# Realisation of Abelian varieties as automorphism groups

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Abstract.

Let $A$ be an Abelian variety over a field $F$. We show that $A$ is isomorphic to the automorphism group scheme of a smooth projective $F$-variety if, and only if, $\mathrm{Aut}_{gp}(\overline{A})$ is finite. This result was proved by Lombardo and Maffei [7] in the case $F = \mathbb{C}$ , and recently by Blanc and Brion [1] in the case of an algebraically closed $F$.

Résumé.

Soit $A$ une variété abélienne sur un corps $F$. On montre que $A$ est isomorphe au schéma en groupes des automorphismes d'une $F$-variété projective et lisse, si et seulement si le groupe des $\overline{F}$-automorphismes de $A$ est fini.
Ce résultat est dû à Lombardo et Maffei [7] lorsque $F = \mathbb{C}$. Il est dû à Blanc et Brion [1] lorsque $F = \overline{F}$ .

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## 1. Introduction.

Let $F$ be a field, with algebraic closure $\overline{F}$. Let $X$ be a projective variety over $F$. The automorphism group functor $\mathbf{Aut}(X)$ is represented by a group scheme, locally of finite type over $F$ ([8], Theorem 3.7). Conversely, given a group scheme $G$, of finite type over $F$, it is natural to ask whether $G$ can be realised as the automorphism group of such an $X$. When $G = A$ is an abelian variety, this question was recently considered in [7]. When $F = \mathbb{C}$, Lombardo and Maffei prove that $A$ is the automorphism group of a projective smooth complex variety, if and only if $\mathrm{Aut}_{gp}(A)$ is finite. They use analytic methods. Their result was extended to $F$ algebraically closed of any characteristic in [1], using algebro-geometric techniques: blowups, Lie algebra computations and modding out actions of finite group schemes. Making a different use of these tools, we provide a generalisation of this result, to the case of all ground fields $F$.

### 1.1. Sketch of our construction.

Let $A/F$ be an abelian variety over a field $F$, such that $G := \mathrm{Aut}_{gp}(\overline{A})$ is finite. We first introduce an integer $n \geq 1$, invertible in $F$, such that $G$ acts faithfully on the $n$-torsion subgroup $A[n](\overline{F})$.
Then, we pick an abelian variety $B_1/F$, enjoying the following properties.

(1) The abelian varieties $A$ and $B_1$ are 'orthogonal', in the sense that

$$\mathrm{Hom}_{gp}(\overline{A}, \overline{B}_1) = 0,$$

where homomorphisms are taken over $\overline{F}$.

(2) There exists an injection (of algebraic $F$-groups)

$$\iota : A[n] \hookrightarrow B_1.$$

We denote by $B_2/F$ the abelian variety fitting into the diagonal extension

$$0 \longrightarrow A[n] \xrightarrow{a \mapsto (a, \iota(a))} A \times B_1 \xrightarrow{\pi} B_2 \longrightarrow 0.$$

Using point (1) above, we prove that automorphisms of (the variety) $B_2$ are diagonal: they come from automorphisms of $A \times B_1$, respecting orbits under the embedded $A[n]$. Next, we build an appropriate smooth closed $F$-subvariety $Y_2 \subset B_2$, stable by translations by $A \simeq \pi(A \times \{0\}) \subset B_2$.
We define a smooth $F$-variety $X$ as the blowup

$$X := \mathrm{Bl}_{Y_2} B_2.$$

The natural arrow

$$A \longrightarrow \mathbf{Aut}(B_2),$$

given by translations, lifts to an arrow

$$\tau : A \longrightarrow \mathbf{Aut}(X).$$

We show that $\tau$ is an isomorphism of algebraic groups over $F$.

## 2. Notation.

2.1. Geometry over $F$. Let $F$ be a field, with algebraic closure $\overline{F}$, and separable closure $F_s \subset \overline{F}$. We denote by $F[\epsilon]$, $\epsilon^2 = 0$, the $F$-algebra of dual numbers. We use it for differential calculus.
By a variety over $F$, we mean a separated $F$-scheme of finite type.
An algebraic $F$-group (or simply $F$-group) is an $F$-group scheme of finite type. It is often assumed to be reduced, hence smooth over $F$.
Let $X$ be a variety over $F$. For a field extension $E/F$, we denote by $X_E := X \times_F E$ the $E$-variety obtained from $X$ by extending scalars. We put $\overline{X} := X \times_F \overline{F}$.
If $X$ is smooth over $F$, we denote by $TX \longrightarrow X$ the tangent bundle of $X$. A global section of the tangent bundle is called a vector field on $X$.
We denote by $\mathrm{Aut}(X)$ the (abstract) group of automorphisms of the $F$-variety $X$, and by $\mathrm{Aut}(\overline{X})$ the group of automorphisms of the $\overline{F}$-variety $\overline{X}$.
If $X/F$ is a projective variety, we denote by $\mathbf{Aut}(X)$ the $F$-group scheme of automorphisms of $X$; it is locally of finite type over $F$. By [2], Lemma 3.1, there is a canonical isomorphism

$$H^0(X, TX) \xrightarrow{\sim} \mathrm{Lie}(\mathbf{Aut}(X)).$$

If an abstract group $G$ acts on a variety $X$, and if $Z \subset X$ is a closed subvariety, we denote by $\mathrm{Stab}_G(Z) \subset G$, or simply by $\mathrm{Stab}(Z) \subset G$ when no confusion arises, the subgroup of transformations leaving $Z$ (globally) invariant.
Let $G/F$ be a group scheme, locally of finite type. In the situation where $G$ acts on $X$, we use the notation $\mathbf{Stab}_G(Z) \subset G$ for the closed $F$-subgroup scheme defined by

$$\mathbf{Stab}_G(Z)(A) = \{g \in G(A), g(Z_A) = Z_A\},$$

for all commutative $F$-algebras $A$. That it is representable follows from [3], II 1.3.6.

2.2. Frobenius and Verschiebung. If $F$ has characteristic $p > 0$, we put

$$X^{(1)} := X \times_{\mathrm{Frob}} F,$$

extension of scalars taken with respect to $\mathrm{Frob} : F \xrightarrow{x \mapsto x^p} F$.
Recall the Frobenius homomorphism

$$\mathrm{Frob}_X : X \longrightarrow X^{(1)};$$

it is a morphism of $F$-varieties, functorial in $X$.
If $X/F$ is an algebraic group, it is a group homomorphism.
If $X$ is a commutative algebraic group, there is the Verschiebung homomorphism

$$\mathrm{Ver}_X : X^{(1)} \longrightarrow X,$$

satisfying $(\mathrm{Ver}_X \circ \mathrm{Frob}_X) = p\mathrm{Id}_X$.
If moreover $X/F$ is a semi-abelian variety, $\mathrm{Ver}_X$ and $\mathrm{Frob}_X$ are isogenies.

2.3. Abelian varieties. If $A$ and $B$ are Abelian varieties over $F$, we denote by $\mathrm{Hom}_{gp}(A, B)$ the group of homomorphisms of algebraic $F$-groups, from $A$ to $B$. We denote by $\mathrm{Hom}_{gp}(\overline{A}, \overline{B})$ the group of homomorphisms of algebraic $\overline{F}$-groups, from $\overline{A}$ to $\overline{B}$. These are finite free $\mathbb{Z}$-modules. We adopt the similar notation for endomorphisms ($\mathrm{End}_{gp}$) and automorphisms ($\mathrm{Aut}_{gp}$). For an integer $n \geq 1$, we denote by $A[n]$ the $n$-torsion of $A$, seen as a finite group scheme over $F$.

2.4. Barycentric operations. Let $A$ be an abelian variety over $F$. Then $A$ comes naturally equipped with barycentric operations with integer coefficients. More precisely, for a positive integer $n$, denote by

$$\mathbf{Z}_1^n \subset \mathbf{Z}^n$$

the subset consisting of integers $\alpha = (\alpha_1, \ldots, \alpha_n)$, with $\alpha_1 + \ldots + \alpha_n = 1$.
For $\alpha \in \mathbf{Z}_1^n$, there is a barycentric operation

$$\mathcal{B}_\alpha : A^n \longrightarrow A,$$

$$(x_1, \ldots, x_n) \mapsto \alpha_1 x_1 + \ldots + \alpha_n x_n.$$

Associativity of the group law of $A$, provides natural associativity relations between the $\mathcal{B}_\alpha$'s, for various $\alpha' s$.
For instance, pick $\alpha = (\alpha_1, \alpha_2) \in \mathbf{Z}_1^2$ and $\gamma = (\gamma_1, \gamma_2) \in \mathbf{Z}_1^2$, and set

$$\delta := (\alpha_1\gamma_1, \alpha_2\gamma_1, \gamma_2) \in \mathbf{Z}_1^3.$$

Then, we have the associativity rule

$$\mathcal{B}_\gamma(\mathcal{B}_\alpha(x_1, x_2), x_3) = \mathcal{B}_\delta(x_1, x_2, x_3).$$

*Remark* 2.1. More generally, these barycentric operations exist for *torsors* under commutative algebraic $F$-groups.

Definition 2.2. *Let $X \subset A$ be an $F$-subvariety. We say that $X$ is stable under all barycentric operations, if the restriction*

$$(\mathcal{B}_\alpha)_{|X^n} : X^n \to A$$

*factors through the closed immersion $X \hookrightarrow A$, for every $n \geq 2$ and every $\alpha \in \mathbf{Z}_1^n$. In this case, we also say that $X$ is barycentric.*

Note that $X$ is barycentric if and only if it is a translate of an algebraic $F$-subgroup $\overrightarrow{X} \subset A$. Checking this fact is left as an exercise for the reader. Of course, $X(F)$ might be empty. If $X$ is geometrically reduced and geometrically connected, so is $\overrightarrow{X}$- hence $\overrightarrow{X}$ is an abelian subvariety of $A$.
Let $A$ and $B$ be two abelian varieties over $F$. Recall the essential fact

$$\mathrm{Hom}_{\overline{F}-var}(\overline{A}, \overline{B}) = B(\overline{F}) \times \mathrm{Hom}_{gp}(\overline{A}, \overline{B}).$$

In particular, morphisms (of varieties) between abelian varieties commute with the barycentric operations $\mathcal{B}_\alpha$.
If $X \subset A$ is a geometrically reduced closed $F$-subvariety, the smallest geometrically reduced barycentric $F$-subvariety containing $X$ is called the barycentric envelope of $X$. We denote it by $\mathcal{E}(X)$.
Assume now that $X$ is geometrically reduced and geometrically connected. Pick $n \geq 1$ and $\alpha \in \mathbf{Z}_1^n$. Consider $\mathcal{B}_\alpha(X^n) \subset A$ as a geometrically reduced and geometrically connected closed subvariety of $A$. Then, if $n$ and $\alpha$ are chosen so that $\mathcal{B}_\alpha(X^n)$ is of maximal dimension, we have $\mathcal{B}_\alpha(X^n) = \mathcal{E}(X)$. Thus, $\mathcal{E}(X)$, being geometrically connected and geometrically reduced, is a translate of an abelian subvariety of $A$.

## 3. Statement of the theorem.

Theorem 3.1. *Let $A$ be an Abelian variety, over a field $F$. The following are equivalent:*
*1) The group $G := \mathrm{Aut}_{gp}(\overline{A})$ is finite.*
*2) There exists a smooth projective $F$-variety $X$, such that $A$ is isomorphic to $\mathbf{Aut}(X)$ (as algebraic groups over $F$).*

Note that 2) $\Rightarrow$ 1) can be checked over $\overline{F}$, which follows from [1], Theorem A. Our task in this paper is to prove the converse implication.

## 4. Auxiliary results.

4.1. Blowups. This section contains two elementary lemmas on automorphisms of blowups, which we provide with short proofs. A good recent reference on this topic, also containing more advanced material, is section 2 of [9].

Lemma 4.1. *Let $Y \hookrightarrow D$ be a closed immersion of smooth $F$-varieties, such that all connected components of $Y$ have codimension $\geq 2$ in $D$.*
*Denote by $\beta : X := \mathrm{Bl}_Y(D) \longrightarrow D$ the blowup of $Y$ inside $D$.*
*The $F$-variety $X$ is smooth.*
*Let $f$ be an automorphism of the $F$-variety $D$. Then, $f$ lifts via $\beta$ to an automorphism of $X$, if and only if $f(Y) = Y$.*

**Proof.** If $f(Y) = Y$, then $f$ lifts to an automorphism of $X$ by the universal property of the blowup.
Conversely, assume that $f$ lifts to an automorphism $\phi$ of $X$, so that we have a commutative square

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d] & X \arrow[d] \\ D \arrow[r, "f"] & D. \end{tikzcd}$$

To check that $f(Y) = Y$, can assume that $F = \overline{F}$. It then suffices to prove that $Y \subset D$ and $f(Y) \subset D$ have the same set of $F$-rational points. This is clear, since the fiber of $\beta$ over a point $x \in D(F)$ is either a point if $s \notin Y(F)$, or a projective space of dimension $\geq 1$ if $x \in Y(F)$. $\square$

Lemma 4.1 has an infinitesimal analogue, as follows.

Lemma 4.2. *Let $Y \hookrightarrow D$ be a closed immersion of smooth $F$-varieties, such that all connected components of $Y$ have codimension $\geq 2$ in $D$.*
*Denote by $\beta : X := \mathrm{Bl}_Y(D) \longrightarrow D$ the blowup of $Y$ inside $D$.*
*Let $s : D \longrightarrow TD$ be a vector field on $D$. Then, $s$ lifts to a vector field on $X$, if and only if $s_{|Y}$ takes values in $TY$.*

**Proof.** Denote by $i : E \hookrightarrow X$ the exceptional divisor. The restriction

$$\beta_{|X-E} : X - E \longrightarrow D - Y$$

is an isomorphism.
We thus have a natural injective $F$-linear arrow

$$\rho : H^0(X, TX) \longrightarrow H^0(D - Y, TD) = H^0(D, TD),$$

$$\sigma \mapsto \sigma_{|X-E}.$$

Note that the equality $H^0(D-Y, TD) = H^0(D, TD)$ follows from the fact that $Y \subset D$ has codimension $\geq 2$. On $E$, we have a natural extension of vector bundles

$$0 \longrightarrow TE \longrightarrow i^*(TX) \longrightarrow N_{E/X} \longrightarrow 0,$$

where $N_{E/X} \simeq \mathcal{O}_E(-1)$ is the normal bundle of $E$ in $X$. Since $Y$ has codimension $\geq 2$ in $D$, we have $H^0(E, O_E(-1)) = 0$. This can be checked on the fibers of $\beta$ over geometric points of $Y$, which are projective spaces of dimension $\geq 1$. Hence, $\sigma_{|E}$ takes values in $TE$. Consequently, $\rho(\sigma)_{|Y}$ takes values in $TY$.
Conversely, let $s : D \longrightarrow TD$ be a vector field on $D$. Then $s$ corresponds to an automorphism $\psi$ of the $F[\epsilon]$-scheme $D \times_F F[\epsilon]$, reducing to the identity at $\epsilon = 0$. Assume that $s_{|Y}$ takes values in $TY$. Then, $\psi$ restricts to an automorphism of the closed subscheme $Z \times_F F[\epsilon] \subset D \times_F F[\epsilon]$. By the universal property (and compatibility with base change) of the blowup, $\psi$ lifts, via $\beta \times_F F[\epsilon]$, to an automorphism of $X \times_F F[\epsilon]$. Equivalenty, $s$ lifts, via $\beta$, to a vector field on $X$.

□

4.2. Hypersections in projective space.
We could not find a reference in the literature for the following result, so that we provide it with a proof.

Proposition 4.3. *Let $S$ be a geometrically irreductible smooth projective $F$-variety, of dimension $\geq 2$. Let $m \geq 1$ be an integer. Then, $S$ contains a geometrically irreductible smooth projective $F$-curve, of genus $g \geq m$.*

**Proof.** Pick a projective embedding $S \subset \mathbb{P}^n$ (everything is over $F$). Let $d \geq 1$ be an integer. Let $H \subset \mathbb{P}^n$ be a degree $d$ hypersurface, given by $h \in H^0(\mathbb{P}^n, \mathcal{O}(d))$. By Bertini's theorem, for $d$ large enough and $h$ general, $S \cap H$ is smooth and geometrically irreductible, of dimension one less than $S$. This version of Bertini's theorem works over any $F$- see [10] and [4] for the delicate case where $F$ is finite. Proceeding by induction, we reduce to the case where $S$ is a surface.
We then take $C := S \cap H$, and show that $g (= h^1(C, \mathcal{O}_C))$ goes to infinity with $d$. To do so, consider the exact sequence of coherent $\mathcal{O}_{\mathbb{P}^n}$-modules

$$0 \to \mathcal{O}_S(-d) \xrightarrow{\times h} \mathcal{O}_S \to \mathcal{O}_C \to 0.$$

Taking Euler characteristics, we get

$$g - 1 = -\chi(\mathcal{O}_C) = \chi(\mathcal{O}_S(-d)) - \chi(\mathcal{O}_S).$$

We conclude using the following fact, applied to $X = S$.
For a closed $m$-dimensional $F$-subvariety $X \subset \mathbb{P}^n$, the association

$$d \mapsto \chi(\mathcal{O}_X(-d))$$

is a degree $m$ polynomial function of $d$. A classical proof is by induction on $m \geq 0$.
□

4.3. (Semi-)abelian varieties. The next Lemma is borrowed from [2], Lemma 5.3. We provide here a different proof. In practice, we will apply it to abelian varieties, in which case it is due to Chow.

Lemma 4.4. *Assume that $F$ has characteristic $p > 0$.*
*Let $A, B$ be semi-abelian varieties over $F$. Then, all elements of $\mathrm{Hom}_{gp}(\overline{A}, \overline{B})$ are defined over the separable closure $F_s \subset \overline{F}$.*

**Proof.** We have to show the following. Let $E/F$ be a purely inseparable algebraic extension. Let $g : A_E \longrightarrow B_E$ be a homomorphism of algebraic groups over $E$. Then $g$ is defined over $F$. Without loss of generality, we can assume that $E/F$ is finite. By induction, we reduce to the case where $E = F(\sqrt[p]{a})/F$ is a primitive purely inseparable extension of height one. Note that $\mathrm{Frob} : E \longrightarrow E$ takes values in $F$. Hence, $g^{(1)} : A_E^{(1)} \longrightarrow B_E^{(1)}$ is defined over $F$. The Frobenius homomorphism

$$\mathrm{Frob}_A : A \longrightarrow A^{(1)}$$

presents $A^{(1)}$ as a quotient of $A$, by a finite (characteristic) sub-$F$-group $\mu_A \subset A$. From the relation

$$\mathrm{Ver}_A \circ \mathrm{Frob}_A = p\mathrm{Id}_A,$$

we deduce $\mu_A \subset A[p]$. Same holds for $B$.
Combining these facts, we get that the $E$-morphism

$$A/\mu_A \longrightarrow B/\mu_B$$

induced by $g$, is defined over $F$. Modding out further, we get that the $E$-morphism

$$A/A[p] \longrightarrow B/B[p],$$

induced by $g$, is defined over $F$.
Via the iso

$$A/A[p] \xrightarrow{\sim} A$$
$$\overline{a} \mapsto pa,$$

this isomorphism is actually $g$ itself. The Lemma is proved. □

Lemma 4.5. *For each $n \geq 2$, there exists an (absolutely) simple $n$-dimensional abelian variety $A$ over $F_s$.*

**Proof.** Since $F_s$ is separably closed, 'simple' is the same as 'absolutely simple', for abelian varieties over $F_s$ (use Lemma 4.4). Without loss of generality, we assume that $F_s$ is the algebraic closure of its prime subfield. Over $\overline{\mathbb{Q}}$, we can then use the existence of abelian surfaces with a prescribed CM type. Over $\overline{\mathbb{F}}_p$, we can use Honda-Tate theory. For concrete constructions, and more general results, we refer to [9], Theorem 1 (where $F_s = \overline{\mathbb{Q}}$), and [6], Theorem 2 (where $F_s = \overline{\mathbb{F}}_p$).

□

Lemma 4.6. *Let $B$ be an abelian variety over $F$, whose simple factors (over $\overline{F}$) are of dimensions $\geq 2$. (Equivalently: all $\overline{F}$-homomorphisms from an elliptic curve to $\overline{B}$ are constant.)*
*Then, there exists a smooth $F$-subvariety $Y \subset B$, which is a disjoint union of smooth $F$-curves, and of a separable closed point, such that*

$$\mathbf{Stab}(Y) = \{\mathrm{Id}\} \subset \mathbf{Aut}(B).$$

**Proof.** Assume first that $B$ is $F$-simple, in the sense that it has no non-trivial proper abelian $F$-subvariety. By Proposition 4.3, we can pick a geometrically irreducible smooth $F$-curve $C \subset B$, of arbitrarily large genus $g \geq 2$.
The group $\mathrm{Aut}(\overline{C})$ is finite. Indeed, $\mathrm{Lie}(\mathbf{Aut}(C))$ is the space of vector fields on $C$, which vanishes since $g \geq 2$.
Let us show that $\mathcal{E}(C) = B$. The barycentric envelope $\mathcal{E}(C)$ is a translate of an abelian subvariety $B' \subset B$. Since $B$ is $F$-simple, we get $B' = B$, hence $\mathcal{E}(C) = B$.
Now, let $g \in \mathbf{Aut}(B)(\overline{F}[\epsilon]) = B(\overline{F}[\epsilon]) \times \mathrm{Aut}_{gp}(\overline{B})$ be such that

$$g_{|C \times_F \overline{F}[\epsilon]} = \mathrm{Id}_{|C \times_F \overline{F}[\epsilon]}.$$

Because $g$ commutes to barycentric operations, $g$ acts as the identity on the closed subscheme

$$\mathcal{E}(C) \times_F \overline{F}[\epsilon] \subset B \times_F \overline{F}[\epsilon].$$

Since $\mathcal{E}(C) = B$, it follows that $g = \mathrm{Id}$. Thus, we get a natural embedding of $F$-group schemes

$$H := \mathbf{Stab}_{\mathbf{Aut}(B)}(C) \hookrightarrow \mathbf{Aut}(C).$$

In particular, $H$ is finite étale over $F$. Let $E/F$ be a finite separable field extension, such that $H(E) = H(\overline{F})$. Denote by

$$\Phi := \bigcup_{h \in H(E), h \neq e} \overline{B}^h \subset \overline{B}$$

be the (strict) closed subscheme, consisting of points fixed by at least one non-trivial element $h \in H(E)$. It is defined over $F$ by Galois descent. There exists a finite separable field extension $L/E$, and a point $b \neq 0 \in B(L)$, which does not lie in $\Phi(L)$, nor in $C(L)$. We then have a separable zero-cycle $[b]$ in the $F$-variety $B$, of degree $[L:F]$. Define $Y \subset B$ as the disjoint union of $[b]$ and $C$. We claim that $Y$ has the required property. Indeed, let $f \in \mathbf{Aut}(B)(\overline{F}[\epsilon])$ be an automorphism stabilizing $Y$- or more accurately, $Y \times_F \overline{F}[\epsilon] \subset B \times_F \overline{F}[\epsilon]$. Then, $f$ permutes the two connected components of the scheme $Y \times_F \overline{F}[\epsilon]$. For dimension reasons, it preserves $C \times_F \overline{F}[\epsilon]$ on the one hand, and $[b] \times_F \overline{F}[\epsilon]$ on the other hand. From the first fact, we know that $f$ belongs to $H(\overline{F})$; in particular, it is defined over $E$, hence over $L$. From the latter fact, we get $f(b) = b$, hence $f = \mathrm{Id}$. The Lemma is proved in this case.

Assume now that $B = B_1 \times \ldots B_n$, where the $B_i$'s are $F$-simple abelian varieties. We can then adapt the preceding proof, as follows. For each $i$, let $C_i \subset B_i$, $L_i/E_i/F$ and $b_i \in B(L_i)$ be as in the first part of the proof. We can fulfill the extra requirements that no $C_i$ passes through 0, and that the $C_i$'s are of different genus (using Proposition 4.3). In particular, when $i \neq j$, $\overline{C}_i$ is not $\overline{F}$-isomorphic to $\overline{C}_j$. We can also assume that $L_i = L$ and $E_i = E$ are independent of $i$. Set

$$b := (b_1, \ldots, b_n) \in B(L).$$

Define $Y$ to be the disjoint union of $[b]$, and of the $n$ curves

$$C_i \simeq \{0\} \times \ldots \times \{0\} \times C_i \times \{0\} \times \ldots \times \{0\} \hookrightarrow B.$$

It is not hard to see, that $Y$ enjoys the required property.

In general, write $B = (\prod_1^r B_j)/\mu$, where $S_1, \ldots, S_r$ are $F$-simple abelian varieties, and where $\mu$ is a finite $F$-subgroup, intersecting trivially each coordinate axis. We can choose

$$Y \hookrightarrow B_1 \times \ldots B_n$$

as in the previous part of the proof, and such that the composite

$$Y \hookrightarrow B_1 \times \ldots B_n \xrightarrow{can} (\prod_1^r B_j)/\mu = B$$

is a closed immersion, identifying $Y$ to a smooth closed subvariety of $B$.

An automorphism of $B$ stabilizing $Y \subset B$ then lifts, via the quotient $can$, to an automorphism of $B_1 \times \ldots B_n$ stabilizing $Y \subset B_1 \times \ldots B_n$. We conclude as before. □

## 5. Proof of the implication 1) ⇒ 2).

Let $A/F$ be an abelian variety, such that $G := \mathrm{Aut}(\overline{A})$ is finite. We give a construction of a smooth projective $F$-variety $X$, such that $A = \mathbf{Aut}(X)$, in several steps.

### 5.1. Construction of $X$.

Denote by $g$ the dimension of $A$.
Let $n \geq 1$ be an integer, invertible in $F$, such that the action of $G$ on $A[n](F_s) \simeq (\mathbb{Z}/n)^{2g}$ is faithful. Such an $n$ exists: use that $G$ is finite, and that torsion points of order prime to $\mathrm{char}(F)$ in $A(\overline{F})$ are Zariski-dense in $A$.
Let $B_s$ be an abelian variety over $F_s$, of dimension $g' \geq g$, such that

$$Hom_{gp}(\overline{A}, \overline{B}) = Hom_{gp}(\overline{B}, \overline{A}) = 0.$$

Since $\overline{A}$ has a finite number of simple components (up to isogeny), which are all defined over $F_s$ by Lemma 4.4, the existence of $B_s$ follows from Lemma 4.5. For example, take for $B_s$ a product of simple abelian varieties, of dimensions greater than that of the simple components of $\overline{A}$.

Let $E/F$, be the finite Galois extension, with group $\Gamma$, which is minimal w.r.t. the following properties.

(1) The extension $E/F$ splits the $F$-group of multiplicative type $A[n]$.
In other words, $A[n](E) \simeq (\mathbb{Z}/n)^{2g}$.
(2) The abelian variety $B_s$ is defined over $E$: there exists an abelian $E$-variety $B_E$, such that $B_E \times_E F_s \simeq B_s$.
(3) Same as (1), for $B_E$: we have $B_E[n](E) \simeq (\mathbb{Z}/n)^{2g'}$.

Using (1), we view $A[n](E)$ as a $(\mathbb{Z}/n)[\Gamma]$-module.
Introduce the Weil restriction of scalars

$$B_1 := R_{E/F}(B_E).$$

Geometrically, we have $\overline{B}_1 \simeq \overline{B}_s^m$, where $m$ is the cardinality of $\Gamma$.
We have

$$B_1[n] = R_{E/F}((\mathbb{Z}/n)^{2g'}),$$

so that $E/F$ splits $B_1[n]$, and $B_1[n](E)$ is a free $(\mathbb{Z}/n)[\Gamma]$-module of rank $2g'$.

Lemma 5.1. *There exists an embedding of $(\mathbb{Z}/n\mathbb{Z})[\Gamma]$-modules*

$$A[n](E) \hookrightarrow B_1[n](E);$$

*that is to say, an embedding of finite étale $F$-group schemes*

$$\iota : A[n] \hookrightarrow B_1[n].$$

**Proof.** We give two (seemingly) different proofs.
The first one uses the perfect duality

$$(.)^\vee := \mathrm{Hom}(., \mathbb{Z}/n),$$

in the category of $(\mathbb{Z}/n)[\Gamma]$-modules. Pick a generating set $t_1, \ldots, t_{2g'}$ of the $\mathbb{Z}/n$-module $A[n](E)^\vee$- which is free of rank $2g \leq 2g'$. Introduce the surjection of $(\mathbb{Z}/n)[\Gamma]$-modules

$$(\mathbb{Z}/n)[\Gamma]^{2g'} \longrightarrow A[n](E)^\vee,$$

$$e_i \mapsto t_i,$$

where $e_i$ denotes the $i$-th element of the canonical basis. Dualizing it yields an injection of $(\mathbb{Z}/n)[\Gamma]$-modules

$$\iota : A[n](E) \longrightarrow (\mathbb{Z}/n)[\Gamma]^{2g'} \simeq B_1[n],$$

concluding the construction.
The second proof is more conceptual. Choose an embedding of constant $E$-group schemes

$$(\mathbb{Z}/n)^{2g} \simeq A_E[n] \hookrightarrow B_E[n] \simeq (\mathbb{Z}/n)^{2g'},$$

which exists simply because $g \leq g'$.
Applying $R_{E/F}$ yields an embedding of $F$-group schemes

$$R_{E/F}(A_E)[n] \hookrightarrow R_{E/F}(B_E)[n] = B_1[n].$$

Composing it with the natural embedding of $F$-groups

$$A[n] \hookrightarrow R_{E/F}(A_E)[n],$$

arising by adjunction from the identity of $A_E[n]$, we get the desired $\iota$. $\square$

Form the exact sequence of algebraic $F$-groups

$$0 \longrightarrow A[n] \overset{a \mapsto (a,\iota(a))}{\longrightarrow} A \times B_1 \overset{\pi}{\longrightarrow} B_2 \longrightarrow 0.$$

Its cokernel $B_2$ is an abelian variety over $F$.
We have $F$-embeddings

$$A \overset{a \mapsto (a,0)}{\hookrightarrow} B_2$$

and

$$B_1 \overset{b_1 \mapsto (0,b_1)}{\hookrightarrow} B_2.$$

Introduce the quotient

$$q : B_2 \longrightarrow B_3 := B_2/A \simeq B_1/\iota(A[n]).$$

Let $Y_3 \subset B_3$ be a smooth $F$-subvariety, enjoying the properties of Lemma 4.6, where we take $B$ to be our $B_3$, and set $Y_3 := Y$.
Put

$$Y_2 := q^{-1}(Y_3).$$

The restriction

$$q_{|Y_2} : Y_2 \longrightarrow Y_3$$

is an $A$-torsor.
We now define

$$X := \mathrm{Bl}_{Y_2}(B_2)$$

to be the blowup of $Y_2$ in $B_2$.

5.2. PROOF THAT $\mathbf{Aut}(X) \simeq A$.
Translating by elements of $A$ inside $B_2$ yields a natural arrow

$$A \longrightarrow \mathbf{Aut}(B_2).$$

Since $Y_2 \subset B_2$ is stable by these translations, we get an induced arrow of $F$-group schemes

$$\tau : A \longrightarrow \mathbf{Aut}(X).$$

It is clear that $\tau$ is an embedding. We are going to show that it is an isomorphism. Let us first check that it induces a bijection

$$A(\overline{F}) \xrightarrow{\sim} \mathrm{Aut}(\overline{X}) = \mathbf{Aut}(X)(\overline{F}).$$

Pick $\phi \in \mathrm{Aut}(\overline{X})$. It induces a birational isomorphism $f_2$ of the $\overline{F}$-variety $\overline{B}_2$, which is a regular isomorphism since $B_2$ is an abelian variety. Thus, we get a commutative diagram

$$\begin{tikzcd} \overline{X} \arrow[r, "\phi"] \arrow[d] & \overline{X} \arrow[d] \\ \overline{B}_2 \arrow[r, "f_2"] & \overline{B}_2, \end{tikzcd}$$

where the vertical arrows are the structure morphism of the blowup.
Using Lemma 4.1, we get $f_2(\overline{Y}_2) = \overline{Y}_2$. We know that

$$f_2(x) = g_2(x) + t_2,$$

where $g \in \mathrm{Aut}_{gp}(\overline{B}_2)$, and $t_2 \in B_2(\overline{F})$. We have to show that $g_2 = \mathrm{Id}$ and $t_2 \in A(\overline{F})$. To do so, we can assume without loss of generality that $t_2 \in B_1(\overline{F})$. We then have to prove $g_2 = \mathrm{Id}$ and

$$t_2 \in A(\overline{F}) \cap B_1(\overline{F}) = \iota(A[n])(\overline{F}).$$

Geometrically, $\overline{B}_1 \simeq \overline{B}_s^m$. Since $Hom_{gp}(\overline{A}, \overline{B}_s) = Hom_{gp}(\overline{B}_s, \overline{A}) = 0$, we get

$$Hom_{gp}(\overline{A}, \overline{B}_1) = Hom_{gp}(\overline{B}_1, \overline{A}) = 0.$$

Therefore $g_2$ leaves $\overline{A} \subset \overline{B}_2$ and $\overline{B}_1 \subset \overline{B}_2$ stable.
We infer that $g_2$ lifts, via $\overline{\pi}$, to a diagonal group automorphism

$$\delta = (h, g_1)$$

of $\overline{A} \times \overline{B}_1$, which automatically leaves the diagonally embedded $\overline{A}[n]$ stable.
Consider the automorphism of $\overline{B}_1$ given by

$$f_1(b_1) := g_1(b_1) + t_2,$$

and the diagonal automorphism of $\overline{A} \times \overline{B}_1$ given by

$$\Delta(a, b_1) := (h(a), f_1(b_1)).$$

Since $\delta$ leaves $\overline{A} \times \iota(\overline{A}[n]) \subset \overline{A} \times \overline{B}_1$ stable, there exists $f_3 \in \mathrm{Aut}(\overline{B}_3)$ such that the diagram

$$\begin{tikzcd} \overline{A} \times \overline{B}_1 \arrow[r, "\Delta"] \arrow[d, "\overline{\pi}"] & \overline{A} \times \overline{B}_1 \arrow[d, "\overline{\pi}"] \\ \overline{B}_2 \arrow[r, "f_2"] \arrow[d, "\overline{q}"] & \overline{B}_2 \arrow[d, "\overline{q}"] \\ \overline{B}_3 \arrow[r, "f_3"] & \overline{B}_3 \end{tikzcd}$$

commutes.

Because $f_2(\overline{Y}_2) = \overline{Y}_2$, we get $f_3(\overline{Y}_3) = \overline{Y}_3$. By Lemma 4.6, we conclude that $f_3 = \mathrm{Id}$. Hence, we have $t_2 \in \iota(A[n])(\overline{F})$ and $g_2 = \mathrm{Id}$. Since $\delta$ preserves the diagonally embedded $\overline{A}[n]$, we get that $h$, restricted to $\overline{A}[n] \subset \overline{A}$, is the identity. Since $G$ acts faithfully on $A[n]$, we conclude that $h = \mathrm{Id}$. Hence, $g_2 = \mathrm{Id}$ as well, and our job is done.

We have proved that $\tau$ induces a bijection on $\overline{F}$-points. If $F$ has characteristic zero, this is enough to conclude that $\tau$ is an isomorphism of algebraic $F$-groups. In general, it remains to check that the $F$-linear map on tangent spaces

$$d_e(\tau) : \mathrm{Lie}(A) \longrightarrow \mathrm{Lie}(\mathbf{Aut}(X))$$

is bijective. Recall that $\mathrm{Lie}(\mathbf{Aut}(X)$ is the space of vector fields on $X$; that is, global section of the tangent bundle $TX \longrightarrow X$. Let

$$s : X \longrightarrow TX$$

be such a section. Restricting $s$ to the complement of the exceptional divisor, we get a global section $\sigma'$ of the tangent bundle of $B_2 - Y_2$. Since $B_2$ is an abelian variety, its tangent bundle is trivial, so that $\sigma'$ is given by an arrow of $F$-varieties

$$\sigma' : B_2 - Y_2 \longrightarrow \mathbb{A}(\mathrm{Lie}(B_2)),$$

with target an affine space of dimension $\dim(B_2)$. Since $Z_2$ has codimension $\geq 2$ in $B_2$, $\sigma'$ extends to a morphism

$$\sigma : B_2 \longrightarrow \mathbb{A}(\mathrm{Lie}(B_2)),$$

which is constant because $B_2/F$ is proper. Write $\sigma = t$, with $t \in \mathrm{Lie}(B_2)$. To conclude, we have to show $t \in \mathrm{Lie}(A)$.
For $y \in B_2(\overline{F})$, denote by

$$\alpha_y : \overline{B}_2 \longrightarrow \overline{B}_2$$

the $\overline{F}$-morphism given by

$$x \mapsto x + y.$$

Recall that the linear isomorphisms

$$d_y\alpha_{-y} : T_y(\overline{B}_2) \xrightarrow{\sim} \mathrm{Lie}(B_2) \otimes_F \overline{F}$$

are used to trivialize the tangent bundle of $B_2$.
Since $\sigma$ lifts to a section of the tangent bundle of the blowup $\mathrm{Bl}_{Y_2}(B_2)$, Lemma 4.2 implies, when $y \in Y_2(\overline{F})$, that $t$ belongs to

$$d_y\alpha_{-y}(T_y(\overline{Y}_2)) \subset \mathrm{Lie}(B_2) \otimes_F \overline{F}.$$

Taking a $y$ lying above (via $\overline{q}$) an isolated separable point of $\overline{Y}_3$, we conclude that $t \in \mathrm{Lie}(A)$, as desired.

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