence 6.5.1 between $({}_1, \ldots, {}_n)$ and $({}_i^1)_{1 \quad i \quad m_1}$, a stability parameter $({}_i^1)_{1 \quad i \quad n_1}$ is associated to $({}_1, \ldots, {}_n)$. This stability parameter provides a parabolic structure

$$F_i = \left\{ s \in F \;\middle|\; |s|_h = O\left(|s(z)|^{{}_i^1}\right) \right\}$$

Note that the holomorphic bundle F is di□erent from the holomorphic bundle E. Even the type of the parabolic structure di□er, E is of type $\_^{0}$ whereas F is of type .

Note that $_0$, the $(1;0)$ part of $_0$, provides an Higgs field:

$$_0 = \frac{1}{2}(A \quad)\frac{dz}{z}:$$

This is the local behaviour of the non-Abelian Hodge theory for the model connection. To summarize, starting from a logarithmic flat connection D_0 with polar part A, a metric h and a parabolic structure we obtain a parabolic Higgs bundle with residue of the Higgs field B and parabolic structure . The relation between those parameters are as described by Simpson [Sim90]

$$\begin{aligned} B &= \tfrac{1}{2}(A \quad) \\ &= \quad \mathrm{Re}\,A: \end{aligned} \tag{6.16}$$

6.5.4 Local description of weighted Sobolev spaces

Definition 6.5.4 (Weighted L^2 spaces). *$r = jzj$ is the radial coordinate on the disk, for real, L^2 is the space of function f on the disk such that $\frac{f}{r^{\ +1}}$ is L^2.*

The hermitian metric h on the vector bundle E induces a metric on End(E) and End(E) ^{1}. The definition of the spaces L^2 extends to section of such bundles using the induced metric. There is an orthogonal decomposition

$$\mathrm{End(E)} = \mathrm{End(E)}_0 \quad \mathrm{End(E)}_1 \tag{6.17}$$

with $\mathrm{End(E)}_0$ the space of endomorphism commuting with A. It induces an orthogonal decomposition

$$^1 \quad \mathrm{End(E)} = \left(^1 \quad \mathrm{End(E)}_0 \right) \quad \left(^1 \quad \mathrm{End(E)}_1 \right)$$

For $f\ 2\ ^1$ End(E) this orthogonal decomposition reads

$$f = f_0 + f_1:$$

Definition 6.5.5 (Sobolev spaces $L^{k;2}$).

$$L^{k;2}\left(^1 \quad \mathrm{End(E)} \right) := \left\{ f\ 2\ L^2 \,\middle|\, r^j f_0;\ \frac{r^j f_1}{r^{k \ \ j}}\ 2\ L^2 \ \text{for}\ 0 \quad j \quad k \right\}$$

with r the covariant derivative with respect to the unitary connection D_0^h.

Definition 6.5.6 (Space of admissible connections). *The space of admissible connections is*

$$A = \left\{ D + a \,\middle|\, a\ 2\ L^{1;2}_{2+} \left(^1 \quad \mathrm{End(E)} \right) \right\}:$$

Remark 6.5.7. *Note that the space of admissible connections is chosen so that the connection $D = D_0 + a$ introduced at the beginning of this section is admissible. Indeed, in the orthonormal trivialization $(e_j)_{1 \ \ j \ \ n}$, the matrix a is strictly block upper triangular. The non zero coe cients strictly above the diagonal have the following form*

$$jzj^{\ i \quad j}\frac{a_{i;j}}{z}$$

with $_i > \ _j$ and $a_{i;j}$ constant. Thus $a\ 2\ L^{1;2}_{2+}$ for small enough parameter:

$$0 < \quad < \ _i \quad _j:$$

6.5.5 Variation of the stability parameters and the metric

In order to pursue the path announced in Diagram (6.15), slightly modify the stability parameter   to a parameter $\tilde{\ }$, a diagonal matrix with coe□cients

$$(\tilde{\ }_1, \tilde{\ }_2, \dots) = \left(\underbrace{\tilde{\ }_{1,1}, \dots, \tilde{\ }_{1,1}}_{1^0_1}, \underbrace{\tilde{\ }_{1,2}, \dots, \tilde{\ }_{1,2}}_{1^0_2}, \dots \right)$$

The associated metric $\tilde{h}$ is defined such that the holomorphic trivialization $(f_j)_{1\ \ j\ \ n}$ of the holomorphic bundle F is orthogonal and

$$|f_j|_{\tilde{h}} = |z|^{\tilde{\ }_j}.$$

This provide an hermitian bundle with orthonormal trivialization $(\tilde{e}_j)_{1\ \ j\ \ n}$ defined by

$$\tilde{e}_j = \frac{f_j}{|z|^{\tilde{\ }_j}}.$$

We follow the same process as before in the opposite direction. $D_0^{\tilde{h}}$ is the $\tilde{h}$-unitary connection with $(0,1)$-part $\overline{\partial}^F$. And

$$\tilde{\ }_0 := \ _0 + \ _0^{\dagger}$$

the adjoint is taken with respect to the metric $\tilde{h}$. Then

$$\tilde{D}_0 := D_0^{\tilde{h}} + \tilde{\ }_0.$$

In the trivialization $(\tilde{e}_j)_{1\ \ j\ \ n}$ it reads

$$\begin{aligned} \tilde{\ }_0 &= \frac{1}{2}(A \quad)\frac{dz}{z} + \frac{1}{2}\left(A^{\dagger} \quad \right)\frac{d\overline{z}}{\overline{z}} \\ D_0^{\tilde{h}} &= d + \frac{1}{2}\tilde{\ }\left(\frac{dz}{z} \quad \frac{d\overline{z}}{\overline{z}}\right) \end{aligned}$$

Setting $\tilde{A} = \tilde{\ } + i\,\mathrm{Im}\,A$ and $\tilde{\ } = \ + \tilde{\ }$ we obtain a canonical form like in Notations 6.5.3

$$D_0^{\tilde{h}} = d + \frac{1}{2}\mathrm{Re}(\tilde{A})\left(\frac{dz}{z} \quad \frac{d\overline{z}}{\overline{z}}\right)$$

$$\tilde{\ }_0 = \frac{1}{2}\left(\tilde{A}\frac{dz}{z} + \tilde{A}^{\dagger}\frac{d\overline{z}}{\overline{z}} \quad \tilde{\ }\frac{dz}{z} \quad \tilde{\ }\frac{d\overline{z}}{\overline{z}}\right)$$

Continuing in the opposite direction, the $(0,1)$-part of $\tilde{D}_0$ defines an holomorphic bundle $\tilde{E}$ with holomorphic trivialization $(\tilde{\ }_j)_{1\ \ j\ \ n}$

$$\tilde{\ }_j := |z|^{\tilde{\ }_j \quad i\,\mathrm{Im}\,\tilde{A}_j}\tilde{e}_j.$$

$\tilde{D}_0$ defines a logarithmic connection on $\tilde{E}$, in the trivialization $(\tilde{\ }_j)_{1\ \ j\ \ n}$ it reads

$$\tilde{D}_0 = d + \tilde{A}\frac{dz}{z}$$

and $\widetilde{A}$ has distinct eigenvalues on each graded of the filtration of type and so does the monodromy of the local system of flat sections.

Let us summarize the local behaviour from Diagram (6.15) in terms of residue. We look at a particular block of size $_j$. The stability parameter associated to the graded of the filtration is specified with over brace. N.A.H stands for non-Abelian Hodge theory.

$$\begin{pmatrix} \overbrace{A_j \ \mathrm{Id}_{\ _1^{j\,0}}}^{j;1} & & \\ 0 & \overbrace{A_j \ \mathrm{Id}_{\ _2^{j\,0}}}^{j;2} & \\ \vdots & 0 & \ddots \end{pmatrix} \xrightarrow{N.A.H} \begin{pmatrix} \overbrace{(A_j \quad _{j;1}) \ \mathrm{Id}_{\ _1^{j\,0}} \qquad\qquad\qquad\qquad\qquad\qquad}^{\frac{1}{j}} & & \\ 0 & (A_j \quad _{j;2}) \ \mathrm{Id}_{\ _2^{j\,0}} & \\ \vdots & 0 & \ddots \end{pmatrix}$$

$$\downarrow \ \mapsto e$$

$$\begin{pmatrix} \overbrace{\widetilde{A}_{j;1} \ \mathrm{Id}_{\ _1^{j\,0}}}^{e_{j;1}} & & \\ 0 & \overbrace{\widetilde{A}_{j;2} \ \mathrm{Id}_{\ _2^{j\,0}}}^{e_{j;2}} & \\ \vdots & 0 & \ddots \end{pmatrix} \xleftarrow{N.A.H} \begin{pmatrix} \overbrace{(A_j \quad _{j;1}) \ \mathrm{Id}_{\ _1^{j\,0}}}^{e_{j;1}} & & \\ 0 & \overbrace{(A_j \quad _{j;2}) \ \mathrm{Id}_{\ _2^{j\,0}}}^{e_{j;2}} & \\ \vdots & 0 & \ddots \end{pmatrix}$$

With $\widetilde{A}_{j;i} = \widetilde{\ }_{j;i} + i \,\mathrm{Im}\, A_j$ and $\widetilde{\ }_{j;i} = \ _{j;i} + \widetilde{\ }_{j;i} \quad \frac{1}{j}$.

6.6 Di eomorphism between moduli spaces

6.6.1 Analytic construction of the moduli spaces

Analytic construction of moduli spaces relies on methods from Kuranishi [Kur65], Atiyah-Hitchin-Singer [AHS78] and Atiyah-Bott [AB83]. In this section we recall the analytic construction of the moduli spaces involved in the parabolic version of non-Abelian Hodge theory. Some particular cases of those moduli spaces were constructed by Konno [Kon93] and Nakajima [Nak96]. However we need more general construction in order to allow not necessarily central action of the residues of the Higgs fields on the graded of the filtration. The construction we follow is the one from Biquard-Boalch [BB04]. Note that a larger family of groups was considered by Biquard, García-Prada, Mundet i Riera [BGM15].

The local canonical model introduced in 6.5.3 is used to represent behaviour of connections near the punctures p_j. Let E a vector bundle on endowed with an hermitian metric h. Notation E refers to a vector bundle from di erential geometry point of view whereas E refers to holomorphic bundle. Let D_0 a model connection such that on the neighborhood of the punctures it coincides with the local model connection of previous subsection. The connection decomposes as

$$D_0 = D_0^h +$$

with D_0^h unitary and self-adjoint with respect to the metric h. We assume for this model connection that in an orthonormal trivialization $(e_i)_{1 \ \ i \ \ n}$ of E near the puncture p_j:

$$D_0^h = d + \frac{1}{2}\mathrm{Re}(A^j)\left(\frac{dz}{z} \quad \frac{d\bar{z}}{\bar{z}}\right)$$

and

$$= \frac{1}{2}\left(A^j\frac{dz}{z} + (A^j)^y\frac{d\bar{z}}{\bar{z}} \quad {}^j\frac{dz}{z} \quad {}^j\frac{d\bar{z}}{\bar{z}}\right)$$

with A^j and ^{j} the residue and the stability parameter for the de Rham moduli space at the puncture p_j. They correspond to the local parameter A and from Section 6.5, they are constant diagonal matrices. The parameters of the de Rham moduli space are chosen so that it corresponds under Riemann-Hilbert correspondence to a resolution of a character varieties with generic monodromies $\widetilde{M}_{\boldsymbol{L};\boldsymbol{P};}$. Therefore connections with such polar parts are necessarily irreducible.

Take r a function strictly positive on the punctured Riemann surface ^{0} such that it coincides with the radial coordinate near each punctures. The global weighted Sobolev space is defined as the local one from 6.5.4 with this positive function r. It is still denoted by $L^{k;2}$ (^{1} End (E)). The space of admissible connections is

$$A = \left\{D_0 + a \,\middle|\, a \;2\; L^{1;2}_{2+}\left({}^1 \quad \mathrm{End(E)}\right)\right\}:$$

This a□ne space is actually endowed with various complex structures. Decomposing according to $(1;0)$-part and $(0;1)$-part $a = a^{1;0} + a^{0;1}$

$$I:a = ia$$

and

$$J:a = i(a^{0;1})^y \quad i(a^{1;0})^y$$

The curvature of an admissible connection $D = D_0 + a$ is denoted by F_D. Consider the complex gauge group

$$G^I = \left\{g \;2\; \mathrm{Aut(E)}\,\middle|\,(D_0^h g)g \;\; {}^1;\; g \;\; {}_0 g \;\; {}^1 \;2\; L^{1;2}_{2+}\right\}$$

It acts on A by

$$g:D := gDg \;\; {}^1 = D \quad (Dg)\,g \;\; {}^1:$$

Next theorem gives an analytic construction of the set of isomorphism classes of parabolic flat connection with prescribed polar part. Later on, this set will be endowed with a manifold structure.

Theorem 6.6.1 (Biquard-Boalch [BB04] Section 8). *The de Rham moduli space of stable flat connection with prescribed polar part on the graded part of the filtration introduced in 6.4.1 is the following set*

$$M^{dR}_{A;} = fD_0 + a \;2\; A\, jF_D = 0g{=}G^I:$$

The stability condition does not appear as it is imposed by the generic choice of eigenvalues of the residue of D_0.

Now starting from $D = D_0 + a \;2\; A$ there is a natural candidate to produce a parabolic Higgs bundle, like in the local model. First decompose D in a unitary part and a self-adjoint part

$$\begin{aligned} D &= D^h + \\ &= D_0^h + \frac{a \quad a^y}{2} + \;\; {}_0 + \frac{a + a^y}{2} \end{aligned}$$

The natural candidate for the underlying holomorphic structure of the parabolic Higgs bundle is, in the orthonormal trivialization $(e_j)_{1 \leq j \leq k}$

$$\overline{\partial}^E = \overline{\partial} - \frac{1}{2}\mathrm{Re}(A)\frac{d\bar{z}}{\bar{z}} + \frac{a^{0,1} - (a^{1,0})^\dagger}{2}.$$

and the Higgs field

$$\Phi = \Phi_0 + \frac{a^{1,0} + (a^{0,1})^\dagger}{2}.$$

This data provides a Higgs bundle if $\overline{\partial}^E \Phi = 0$, equivalently if the pseudo curvature G_D vanishes. Note that the complex structure J is compatible with the Higgs bundles point of view. Indeed if Φ is the Higgs field associated to D then $i\Phi$ is the Higgs field associated to $J.D$. The complex gauge group acts on the Higgs bundles structures by

$$g.(\overline{\partial}^E, \Phi) := (g\overline{\partial}^E g^{-1}, g\Phi g^{-1}).$$

Next theorem gives an analytic construction of the set of isomorphism classes of parabolic Higgs bundles with prescribed residue. Later on, this set will be endowed with a manifold structure.

Theorem 6.6.2 (Biquard-Boalch [BB04] Section 7). *The Dolbeault moduli space of stable parabolic Higgs bundles with prescribed polar part on the graded part of the filtration introduced in 6.4.3 is the following set*

$$\mathcal{M}^{Dol}_{B,} = \left\{ D_0 + a \in \mathcal{A} \,\middle|\, \overline{\partial}^E \Phi = 0 \right\} / \mathcal{G}^J.$$

The stability condition does not appear as it is imposed by the generic choice of eigenvalues of the residue. As a group $\mathcal{G}^J$ is just $\mathcal{G}^I$, we change the upper index to precise which action is considered, the I-linear action or the J-linear action.

The non-Abelian Hodge theory gives a correspondence between Dolbeault and de Rham moduli spaces. The parameters are intertwined as in the local model. A nice way to state this correspondence is with hyperkähler geometry. Introduce the unitary gauge group

$$\mathcal{G} = \left\{ g \in U(E) \,\middle|\, (D_0 g) g^{-1} \in L^{1,2}_{2+} \right\}.$$

Consider the moduli space

$$\mathcal{M} = \left\{ D \in \mathcal{A} \,\middle|\, \overline{\partial}^E \Phi = 0,\ F_D = 0 \right\} / \mathcal{G}.$$

The equations defining $\mathcal{M}$ can be interpreted as vanishing of an hyperkähler moment map. Then the moduli space $\mathcal{M}$ is an hyperkähler reduction as in [Hit+87].

Theorem 6.6.3 (Biquard-Boalch [BB04] Theorem 5.4). *The moduli space $\mathcal{M}$ carries an hyperkähler manifold structure.*

Proof. The deformation theory for the moduli space $\mathcal{M}$ at a point $[D]$ is encoded in the following complex

$$L^{2,2}_{2+}(\mathfrak{u}(E)) \xrightarrow{D} L^{1,2}_{2+}(\Omega^1 \otimes \mathrm{End}\,E) \xrightarrow{D + D} L^2_{2+}((\Omega^2 \otimes \mathrm{End}\,E) \oplus i\mathfrak{u}(E)).$$

D is the formal adjoint of D with respect to the L^2 inner product and the metric h. The analytic study of this complex is detailed in [BB04]. Its first cohomology group is represented by the harmonic space H^1 $L^{1;2}_{2+}$ (^{1} End E). The Kuranishi slice at $[D]$ is defined by

$$S_D := \{D + a \mid \mathrm{Im}(D\ a) = 0;\ G_{D+a} = 0;\ F_{D+a} = 0\}. \tag{6.18}$$

Taking a small enough neighborhood of D in the Kuranishi slice, one obtains a finite dimensional manifold transverse to the G-orbits. The Kuranishi map provides an isomorphism between a neighborhood of 0 in H^1 and a neighborhood of D in the Kuranishi slice, see Konno [Kon93, Lemma 3.8, Theorem 3.9]. This provides an hyperkähler manifold structure on the moduli space. □

Now the non-Abelian Hodge theory can be described the following way.

Theorem 6.6.4 (Biquard-Boalch [BB04] Theorem 6.1). *The manifold M endowed with the complex structure I is the moduli space $M^{dR}_{A;}$.*

The manifold M endowed with the complex structure J is the moduli space $M^{Dol}_{B;}$.

6.6.2 Construction of the diffeomorphisms

Theorem 6.6.5 (Riemann-Hilbert correspondence). *The moduli space $M^{dR}_{A;}$ is complex analytically isomorphic to a resolution of character varieties $\widetilde{M}_{\mathbf{L};\mathbf{P};}$.*

Proof. As explained in 6.4.6, $\widetilde{M}_{\mathbf{L};\mathbf{P};}$ is nothing but the moduli space of filtered local systems with prescribed graded part of the monodromy around the punctures. Filtered version of the Riemann-Hilbert correspondence is established as an equivalence of category by Simpson [Sim90]. Yamakawa [Yam08] proved that it is a diffeomorphism using a particular construction of the de Rham moduli space from Inaba [Ina13]. The same argument apply with the de Rham moduli space endowed with the manifold structure from M. Starting from a flat connection, the associated local system is obtained by taking flat sections i.e. solving a differential equation. When the parameters of the equation vary complex analytically, so does the solution. □

Then $M^{dR}_{A;}$ and $M^{Dol}_{B;}$ are diffeomorphic as both are M with a particular complex structure. The first line in the path announced in Diagram 6.15 is now constructed. The second line is obtained exactly like the first, but in the other direction. It remains to describe the vertical arrow between two Dolbeault moduli spaces $M^{Dol}_{B;}$ and $M^{Dol}_{B;e}$. This is given by Biquard, García-Prada, Mundet i Riera [BGM15, Theorem 7.10]. The construction of the diffeomorphism is detailed in the remaining of the section.

Because of genericity of the eigenvalues of the residue, the stability parameter is irrelevant. The parameter can be changed to a stability parameter $\tilde{}$ with different values for each graded of the filtration. Namely one can chose $\tilde{}$ such that the associated matrix satisfies $Z_{\mathrm{GL}_n}(\tilde{}^{\,i}) = Z_{\mathrm{GL}_n}(B^i)$ and such that the parabolic degree remains 0. The local behaviour near each puncture is described by the right hand side of the diagram at the end of 6.5.5.

We introduce the following notation

$$_i := \tilde{}_i \quad _i.$$

For the construction of the di eomorphism in Theorem 6.6.7, it will be conveniant to assume

$$\max_{i;j} j \quad _i \qquad _j j <$$

with the parameter appearing in the weighted Sobolev space $L^{1;2}_{2+}$.

Proposition 6.6.6. *For such choice of parameter there is a natural bijection between* $M^{Dol}_{B;}$ *and* $M^{Dol}_{B;e}$*.*

Proof. $M^{Dol}_{B;}$ classifies isomorphism classes of parabolic Higgs bundles with parabolic structure at p_j

$$0 = F^j_0 \; (\; F^j_1 \; (\qquad (\; F^j_{n_j} = F^j$$

and with the residue of the Higgs fields preserving this filtration and acting as a semisimple endomorphism B^j_i on the graded spaces

$$F^j_i = F^j_{i \; 1}:$$

Such spaces decomposes as direct sum of eigenspaces for B^j_i. After ordering the eigenvalues, we obtain a uniquely determined refinement of the initial parabolic structure:

$$0 = \widetilde{F}^j_0 \; (\; \widetilde{F}^j_1 \; (\qquad (\; \widetilde{F}^j_{m_j} = F^j:$$

Then the residue of the Higgs field acts as a central endomorphism on the graded $\widetilde{F}^j_i = \widetilde{F}^j_{i \; 1}$. This gives a map $f : M^{Dol}_{B;} \; ! \; M^{Dol}_{B;e}$. Stability is not an issue as the polar part of the residue is generic. The map forgetting part of the filtration is an inverse so that there is a natural bijection between both moduli spaces. □

Before proving that this bijection is a di eomorphism the manifold structure on $M^{Dol}_{B;e}$ is detailed. It is constructed just like $M^{Dol}_{B;}$ but with di erent parameters.

Similarly to M, construct a moduli space $M_{\tilde{h}}$. Instead of the initial metric h, we use a metric $\widetilde{h}$, similar to the local model from 6.5.5. Namely it is chosen so that near each puncture it admits as an orthonormal trivialization $(\widetilde{e}_i)_{1 \; i \; n}$ with

$$\widetilde{e}_i = r^{\; i} e_i:$$

Where $(e_i)_{1 \; i \; n}$ is the orthnormal trivialization with respect to h near the puncture and $_i = \widetilde{\;}_i \qquad _i$.

First we construct $\widetilde{D}_0$, a starting point to construct an a ne space of admissible connections. Recall that

$$D_0 = D^h_0 + \quad _0$$

with D^h_0 a h-unitary connection and $_0$ self-adjoint with respect to h. Take $D^{h^{00}}_0$ the $(0;1)$-component of D^h_0 and $^{1;0}_0$ the $(1;0)$-component of $_0$. There exists a unique $D^{\tilde{h}^0}_0$ such that $D^{\tilde{h}^0}_0 + D^{h^{00}}_0$ is $\widetilde{h}$-unitary. Let $^{1;0\tilde{y}}_0$ the adjoint of $^{1;0}_0$ with respect to the metric $\widetilde{h}$. Then $\widetilde{D}_0$ is defined by

$$\widetilde{D}_0 := D^{\tilde{h}^0}_0 + D^{h^{00}}_0 + \quad ^{1;0}_0 + \quad ^{1;0\tilde{y}}_0:$$

Near the puncture, in the trivialization $(\widetilde{e}_i)_{1 \; i \; n}$, the connection $\widetilde{D}_0$ behaves exactly like the local model with the same name introduced in 6.5.5. Define the a ne space of admissible connections with respect to $\widetilde{D}_0$ and the metric $\widetilde{h}$.

$$A_{\tilde{h}} := \left\{ \widetilde{D}_0 + \tilde{a} \, j \, \tilde{a} \; 2 \; L^{1;2}_{2+e} \left(\; ^1 \quad \mathrm{End(E)} \right) \right\}$$

The weighted Sobolev space $L^{1;2}_{\ 2+e}(\ ^1 \ \ \mathrm{End(E)})$ is also defined using the metric $\widetilde{h}$. Moreover notice that we do not chose the same parameter for A and for $A_{\widetilde{h}}$. It will be convenient to chose $\widetilde{\ }$ such that

$$0 < \widetilde{\ } < \max_{i;j} j \ _i \ \ _j j: \tag{6.19}$$

With this set up, we are ready to prove that the bijection from previous proposition is a di□eomorphism.

Theorem 6.6.7. *The natural bijection between $M^{Dol}_{B;}$ and $M^{Dol}_{B;e}$ is a di eomorphism.*

Proof. $M^{Dol}_{B;}$ is identified with the manifold M with the complex structure J.

Take an element in $M^{Dol}_{B;}$ identified with an element $[D] \ 2 \ M$. $[D]$ is the class of $D = D_0 + a$ an admissible connection with vanishing curvature and pseudo-curvature. By construction of the manifold structure, a neighborhood of $[D]$ in M is di□eomorphic with a neighborhood of D in the Kuranishi slice S_D defined in (6.18). We shall prove that the bijection from Proposition 6.6.6 induces a smooth map from a neighborhood of D in S_D to $A_{\widetilde{h}}$.

First we describe the image of the connection D, it is obtained exactly the same way $\widetilde{D}_0$ is obtained from D_0. It decomposes as a connection h-unitary plus a hermitian part

$$D = D_0^h + \frac{a \quad a^y}{2} + \ \ _0 + \frac{a + a^y}{2}:$$

It can be decomposed further in components of type $(1;0)$ and $(0;1)$. Then the $(0;1)$-component of the h-unitary part is

$$\overline{@}^F = D_0^{h00} + \frac{a^{0;1} \quad a^{1;0y}}{2}$$

and the $(1;0)$-component of the self-adjoint part is

$$= \ \ ^{1;0} + \frac{a^{1;0} + a^{0;1y}}{2}:$$

The parabolic Higgs bundle associated to D is $(\overline{@}^F; \ )$. Now we switch to the metric $\widetilde{h}$. Near each puncture, in the $\widetilde{h}$-orthonormal trivialization $(\widetilde{e}_i)_{1 \ i \ n}$

$$\overline{@}^F = D_0^{h00} + \left(\frac{\widetilde{\ }}{2}\right)\frac{d\overline{z}}{\overline{z}} + \widetilde{H}\frac{a^{0;1} \quad a^{1;0y}}{2}\widetilde{H}^{\ 1}$$

and

$$= \ \ ^{1;0} + \widetilde{H}\frac{a^{1;0} + a^{0;1y}}{2}\widetilde{H}^{\ 1}:$$

with $\widetilde{H}$ a diagonal matrix with coe□cients $r^{\ i}$. Using the metric $\widetilde{h}$ we construct $D^0_{\widetilde{h}}$ such that $D^0_{\widetilde{h}} + \overline{@}^F$ is $\widetilde{h}$-unitary. And $\ ^{\widetilde{y}}$ the adjoint of with respect to $\widetilde{h}$. We want to prove that

$$D^0_{\widetilde{h}} + \overline{@}^F + \ \ + \ ^{\widetilde{y}}$$

belongs to the space of admissible connections $\mathcal{A}_{\theta}$. Let

$$\tilde{a} := D^{0}_{\theta} + \overline{\partial}^{F} + \quad + \quad \varphi \quad \tilde{D}_0.$$

Components of $\tilde{a}$ are obtained from components of a by multiplication by $r^{\ i \quad j}$. Thus for $\tilde{\ }$ small enough (6.19), $\tilde{a}$ belongs to $L^{1,2}_{\ 2+\epsilon}$. Therefore the bijection from $\mathcal{M}^{Dol}_{B,}$ to $\mathcal{M}^{Dol}_{B,\epsilon}$ comes from a map

$$\begin{array}{ccc} \{D_0 + a \in \mathcal{A} \mid F_{D_0+a} = G_{D_0+a} = 0\} & \to & \left\{\tilde{D}_0 + \tilde{a} \in \mathcal{A}_{\theta} \,\middle|\, G_{\tilde{D}_0+\tilde{a}} = 0\right\} \\ D_0 + a & \mapsto & \tilde{D}_0 + \tilde{a}. \end{array}$$

This restricts to a diffeomorphism from a neighborhood of D in the Kuranishi slice S_D to a manifold transverse to the $\mathcal{G}^J$-orbits in a neighborhood of D. Therefore the map $\mathcal{M}^{Dol}_{B,} \to \mathcal{M}^{Dol}_{B,\epsilon}$ is a diffeomorphism. □

To finish, let us detail the last step at the bottom left corner of Diagram (6.15). Applying successively non-Abelian Hodge theory and Riemann-Hilbert correspondence, the moduli space $\mathcal{M}^{Dol}_{B,\epsilon}$ is diffeomorphic to a moduli space of filtered local system $\widetilde{\mathcal{M}}_{\mathbf{L},\mathbf{P},\epsilon}$. The parameters are such that $Z_{\mathrm{GL}_n}(\tilde{\ }^{j}) = L^j$ for $1 \leq j \leq k$. The map $p^{\epsilon} : \widetilde{\mathcal{M}}_{\mathbf{L},\mathbf{P},} \to \mathcal{M}_{\mathcal{S}}$ from 3.5.11 is an isomorphism. $\mathcal{M}_{\mathcal{S}}$ is the character variety with monodromy at the puncture p_j in S_j the conjugacy class of $\tilde{\ }^{j}$.

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