Lifting vector bundles to Witt vector bundles

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Abstract. Let X be a scheme. Let $r \geq 2$ be an integer. Denote by $\mathbf{W}_r(X)$ the scheme of Witt vectors of length r, built out of X. We are concerned with the question of extending (=lifting) vector bundles on X, to vector bundles on $\mathbf{W}_r(X)$—promoting a systematic use of Witt modules and Witt vector bundles. To begin with, we investigate two elementary but significant cases, in which the answer to this question is positive: line bundles, and the tautological vector bundle of a projective bundle over an affine base. We then offer a simple (re)formulation of classical results in deformation theory of smooth varieties over a field k of characteristic $p > 0$, and extend them to *reduced* k-schemes. Some of these results were recently recovered, in another form, by Stefan Schröer. As an application, we prove that the tautological vector bundle of the Grassmannian $\mathrm{Gr}_{\mathbb{F}_p}(m,n)$ does not extend to $\mathbf{W}_2(\mathrm{Gr}_{\mathbb{F}_p}(m,n))$, if $2 \leq m \leq n-2$. To conclude, we establish a connection to the work of Zdanowicz, on non-liftability of some projective bundles.

Contents

1. Introduction

Let p be a prime, and let X be a scheme, not necessarily of characteristic p. For any $r \geq 1$, denote by $\mathbf{W}_r(X)$ the scheme of p-typical Witt vectors of length r, built out of X. Let V be a vector bundle over X. This paper mainly deals with

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Question 1.1. Is V the restriction to X of a vector bundle defined over $\mathbf{W}_r(X)$?

To fix ideas, until the end of the introduction, assume that X is an $\mathbb{F}_p$-scheme. The closed immersion $X \hookrightarrow \mathbf{W}_r(X)$ can then be thought of as a universal thickening of X, of characteristic p^r. Extending V to $\mathbf{W}_r(X)$ is, in a sense made precise in Section 5, the "universal" deformation problem for V.

In order to tackle Question 1.1, we introduce in Section 3 the notions of *Witt modules* and *Witt vector bundles*. Focusing on these objects (which are not at all new) provides a novel viewpoint on some problems in deformation theory. Rather than "extending V to a vector bundle over $\mathbf{W}_r(X)$", we say "lifting V to a $\mathbf{W}_r$-bundle *over* X"—both formulations being of course equivalent. Doing so, the base scheme X remains unchanged, while considering a larger class of sheaves of modules over it. Section 3 is devoted to laying foundations for this theory.

Section 4 elaborates on a positive elementary answer to Question 1.1, for line bundles. The starting point here, is that every line bundle L admits a natural lift to a $\mathbf{W}_r$-bundle: its r-th Teichmüller lift $\mathbf{W}_r(L)$. Section 5 deals with lifting the tautological vector bundle on $Y = \mathbb{P}_X(V)$, the projective space of a vector bundle V over an affine $\mathbb{F}_p$-base X, respecting the tautological exact sequence. The answer is positive again—see Theorem 5.7.

In Section 6, we use the point of view of Witt modules to provide a unified functorial formulation of three questions in deformation theory:

(1) Let X be a scheme over a perfect field k, of characteristic p. Lift X to a flat $\mathbf{W}_2(k)$-scheme X_2 (a problem made famous by the article [DI]).
(2) Same as (1), with the strong extra constraint, that the Frobenius of X lifts to X_2 (see the nice paper [MS]).
(3) Lift a vector bundle over X to a vector bundle over $\mathbf{W}_2(X)$—the main topic of this paper.

To achieve this, we describe equivalences of categories interpreting these questions in the common framework of Witt modules, see Propositions 6.13, 6.17 and 6.26. These hold for any *reduced* X, thus providing a gain of generality with respect to classical results, which require that X be *smooth* over k.

Several applications (some classical, some new) of these equivalences are given.

- Proof that (1) has an affirmative and functorial answer, for any Frobenius-split reduced k-scheme—see Corollary 6.14 (for an introduction to Frobenius splitting, see [MR]).
- Explicit 2-extensions realizing the classical cohomological obstructions in deformation theory—see Corollaries 6.15 and 6.27.
- In Theorem 7.1, it is shown that the tautological vector bundle on the Grassmannian $\mathrm{Gr}(m, n)$ never lifts to a $\mathbf{W}_2$-bundle, if $2 \leq m \leq n-2$. Two different proofs are given.
- Finally, using the previous item, Section 8 provides a new proof, and a slight strenghtening, of a result of Zdanowicz ([Z], Theorem 6.5).

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2. Conventions

All schemes are assumed to be quasi-compact and quasi-separated.

3. Witt modules and $\mathbf{W}_r$-bundles

For each integer $r \geq 1$, and for each commutative ring A, denote by $\mathbf{W}_r(A)$ the ring of p-typical Witt vectors of A, of length r. Note that $\mathbf{W}_r$ itself might be considered as a ring scheme defined over $\mathbb{Z}$, isomorphic, as a scheme, to the r-dimensional affine space $\mathbb{A}^r$. For details on the construction of Witt vectors (adopting different viewpoints) we refer to [Bo], [Ha], [Le], and [Se1].
Fix a scheme X, not necessarily of characteristic p. Cover it by affine open subschemes $U_i = \mathrm{Spec}(A_i)$. Then, the affine schemes $\mathrm{Spec}(\mathbf{W}_r(A_i))$ glue, giving rise to $\mathbf{W}_r(X)$, the scheme of p-typical Witt vectors of length r, built out of X.

3.1. Basic definitions.

The following definition is classical, see [Se2].

Definition 3.1. *Let $r \geq 1$ be an integer. The association*

$$U \mapsto \mathbf{W}_r(\mathcal{O}_X(U)),$$

defines a sheaf of commutative rings on X. Denote it by $\mathbf{W}_r(\mathcal{O}_X)$.

As a sheaf of sets, $\mathbf{W}_r(\mathcal{O}_X)$ is represented by the r-dimensional affine space $\mathbb{A}^r_X$; it is thus also a sheaf for the fppf topology. If $1 \leq s < r$ are integers, denote by

$$\pi_{r,s} : \mathbf{W}_r(\mathcal{O}_X) \longrightarrow \mathbf{W}_s(\mathcal{O}_X)$$

the quotient, fitting into an exact sequence of sheaves of commutative rings on X

$$0 \longrightarrow \mathbf{W}_{r-s}(\mathcal{O}_X) \xrightarrow{i_{r-s,r}} \mathbf{W}_r(\mathcal{O}_X) \xrightarrow{\pi_{r,s}} \mathbf{W}_s(\mathcal{O}_X) \longrightarrow 0.$$

Definition 3.2. *Let $r \geq 1$ be an integer. The association*

$$U \mapsto \mathbf{W}_r(\mathcal{O}_X(U))^\times,$$

defines a sheaf of Abelian groups on X. It is given by (U-points of) an affine and smooth $\mathbb{Z}$-group scheme, denoted by $\mathbf{W}_r^\times$.

Note that $\mathbf{W}_1^\times = \mathbb{G}_m$. The Teichmüller representative is given by the formula

$$\begin{array}{cccc} \tau_r : & \mathbf{W}_1^\times & \longrightarrow & \mathbf{W}_r^\times, \\ & x & \longmapsto & (x, 0, 0, \dots, 0). \end{array}$$

We denote this homomorphism of $\mathbb{Z}$-groups schemes by τ, if the dependence in r is clear. It is a natural splitting of the quotient map $\mathbf{W}_r^\times \longrightarrow \mathbb{G}_m$.

Remark 3.3. Let R be an $\mathbb{F}_p$-algebra. For all $t \in R$, the following formula holds:

$$p\tau_{r+1}(t) = i_{r,r+1}(\tau_r(t^p)) \in \mathbf{W}_{r+1}(R).$$

This can be used to check that, for all $x, y \in \mathbf{W}_r(R)$,

$$i_{r,r+1}(x)i_{r,r+1}(y) = p\, i_{r,r+1}(xy),$$

where multiplication on the left (resp. right), is that of $\mathbf{W}_{r+1}(R)$ (resp. $\mathbf{W}_r(R)$).

Let $r \geq 1$ be an integer. There is an isomorphism of affine $\mathbb{Z}$-group schemes

$$\begin{aligned} \mathbb{G}_m \times (1+\mathbf{W}_r)^\times &\xrightarrow{\sim} \mathbf{W}_{r+1}^\times \\ (a, 1+x) &\longmapsto \tau_{r+1}(a)(1+x). \end{aligned}$$

(Strictly speaking, one should write $(1 + i_{r,r+1}(\mathbf{W}_r))^\times$ instead of $(1 + \mathbf{W}_r)^\times$.) This formula, as well as the next ones, are given at the level of functors of points.

As a morphism of linear algebraic $\mathbb{F}_p$-groups, the logarithm is well-defined:

$$\begin{aligned} \log : (1+\mathbf{W}_r)^\times &\longrightarrow (\mathbf{W}_r, +), \\ (1-x) &\longmapsto -x - \frac{p}{2}x^2 - \frac{p^2}{3}x^3 - \frac{p^3}{4}x^4 - \dots \end{aligned}$$

If p is odd, it is an isomorphism of linear algebraic $\mathbb{F}_p$-groups, with inverse

$$\begin{aligned} \exp : (\mathbf{W}_r, +) &\longrightarrow (1+\mathbf{W}_r)^\times, \\ t &\longmapsto 1 + t + \frac{p}{2!}t^2 + \frac{p^2}{3!}t^3 + \dots \end{aligned}$$

The sums occuring above are well-defined, since $\frac{p^i}{i}$ (resp. $\frac{p^i}{i!}$ when p is odd) is a p-adic integer, for all $i \geq 0$. Moreover, these sums are finite, because x is nilpotent.

If $p = 2$, the logarithm is an isogeny of degree two, with kernel $\{1, -1\}$. Over $\mathbb{F}_2$, the algebraic group $(1 + \mathbf{W}_r)^\times$ is then isomorphic to the middle term of the pull-back of the exact sequence

$$0 \longrightarrow \mathbf{W}_{r-1} \longrightarrow \mathbf{W}_r \longrightarrow \mathbb{G}_a \longrightarrow 0,$$

by the Lang isogeny

$$(\mathrm{frob} - \mathrm{Id}) : \mathbb{G}_a \longrightarrow \mathbb{G}_a.$$

Thus, from the point of view of linear algebraic groups, the multiplicative group scheme $\mathbf{W}_{r+1}^\times$, over $\mathbb{F}_p$, is nothing new.
However, over $\mathbb{Z}$, the morphisms log and exp above are not defined, and $\mathbf{W}_{r+1}^\times$ is much more intriguing, as the following remark shows.

Remark 3.4. Put

$$\mathbb{G}_{\mathrm{a/m}} := \mathrm{Ker}(\mathbf{W}_2^\times \longrightarrow \mathbf{W}_1^\times).$$

it is a smooth affine group scheme over $\mathbb{Z}$. Its generic fiber $\mathbb{G}_{\mathrm{a/m}} \times_{\mathbb{Z}} \mathbb{Q}$ is isomorphic to $\mathbb{G}_{\mathrm{m}}$, whereas its special fiber $\mathbb{G}_{\mathrm{a/m}} \times_{\mathbb{Z}} \mathbb{F}_p$ is isomorphic to $\mathbb{G}_{\mathrm{a}}$. More generally, consider

$$D_r := \mathrm{Ker}(\mathbf{W}_{r+2}^\times \longrightarrow \mathbf{W}_2^\times);$$

Its generic fiber is a split algebraic torus of dimension r, whereas its special fiber is isomorphic to $(\mathbf{W}_r, +)$ (also for $p = 2$). These are simple examples of deformations of (additive groups of) Witt vectors to (split) algebraic tori, in the spirit of [TO].
Let Λ be a discrete valuation ring, of mixed characteristic $(0, p)$ (e.g. $\Lambda = \mathbb{Z}_p$). Over Λ, one may attempt to twist D_r (by a suitable $\mathrm{Aut}(D_r)$-torsor), in order to obtain more interesting deformations of Witt vectors, to non-split algebraic tori. We will not follow this track here, but the interested reader is welcome to do so.

We now introduce Witt modules and $\mathbf{W}_n$-bundles, and their first properties.

Definition 3.5. *Assume that $X = \mathrm{Spec}(A)$ is affine. Let $r \geq 1$ be a positive integer. Let M be a $\mathbf{W}_r(A)$-module. The formula*

$$U \mapsto M \otimes_{\mathbf{W}_r(A)} \mathbf{W}_r(\mathcal{O}_X(U))$$

defines a presheaf (for the Zariski topology) on X. Denote by $\tilde{M}$ the associated sheaf. It is a sheaf of $\mathbf{W}_r(\mathcal{O}_X)$-modules.

Definition 3.6 (Witt module). *Let $r \geq 1$ be a positive integer. A Witt module of height r over X is a sheaf of $\mathbf{W}_r(\mathcal{O}_X)$-modules, which is locally isomorphic to a sheaf of the shape $\tilde{M}$ (cf. Definition 3.5).*

Definition 3.7 ($\mathbf{W}_r$-bundle). *Let $r, n \geq 1$ be two positive integers. A $\mathbf{W}_r$-bundle over X of rank n is a Witt module of height r, locally isomorphic to $\mathbf{W}_r(\mathcal{O}_X)^n$.*

Remark 3.8. A Witt module of height 1 over X is a quasi-coherent $\mathcal{O}_X$-module, while a a $\mathbf{W}_1$-bundle is a vector bundle over X. Witt modules of height r (resp. a $\mathbf{W}_r$-bundle) are simply modules (resp. vector bundles) over the scheme $\mathbf{W}_r(X)$.

Notation 3.9. Let V_r/X be a $\mathbf{W}_r$-bundle over X. Denote by $V_r^\vee$ the sheaf

$$\underline{\mathrm{Hom}}_{\mathbf{W}_r(\mathcal{O}_X)-\mathrm{Mod}}(V, \mathbf{W}_r(\mathcal{O}_X));$$

it is a $\mathbf{W}_r$-bundle over X.

If $s \leq r$ is a positive integer, define the $\mathbf{W}_s$-bundle

$$V_r \otimes_{\mathbf{W}_r} \mathbf{W}_s$$

as the sheaf associated to the presheaf

$$U \mapsto V(U) \otimes_{\mathbf{W}_r(\mathcal{O}_X(U))} \mathbf{W}_s(\mathcal{O}_X(U)).$$

3.2. Frobenius twists, for $\mathbb{F}_p$-schemes. Keep notation of the previous paragraph. Assume moreover, that X is an $\mathbb{F}_p$-scheme. Denote by

$$\mathrm{frob}_X : X \longrightarrow X,$$

the (absolute) Frobenius of X, given by raising functions to their p-th power. By functoriality, it induces an arrow

$$\mathbf{W}_r(X) \longrightarrow \mathbf{W}_r(X),$$

for each $r \geq 1$, which we still denote by frob_X, or just by frob.

If M is a $\mathbf{W}_r(\mathcal{O}_X)$-module, we set, for each $n \geq 1$,

$$M^{(n)} := (\mathrm{frob}^n)^*(M).$$

Considering M as a $\mathbf{W}_s(\mathcal{O}_X)$-module, for some $s > r$, does not affect the formation of $M^{(n)}$. This follows from the commutative diagram

$$\begin{tikzcd} \mathbf{W}_r(X) \arrow[r, "\mathrm{frob}^n"] \arrow[d] & \mathbf{W}_r(X) \arrow[d] \\ \mathbf{W}_s(X) \arrow[r, "\mathrm{frob}^n"] & \mathbf{W}_s(X), \end{tikzcd}$$

where the vertical arrows are the natural immersions. The notation $M^{(n)}$ thus makes sense for Witt modules.

For $s > r$, let V_s be a $\mathbf{W}_s$-bundle. Then $V_s^{(n)}$ is a $\mathbf{W}_s$-bundle as well. Moreover, there is an exact sequence of Witt modules on X

$$\mathcal{E}_r(V_s) : 0 \longrightarrow (\mathrm{frob}^r)_*(V_{s-r}^{(r)}) \longrightarrow V_s \xrightarrow{\pi_{s,r}} V_r \longrightarrow 0.$$

If $s = r+1$, and if X is regular, then $(\mathrm{frob}^r)_*(V_{s-r}^{(r)})$ is a vector bundle on X.

Recall that, in the case where $r = 1$ and L is a line bundle, $L^{(1)}$ is naturally isomorphic to $L^{\otimes p}$. This fundamental fact does not extend to vector bundles of dimension greater than 2, nor to $\mathbf{W}_2$-line bundles.

3.3. Frobenius twists, general case. Consider the morphism of schemes

$$\mathrm{Frob} : \mathbf{W}_r(X) \longrightarrow \mathbf{W}_{r+1}(X),$$

that functorially arises from the morphism of ring schemes over $\mathbb{Z}$ (cf. [DK, Definition 1.1])

$$\mathrm{Frob} : \mathbf{W}_{r+1} \longrightarrow \mathbf{W}_r.$$

Note that, for $r = 1$, it is given by the Witt polynomial

$$(x_0, x_1) \mapsto x_0^p + px_1.$$

For a $\mathbf{W}_{r+1}$-module M_{r+1} over X, $\mathrm{Frob}^*(M_{r+1})$ is a $\mathbf{W}_r$-module over X. Note in particular the length shift by (-1) in this general notion of Frobenius pull-back.
If X is an $\mathbb{F}_p$-scheme, then Frob equals the composite

$$\mathbf{W}_r(X) \xrightarrow{\mathrm{frob}} \mathbf{W}_r(X) \overset{nat}{\hookrightarrow} \mathbf{W}_{r+1}(X),$$

where *nat* is the natural nilpotent immersion. Thus, $\mathrm{Frob}^*(M_{r+1})$ depends only on $M_r := M_{r+1} \otimes_{\mathbf{W}_{r+1}} \mathbf{W}_r$, and coincides with $M_r^{(1)} = \mathrm{frob}^*(M_r)$.
To avoid confusion, we reserve the notation $(\cdot)^{(1)}$ for the Frobenius pull-back of a $\mathbf{W}_r$-bundle over an $\mathbb{F}_p$-scheme X, via $\mathrm{frob} : \mathbf{W}_r(X) \longrightarrow \mathbf{W}_r(X)$.
For $s > r$ and V_s a $\mathbf{W}_s$-bundle, $(\mathrm{Frob}^r)^*(V_s)$ is a $\mathbf{W}_{s-r}$-bundle and there is an exact sequence of Witt modules on X

$$\mathcal{E}_r(V_s) : 0 \longrightarrow (\mathrm{Frob}^r)_*(\mathrm{Frob}^r)^*(V_s) \longrightarrow V_s \xrightarrow{\pi_{s,r}} V_r \longrightarrow 0.$$

3.4. Yoga of extensions. In this and all subsequent sections, we will make constant use of the notions of extensions of Witt modules, endowed with the following classical notions: Baer sum, pullback, pushforward, change of the base. We briefly recall these, without proofs.

Definition 3.10. *(Extensions and operations on them)*
Let $\mathcal{A}$ be an Abelian category. Let

$$\mathcal{E} : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

be an exact sequence in $\mathcal{A}$, thought of as an extension of C by A.

Let $f : A \longrightarrow A'$ be a morphism in $\mathcal{A}$. Then $f_*(\mathcal{E})$ is the unique extension fitting in a commutative diagram

$$\begin{tikzcd}
\mathcal{E} : 0 \arrow[r] & A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\
f_*(\mathcal{E}) : 0 \arrow[r] & A' \arrow[r] & D \arrow[r] & C \arrow[r] & 0.
\end{tikzcd}$$

We refer to $f_*(\mathcal{E})$ as the pushforward of $\mathcal{E}$ by f.

Let $g : C' \longrightarrow C$ be a morphism in $\mathcal{A}$. Then $g^*(\mathcal{E})$ is the unique extension fitting in a commutative diagram

$$\begin{tikzcd}
g^*(\mathcal{E}) : 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d, "g"] & 0 \\
\mathcal{E} : 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0.
\end{tikzcd}$$

We refer to $g^*(\mathcal{E})$ as the pullback of $\mathcal{E}$ by g.

Let

$$\mathcal{E}' : 0 \longrightarrow A' \longrightarrow B' \longrightarrow C' \longrightarrow 0,$$

be another extension in $\mathcal{A}$. Then one can easily show that the data of a commutative diagram

$$\begin{tikzcd}
\mathcal{E} : 0 \arrow[r] & A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, "g"] & 0 \\
\mathcal{E}' : 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0,
\end{tikzcd}$$

is equivalent to an isomorphism (of extensions of C by A') $f_*(\mathcal{E}) \xrightarrow{\sim} g^*(\mathcal{E}')$.

Remark 3.11. Let

$$\mathcal{E} : 0 \longrightarrow \mathcal{O}_X \longrightarrow B \longrightarrow C \longrightarrow 0,$$

be an extension of (quasi-coherent) modules over a scheme X, of characteristic p. In the sequel, the notation $\mathrm{frob}_*(\mathcal{E})$ may refer either to the pushforward

$$\mathrm{frob}_*(\mathcal{E}) : 0 \longrightarrow \mathrm{frob}_*(\mathcal{O}_X) \longrightarrow \tilde{B} \longrightarrow C \longrightarrow 0,$$

in the sense of the definition above, or to the extension

$$\mathrm{frob}_*(\mathcal{E}) : 0 \longrightarrow \mathrm{frob}_*(\mathcal{O}_X) \longrightarrow \mathrm{frob}_*(B) \longrightarrow \mathrm{frob}_*(C) \longrightarrow 0,$$

obtained by applying the exact functor (on $\mathcal{O}_X$-modules) frob_*. This ambiguity is nothing serious: the meaning of $\mathrm{frob}_*(\mathcal{E})$ is always clear from the context. The same goes for the notation $\mathrm{frob}^*(\mathcal{E})$.

3.5. The arrow κ. Assume given an extension of Abelian groups

$$0 \longrightarrow A \longrightarrow B \xrightarrow{\pi} C \longrightarrow 0,$$

where A and C are killed by p (hence B is killed by p^2). By assumption, for any $c \in C$, setting

$$\kappa(c) := pb,$$

where b is such that $\pi(b) = c$ gives rise to a well-defined morphism

$$\kappa : C \longrightarrow A.$$

Furthermore, κ is trivial if and only if $pB = 0$, and it is an isomorphism if and only if B is a lift of the $\mathbb{F}_p$-vector space C to a free $(\mathbb{Z}/p^2\mathbb{Z})$-module.

One can clearly "sheafify" the construction of the arrow κ in the following way.
Let $\mathcal{S}$ be a site, and let

$$\mathcal{E} : 0 \longrightarrow A \longrightarrow B \xrightarrow{\pi} C \longrightarrow 0,$$

be an exact sequence of sheaves of Abelian groups on $\mathcal{S}$. Assume given an endomorphism $f \in \mathrm{Hom}_{\mathcal{S}}(B, B)$, leaving A stable. It thus induces an endomorphism of C, still denoted by f. Assuming that $f(A) = f(C) = 0$, define a homomorphism

$$\kappa = \kappa_{\mathcal{E},f} \in \mathrm{Hom}_{\mathcal{S}}(C, A),$$

through the following process. Let Y be an object of $\mathcal{S}$. For any section $s \in C(Y)$, pick a covering (Y_i) of Y such that, for each i, the restriction $s_i = s_{|Y_i}$ lifts to $t_i \in B(Y_i)$. Then $u_i := f(t_i)$ belongs to $A(Y_i)$. The u_i hence glue to give an element in $A(Y)$, depending only on s. Denote it by $\kappa(s)$. Note that $\kappa = 0$ if and only if $f = 0$.

Now if $\mathcal{S}$ is either the Zariski, étale or fppf site, big or small, of a scheme X of characteristic p, consider the exact sequence of Witt modules over X

$$\mathcal{E} : 0 \longrightarrow \mathrm{Frob}_*(\mathcal{O}_X) \longrightarrow \mathbf{W}_2(\mathcal{O}_X) \xrightarrow{\pi} \mathcal{O}_X \longrightarrow 0.$$

and take f to be multiplication by p. Then, the homomorphism $\kappa_{\mathcal{E},p}$ is given by the Frobenius

$$\mathrm{Frob}: \mathcal{O}_X \xrightarrow{x \mapsto x^p} \mathrm{Frob}_*(\mathcal{O}_X).$$

Now assume that X is defined over a perfect field k and that X/k admits a lift X_2, flat over $\mathbf{W}_2(k)$. Recall that a $\mathbf{W}_2(k)$-module is flat, if and only if it is free. Consider the exact sequence

$$0 \longrightarrow p\mathcal{O}_{X_2} \longrightarrow \mathcal{O}_{X_2} \xrightarrow{can} \mathcal{O}_X \longrightarrow 0,$$

whose kernel is a square-zero ideal of $\mathcal{O}_{X_2}$. If f is multiplication by p, then κ is an isomorphism.

3.6. Lifting Witt vector bundles.

In this section, X is an arbitrary scheme.

Definition 3.12. *Let $r \geq 1$ be an integer. Let V_r/X be a $\mathbf{W}_r$-bundle.*

- *Let $s > r$ be an integer. A lifting of V_r to a $\mathbf{W}_s$-bundle is the data of a $\mathbf{W}_s$-bundle V_s on X, together with an isomorphism (of $\mathbf{W}_r$-bundles)*

$$f_r : V_s \otimes_{\mathbf{W}_s} \mathbf{W}_r \xrightarrow{\sim} V_r.$$

  If X is an $\mathbb{F}_p$-scheme, one also says that V_s is a lift of V_r to p^s-torsion.

- *A complete lifting of V_r is the data, for each $s > r$, of a $\mathbf{W}_s$-bundle V_s on X, together with isomorphisms (of $\mathbf{W}_s$-bundles)*

$$f_s : V_{s+1} \otimes_{\mathbf{W}_{s+1}} \mathbf{W}_s \xrightarrow{\sim} V_s.$$

The notion of a complete lifting of V_r is the same as that of a lift of V_r to a $\mathbf{W}_\infty$-bundle, where $\mathbf{W}_\infty = \varprojlim \mathbf{W}_r$. In the case where $r = 1$ and $X = \mathrm{Spec}(\mathbb{F}_p)$, a complete lifting for V_1 is then the data of a free $\mathbb{Z}_p$-module of finite rank V_∞, together with an isomorphism of $\mathbb{F}_p$-vector spaces $V_\infty/p \xrightarrow{\sim} V_1$.

Remark 3.13. The existence of (unrelated) lifts of V_r to p^s-torsion, for each $s > r$, need not imply the existence of a complete lifting of V_r.

Remark 3.14. Assume that X is an $\mathbb{F}_p$-scheme. Liftability of a $\mathbf{W}_r$-bundle V_r to a $\mathbf{W}_{r+1}$-bundle is then equivalent to the vanishing of a natural class in the coherent cohomology group $H^2(X, \mathrm{End}(V_1)^{(r)})$. More generally, assume that $r \leq s \leq 2r$. Then, lifting V_r to p^s-torsion is obstructed by a class in $H^2(X, \mathrm{End}(V_{s-r})^{(r)})$, where $V_{s-r} := V_r \otimes_{\mathbf{W}_r} \mathbf{W}_{s-r}$. One checks this using the exact sequence $\mathcal{E}_r(V_s)$ of Section 3.2.

Remark 3.15. In general, liftability of a vector bundle on X to a $\mathbf{W}_2$-bundle, is a highly non-abelian problem.

4. The Teichmüller representative lifts line bundles

The Teichmüller representative yields natural lifts for line bundles. This construction is detailed below and works over an arbitrary scheme X. However, some results are specific to $\mathbb{F}_p$-schemes—the case of interest for applications later on.

4.1. Lifts of line bundles.

Definition 4.1. *Let L be a line bundle over an arbitrary scheme X. Denote by*

$$P := \underline{\mathrm{Isom}}(L, \mathcal{O}_X) = \mathbb{A}(L) - \{0\},$$

the corresponding $\mathbb{G}_m$-torsor over X. Let $r \geq 1$ be an integer. Then

$$P_r := (\tau_r)_*(P),$$

is a $\mathbf{W}_r^\times$-torsor over X.

Twisting the trivial invertible $\mathbf{W}_r$-module $\mathbf{W}_r(\mathcal{O}_X)$ by P_r yields an invertible $\mathbf{W}_r$-module. We denote it by $\mathbf{W}_r(L)$. It is the r-th Teichmüller lift of L.

Remark 4.2. Note that, for a group scheme G, twisting G-schemes by G-torsors usually requires extra assumptions—e.g. quasiprojectivity. Here, for $G = \mathbf{W}_r^\times$, no assumption is needed since line bundles are locally trivial for the Zariski topology.

Remark 4.3. There is a natural isomorphism of $\mathbf{W}_r$-line bundles

$$\mathrm{Frob}^*(\mathbf{W}_{r+1}(L)) = \mathbf{W}_r(L^{\otimes p}),$$

arising functorially from the fact that the composite

$$\mathbf{W}_1^\times \xrightarrow{\tau_{r+1}} \mathbf{W}_{r+1}^\times \xrightarrow{\mathrm{Frob}} \mathbf{W}_r^\times,$$

coincides with

$$\mathbf{W}_1^\times \xrightarrow{x \mapsto x^p} \mathbf{W}_1^\times \xrightarrow{\tau_r} \mathbf{W}_r^\times.$$

The sequence $(\mathbf{W}_r(L))_{r\geq 1}$, together with the natural isomorphisms

$$\mathbf{W}_{r+1}(L) \otimes_{\mathbf{W}_{r+1}} \mathbf{W}_r \simeq \mathbf{W}_r(L),$$

form a natural complete lifting for L, which is functorial in the following sense.

Pick integers $s > r \geq 1$. Consider the exact sequence of Witt modules on X

$$0 \longrightarrow (\mathrm{Frob}^r)_*(\mathbf{W}_{s-r}(\mathcal{O}_X)) \xrightarrow{i = i_{s-r,s}} \mathbf{W}_s(\mathcal{O}_X) \xrightarrow{\pi = \pi_{s,r}} \mathbf{W}_r(\mathcal{O}_X) \longrightarrow 0.$$

Written as such, the arrows i and π are $\mathbb{G}_m$-equivariant, where $\mathbb{G}_m$ acts on $\mathbf{W}_r(\mathcal{O}_X)$ via the multiplicative section τ_r.

Twisting by the $\mathbb{G}_m$-torsor P associated to a line bundle L over X, one gets an exact sequence of Witt modules on X

$$0 \longrightarrow (\mathrm{Frob})^r_*(\mathbf{W}_{s-r}(L^{\otimes p^r})) \longrightarrow \mathbf{W}_s(L) \xrightarrow{\pi_L = \pi_{L,s,r}} \mathbf{W}_r(L) \longrightarrow 0,$$

which coincides with the sequence $\mathcal{E}_r(\mathbf{W}_s(L))$ from the end of Section 3.3.

We describe now the space of lifts of a line bundle to a $\mathbf{W}_2$-line bundle, over an $\mathbb{F}_p$-base.

Proposition 4.4. *Assume that X is an $\mathbb{F}_p$-scheme. Let X be a scheme and L_1 be a line bundle over X. Lifts of L_1 to a $\mathbf{W}_2$-bundle L_2 over X are in natural bijection with $\mathbb{G}_a$-torsors over X. Hence, the set of isomorphism classes of such L_2's is in natural bijection with $H^1(X, \mathcal{O}_X)$.*

Proof. Let L_2 be a lift of L_1, to a $\mathbf{W}_2$-bundle over X. It fits into the extension of $\mathbf{W}_2$-modules over X

$$\mathcal{E}_1(L_2) : 0 \longrightarrow \mathrm{frob}_*(L_1^{\otimes p}) = \mathrm{frob}_*(L_1^{(1)}) \longrightarrow L_2 \longrightarrow L_1 \longrightarrow 0.$$

In particular, taking $L_2 = \mathbf{W}_2(L_1)$ yields

$$\mathcal{E}_1(\mathbf{W}_2(L_1)) : 0 \longrightarrow \mathrm{frob}_*(L_1^{\otimes p}) \longrightarrow \mathbf{W}_2(L_1) \longrightarrow L_1 \longrightarrow 0.$$

The κ arrow (Section 3.5) of both extensions is frob_{L_1}. Form the Baer difference

$$\Delta(L_2) := \mathcal{E}_1(L_2) - \mathcal{E}_1(\mathbf{W}_2(L_1)) : 0 \longrightarrow \mathrm{frob}_*(L_1^{\otimes p}) \longrightarrow D(L_2) \longrightarrow L_1 \longrightarrow 0.$$

Since the arrow κ commutes with Baer sum, it is trivial for $\Delta(L_2)$. Its middle term $D(L_2)$ is thus a quasi-coherent $\mathcal{O}_X$-module, so that $\Delta(L_2)$ is actually an extension of quasi-coherent $\mathcal{O}_X$-modules. Using the adjunction between frob_* and frob^*, $\Delta(L_2)$ gives rise to an extension of vector bundles over X

$$\epsilon(L_2) : 0 \longrightarrow L_1^{\otimes p} \longrightarrow E(L_2) \longrightarrow L_1^{\otimes p} \longrightarrow 0.$$

Applying $\cdot \otimes L_1^{\otimes -p}$ yields an extension of vector bundles over X

$$\alpha(L_2) := \epsilon(L_2) \otimes L_1^{\otimes -p} : 0 \longrightarrow \mathcal{O}_X \longrightarrow A(L_2) \longrightarrow \mathcal{O}_X \longrightarrow 0;$$

that is to say, a $\mathbb{G}_a$-torsor over X.

This construction is reversible, providing an equivalence between liftings of L_1 to a $\mathbf{W}_2$-line bundle L_2, and $\mathbb{G}_a$-torsors over X, which are classified by $H^1(X, \mathcal{O}_X)$. $\square$

Remark 4.5. Alternatively, one can prove Proposition 4.4 in the following way. Recall that $H^1(S, \mathbf{W}_r^\times)$ classifies $\mathbf{W}_r$-line bundles over S. Consider the natural isomorphism of $\mathbb{F}_p$-group schemes

$$\mathbb{G}_a \times \mathbb{G}_m \xrightarrow{\sim} \mathbf{W}_2^\times,$$

and apply $H^1(S, .)$, to get an isomorphism

$$H^1(S, \mathcal{O}_S) \oplus H^1(S, \mathbb{G}_m) \xrightarrow{\sim} H^1(S, \mathbf{W}_2^\times).$$

This point of view generalizes to arbitrary schemes as follows. Over $\mathbb{Z}$, consider the affine group scheme $\mathbb{G}_{a/m}$ defined in Remark 3.4 Let L_1 be a line bundle over an arbitrary scheme X. Then one can prove that isomorphism classes of lifts of L_1 to a $\mathbf{W}_2$-line bundle L_2 are in natural bijection with $H^1_{\mathrm{Zar}}(X, \mathbb{G}_{a/m})$.

4.2. The Teichmüller section for line bundles. Keep notation of the preceding paragraph. The surjection $\pi : \mathbf{W}_s \longrightarrow \mathbf{W}_r$ for $s > r$ has a natural multiplicative scheme-theoretic section,

$$\begin{aligned} \tau (= \tau_{r,s}) : \mathbf{W}_r &\longrightarrow \mathbf{W}_s, \\ (x_1, \ldots, x_r) &\longmapsto (x_1, \ldots, x_r, 0, \ldots, 0). \end{aligned}$$

This section is $\mathbb{G}_m$-equivariant, for the $\mathbb{G}_m$-action on $\mathbf{W}_r$ (resp. on $\mathbf{W}_s$) given by $\tau_{1,r}$ (resp. $\tau_{1,s}$). Twisting by the torsor P associated to L then yields a natural, scheme-theoretic section of π_L, its Teichmüller section, denoted by

$$\tau_L : \mathbf{W}_r(L) \longrightarrow \mathbf{W}_s(L).$$

Note that τ_L is of course not additive. Taking global sections, one gets exactness of the sequence of abelian groups

$$0 \longrightarrow H^0(X, (\mathrm{Frob}^r)^*(\mathbf{W}_s(L))) \longrightarrow H^0(X, \mathbf{W}_s(L)) \longrightarrow H^0(X, \mathbf{W}_r(L)) \longrightarrow 0.$$

Furthermore, assume that G is a finite group acting on X by scheme automorphisms, and semi-linearly on L; in other words, that L is a G-linearized line bundle. Then, the natural map on G-invariant sections

$$H^0(G, H^0(X, \mathbf{W}_s(L))) \longrightarrow H^0(G, H^0(X, \mathbf{W}_r(L)))$$

is onto as well. Indeed, the Teichmüller section τ_L is then G-equivariant.

5. Lifting the tautological vector bundle of a projective space

5.1. Two facts about lifting vector bundles. The following Lemma asserts that being liftable is invariant under twists.

Lemma 5.1. *Let X be a scheme, let V be a vector bundle on X, and let L be a line bundle on X. Let $r \geq 2$ be an integer. Then, V lifts to a $\mathbf{W}_r$-bundle if and only if $V \otimes L$ does.*

Proof. Let V_r be a lift of $V = V_1$ to a $\mathbf{W}_r$-bundle over X. By Section 4, the line bundle L admits the lift $\mathbf{W}_r(L)$. Then, $V_r \otimes \mathbf{W}_r(L)$ is a lift of $V \otimes L$. $\square$

In some sense, liftability of vector bundles to Witt vector bundles is "the hardest" existence problem in the deformation theory of vector bundles. In characteristic p, this is made precise in Lemma 5.5 below.

Definition 5.2. *Let X be a scheme and let $i : X \hookrightarrow Y$ be a closed immersion.*

We say that i is a p-elementary thickening of X, if the sheaf of ideals $I \subset \mathcal{O}_Y$ defining X satisfies $I^p = pI = 0$.

Let $r \geq 1$ be an integer. We say that i is a p-thickening of depth r of X if it can be written as a composite

$$s : X = X_0 \hookrightarrow X_1 \hookrightarrow \ldots \hookrightarrow X_{r-1} \hookrightarrow X_r = Y,$$

of r p-elementary thickenings.

Example 5.3. If X is a smooth variety over a perfect field k of characteristic p, and if X_{r+1} is a lift of X to a scheme X_{r+1}, flat over $\mathbf{W}_{r+1}(k)$, then the closed immersion $X \hookrightarrow X_{r+1}$ is a p-thickening of depth r.

Lemma 5.4. *Let X be an arbitrary scheme and let $i : X \hookrightarrow Y$ be a closed immersion. Then i is a p-thickening of depth r of X, if and only if the sheaf of ideals $I \subset \mathcal{O}_Y$ defining X satisfies $p^{r-i} I^{p^i} = 0$, for every $i = 0, \ldots, r$.*

Proof. Assume that I satisfies the conditions of the Lemma. For $j = 0, \ldots, r$, define the sheaf of ideals

$$J_j := (I^{p^j} + pI^{p^{j-1}} + p^2 I^{p^{j-2}} + \ldots + p^j I) \subset \mathcal{O}_Y.$$

Let $Z_j \hookrightarrow Y$ be the closed subscheme defined by J_j. It is straightforward to check that

$$(J_j^p + pJ_j) \subset J_{j+1} \subset J_j,$$

just using $p \geq 2$ (that p is prime is not needed here). Hence, the closed immersion $Z_j \hookrightarrow Z_{j+1}$ is a p-elementary thickening. To conclude, observe that $Z_0 = X$ and $Z_r = Y$. The converse implication is straightforward as well. $\square$

Lemma 5.5. *Assume that X is an $\mathbb{F}_p$-scheme. Let $r \geq 1$ be an integer, and let $i : X \hookrightarrow Y$ be a p-thickening of X, of depth r. Let V be a vector bundle on X. Assume that V admits a lift to a $\mathbf{W}_{r+1}$-bundle on X. Then, $(\mathrm{frob}^r)^*(V)$ extends (via i) to a vector bundle on Y.*

Proof. Denote by

$$f : X \longrightarrow \mathbf{W}_{r+1}(X),$$

the natural thickening. There is a natural morphism

$$F_r : Y \longrightarrow \mathbf{W}_{r+1}(X),$$

such that

$$F_r \circ i = f \circ \mathrm{frob}^r.$$

This fact is classical, but we could not find a suitable reference in the literature. Laurent Fargues kindly suggested a quick proof, which we now provide.

By glueing, it is straightforward to reduce to the affine case $X = \mathrm{Spec}(A)$. Let $I \subset A$ be the ideal defining X. By Lemma 5.4, it satisfies, for every $i = 0, \dots, r$,

$$p^{r-i} I^{p^i} = 0. \tag{1}$$

Consider the natural ring homomorphism given by the Witt polynomial

$$\begin{aligned} \Phi_r : \mathbf{W}_{r+1}(A) &\longrightarrow A, \\ x := (x_0, \dots, x_r) &\longmapsto x_0^{p^r} + p x_1^{p^{r-1}} + p^2 x_2^{p^{r-2}} + \dots + p^r x_r. \end{aligned}$$

By (1), Φ_r vanishes when all x_i belong to I. Equivalently, it vanishes when x belongs to the kernel of the natural exact sequence of abelian groups

$$0 \longrightarrow \mathbf{W}_{r+1}(I) \longrightarrow \mathbf{W}_{r+1}(A) \xrightarrow{\pi} \mathbf{W}_{r+1}(A/I) \longrightarrow 0.$$

Thus, Φ_r factors via π, to a ring homomorphism

$$\phi_r : \mathbf{W}_{r+1}(A/I) \longrightarrow A,$$

proving the claim (set $F_r := \mathrm{Spec}(\phi_r)$).

Via F_r^*, the existence of an extension of V to (a vector bundle over) $\mathbf{W}_{r+1}(X)$ then implies that of an extension of $(\mathrm{frob}^r)^*(V)$ to Y. $\square$

From now on, and until the end of this section, X is a scheme of characteristic $p > 0$, and V/X is a vector bundle, of constant rank $n \geq 1$. Denote by

$$f : \mathbb{P}(V) \longrightarrow X$$

the associated projective bundle, parametrizing *quotient* line bundles of V.

Definition 5.6. *Denote by $\mathcal{H}_V$, or simply by $\mathcal{H}$ if the dependence in V is clear, the tautological vector bundle on $\mathbb{P}(V)$.*
There is the tautological exact sequence of vector bundles on $\mathbb{P}(V)$

$$\mathcal{T}(= \mathcal{T}_V) : 0 \longrightarrow \mathcal{H} \longrightarrow f^*(V) \longrightarrow \mathcal{O}(1) \longrightarrow 0.$$

Theorem 5.7. *Assume that the $\mathbb{F}_p$-scheme X is affine. Then V lifts completely. Moreover, if we choose a complete lifting $(V_r)_{r\geq 1}$ of V, over X, then there exists a complete lifting $(\mathcal{H}_r)_{r\geq 1}$ of $\mathcal{H}$, over $\mathbb{P}(V)$, such that each $\mathcal{H}_r$ fits into an exact sequence of $\mathbf{W}_r$-bundles on $\mathbb{P}(V)$*

$$\mathcal{T}_r : 0 \longrightarrow \mathcal{H}_r \longrightarrow f^*(V_r) \longrightarrow \mathbf{W}_r(\mathcal{O}(1)) \longrightarrow 0,$$

and such that these exact sequences are compatible. In other terms, for each $r \geq 1$, there is a commutative diagram

$$\begin{tikzcd}
\mathcal{T}_{r+1} : 0 \arrow[r] & \mathcal{H}_{r+1} \arrow[r] \arrow[d] & f^*(V_{r+1}) \arrow[r, "\rho_{r+1}"] \arrow[d] & \mathbf{W}_{r+1}(\mathcal{O}(1)) \arrow[r] \arrow[d, "\pi_{r+1,r}"] & 0 \\
\mathcal{T}_r : 0 \arrow[r] & \mathcal{H}_r \arrow[r] & f^*(V_r) \arrow[r, "\rho_r"] & \mathbf{W}_r(\mathcal{O}(1)) \arrow[r] & 0,
\end{tikzcd}$$

where the middle vertical arrow is the natural one.

Proof. Assume that V_r is a given lift of V, to a $\mathbf{W}_r$-bundle. The obstruction to lifting V_r to a $\mathbf{W}_{r+1}$-bundle V_{r+1} lies in

$$\mathrm{Ext}^2_{\mathcal{O}_X-\mathrm{Mod}}(V, (\mathrm{frob}^r)_*(V^{(r)})) = H^2(X, \mathrm{End}(V)^{(r)}).$$

This cohomology group vanishes since X is affine, whence the first claim.

Let us prove the second claim: assume that $\mathcal{H}_r$, together with the extension

$$\mathcal{T}_r : 0 \longrightarrow \mathcal{H}_r \longrightarrow f^*(V_r) \xrightarrow{\rho_r} \mathbf{W}_r(\mathcal{O}(1)) \longrightarrow 0,$$

has been constructed. There is a natural duality isomorphism

$$\underline{\mathrm{Hom}}_{\mathbf{W}_r(\mathcal{O}_X)-\mathrm{Mod}}(f^*(V_r), \mathbf{W}_r(\mathcal{O}(1))) = f^*(V_r)^\vee(1) \xrightarrow{\sim} f^*(V_r^\vee)(1).$$

The surjection ρ_r thus corresponds to a global section

$$s_r \in H^0(\mathbb{P}(V), f^*(V_r^\vee)(1)).$$

One would like to lift it, through the epimorphism of the exact sequence (on $\mathbb{P}(V)$)

$$0 \longrightarrow \mathrm{frob}_*^r(f^*(V^{(r)\vee}))(1) \longrightarrow f^*(V_{r+1}^\vee)(1) \longrightarrow f^*(V_r^\vee)(1) \longrightarrow 0.$$

The obstruction to do so is a class

$$c \in H^1(\mathbb{P}(V), f^*(V^{\vee(r)})(p^r)) = 0.$$

(To get this vanishing result, use $H^1_{\mathrm{Zar}}(\mathbb{P}(V), \mathcal{O}(p^r)) = 0$, together with the projection formula.)
Hence, s_r can be lifted to a global section

$$s_{r+1} \in H^0(\mathbb{P}(V), f^*(V_{r+1}^\vee)(1)).$$

Dualizing, it corresponds to a homomorphism

$$\rho_{r+1} : f^*(V_{r+1}) \longrightarrow \mathbf{W}_{r+1}(\mathcal{O}(1)),$$

lifting ρ_r. Since ρ_r is surjective, ρ_{r+1} is surjective as well (using Nakayama's Lemma). Define the $\mathbf{W}_{r+1}$-bundle $\mathcal{H}_{r+1}$ to be its kernel. Existence of the required commutative diagram is automatic. The result is proved. □

6. (Re)visiting equivalences of categories in deformation theory

Let k be a perfect field of characteristic p and X be a k-scheme. Recall that frob denotes the (absolute) Frobenius of X. We present a functorial description of three problems in deformation theory, using the point of view of Witt modules:

(1) Lift X to a scheme X_2, flat over $\mathbf{W}_2(k)$.

(2) Lift X to a scheme X_2, flat over $\mathbf{W}_2(k)$, *together with its Frobenius morphism.*

(3) Lift a given vector bundle V/X to a $\mathbf{W}_2$-bundle V_2.

In (2), one requires the existence of an endomorphism F_2 of X_2, whose mod p reduction is the Frobenius of X. This is a very strong extra requirement. Since k is perfect, such an F_2 is automatically compatible with the Frobenius of $\mathbf{W}_2(k)$.

Problems (1) and (2) have been the subject of sustained investigation by many authors—see, for instance, the seminal papers [DI] and [MS]. Most related results below are known. Nevertheless, our approach, via the extension $\mathcal{E}\mathbf{W}_2(X)$, is new.

Problem (3) is relatively new.

6.1. Recollections on deformation theory. Let us first recall some well-known concepts and facts from deformation theory. For details, see [Il].

Definition 6.1. *Let X be a scheme over a base B, and let $\mathcal{M}$ be a coherent $\mathcal{O}_X$-module. A square-zero extension of X by $\mathcal{M}$ (in the category of B-schemes) is the data of a B-scheme Y, together with a closed embedding*

$$i : X \longrightarrow Y,$$

defined by an Ideal $\mathcal{I} \subset \mathcal{O}_Y$ of square zero, equipped with an isomorphism of $\mathcal{O}_X$-modules

$$f : \mathcal{I} \xrightarrow{\sim} \mathcal{M}.$$

Square-zero extensions of X by $\mathcal{M}$ (in the category of B-schemes) form a category (where morphisms are isomorphisms), which we denote by $\mathbf{Exal_B}(X, \mathcal{M})$.

Remark 6.2. The previous definition implicitly uses the fact that the $\mathcal{O}_Y$-module $\mathcal{I}$ is actually an $\mathcal{O}_X$-module.

In short, a square-zero extension can be thought of as an extension

$$0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{O}_Y \xrightarrow{\pi} \mathcal{O}_X \longrightarrow 0,$$

in which $\mathcal{M}$ is a square-zero ideal, and π is a homomorphism of $\mathcal{O}_B$-algebras.

Definition 6.3. *Let X be a scheme, and let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{O}_X$-modules. Denote by $\mathbf{Ext}^1_{\mathcal{O}_X}(\mathcal{N}, \mathcal{M})$ the category whose objects are extensions of $\mathcal{O}_X$-modules*

$$0 \longrightarrow \mathcal{M} \longrightarrow E \longrightarrow \mathcal{N} \longrightarrow 0,$$

morphisms being morphisms of exact sequences, which are identity on $\mathcal{M}$ on $\mathcal{N}$.

Remark 6.4. Morphisms in $\mathbf{Ext}^1_{\mathcal{O}_X}(\mathcal{N}, \mathcal{M})$ are isomorphisms, and the group of automorphisms of any object is naturally isomorphic to $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{N}, \mathcal{M})$.

Proposition 6.5 ([Il]). *Let X be a smooth k-variety, $\mathcal{M}$ be an $\mathcal{O}_X$-module and $\Omega^1_{X/k}$ be the sheaf of Kähler differentials of X over k.*
There is an equivalence of categories

$$\mathbf{Exal}_k(X, \mathcal{M}) \xrightarrow{\sim} \mathbf{Ext}^1_{\mathcal{O}_X}(\Omega^1_{X/k}, \mathcal{M})$$

that is compatible with Baer sum.

6.2. An equivalence of categories for Problem (1).
To begin with, assume that X/k is smooth. One can then replace "flat" by "smooth" in (1). By deformation theory, there exists a natural class

$$\mathrm{Obs}(X_2) \in \mathrm{Ext}^2_{\mathcal{O}_X-\mathrm{Mod}}(\Omega^1_{X/k}, \mathcal{O}_X),$$

which vanishes if, and only if, Problem (1) has a positive answer.

In what follows, we give a simple functorial interpretation of this class. Note that, in Proposition 2.2 of the recent article [Yo], a similar goal is achieved. The approach chosen there is quite different from ours: a main input in its proof is Proposition 1 of the Appendix of [MS], which is concerned with our Problem (2).

From now on, we remove the smoothness assumption on X/k.

Lemma 6.6. *Let X be a k-scheme. The data of a lifting of X/k to a scheme X_2, flat over $\mathbf{W}_2(k)$, is equivalent to that of a square-zero extension (of schemes over $\mathbf{W}_2(k)$)*

$$(\mathcal{E} : 0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_X \longrightarrow 0) \in \mathbf{Exal}_{\mathbf{W}_2(k)}(X, \mathcal{O}_X),$$

such that $\kappa_{\mathcal{E},p} = \mathrm{Id}$ (cf. Section 3.5 for the definition of κ).

Proof. The statement follows immediately from the following observation. If M is a $\mathbf{W}_2(k)$-module, then it is flat if and only if it is free. Equivalently, for the exact sequence

$$\mathcal{M} : 0 \longrightarrow pM \longrightarrow M \longrightarrow M/pM \longrightarrow 0,$$

the connecting arrow $\kappa_{\mathcal{M},p} : M/pM \longrightarrow pM$ is an isomorphism. □

Remark 6.7. In Lemma 6.6, it is crucial to work in the category of $\mathbf{W}_2(k)$-schemes, and hence consider square-zero extensions in $\mathbf{Exal}_{\mathbf{W}_2(k)}(X, \mathcal{O}_X)$.

DEFINITION 6.8. *Using the description of the previous Lemma, the liftings of X/k to a scheme flat over $\mathbf{W}_2(k)$ form a category (with morphisms being isomorphisms). It is the full subcategory of $\mathbf{Exal}_{\mathbf{W}_2(k)}(X, \mathcal{O}_X)$ consisting of extensions having $\kappa = \mathrm{Id}$. Denote it by $\mathcal{L}_{\mathbf{W}_2(k)}(X)$, or simply by $\mathcal{L}_2(X)$.*

The notation $\mathcal{L}_2(X)$ is unambiguous, thanks to the following Lemma.

LEMMA 6.9. *Let X be a k-scheme. Then, the forgetful functor*

$$\mathcal{L}_{\mathbf{W}_2(k)}(X) \longrightarrow \mathcal{L}_{\mathbb{Z}/p^2\mathbb{Z}}(X)$$

is an equivalence of categories.

Proof. It is enough to deal with the affine case $X = \mathrm{Spec}(A)$. Let A_2 be a lift of A to a flat (i.e. free) $(\mathbb{Z}/p^2\mathbb{Z})$-algebra. There is a natural isomorphism of k-vector spaces

$$\begin{aligned} A_2/pA_2 = A &\xrightarrow{\sim} pA_2, \\ \overline{x} &\mapsto px. \end{aligned}$$

Since k is perfect, A_2 can be turned into a $\mathbf{W}_2(k)$-algebra in a unique way, by the classical formula

$$\begin{aligned} \mathbf{W}_2(k) &\longrightarrow A_2 \\ \tau(x^p) &\mapsto \tilde{x}^p, \end{aligned}$$

where $\tilde{x} \in A_2$ is any lift of $x \in k \subset A$. Furthermore, let $(E_i)_{i\in I} \in A_2^I$ be a family, such that $(e_i := \overline{E_i})_{i\in I}$ is a k-basis of A. Let us check that $(E_i)_{i\in I}$ is a basis of the $\mathbf{W}_2(k)$-module A_2. That $(E_i)_{i\in I}$ is generating, is a straightforward two-step dévissage. This is actually (and more generally) a variant of Nakayama's Lemma, for possibly non-finitely generated modules over Artinian local rings (here $\mathbf{W}_2(k)$). Suppose now there is a relation

$$\sum_{i\in I} \tilde{x}_i E_i = 0,$$

where $\tilde{x}_i \in A_2$ is non-zero for finitely many $i \in I$. Since (e_i) is a k-basis of A and k is perfect, one may write $\tilde{x}_i = p\tau(x_i)$ for $x_i \in k$. From the relation

$$p \sum \tau(x_i) E_i = 0 \in A_2,$$

and via the isomorphism $pA_2 \xrightarrow{\sim} A$, one gets

$$\sum x_i e_i = 0 \in A,$$

hence $x_i = 0$, implying that $\tilde{x}_i = 0$. Thus, A_2 is a free $\mathbf{W}_2(k)$-module.
By uniqueness of the $\mathbf{W}_2(k)$-algebra structure, a lift of a morphism of k-algebras $A \longrightarrow A'$ to a morphism of flat $(\mathbb{Z}/p^2\mathbb{Z})$-algebras $A_2 \longrightarrow A_2'$, is automatically $\mathbf{W}_2(k)$-linear. The Lemma is proved. □

Given this last result, the reader may thus do one of the following.

(1) Bluntly assume that $k = \mathbb{F}_p$ everywhere;

(2) Go on with an arbitrary perfect field k, but not worry about checking $\mathbf{W}_2(k)$-linearity of homomorphisms.

From now on, assume that X is a *reduced* k-scheme. The next definition is the key prerequisite for our equivalence of categories.

Recall that, for the natural exact sequence

$$\mathcal{E}\mathbf{W}_2(X) : 0 \longrightarrow \mathrm{frob}_*(\mathcal{O}_X) \longrightarrow \mathbf{W}_2(\mathcal{O}_X) \longrightarrow \mathcal{O}_X \longrightarrow 0,$$

one has

$$\kappa_{\mathcal{E}\mathbf{W}_2(X),p} = \mathrm{frob} : \mathcal{O}_X \longrightarrow \mathrm{frob}_*(\mathcal{O}_X),$$

see the end of Section 3.5.

Definition 6.10. *Denote by*

$$\mathcal{E}F(X) : 0 \longrightarrow \mathcal{O}_X \xrightarrow{\mathrm{frob}} \mathrm{frob}_*(\mathcal{O}_X) \xrightarrow{d} \mathrm{frob}_*(B^1_X) \longrightarrow 0$$

the natural sequence of $\mathcal{O}_X$-modules, in which $\mathrm{frob}_*(B^1_X)$ is the cokernel of Frobenius. This notation is in accordance with the usual one.

The pushforward of $\mathcal{E}\mathbf{W}_2(X)$ by d is a square-zero extension, denoted by

$$\mathrm{C}\mathbf{W}_2(X) : 0 \longrightarrow \mathrm{frob}_*(B^1_X) \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

Clearly, it has $\kappa = 0$. Hence, we have $p\mathcal{O}_Y = 0$. In other words,

$$\mathrm{C}\mathbf{W}_2(X) \in \mathbf{Exal}_k(X, \mathrm{frob}_*(B^1_X)),$$

is a square-zero extension of X in the category of k-schemes.

If X/k is smooth, by Proposition 6.5 we get an extension of $\mathcal{O}_X$-modules

$$C\Omega(X) : 0 \longrightarrow \mathrm{frob}_*(B^1_X) \longrightarrow E \longrightarrow \Omega^1_{X/k} \longrightarrow 0,$$

corresponding to $\mathrm{C}\mathbf{W}_2(X)$.

Note that the extensions $\mathcal{E}F(X)$, $\mathrm{C}\mathbf{W}_2(X)$ and $C\Omega(X)$ *naturally* depend on X.

Remark 6.11. The extension $C\Omega(X)$ is denoted this way as it is given by the Cartier operator, when X/k is smooth. Note that $\mathrm{C}\mathbf{W}_2(X)$ (for Cartier-Witt) actually makes sense for any *reduced* X/k.

We now state and prove the promised equivalence of categories. Recall that X is any reduced k-scheme.

Definition 6.12. *Let $d : \mathrm{frob}_*(\mathcal{O}_X) \longrightarrow \mathrm{frob}_*(B^1_X)$ be as in Definition 6.10. Denote by $\tilde{\mathcal{L}}_2(X)$ the category whose objects are pairs $(\mathcal{E}, f)$, consisting of a square-zero extension*

$$(\mathcal{E} : 0 \longrightarrow \mathrm{frob}_*(\mathcal{O}_X) \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_X \longrightarrow 0) \in \mathbf{Exal}_k(X, \mathrm{frob}_*(\mathcal{O}_X)),$$

together with an isomorphism (in $\mathbf{Exal}_k(X, \mathrm{frob}_*(B^1_X))$)

$$f : d_*(\mathcal{E}) \xrightarrow{\sim} \mathrm{C}\mathbf{W}_2(X).$$

Morphisms in $\tilde{\mathcal{L}}_2(X)$ are (iso)morphisms commuting to the given isomorphisms.

Proposition 6.13. *Let X be a reduced scheme over k. There is an equivalence of categories*

$$\Phi : \mathcal{L}_2(X) \xrightarrow{\sim} \tilde{\mathcal{L}}_2(X).$$

Proof. Let
$$(\mathcal{E} : 0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{X_2} \longrightarrow \mathcal{O}_X \longrightarrow 0) \in \mathcal{L}_2(X)$$
be a lift of X to a scheme flat over $\mathbf{W}_2(k)$. One has $\kappa_{\mathcal{E},p} = \mathrm{Id}$ (cf. Section 3.5 for the definition of κ). The pushforward
$$\mathrm{frob}_*(\mathcal{E}) : 0 \longrightarrow \mathrm{frob}_*(\mathcal{O}_X) \longrightarrow \tilde{\mathcal{O}}_{X_2} \longrightarrow \mathcal{O}_X \longrightarrow 0,$$
and
$$\mathcal{E}\mathbf{W}_2(X) : 0 \longrightarrow \mathrm{frob}_*(\mathcal{O}_X) \longrightarrow \mathbf{W}_2(\mathcal{O}_X) \longrightarrow \mathcal{O}_X \longrightarrow 0,$$
are both square-zero extensions of X by $\mathrm{frob}_*(\mathcal{O}_X)$, in the category of schemes over $\mathbf{W}_2(k)$. They both have $\kappa = \mathrm{frob}$, so that their Baer difference $\mathcal{E}\mathbf{W}_2(X) - \mathrm{frob}_*(\mathcal{E})$ is a square-zero extension
$$\mathcal{F} : 0 \longrightarrow \mathrm{frob}_*(\mathcal{O}_X) \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_X \longrightarrow 0,$$
in the category of k-schemes (indeed, the κ arrow commutes with Baer sum, so that $\mathcal{F}$ has $\kappa = 0$). Because $d \circ \mathrm{frob} = 0$, the extension $d_*(\mathrm{frob}_*(\mathcal{E}))$ has a natural trivialization, so that the square-zero extension $d_*(\mathcal{F})$ is naturally isomorphic to $d_*(\mathcal{E}\mathbf{W}_2(X)) = \mathrm{C}\mathbf{W}_2(X)$. Denoting the natural isomorphism by f, the assignment
$$\mathcal{E} \longrightarrow (\mathcal{F}, f)$$
gives rise to a functor $\Phi : \mathcal{L}_2(X) \longrightarrow \tilde{\mathcal{L}}_2(X)$.

A quasi-inverse of Φ is obtained as follows. Pick an object $(\mathcal{F}, f)$ in $\tilde{\mathcal{L}}_2(X)$. View
$$\mathcal{F} : 0 \longrightarrow \mathrm{frob}_*(\mathcal{O}_X) \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_X \longrightarrow 0,$$
as a square-zero extension in $\mathbf{Exal}_{\mathbf{W}_2(k)}(X, \mathrm{frob}_*(\mathcal{O}_X))$, having $\kappa = 0$. Then, the Baer difference
$$\tilde{\mathcal{E}} := (\mathcal{E}\mathbf{W}_2(X) - \mathcal{F}) \in \mathbf{Exal}_{\mathbf{W}_2(k)}(X, \mathrm{frob}_*(\mathcal{O}_X)),$$
has $\kappa = \mathrm{frob}$, and $d_*(\tilde{\mathcal{E}})$ is equipped with the trivialization induced by f. Therefore, it naturally yields a square-zero extension
$$\mathcal{E} : 0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{X_2} \longrightarrow \mathcal{O}_X \longrightarrow 0,$$
in $\mathbf{Exal}_{\mathbf{W}_2(k)}(X, \mathcal{O}_X)$, together with a natural isomorphism
$$\mathrm{frob}_*(\mathcal{E}) \xrightarrow{\sim} \tilde{\mathcal{E}}.$$
The extension $\mathcal{E}$ then has $\kappa = \mathrm{Id}$, hence belongs to $\mathcal{L}_2(X)$. The assignment $(\mathcal{F}, f) \mapsto \mathcal{E}$ defines a functor Ψ in the reverse direction which is the inverse of Φ. □

We now present two consequences of the preceding result. The first one is the existence of liftings of Frobenius-split k-schemes, a question which has been studied by several authors. See for instance the recent preprint [Yo, Theorem 4.4], where X/k is assumed to be smooth, but where Frobenius-splitting is replaced by a weaker notion.

Corollary 6.14 (Lifting of Frobenius-split k-schemes). *Let X be a reduced k-scheme. Assume that X is Frobenius-split, i.e. that the exact sequence of coherent $\mathcal{O}_X$-modules*
$$\mathcal{E}F(X) : 0 \longrightarrow \mathcal{O}_X \longrightarrow \mathrm{frob}_*(\mathcal{O}_X) \longrightarrow \mathrm{frob}_*(B^1_X) \longrightarrow 0,$$
splits. Then, X admits a lift to a scheme X_2, flat over $\mathbf{W}_2(k)$. More precisely, every splitting of $\mathcal{E}F(X)$ naturally determines such an X_2.

Proof. Denote by $s : \mathrm{frob}_*(B^1_X) \longrightarrow \mathrm{frob}_*(\mathcal{O}_X)$ the splitting of $\mathcal{E}F(X)$ and put $\mathcal{F} := s_*(\mathrm{C}\mathbf{W}_2(X))$. It is then clear that $\mathcal{F}$ is a square-zero extension and that $d_*(\mathcal{F})$ is isomorphic to $\mathrm{C}\mathbf{W}_2(X)$. Conclude by applying Proposition 6.13. □

For another proof of the next corollary, see [Sch, Thm 9.5, Prop 1.1].

COROLLARY 6.15. *Assume that X/k is smooth. Then giving a lift of X to a scheme X_2, smooth over $\mathbf{W}_2(k)$, is equivalent to giving an extension of $\mathcal{O}_X$-modules*

$$\mathcal{E} : 0 \longrightarrow \mathrm{frob}_*(\mathcal{O}_X) \longrightarrow E \longrightarrow \Omega^1_{X/k} \longrightarrow 0,$$

together with an isomorphism (in $\mathbf{Ext}^1_{\mathcal{O}_X}(\Omega^1_{X/k}, \mathrm{frob}_*(B^1_X))$)

$$d_*(\mathcal{E}) \xrightarrow{\sim} C\Omega(X),$$

where $C\Omega(X) \in \mathbf{Ext}^1_{\mathcal{O}_X}(\Omega^1_{X/k}, \mathrm{frob}_*(B^1_X))$ is the extension of Definition 6.10.

In particular, such a lift X_2 exists if and only if the cup-product

$$\mathcal{E}F(X) \cup C\Omega(X) \in \mathbf{Ext}^2_{\mathcal{O}_X}(\Omega^1_{X/k}, \mathcal{O}_X),$$

vanishes.

Proof. The first part of the Corollary is a translation of the equivalence of categories in Proposition 6.13, using that of Proposition 6.5. The second part follows from the first, applying standard considerations in homological algebra. □

Note that the cup-product $\mathcal{E}F(X) \cup C\Omega(X) \in \mathbf{Ext}^2_{\mathcal{O}_X}(\Omega^1_{X/k}, \mathcal{O}_X)$ of the previous Corollary is the obstruction $\mathrm{Obs}(X_2)$, given by classical deformation theory.

6.3. AN EQUIVALENCE OF CATEGORIES FOR PROBLEM (2). We move on towards a functorial description of Problem (2): lifting k-schemes together with their Frobenius morphism.

PROPOSITION 6.16. *Let X be a reduced k-scheme. Let $X_2 \in \mathcal{L}_2(X)$ be a lift of X, viewed as a square-zero extension*

$$\mathcal{E} : 0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{X_2} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

in $\mathbf{Exal}_{\mathbf{W}_2(k)}(X, \mathcal{O}_X)$, having $\kappa = \mathrm{Id}$. Then the following hold.

(1) *Consider the pullback $\mathrm{frob}^*(\mathcal{E}) \in \mathbf{Exal}_{\mathbf{W}_2(k)}(X, \mathrm{frob}_*(\mathcal{O}_X))$. It is the extension whose middle term is the sheaf of $\mathbf{W}_2(k)$-algebras $\mathcal{O}_Y$, defined as the fibered product*

$$\begin{tikzcd} \mathcal{O}_Y \arrow[r] \arrow[d] & \mathcal{O}_X \arrow[d, "\mathrm{frob}"] \\ \mathcal{O}_{X_2} \arrow[r] & \mathcal{O}_X. \end{tikzcd}$$

Then, this extension is naturally isomorphic to $\mathcal{E}\mathbf{W}_2(X)$.

(2) *The data of a lift of $X \xrightarrow{\mathrm{frob}} X$, to a morphism $X_2 \xrightarrow{F_2} X_2$ is equivalent to the data of an isomorphism of square-zero extensions in $\mathbf{Exal}_{\mathbf{W}_2(k)}(X, \mathrm{frob}_*(\mathcal{O}_X))$,*

$$\mathrm{frob}_*(\mathcal{E}) \xrightarrow{\sim} \mathcal{E}\mathbf{W}_2(X).$$

(3) *Assume that X is a smooth k-scheme. Denote by T_X its tangent bundle. Then, there exists a natural class*

$$\mathrm{obs}(F_2) \in H^1(X, \mathrm{frob}^*(T_X)),$$

which vanishes if and only if there exists an F_2, as in item (2).

Proof. By glueing it is enough for (1) and (2) to deal with the case where $X = \mathrm{Spec}(A)$ is affine. Then $X_2 = \mathrm{Spec}(A_2)$, for some $\mathbf{W}_2(k)$-algebra A_2, free as a $\mathbf{W}_2(k)$-module. There is a commutative diagram

$$\begin{tikzcd}
\mathcal{E}\mathbf{W}_2(A) : 0 \arrow[r] & \mathrm{frob}_*(A) \arrow[r] \arrow[d, "\mathrm{Id}"] & \mathbf{W}_2(A) \arrow[r] \arrow[d, "f_2"] & A \arrow[r] \arrow[d, "\mathrm{frob}"] & 0 \\
\mathcal{E} : 0 \arrow[r] & A \arrow[r] & A_2 \arrow[r] & A \arrow[r] & 0,
\end{tikzcd}$$

where f_2 is the natural ring homomorphism. The slightly abusive notation

$$\mathrm{Id} : \mathrm{frob}_*(A) \longrightarrow A$$

makes sense, remembering that $\mathrm{frob}_*(A) = A$, as Abelian groups. The existence of this diagram proves item (1). For item (2), observe that a lift of $A \xrightarrow{\mathrm{frob}} A$, to a ring homomorphism $A_2 \xrightarrow{F_2} A_2$, is equivalent to a commutative diagram

$$\begin{tikzcd}
\mathcal{E} : 0 \arrow[r] & A \arrow[r] \arrow[d, "\mathrm{frob}"] & A_2 \arrow[r] \arrow[d, "F_2"] & A \arrow[r] \arrow[d, "\mathrm{frob}"] & 0 \\
\mathcal{E} : 0 \arrow[r] & A \arrow[r] & A_2 \arrow[r] & A \arrow[r] & 0,
\end{tikzcd}$$

in other words, an isomorphism (in $\mathbf{Exal}_{\mathbf{W}_2(k)}(A, \mathrm{frob}_*(A))$)

$$\mathrm{frob}_*(\mathcal{E}) \xrightarrow{\sim} \mathrm{frob}^*(\mathcal{E}).$$

By item (1), the right side is naturally isomorphic to $\mathcal{E}\mathbf{W}_2(A)$, whence the claim. Keeping the same notations we now prove item (3): given a smooth k-scheme X, form the Baer difference

$$\Delta := (\mathrm{frob}_*(\mathcal{E}) - \mathcal{E}\mathbf{W}_2(\mathcal{O}_X)) : 0 \longrightarrow \mathrm{frob}_*(\mathcal{O}_X) \longrightarrow \mathcal{D} \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

Since formation of the connecting arrow κ commutes to Baer sum, one has $\kappa_\Delta = \mathrm{frob} - \mathrm{frob} = 0$, thus Δ belongs to $\mathbf{Exal}_k(X, \mathrm{frob}_*(\mathcal{O}_X))$. Since X/k is smooth, it is given by an extension of $\mathcal{O}_X$-modules

$$0 \longrightarrow \mathrm{frob}_*(\mathcal{O}_X) \longrightarrow * \longrightarrow \Omega^1_{X/k} \longrightarrow 0,$$

see Proposition 6.5. Recalling that the vector bundles T_X and $\Omega^1_{X/k}$ are dual to each other, one gets

$$\begin{aligned}
\mathrm{Ext}^1_{\mathcal{O}_X-\mathrm{Mod}}(\Omega^1_{X/k}, \mathrm{frob}_*(\mathcal{O}_X)) &= H^1(X, (\Omega^1_{X/k})^\vee \otimes_{\mathcal{O}_X} \mathrm{frob}_*(\mathcal{O}_X)) \\
&= H^1(X, T_X \otimes_{\mathcal{O}_X} \mathrm{frob}_*(\mathcal{O}_X)) = H^1(X, \mathrm{frob}^*(T_X)),
\end{aligned}$$

where the last equality uses the projection formula. The claim follows. $\square$

Proposition 6.17 (Obstruction for Problem (2)). *Let X/k be a reduced scheme. The data of a lift of X to a scheme X_2, flat over $\mathbf{W}_2(k)$, together with a lift $\mathrm{frob}_2 : X_2 \longrightarrow X_2$ of the Frobenius of X, is equivalent to that of a splitting of the square-zero extension*

$$C\mathbf{W}_2(X) \in \mathbf{Exal}^1_k(X, \mathrm{frob}_*(B^1_X)).$$

If X/k is smooth, this is equivalent to the data of a splitting of the extension

$$C\Omega(X) \in \mathbf{Ext}^1_{\mathcal{O}_X}(\Omega^1_{X/k}, \mathrm{frob}_*(B^1_X)).$$

Proof. Use the equivalence of categories provided in Proposition 6.13. Keeping the notation of its proof, we see by Proposition 6.16 that the data of a lift of X, flat over $\mathbf{W}_2(k)$, together with its Frobenius, amounts to specifying an isomorphism $\mathrm{frob}_*(\mathcal{E}) \xrightarrow{\sim} \mathcal{E}\mathbf{W}_2(X)$; that is, a splitting of $C\mathbf{W}_2(X)$. To prove the last assertion (when X/k is smooth), remember that $C\Omega(X)$ then corresponds to $C\mathbf{W}_2(X)$, through the equivalence of Proposition 6.5. $\square$

6.4. An equivalence of categories for Problem (3). The approach taken here is, *mutatis mutandis*, the same as that used to tackle Problem (1). Some proofs are thus left to the reader. Symmetric and divided powers of modules are freely used below. These are polynomial functors, characterized by a universal property (see [Fe] for details).

Let X be a scheme over k. Let V be a vector bundle over X. Denote by

$$f : P := \mathbb{P}_X(V) \longrightarrow X$$

the projective bundle of V. Denote by

$$\mathrm{ad} : V \longrightarrow \mathrm{frob}_*(\mathrm{frob}^*(V))$$

the $\mathcal{O}_X$-linear (first) adjunction morphism. Recall that, if $\mathcal{M}$ a quasi-coherent $\mathcal{O}_X$-module, we have an adjunction isomorphism

$$\mathbf{Ext}^1_{\mathcal{O}_X}(\mathrm{frob}^*(V), \mathcal{M}) \xrightarrow{\sim} \mathbf{Ext}^1_{\mathcal{O}_X}(V, \mathrm{frob}_*(\mathcal{M})),$$

given by applying the exact functor frob_*, followed by pulling back by ad. To get its inverse, apply frob^*, and push forward by the (second) $\mathcal{O}_X$-linear adjunction

$$\mathrm{frob}^*(\mathrm{frob}_*(V)) \longrightarrow V.$$

Definition 6.18. *There are morphisms (of vector bundles over X)*

$$\begin{aligned} \mathrm{Ver}_V : \mathrm{frob}^*(V) &\longrightarrow \mathrm{Sym}^p_{\mathcal{O}_S}(V) \\ v \otimes 1 &\longmapsto v^p \end{aligned}$$

and

$$\begin{aligned} \mathrm{frob}_V : \Gamma^p_{\mathcal{O}_S}(V) &\longrightarrow \mathrm{frob}^*(V) \\ [v]_p &\longmapsto v \otimes 1 \end{aligned}$$

called the Verschiebung *and the* Frobenius *of V, which fit into exact sequences*

$$\mathcal{E}\mathrm{Ver}(V) : 0 \longrightarrow \mathrm{frob}^*(V) \xrightarrow{\mathrm{Ver}_V} \mathrm{Sym}^p_{\mathcal{O}_X}(V) \xrightarrow{q_V} \overline{\mathrm{Sym}}^p_{\mathcal{O}_X}(V) \longrightarrow 0$$

and

$$\mathcal{E}\mathrm{frob}(V) : 0 \longrightarrow \overline{\Gamma}^p_{\mathcal{O}_X}(V) \longrightarrow \Gamma^p_{\mathcal{O}_X}(V) \xrightarrow{\mathrm{frob}_V} \mathrm{frob}^*(V) \longrightarrow 0,$$

where $\overline{\mathrm{Sym}}^p_{\mathcal{O}_X}(V)$ (resp. $\overline{\Gamma}^p_{\mathcal{O}_X}(V)$) is defined as as the cokernel of Ver_V (resp. kernel of frob_V).

Remark 6.19. The dual of the exact sequence (of vector bundles over X) $\mathcal{E}\mathrm{Ver}(V^\vee)$ is naturally isomorphic to $\mathcal{E}\mathrm{frob}(V)$.

Lemma 6.20. *There is a natural isomorphism (of vector bundles over X)*

$$\Phi_V : \overline{\mathrm{Sym}}^p_{\mathcal{O}_X}(V) \xrightarrow{\sim} \overline{\Gamma}^p_{\mathcal{O}_X}(V).$$

In what follows, we may tacitly use this result to identify these vector bundles.

Proof. Consider the natural homomorphism

$$\alpha_V : \mathrm{Sym}^p_{\mathcal{O}_X}(V) \longrightarrow \Gamma^p_{\mathcal{O}_X}(V),$$

defined on sections by the formula

$$v_1 v_2 \dots v_p \mapsto [v_1]_1 \dots [v_p]_1.$$

It takes values in $\overline{\Gamma}^p_{\mathcal{O}_X}(V)$, and vanishes on $\mathrm{Im}(\mathrm{Ver}_V)$ because of the identity

$$[v]_1^p = p![v]_p = 0,$$

which follows by induction from the rule

$$[x]_i[x]_j = \binom{i+j}{i}[x]_{i+j},$$

which is part of the definition of divided powers (see [Fe]).

The resulting homomorphism

$$\overline{\mathrm{Sym}}^p_{\mathcal{O}_X}(V) \longrightarrow \overline{\Gamma}^p_{\mathcal{O}_X}(V)$$

is an isomorphism. To check this, one can assume that $V = \mathcal{O}_X^d$ is the trivial rank d vector bundle, and that $X = \mathrm{Spec}(A)$ is affine. The rest of the verification is then straightforward. Indeed, the description of Sym^p and Γ^p as polynomial functors then presents both $\overline{\mathrm{Sym}}^p_{\mathcal{O}_X}(V)$ and $\overline{\Gamma}^p_{\mathcal{O}_X}(V)$ as trivial vector bundles, with respective canonical basis

$$e_1^{a_1} \ldots e_p^{a_p}$$

and

$$[e_1]_{a_1} \ldots [e_p]_{a_p},$$

both indexed by proper partitions

$$a_1 + \ldots + a_p = p.$$

Here "proper" means that at least two a_i'are non-zero. Since $(p-1)!$ is invertible, α_V maps each $e_1^{a_1} \ldots e_p^{a_p}$ to an invertible multiple of $[e_1]_{a_1} \ldots [e_p]_{a_p}$ and the Lemma is proved. □

Remark 6.21. By adjunction, we have a natural isomorphism

$$\mathbf{Ext}^1_{\mathcal{O}_X}(\mathrm{frob}^*(V), \overline{\Gamma}^p_{\mathcal{O}_X}(V)) \xrightarrow{\sim} \mathbf{Ext}^1_{\mathcal{O}_X}(V, \mathrm{frob}_*(\overline{\Gamma}^p_{\mathcal{O}_X}(V))),$$

through which $\mathcal{E}\mathrm{frob}(V)$ corresponds to an extension

$$\overline{\mathcal{E}}\mathrm{frob}(V) : 0 \longrightarrow \mathrm{frob}_*(\overline{\Gamma}^p_{\mathcal{O}_X}(V)) \longrightarrow \Phi(V) \longrightarrow V \longrightarrow 0.$$

To get a mod p^2 avatar of $\overline{\mathcal{E}}\mathrm{frob}(V)$, one uses Teichmüller lifts of line bundles. Recall (Section 4) the natural exact sequence of Witt modules on $P := \mathbb{P}(V)$

$$0 \longrightarrow \mathrm{frob}_*(\mathcal{O}_P(p)) \longrightarrow \mathbf{W}_2(\mathcal{O}_P(1)) \longrightarrow \mathcal{O}_P(1) \longrightarrow 0.$$

Applying f_*, one gets an exact sequence of Witt modules on X

$$0 \longrightarrow \mathrm{frob}_*(\mathrm{Sym}^p_{\mathcal{O}_X}(V)) \longrightarrow f_*(\mathbf{W}_2(\mathcal{O}_P(1))) \xrightarrow{\Phi_V} V \longrightarrow 0.$$

Surjectivity of the last arrow follows from the (effect on global sections of the) Teichmüller section $\tau_{\mathcal{O}_P(1)}$; see Section 4.2.

Definition 6.22. *The exact sequence of $\mathbf{W}_2(\mathcal{O}_X)$-modules on X*

$$0 \longrightarrow \mathrm{frob}_*(\mathrm{Sym}^p_{\mathcal{O}_X}(V)) \xrightarrow{i_V} f_*(\mathbf{W}_2(\mathcal{O}_P(1))) \xrightarrow{\Phi_V} V \longrightarrow 0,$$

is denoted by $\overline{\mathcal{E}}\mathbf{W}_2(V)$. Denote by τ_V the natural (sheaf-theoretic) section of Φ_V induced by the Teichmüller section $\tau_{\mathcal{O}_P(1)}$.

Remark 6.23. Assume that $X = \mathrm{Spec}(A)$ is affine. Denote by B the symmetric algebra $\mathrm{Sym}_A(V) = \bigoplus_{i=0}^{\infty} \mathrm{Sym}_A^i(V)$. One can also obtain $\overline{\mathcal{E}}\mathbf{W}_2(V)$ from the exact sequence

$$0 \longrightarrow \mathrm{frob}_*(B) \longrightarrow \mathbf{W}_2(B) \longrightarrow B \longrightarrow 0,$$

by pulling it back by the inclusion $V \longrightarrow B$, and pushing it forward by the projection $\mathrm{frob}_*(B) \longrightarrow \mathrm{frob}_*(\mathrm{Sym}_A^p(V))$.

LEMMA 6.24. *One has the formula*

$$\tau_V(x+y) = \tau_V(x) + \tau_V(y) + i_V\left(\sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} x^i y^{p-i}\right).$$

Proof. Follows from th fact that the same formula defines addition on $\mathbf{W}_2$. □

LEMMA 6.25. *The extension $\overline{\mathcal{E}}\mathbf{W}_2(V)$ has $\kappa := \kappa_{\overline{\mathcal{E}}\mathbf{W}_2(V),p}$ given by the map adjoint to Ver_V. Concretely, it is given by*

$$\begin{array}{ccc} V & \longrightarrow & \mathrm{frob}_*(\mathrm{Sym}^p_{\mathcal{O}_X}(V)). \\ x & \longmapsto & x^p \end{array}$$

There is a natural commutative diagram of $\mathbf{W}_2(\mathcal{O}_S)$-modules

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{frob}_*(\mathrm{Sym}^p_{\mathcal{O}_X}(V)) \arrow[r] \arrow[d, "\frac{1}{(p-1)!}\mathrm{frob}_*(q_V)"] & f_*(\mathbf{W}_2(\mathcal{O}_P(1))) \arrow[r] \arrow[d, "F"] & V \arrow[r] \arrow[d, "\mathrm{ad}"] & 0 \\ 0 \arrow[r] & \mathrm{frob}_*(\overline{\mathrm{Sym}}^p_{\mathcal{O}_X}(V)) \arrow[r] & \mathrm{frob}_*(\Gamma^p_{\mathcal{O}_X}(V)) \arrow[r] & \mathrm{frob}_*(\mathrm{frob}^*(V)) \arrow[r] & 0, \end{tikzcd}$$

where the upper line is $\overline{\mathcal{E}}\mathbf{W}_2(V)$, the lower line is $\mathrm{frob}_*(\mathcal{E}\mathrm{frob}(V))$, and where F is defined (on sections) by the formula

$$F(\tau_V(x)) = [x]_p,$$

for $x \in V$.

Consequently, $(\mathrm{frob}_*(q_V))_*(\overline{\mathcal{E}}\mathbf{W}_2(V))$ is naturally isomorphic to $-\overline{\mathcal{E}}\mathrm{frob}(V)$.

Proof. Can assume $X = \mathrm{Spec}(A)$ is affine. The key point here is that the formula giving F makes sense, and indeed defines a homomorphism of Witt modules. This follows from the previous Lemma, combined to the formula, in $\Gamma_A^p(V)$,

$$[x+y]_p = [x]_p + [y]_p + \sum_{i=1}^{p-1} [x]_i [y]_{p-i}.$$

□

THEOREM 6.26 (An equivalence of categories for Problem (3)). *To give a lift of V to a $\mathbf{W}_2$-bundle V_2 on X is equivalent to give an extension*

$$\mathcal{F} \in \mathbf{Ext}^1_{\mathcal{O}_X}(\mathrm{frob}^*(V), \mathrm{Sym}^p_{\mathcal{O}_X}(V)),$$

together with an isomorphism

$$(q_V)_*(\mathcal{F}) \xrightarrow{\sim} \mathcal{E}\mathrm{frob}(V),$$

in $\mathbf{Ext}^1_{\mathcal{O}_X}(\mathrm{frob}^*(V), \overline{\mathrm{Sym}}^p_{\mathcal{O}_X}(V))$.

Proof. Similar to that of Proposition 6.13. □

In purely cohomological terms, one immediately deduces the following corollary.

Corollary 6.27. *The obstruction* $\mathrm{Obs}(V_2)$ *to lifting* V *to a* $\mathbf{W}_2$*-bundle on* X *is the element of* $\mathrm{Ext}^2_{\mathcal{O}_X}(\mathrm{frob}^*(V), \mathrm{frob}^*(V))$ *represented by the* 2*-extension*

$$0 \longrightarrow \mathrm{frob}^*(V) \longrightarrow \mathrm{Sym}^p_{\mathcal{O}_X}(V) \longrightarrow \Gamma^p_{\mathcal{O}_X}(V) \longrightarrow \mathrm{frob}^*(V) \longrightarrow 0,$$

defined as the cup-product of $\mathcal{E}\mathrm{frob}(V)$ *and* $\mathcal{E}\mathrm{Ver}(V)$*.*

6.5. Relating Problems (2) and (3).

Proposition 6.28. *Let* X *be a smooth variety, over a perfect field* k*,* $\mathrm{char}(k) = p$*. Denote by* T_X *the tangent bundle of* X/k*. Let* V *be a vector bundle on* X*.*
Consider the following assertions.

(a) Lift the variety X *to a smooth scheme over* $\mathbf{W}_2(k)$*, together with its Frobenius.*

(b) Lift the variety X *to a smooth scheme* $X_2/\mathbf{W}_2(k)$*, in such a way that* $\mathrm{frob}^*(V)$ *lifts, to a vector bundle on* X_2*.*

(c) Lift the vector bundle V/X *to a* $\mathbf{W}_2$*-bundle on* X*.*

Then (a) *implies* (b) *for* $V = T_X$*, and* (b) *implies* (c)*.*

Proof. The first implication is straightforward. Indeed, let $X_2 \longrightarrow \mathrm{Spec}(\mathbf{W}_2(k))$ be a smooth morphism, lifting $X \longrightarrow \mathrm{Spec}(k)$. Denote by V_{X_2} its tangent bundle. It is a vector bundle over X_2. Let $X_2 \xrightarrow{F_2} X_2$ be a lift of $X \xrightarrow{\mathrm{frob}} X$. Then, $F_2^*(V_{X_2})$ is a lift of $\mathrm{frob}^*(V)$, to a vector bundle on X_2.
Let us prove (b)$\Longrightarrow$(c). Let $V^{[1]}_{X_2}$ be a lift of $\mathrm{frob}^*(V)$, to a vector bundle over X_2. There is the natural reduction sequence of $\mathcal{O}_{X_2}$-modules

$$0 \longrightarrow \mathrm{frob}^*(V) \longrightarrow V^{[1]}_{X_2} \longrightarrow \mathrm{frob}^*(V) \longrightarrow 0.$$

Recall that, at the level of topological spaces, $X_2 = X$ and $X \xrightarrow{\mathrm{frob}} X$ is the identity. Consider the extension above, merely as an extension of Zariski sheaves on X. One may then replace $\mathrm{frob}^*(V)$ by $\mathrm{frob}_*(\mathrm{frob}^*(V))$. Form the pull-back diagram of Zariski sheaves on X

$$\begin{tikzcd}
\mathcal{E}V_2 : 0 \arrow[r] & \mathrm{frob}_*(\mathrm{frob}^*(V)) \arrow[r] \arrow[d, equal] & V_2 \arrow[r] \arrow[d] & V \arrow[r] \arrow[d, "\mathrm{frob}_V"] & 0 \\
0 \arrow[r] & \mathrm{frob}_*(\mathrm{frob}^*(V)) \arrow[r] & V^{[1]}_{X_2} \arrow[r] & \mathrm{frob}_*(\mathrm{frob}^*(V)) \arrow[r] & 0,
\end{tikzcd}$$

where frob_V is the natural $\mathcal{O}_X$-linear arrow. Using the first commutative diagram of the proof of Proposition 6.16, one sees that V_2 has a natural structure of a $\mathbf{W}_2(\mathcal{O}_X)$-module, for which $\mathcal{E}V_2$ is an extension of $\mathbf{W}_2(\mathcal{O}_X)$-modules. It is straightforward to check, that its associated connecting arrow $V \xrightarrow{\kappa} \mathrm{frob}_*(\mathrm{frob}^*(V))$ equals frob_V. Hence, V_2 is a lift of V to a $\mathbf{W}_2$-bundle, and $\mathcal{E}V_2$ is the reduction sequence of V_2. □

Remark 6.29. Assume there exists a lift X_2, as in (b) above. Then liftability of V to a $\mathbf{W}_2$-bundle on X, implies that of $\mathrm{frob}^*(V)$ to a vector bundle on X_2. This is a particular case of Lemma 5.5.

7. Grassmannians whose tautological bundle does not lift

In this section, Grassmannian varieties $\mathrm{Gr}(m, n)$ are considered over a base of characteristic p. Our goal is the next Theorem, proved by applying the equivalences of categories offered in the previous section, to tautological bundles on $\mathrm{Gr}(m, n)$.

Theorem 7.1. *Let m and n be two integers, with $2 \leq m \leq n-2$.*

Then, the tautological vector bundle V of $\mathrm{Gr}(m,n)$ does not lift to a $\mathbf{W}_2$-bundle. Neither do its Frobenius twists $V^{(s)}$, for all $s \geq 0$.

Remark 7.2. In the Proposition, one can assume without loss of generality that the base is a perfect field k. Using that formation of coherent cohomology of varieties commutes with change of the base field k, one could actually reduce to $k = \mathbb{F}_p$.

7.1. First proof of Theorem 7.1.

We give here a first proof of Theorem 7.1 that uses Proposition 6.26 and some explicit computations that we carry in Section 9. We only proof the case where $s = 0$, the general case being the same.

Let k be a perfect field of characeristic p. Let E be a k-vector space of dimension n and let $X := \mathrm{Gr}(m,E)$ be the corresponding Grasmannian, which parametrizes m-dimensional subspaces of E. Denote by $f : X \longrightarrow \mathrm{Spec}(k)$ its structure morphism and by V its tautological bundle. There is an exact sequence

$$0 \longrightarrow V \longrightarrow f^*(E) \longrightarrow W \longrightarrow 0,$$

whose cokernel is a vector bundle W on X. One knows (see Section 5) that V admits a lifting tower, in the (dual) cases $m = 1$ or $m = n-1$.

Assume now that $2 \leq m \leq n-2$. The goal is to show that V does not lift to a $\mathbf{W}_2$-bundle on X. Consider the vector bundle extension

$$\mathcal{E}\mathrm{frob}(V) : 0 \longrightarrow \overline{\mathrm{Sym}}^p_{\mathcal{O}_X}(V) \longrightarrow \Gamma^p_{\mathcal{O}_X}(V) \xrightarrow{\mathrm{frob}_V} \mathrm{frob}^*(V) \longrightarrow 0.$$

Assume that V can be lifted to a $\mathbf{W}_2$-bundle. By Proposition 6.26, this means that $\mathcal{E}\mathrm{frob}(V)$ admits a lift to an extension

$$\mathcal{F} : 0 \longrightarrow \mathrm{Sym}^p_{\mathcal{O}_X}(V) \longrightarrow F \longrightarrow \mathrm{frob}^*(V) \longrightarrow 0.$$

By item 2) of Lemma 9.4 (for $r = 1$), $\mathcal{F}$ would be split, hence so would $\mathcal{E}\mathrm{frob}(V)$. But this contradicts item 4) of the same Lemma (again for $r = 1$).

7.2. A second proof for Theorem 7.1.

We give now a second proof that promotes the use of Teichmüller lifts of line bundles. Though more elementary than the first one, it relies on the specific fact that our base X is a projective homogeneous space of a reductive algebraic group. Here again, we assume for simplicity that $s = 0$, the proof of the general case being the same, and we use computations from Section 9.

Keep notation of the preceding section. In particular, one has $2 \leq m \leq n-2$ and $X = \mathrm{Gr}(m,E)$ with structure morphism $f : X \longrightarrow \mathrm{Spec}(k)$.

Assume that V admits a lift to a $\mathbf{W}_2$-bundle on X. Consider the arrow

$$h : \mathrm{Fl}(E) = \mathrm{Fl}(1,2,\ldots,n-1,E) \longrightarrow X,$$

where $\mathrm{Fl}(E)$ denotes the complete flag scheme of E (see Definition 9.1). Denote the tautological filtration on $\mathrm{Fl}(E)$ by

$$0 \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset \ldots \subset \mathcal{V}_n = E,$$

and its graded pieces by $\mathcal{L}_i := \mathcal{V}_i/\mathcal{V}_{i-1}$. Thus, over $\mathrm{Fl}(E)$, $h^*(V) = \mathcal{V}_m$, so that $\mathcal{V}_m$ admits a lift, to a $\mathbf{W}_2$-bundle $\mathcal{V}_{m,2}$ over $\mathrm{Fl}(E)$.

Assume that $m \geq 3$. We claim that the natural quotient

$$\pi_m : \mathcal{V}_m \longrightarrow \mathcal{L}_m,$$

lifts to a surjection of $\mathbf{W}_2$-bundles

$$\pi_{m,2} : \mathcal{V}_{m,2} \longrightarrow \mathbf{W}_2(\mathcal{L}_m).$$

To prove it, one proceeds as in the proof of Theorem 5.7. The space of such lifts is a torsor under the vector bundle $\mathcal{V}_m^{\vee(1)} \otimes \mathcal{L}_m^{(1)}$. By item (2) of Lemma 9.5, this torsor is trivial. The claim follows.

The kernel of $\pi_{m,2}$ is then a lift of $\mathcal{V}_{m-1}$ to a $\mathbf{W}_2$-bundle $\mathcal{V}_{m-1,2}$. By descending induction on m, one infers that $\mathcal{V}_2$ lifts, to a $\mathbf{W}_2$-bundle $\mathcal{V}_{2,2}$ over $\mathrm{Fl}(E)$, and that the arrow

$$\pi_2 : \mathcal{V}_2 \longrightarrow \mathcal{L}_2$$

lifts to a surjection of $\mathbf{W}_2$-bundles

$$\pi_{2,2} : \mathcal{V}_{2,2} \longrightarrow \mathbf{W}_2(\mathcal{L}_2),$$

whose kernel $\mathcal{L}_{1,2}$ is a $\mathbf{W}_2$-line bundle, lifting $\mathcal{L}_1$. By Proposition 4.4, the space of such lifts is a principal homogeneous space of $H^1(\mathrm{Fl}(E), \mathcal{O}_{\mathrm{Fl}(E)})$, pointed by $\mathbf{W}_2(\mathcal{L}_1)$. As $H^1(\mathrm{Fl}(E), \mathcal{O}_{\mathrm{Fl}(E)}) = 0$, one sees that $\mathcal{L}_{1,2}$ is isomorphic to $\mathbf{W}_2(\mathcal{L}_1)$.
Altogether, we have built an extension of $\mathbf{W}_2$-bundles

$$\mathcal{E}_{1,2,2} := 0 \longrightarrow \mathbf{W}_2(\mathcal{L}_1) \longrightarrow \mathcal{V}_{2,2} \longrightarrow \mathbf{W}_2(\mathcal{L}_2) \longrightarrow 0,$$

lifting the tautological extension

$$\mathcal{E}_{1,2} : 0 \longrightarrow \mathcal{L}_1 \longrightarrow \mathcal{V}_2 \longrightarrow \mathcal{L}_2 \longrightarrow 0.$$

The natural exact sequence of Witt modules

$$0 \longrightarrow (\mathcal{L}_2^\vee \otimes \mathcal{L}_1)^{(1)} \longrightarrow \mathbf{W}_2(\mathcal{L}_2^\vee \otimes \mathcal{L}_1) \longrightarrow \mathcal{L}_2^\vee \otimes \mathcal{L}_1 \longrightarrow 0$$

admits a sheaf-theoretic section: the Teichmüller section τ. Thus, the sequence

$$0 \longrightarrow H^1(X, (\mathcal{L}_2^\vee \otimes \mathcal{L}_1)^{(1)}) \xrightarrow{\iota} H^1(X, \mathbf{W}_2(\mathcal{L}_2^\vee \otimes \mathcal{L}_1)) \longrightarrow H^1(X, (\mathcal{L}_2^\vee \otimes \mathcal{L}_1)^{(1)})$$

is exact. Therefore, the set of isomorphism classes of lifts of $\mathcal{E}_{1,2}$ to an extension $\mathcal{E}_{1,2,2}$ as above, is a principal homogeneous space of

$$\mathrm{Ext}^1(\mathcal{L}_2^{(1)}, \mathcal{L}_1^{(1)}) = H^1(X, (\mathcal{L}_2^\vee \otimes \mathcal{L}_1)^{(1)}).$$

By point (3) of Lemma 9.5, this is a one-dimensional k-vector space, with generator (the class of) $\mathcal{E}_{1,2}^{(1)}$. Note that the computation (3) of Lemma 9.5 remains valid over an arbitrary base ring of characteristic p. Hence, the result above remains valid after changing the base, from k to an arbitrary k-algebra.

Set $G := \mathrm{GL}_k(E)$ for the linear algebraic k-group of linear automorphisms of E. Consider $\mathrm{Fl}(E)$ as a projective homogenous space of G, over k. For this structure, the tautological vector bundles $\mathcal{V}_i$, as well as the line bundles $\mathcal{L}_i$, are naturally G-linearized. By functoriality of the Teichmüller lift of line bundles, the $\mathbf{W}_2$-bundles $\mathbf{W}_2(\mathcal{L}_i)$, over $\mathrm{Fl}(E)$, are G-linearized as well. Thus, $\mathcal{E}_{1,2,2}$ is an extension between the G-linearized $\mathbf{W}_2$-bundles $\mathbf{W}_2(\mathcal{L}_2)$ and $\mathbf{W}_2(\mathcal{L}_1)$. However, so far, its middle term $\mathcal{V}_{2,2}$ need not admit a G-linearization. We are going to show that this can in fact be done, in the strongest possible sense.

Denote by $k' := k[G]$ the function ring of G, and by $g \in G(k')$ the point corresponding to the identity of G. Change the base ring, from k to k'. Set

$$\mathrm{Fl}(E)' := \mathrm{Fl}(E) \times_k k',$$

and similarly for other objects, and work over k'. Consider the base-changed extension

$$\mathcal{E}'_{1,2,2} := 0 \longrightarrow \mathbf{W}_2(\mathcal{L}'_1) \longrightarrow \mathcal{V}'_{2,2} \longrightarrow \mathbf{W}_2(\mathcal{L}'_2) \longrightarrow 0,$$

of vector bundles over $\mathrm{Fl}(E)'$. Set

$$\begin{array}{cccc} g^*: & \mathrm{Fl}(E)' & \longrightarrow & \mathrm{Fl}(E)' \\ & \nabla & \longmapsto & (g^{-1}).\nabla \end{array}$$

where the presence of an exponent -1 guarantees that pulling back by (scheme-theoretic) points of G, provides a *left action* of G, on $\mathbf{W}_2$-bundles.

Consider the extension

$$g^*(\mathcal{E}'_{1,2,2}) := 0 \longrightarrow \mathbf{W}_2(\mathcal{L}'_1) \xrightarrow{\sim} g^*(\mathbf{W}_2(\mathcal{L}'_1)) \longrightarrow g^*(\mathcal{V}'_{2,2}) \longrightarrow g^*(\mathbf{W}_2(\mathcal{L}'_2)) \xrightarrow{\sim} \mathbf{W}_2(\mathcal{L}'_2) \longrightarrow 0,$$

where isomorphisms are given by the natural G-linearizations. The set of isomorphism classes of lifts of $\mathcal{E}'_{1,2}$ to $\mathbf{Ext}^1_{G,2}(\mathbf{W}_2(\mathcal{L}'_2), \mathbf{W}_2(\mathcal{L}'_1))$ is a principal homogeneous space of k'. Precisely, there exists a unique element $\lambda \in k'$, such that

$$g^*(\mathcal{E}'_{1,2,2}) - \mathcal{E}'_{1,2,2} \simeq \lambda\iota(\mathcal{E}^{'(1)}_{1,2}) \in \mathbf{Ext}^1_{G,2}(\mathbf{W}_2(\mathcal{L}'_2), \mathbf{W}_2(\mathcal{L}'_1)).$$

Considered as a morphism of k-varieties,

$$\lambda : G \longrightarrow \mathbb{G}_a$$

is then a Hochschild 1-cocycle, i.e. a group homomorphism. Since

$$\mathrm{Hom}(G, \mathbb{G}_a) = 0,$$

it is trivial. Hence, the extensions (of $\mathbf{W}_2$-bundles over $\mathrm{Fl}(E)'$) $g^*(\mathcal{E}'_{1,2,2})$ and $\mathcal{E}'_{1,2,2}$ are isomorphic. Let

$$\phi_g : g^*(\mathcal{E}'_{1,2,2}) \longrightarrow \mathcal{E}'_{1,2,2}$$

be an isomorphism of extensions. Denote the coordinates of $G \times_k G$ by (g_1, g_2), set $k'' := k[G \times_k G] = k' \otimes_k k'$ and $\mathrm{Fl}(E)'' = \mathrm{Fl}(E) \times_k k''$. Working over k'', the formula

$$\phi_{g_1} \circ (g_1)^*(\phi_{g_2}) \circ \phi^{-1}_{g_1 g_2}$$

defines an automorphism of the extension $\mathcal{E}''_{1,2,2}$; that is, an element of

$$\mathrm{Hom}_{\mathrm{Fl}(E)''}(\mathbf{W}_2(\mathcal{L}''_2), \mathbf{W}_2(\mathcal{L}''_1)),$$

which is trivial by a straightforward two-step dévissage, using point (4) of Lemma 9.5 below. Thus, the cocycle condition

$$\phi_{g_1} \circ (g_1)^*(\phi_{g_2}) = \phi_{g_1 g_2}$$

identically holds. In other words, ϕ_g provides a G-linearization of $\mathcal{E}_{1,2,2}$.
Choose a k-basis of E, i.e. a direct sum decomposition of k-vector spaces

$$E = L_1 \bigoplus L_2 \bigoplus \ldots \bigoplus L_n$$

with $L_i = k$ for all i. It determines an embbedding of k-groups

$$\mathbb{G}_a \subset \mathrm{GL}_k(E) \simeq \mathrm{GL}_n,$$

identifying $\mathbb{G}_a$ to the subgroup consisting of strictly upper triangular matrices

$$\begin{pmatrix} 1 & x & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}.$$

The filtration associated to the chosen basis gives a $\mathbb{G}_a$-invariant k-rational point

$$\nabla \in \mathrm{Fl}(E)(k).$$

To conclude the proof, let us show how specialising at ∇ yields a contradiction.
The fiber of $\mathcal{E}_{1,2}$, at ∇, is the natural (non-split) extension of $\mathbb{G}_a$-modules over k,

$$E_{1,2} : 0 \longrightarrow L_1 \longrightarrow L_1 \oplus L_2 \longrightarrow L_2 \longrightarrow 0,$$

where $\mathbb{G}_a$ acts as the vector group of the one-dimensional k-vector space $\mathrm{Hom}(L_2, L_1)$. Similarly, the fiber of $\mathcal{E}_{1,2,2}$ at ∇ is a $\mathbb{G}_a$-linearized lift of $E_{1,2}$, to a $\mathbb{G}_a$-linearised extension of $\mathbf{W}_2(k)$-modules

$$E_{1,2,2} : 0 \longrightarrow \mathbf{W}_2(L_1) \longrightarrow \mathbf{W}_2(L_1) \oplus \mathbf{W}_2(L_2) \longrightarrow \mathbf{W}_2(L_2) \longrightarrow 0,$$

with trivial action of $\mathbb{G}_a$ on both graded pieces. This action is simply given by a homomorphism of linear algebraic k-groups

$$\alpha_2 : \mathbb{G}_a \longrightarrow \mathbf{W}_2(L_2^\vee \otimes L_1) \simeq R_{\mathbf{W}_2(k)/k}(\mathbb{G}_{a,\mathbf{W}_2(k)}).$$

Here $\mathbf{W}_2(L_2^\vee \otimes L_1)$ is considered as a two-dimensional group scheme over k, isomorphic to the Greenberg transfer (see [BGA]) of $\mathbb{G}_a$, from $\mathbf{W}_2(k)$ to k.
That $E_{1,2,2}$ lifts $E_{1,2}$, means that the composite arrow of algebraic k-groups

$$\mathbb{G}_a \xrightarrow{\alpha_2} R_{\mathbf{W}_2(k)/k}(\mathbb{G}_{a,\mathbf{W}_2(k)}) \xrightarrow{\rho} \mathbb{G}_a$$

is the identity. (Here ρ is the natural reduction homomorphism.)
Taking k-points, one gets a factorisation of Id_k, as a composite of homomorphism of additive groups

$$k \xrightarrow{\alpha_2(k)} \mathbf{W}_2(k) \longrightarrow k,$$

where the right arrow is the reduction homomorphism. Such a splitting does not exist, because $p \neq 0 \in \mathbf{W}_2(k)$. The proof is complete.

8. Application: non-liftability of some projective bundles

Theorem 7.1 is close to Theorem 6.5 of [Z]. The connection is made explicit below. The result is valid when $\mathbb{F}_p$ is replaced by an arbitrary field k of characteristic p, with the same proof.

Theorem 8.1. *(see* [Z], *Theorem 6.5)*
Let m, n be integers, with $2 \leq m \leq n-2$. Denote by V the tautological vector bundle on the Grassmannian

$$X := \mathrm{Gr}(m, n) \longrightarrow \mathrm{Spec}(\mathbb{F}_p).$$

Let $r \geq 1$ be an integer. Consider the projective bundle

$$f : \mathbb{P}(V^{(r)}) \longrightarrow X.$$

Then, the following holds.

(1) *Let R be a ring where $p \neq 0 \in R$ and $p \notin R^\times$. Then, the relative scheme*

$$\mathbb{P}(V^{(r)}) \times_{\mathrm{Spec}(\mathbb{F}_p)} \mathrm{Spec}(R/p) \longrightarrow \mathrm{Spec}(R/p)$$

does not lift to a scheme flat over $\mathrm{Spec}(R)$*.*

(2) *Let R be a p-adically complete local ring with residue field $\mathbb{F}_p$.*
If $p \neq 0 \in R$, then $\mathbb{P}(V^{(r)})$ does not lift to a scheme flat over $\mathrm{Spec}(R)$*.*

Proof. We prove item (1). Set $Y := \mathbb{P}(V^{(r)})$. Without loss of generality, one can assume that R is of finite-type over $\mathbb{Z}$, hence Noetherian. One can then also assume that $\mathrm{Spec}(R)$ is connected. Then $p \notin p^2R$, for otherwise $pR \subset R$ would be an idempotent ideal, hence generated by an idempotent element by Nakayama's Lemma, contradicting the connectedness of $\mathrm{Spec}(R)$. Replacing R with R/p^2, one may assume $p^2 = 0 \in R$.
The obstruction to lifting Y then lies in the coherent cohomology group

$$H^2(Y \times_{\mathrm{Spec}(\mathbb{F}_p)} \mathrm{Spec}(R/p), T_{Y/\mathbb{F}_p} \otimes_{\mathbb{F}_p} pR),$$

where $T_{Y/\mathbb{F}_p}$ stands for the tangent bundle of $Y \longrightarrow \mathrm{Spec}(\mathbb{F}_p)$. Using that the projection

$$Y \times_{\mathrm{Spec}(\mathbb{F}_p)} \mathrm{Spec}(R/p) \longrightarrow Y$$

is affine, this group is identified to

$$H^2(Y, T_{Y/\mathbb{F}_p} \otimes_{\mathbb{F}_p} pR).$$

Since the arrow of $\mathbb{F}_p$-vector spaces

$$\begin{array}{ccc} \mathbb{F}_p & \longrightarrow & pR \\ 1 & \longmapsto & p \end{array}$$

is a (split) injection by assumption, liftability over R implies liftability over $\mathbb{Z}/p^2$. Thenceforward, assume $R = \mathbb{Z}/p^2$.
Assume that Y_2 is a lift of Y, flat over $\mathbb{Z}/p^2$. Over Y, there is the tautological extension of vector bundles

$$\mathcal{E} : 0 \longrightarrow \mathcal{H} \longrightarrow f^*(V^{(r)}) \longrightarrow \mathcal{O}(1) \longrightarrow 0.$$

The obstruction to lifting $\mathcal{H}$ to a vector bundle over Y_2 lies in

$$H^2(Y, \mathrm{End}(\mathcal{H})) = \mathrm{Ext}^2_Y(\mathcal{H}, \mathcal{H}).$$

This cohomology group vanishes. To see why, use Leray's spectral sequence and the last two items of Lemma 9.3 here below. Thus, $\mathcal{H}$ lifts to a vector bundle $\mathcal{H}_2$ over Y_2. Similarly, since $H^2(Y, \mathcal{O}_Y) = 0$, the line bundle $\mathcal{O}(1)$ lifts to a line bundle $\mathcal{O}_2(1)$ over Y_2 (in fact unique up to isomorphism, since $H^1(Y, \mathcal{O}_Y) = 0$).

Claim: $\mathcal{E}$ lifts to an extension of vector bundles over Y_2,

$$\mathcal{E}_2 : 0 \longrightarrow \mathcal{H}_2 \longrightarrow V_2^{[r]} \longrightarrow \mathcal{O}_2(1) \longrightarrow 0.$$

Indeed, the obstruction to its existence lies in

$$H^2(Y, \mathcal{H}(-1)) = \mathrm{Ext}^2_Y(\mathcal{O}(1), \mathcal{H}),$$

and items (2) and (3) of Lemma 9.3 imply that this group vanishes. This proves the existence of $\mathcal{E}_2$. Then $V_2^{[r]}$ is a lift of $f^*(V^{(r)})$, to a vector bundle over Y_2. Using Proposition 6.28, since $r \geq 1$, one sees that $f_*(V^{(r-1)})$ lifts to a $\mathbf{W}_2$-bundle over Y. Denoting by $W_2^{[r-1]}$ such a lift, one gets an extension of $\mathbf{W}_2$-modules over Y,

$$0 \longrightarrow \mathrm{frob}_*(f^*(V^r)) \longrightarrow W_2^{[r-1]} \longrightarrow f^*(V^{(r-1)}) \longrightarrow 0,$$

having $\kappa = \mathrm{frob}$. Applying f_* to this extension, using the projection formula and $R^1 f_*(\mathcal{O}_Y) = 0$, one gets the extension of $\mathbf{W}_2$-modules over X

$$0 \longrightarrow \mathrm{frob}_*(V^r) \longrightarrow f_*(W_2^{[r-1]}) \longrightarrow V^{(r-1)} \longrightarrow 0,$$

having $\kappa = \mathrm{frob}$ as well. Thus, $f_*(W_2^{[r-1]})$ is a lift of $V^{(r-1)}$, to a $\mathbf{W}_2$-bundle over X. This contradicts Theorem 7.1.
To finish the proof, it remains to prove that (1) implies (2). It suffices to prove that Y is a rigid $\mathbb{F}_p$-variety (i.e. that it has no nontrivial deformations), for then any deformation of Y over R/p has to be trivial. One needs to show that $H^1(Y, T_{Y/\mathbb{F}_p}) = 0$. The first fundamental sequence for $f : Y \longrightarrow X$ (a smooth morphism of smooth $\mathbb{F}_p$-varieties) provides an exact sequence of vector bundles over Y

$$0 \longrightarrow T_{Y/X} \longrightarrow T_{Y/\mathbb{F}_p} \longrightarrow f^*(T_{X/\mathbb{F}_p}) \longrightarrow 0.$$

By dévissage, it is enough to show that $H^1(Y, f^*(T_{X/\mathbb{F}_p}))$ and $H^1(Y, T_{Y/X})$ both vanish. Since f is a projective bundle, $f_*(\mathcal{O}_Y) = \mathcal{O}_X$ and $R^1 f_*(\mathcal{O}_Y) = 0$. Using the projection formula and Leray's spectral sequence, vanishing of the first group boils down to that of $H^1(X, T_{X/\mathbb{F}_p})$. Recall that the tautological extension of vector bundles over X,

$$0 \longrightarrow V \longrightarrow \mathcal{O}_X^n \longrightarrow W \longrightarrow 0,$$

gives rise to a natural iso

$$T_{X/\mathbb{F}_p} = \mathrm{Hom}_{\mathcal{O}_X}(V, W).$$

The vanishing of $H^1(X, \mathrm{Hom}_{\mathcal{O}_X}(V, W))$ then follows from item (1) of Lemma 9.4. To prove vanishing of $H^1(Y, T_{Y/X})$, it suffices to prove that of $H^1(X, f^*(T_{Y/X}))$ and $R^1 f^*(T_{Y/X})$. Since f is the projective bundle of $V^{(r)}$, item (4) of Lemma 9.3 provides an exact sequence

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\lambda \mapsto \lambda \mathrm{Id}} \mathrm{End}(V^{(r)}) \longrightarrow f^*(T_{Y/X}) \longrightarrow 0.$$

Vanishing of $H^1(X, f^*(T_{Y/X}))$ follows by dévissage, using that of $H^2(X, \mathcal{O}_X)$ (classical) and $H^1(X, \mathrm{End}(V^{(r)}))$ (item (5) of Lemma 9.5). Vanishing of $R^1 f^*(T_{Y/X})$ is item (5) of Lemma 9.3, for $i = 1$. $\square$

9. Some computations in coherent cohomology of flag schemes.

Let us deal with the (more or less classical) cohomological computations used in this paper. Let V be a vector bundle of rank $n \geq 2$, over a scheme S.
In many applications, S is just the spectrum of a field of characteristic p.

9.1. Flag schemes and classical cohomological tools.

Definition 9.1 (Flag schemes).
Let

$$1 \leq n_1 < \ldots < n_s < n$$

be a strictly increasing sequence of integers. Denote by

$$F(= F_{n_1, \ldots, n_s}) : \mathrm{Fl}(n_1, \ldots, n_s, V) \longrightarrow S$$

the scheme of flags of sub-bundles of V*, of dimensions* $n_1, \ldots, n_s$*. Denote by*

$$0 \subset \mathcal{V}_{n_1} \subset \ldots \subset \mathcal{V}_{n_s} \subset \mathcal{V}_n = F^*(V)$$

the tautological flag over $\mathrm{Fl}(n_1, \ldots, n_s, V)$*.*

Denote $\mathrm{Fl}(1, 2, \ldots, n-1, V)$ *simply by* $\mathrm{Fl}(V)$*; it is the scheme of complete flags of the vector bundle* V*. For* $1 \leq i \leq j \leq n$*, denote by*

$$\mathcal{V}_{j/i} := \mathcal{V}_j / \mathcal{V}_i$$

and

$$\mathcal{L}_i := \mathcal{V}_i / \mathcal{V}_{i-1}$$

the natural quotients.
For an arbitrary sequence of relative integers $a_1, \ldots, a_n$*, put*

$$\mathcal{O}(a_1, \ldots, a_n) := \mathcal{L}_1^{\otimes a_1} \otimes \ldots \otimes \mathcal{L}_n^{\otimes a_n}.$$

Proposition 9.2. *Denote by*

$$f : \mathbb{P}(V) \longrightarrow S$$

the projective bundle of V*, by* $\mathcal{O}(1)$ *its twisting sheaf, and*

$$F : \mathrm{Fl}(V) \longrightarrow S$$

its complete flag scheme. (Note that $\mathcal{O}(1) = \mathcal{L}_n$*, with the notation above.)*
Let $m \geq 0$ *and* $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ *be integers. The following holds.*

(1) $f_*(\mathcal{O}(m)) = \mathrm{Sym}^m(V)$*, and* $R^i f_*(\mathcal{O}(m)) = 0$*, for* $i \geq 1$*.*
(2) $R^i f_*(\mathcal{O}(-m)) = 0$ *for* $i < n - 1$*.*

(3) $R^{n-1}f_*(\mathcal{O}(-m)) = 0$, *for* $0 \le m \le n-1$.
(4) $R^{n-1}f_*(\mathcal{O}(-m)) = \Gamma^{m-n}(V^\vee) \otimes \mathrm{Det}(V)^\vee$, *for* $m \ge n$.
(5) Computations similar to (1)-(4), but over $\mathrm{Fl}(V)$ *in place of* $\mathbb{P}(V)$. *Just replace* $\mathcal{O}(*)$ *by* $\mathcal{O}(0,\dots,0,*)$, *or dually* $\mathcal{O}(-*,0,\dots,0)$. *When* $n = 2$, *one gets a natural complete flag on the vector bundle*

$$F^*(R^1F_*(\mathcal{O}(m,0))).$$

Its graded pieces are the line bundles $\mathcal{O}(n_1,n_2)$, $n_1, n_2 \ge 1$, $n_1 + n_2 = m$.
(6) If a *is not an increasing sequence, then*

$$F_*(\mathcal{O}(a_1,\dots,a_n)) = 0.$$

(7) If a *is an increasing sequence, then, for all* $i \ge 1$,

$$R^i(F_*)(\mathcal{O}(a_1,\dots,a_n)) = 0.$$

Proof. For items (1) to (4), see [SP, Tag 30.8]. Item (5) follows, using Leray's spectral sequence and the projection formula, w.r.t. the factorisation

$$\mathrm{Fl}(V) \longrightarrow \mathbb{P}(V) \longrightarrow S.$$

For items (6) and (7), see [B, Proposition 1.4.5]. □

9.2. Some computations.

Lemma 9.3. *Consider the projective bundle*

$$f : \mathbb{P}(V) \longrightarrow S$$

with tautological extension

$$\mathcal{E} : 0 \longrightarrow \mathcal{V}_{n-1} \longrightarrow \mathcal{V}_n (= f^*(V)) \longrightarrow \mathcal{L}_n (= \mathcal{O}(1)) \longrightarrow 0.$$

The following equalities hold.

(1) $R^if_*(\mathcal{V}_{n-1}) = 0$, *for all* $i \ge 0$.

(2) $R^if_*(\mathcal{V}_{n-1}(-1)) = 0$, *for all* $i \ne 1$.

(3) $R^1f_*(\mathcal{V}_{n-1}(-1)) = \mathcal{O}_S$, *with generator given by* $\mathcal{E}$.

(4) $f_*(\mathcal{V}_{n-1}^\vee(1)) = \mathrm{End}(V)/\mathcal{O}_S\mathrm{Id}$.

(5) $R^if_*(\mathcal{V}_{n-1}^\vee(1)) = 0$, *for all* $i \ge 1$.

(6) $R^if_*(\mathrm{End}(\mathcal{V}_{n-1})) = 0$, *for* $i \ge 1$.

(7) $f_*(\mathrm{End}(\mathcal{V}_{n-1})) = \mathcal{O}_S$, *with generator given by* $\mathrm{Id}_{\mathcal{V}_{n-1}}$.

Proof. Applying f_* to the surjection of vector bundles over $\mathbb{P}(V)$

$$f^*(V) \longrightarrow \mathcal{O}(1),$$

one gets

$$\mathrm{Id} : V \longrightarrow V.$$

Item (1) follows, applying f_* to $\mathcal{E}$, together with the projection formula, and the well-known equalities

$$f_*(\mathcal{O}_{\mathbb{P}(V)}) = \mathcal{O}_S,$$

and

$$R^if_*(\mathcal{O}_{\mathbb{P}(V)}) = 0,$$

for all $i \geq 1$.
To get items (2) to (5), consider the extensions

$$\mathcal{E}(-1) : 0 \longrightarrow \mathcal{V}_{n-1}(-1) \longrightarrow f^*(V)(-1) \longrightarrow \mathcal{O}_{\mathbb{P}(V)} \longrightarrow 0$$

and

$$\mathcal{E}^\vee(1) : 0 \longrightarrow \mathcal{O}_Y \longrightarrow f^*(V^\vee)(1) \longrightarrow \mathcal{V}_{n-1}^\vee(1) \longrightarrow 0.$$

Apply $f_*(.)$ to these, using the projection formula, and the vanishing of

$$f_*(\mathcal{O}(-1)), R^i f_*(\mathcal{O}(1)), R^i f_*(\mathcal{O}(-1)),$$

which holds for all $i \geq 1$. Introduce the extension

$$\mathcal{E}^\vee \otimes \mathcal{V}_{n-1} : 0 \longrightarrow \mathcal{V}_{n-1}(-1) \longrightarrow f^*(V^\vee) \otimes \mathcal{V}_{n-1} \longrightarrow \mathrm{End}(\mathcal{V}_{n-1}) \longrightarrow 0.$$

Applying the same technique as above, one gets items (6) and (7).

□

One can perform similar computations on Grassmannians and other flag schemes. If S is an $\mathbb{F}_p$-scheme, some items above remain valid for Frobenius twists- depending on the cohomological degree and on the dimension of the tautological bundles. Positive examples are treated in a systematic manner in the Lemmas below.

LEMMA 9.4. *Assume that S is an $\mathbb{F}_p$-scheme.*
For $1 \leq m \leq n-2$, consider the Grassmannian

$$f : \mathrm{Gr}(m, V) \longrightarrow S,$$

over which there is the tautological extension of vector bundles

$$\mathcal{E} : 0 \longrightarrow \mathcal{V}_m \longrightarrow f^*(V) \longrightarrow \mathcal{V}_{n/m} \longrightarrow 0.$$

The following is true.

1. *For all $i \geq 1$, $R^i f_*(\mathcal{V}_m^\vee \otimes \mathcal{V}_{n/m}) = 0$.*
2. *For all $r \geq 0$, $\mathrm{Ext}^1_{\mathcal{O}_{\mathrm{Gr}(m,V)}}(\mathcal{V}_m^{(r)}, \mathrm{Sym}^{p^r}(\mathcal{V}_m)) = 0$.*
3. *For all $i \geq 0$, $f_*(\mathrm{End}(\Gamma^i(\mathcal{V}_m))) = \mathcal{O}_S \mathrm{Id}$.*
4. *For all $r \geq 1$, $\mathrm{Hom}_{\mathcal{O}_{\mathrm{Gr}(m,V)}}(\mathcal{V}_m^{(r)}, \Gamma^{p^r}(\mathcal{V}_m)) = 0$.*

Moreover, items (2), (3) and (4) hold, replacing $\mathcal{V}_m$ by $\mathcal{V}_m^{(s)}$, for all $s \geq 1$.

Proof. Let us prove item (1). Consider the factorisation

$$\mathrm{Fl}(V) \xrightarrow{g} \mathrm{Gr}(m, V) \longrightarrow S,$$

where g is the composite of the complete flag schemes of $\mathcal{V}_m$ and $\mathcal{V}_{n/m}$. One has $g_*(\mathcal{O}_{\mathrm{Fl}(V)}) = \mathcal{O}_{\mathrm{Gr}(m,V)}$, and $R^i g_*(\mathcal{O}_{\mathrm{Fl}(V)}) = 0$ for all $i \geq 1$. This is easily derived from the similar well-known formulas for projective bundles.
Hence, using the projection formula, it suffices to prove (1) after base-change to $\mathrm{Fl}(V)$ (that is to say, with F in place of f). There, the vector bundle $\mathcal{V}_m^\vee \otimes \mathcal{V}_{n/m}$ acquires a complete filtration, whose graded pieces are line bundles of the shape $\mathcal{O}(0, \ldots, 0, -1, 0, \ldots, 0, 1, 0, \ldots, 0)$. These have $R^i F_*(.) = 0$, for all $i \geq 1$. Indeed, for the line bundle $\mathcal{O}(-1, 0, \ldots, 0, 1)$, this is item (7) of Lemma 9.2. In all other cases, the sequence of integers inside $\mathcal{O}(.)$ contains two consecutive terms, which are either $(0, -1)$ or $(1, 0)$. Say their indices are $j, j+1$, so that the corresponding line bundles on $\mathrm{Fl}(V)$ are $\mathcal{L}_j$ and $\mathcal{L}_{j+1}$ Consider the factorisation

$$\mathrm{Fl}(V) \xrightarrow{h} \mathrm{Fl}(1, \ldots, j-1, j+1, j+2, \ldots, n-1, V) \longrightarrow S.$$

Then h is the $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{V}_{j+1/j-1})$, with twisting sheaf $\mathcal{O}(1) := \mathcal{L}_{j+1}$. Recall that, for all $i \geq 0$, one has

$$R^i h_*(\mathcal{L}_{j+1}^{-1}) = R^i h_*(\mathcal{O}(-1)) = 0,$$

and similarly

$$R^i h_*(\mathcal{L}_j) = R^i h_*(\mathrm{Det}(\mathcal{V}_{j+1/j-1})(-1)) = 0,$$

by the projection formula. Using Leray's spectral sequence for the factorisation above, one indeed concludes that $R^i F_*(.) = 0$, for all graded pieces and for all $i \geq 1$. The claim follows by dévissage on the filtration. Let us prove items (2), (3) and (4) as stated (i.e. when $s = 0$). The proof for $s \geq 1$ is the same. For (2), arguing as above, one gets that

$$\mathrm{Ext}^1_{\mathcal{O}_{\mathrm{Gr}(m,V)}}(\mathcal{V}_m^{(r)}, \mathrm{Sym}^{p^r}(\mathcal{V}_m)) = \mathrm{Ext}^1_{\mathcal{O}_{\mathrm{Fl}(V)}}(\mathcal{V}_m^{(r)}, \mathrm{Sym}^{p^r}(\mathcal{V}_m)).$$

Using the projection formula and item (1) of Proposition 9.2, one then sees that

$$\mathrm{Ext}^1_{\mathcal{O}_{\mathrm{Fl}(V)}}(\mathcal{V}_m^{(r)}, \mathrm{Sym}^{p^r}(\mathcal{V}_m)) = \mathrm{Ext}^1_{\mathcal{O}_{\mathrm{Fl}(V)}}(\mathcal{V}_m^{(r)}, \mathcal{L}_m^{p^r}).$$

By dévissage along the natural filtration of $\mathcal{V}_m$, it remains to prove vanishing of

$$\mathrm{Ext}^1_{\mathcal{O}_{\mathrm{Fl}(V)}}(\mathcal{L}_i^{p^r}, \mathcal{L}_m^{p^r})) = H^1(\mathrm{Fl}(V), \mathcal{L}_i^{-p^r} \otimes \mathcal{L}_m^{p^r}),$$

for $i = 1, \ldots, m$. This is clear if $i = m$. If $i < m$, consider the factorisation

$$\mathrm{Fl}(V) \longrightarrow \mathrm{Fl}(1, \ldots, m-1, m, V) \xrightarrow{h} \mathrm{Fl}(1, \ldots, m-1, V) \longrightarrow S,$$

where h is the projective bundle of the vector bundle dual to $\mathcal{V}_n/\mathcal{V}_{m-1}$. Its twisting sheaf is $\mathcal{O}(1) := \mathcal{L}_m^{-1}$. Since $m \leq n-2$, one has

$$R^j h_*(\mathcal{L}_m^{p^r}) = R^j h_*(\mathcal{O}(-p^r)) = 0,$$

for $j = 0, 1$, by item (2) of Proposition 9.2. Once more, we conclude using Leray spectral sequence and projection formula. By duality (between Sym and Γ), the equality in item (3) is equivalent to

$$f_*(\mathrm{End}(\mathrm{Sym}^i(\mathcal{V}_m))) = \mathcal{O}_S \mathrm{Id},$$

which we now prove. As above, this first reduces to

$$f_*(\mathrm{End}_{\mathrm{Fl}(V)}(\mathrm{Sym}^i(\mathcal{V}_m))) = \mathcal{O}_S \mathrm{Id},$$

then to

$$f_*(\mathrm{Sym}^i(\mathcal{V}_m)^\vee \otimes \mathcal{L}_m^i) = \mathcal{O}_S \pi_m^i,$$

where $\pi_m : \mathcal{V}_m \longrightarrow \mathcal{L}_m$ is the natural surjection. The vector bundle $\mathrm{Sym}^i(\mathcal{V}_m)$ has a natural complete filtration, with graded pieces the line bundles

$$\mathcal{O}(a_1, \ldots, a_m, 0, \ldots, 0),$$

one for each partition $a_1 + \ldots + a_{m-1} + a_m = i$. By dévissage, it suffices to prove

$$f_*(\mathcal{O}_{\mathrm{Fl}(V)}) = \mathcal{O}_S,$$

for the partition $0 + \ldots + 0 + i = i$, and

$$f_*(\mathcal{O}(-a_1, \ldots, \ldots, -a_{m-1}, i - a_m, 0 \ldots, 0)) = 0,$$

for all other partitions. The former is clear. For the latter, observe that $i - a_m < 0$, and use item (6) of Proposition 9.2.
To prove item (4), pick

$$\phi \in \mathrm{Hom}_{\mathcal{O}_{\mathrm{Gr}(m,V)}}(\mathcal{V}_m^{(r)}, \Gamma^{p^r}(\mathcal{V}_m)).$$

Consider the composite

$$\psi : \Gamma^{p^r}(\mathcal{V}_m) \xrightarrow{\mathrm{frob}^r_{\mathcal{V}_m}} \mathcal{V}_m^{(r)} \xrightarrow{\phi} \Gamma^{p^r}(\mathcal{V}_m).$$

By item (3), one gets $\psi = \lambda \mathrm{Id}$, where

$$\lambda \in H^0(S, \mathcal{O}_S) = H^0(\mathrm{Gr}(m,V), \mathcal{O}_{\mathrm{Gr}(m,V)}).$$

It suffices to prove $\lambda = 0$, for then the surjectivity of $\mathrm{frob}^r_{\mathcal{V}_m}$ implies $\phi = 0$. To finish, observe that vanishing of λ can be checked over each affine open $U \subset \mathrm{Gr}(m,V)$, where the vector bundle $\mathcal{V}_m$ is trivial. Over such a $U = \mathrm{Spec}(R)$, choosing an R-basis of $\mathcal{V}_m$ identifies the map of R-modules

$$\mathrm{frob}^r_{\mathcal{V}_m} : \Gamma^{p^r}(\mathcal{V}_m) \longrightarrow \mathcal{V}_m^{(r)}$$

to a projection

$$\mathrm{pr} : R^M \xrightarrow{(x_1,\ldots,x_M) \longrightarrow (x_1,\ldots,x_m)} R^m.$$

It then becomes obvious that $\lambda \mathrm{Id}_{R^M}$ factors through pr, if and only if $\lambda = 0$. □

Lemma 9.5. *Assume that S is an $\mathbb{F}_p$-scheme. Let $1 \le m \le n-2$ be an integer. Denote by*

$$F : \mathrm{Fl}(V) \longrightarrow S$$

the complete flag scheme of V, and by

$$\pi_m : \mathcal{V}_m \longrightarrow \mathcal{L}_m$$

the natural surjection of vector bundles over $\mathrm{Fl}(V)$*, with kernel $\mathcal{V}_{m-1}$.*
Denote by

$$\mathcal{E}_{m-1,m} : 0 \longrightarrow \mathcal{V}_{m-1} \longrightarrow \mathcal{V}_m \xrightarrow{\pi_m} \mathcal{L}_m \longrightarrow 0$$

the tautological extension of vector bundles over $\mathrm{Fl}(V)$*.*

Let $r \ge 0$ be an integer. The following equalities hold.

(1) $F_*(\mathcal{V}_m^{(r)\vee} \otimes \mathcal{L}_m^{(r)}) = \mathcal{O}_S \pi_m^{(r)}$.
(2) $R^1F_*(\mathcal{V}_m^{(r)\vee} \otimes \mathcal{L}_m^{(r)}) = 0$.
(3) $R^1F_*(\mathcal{L}_m^{(r)\vee} \otimes \mathcal{V}_{m-1}^{(r)}) = \mathcal{O}_S \mathcal{E}^{(r)}_{m-1,m}$.
(4) $F_*(\mathcal{L}_m^{(r)\vee} \otimes \mathcal{V}_{m-1}^{(r)}) = 0$.
(5) $R^1F_*(\mathrm{End}(\mathcal{V}_m^{(r)})) = 0$.

Proof. Proofs of items (1) and (2), are the same as that of items (2) and (3) of Lemma 9.4. For $m = 2$, using Leray's spectral sequence w.r.t.

$$\mathrm{Fl}(V) \xrightarrow{f} \mathrm{Fl}(3,4,\ldots,n-1,V) \longrightarrow S,$$

one deduces (3) and (4) from Lemma 9.6 here below (applied to the vector bundle $\mathcal{V}_3$, over $\mathrm{Fl}(3,4,\ldots,n-1,V)$). For $m = 3$, consider (the long exact sequence in cohomology obtained by applying $f_*(.)$ to) the natural extension

$$0 \longrightarrow \mathcal{L}_3^{(r)\vee} \otimes \mathcal{V}_2^{(r)} \longrightarrow \mathcal{L}_3^{(r)\vee} \otimes \mathcal{V}_3^{(r)} \longrightarrow \mathcal{O}_{\mathrm{Fl}(V)} \longrightarrow 0.$$

By item (2) of Proposition 9.2, and the projection formula, one gets, for $i = 0, 1$,

$$R^i f_*(\mathcal{L}_3^{(r)\vee} \otimes \mathcal{V}_3^{(r)}) = R^i f_*(\mathcal{L}_3^{-p^r}) \otimes \mathcal{V}_3^{(r)} = 0.$$

Conclude using Leray's spectral sequence. The proof for $m \ge 4$ is similar.

Let us prove item (5). The case $m = 1$ boils down to vanishing of $R^1F_*(\mathcal{O}_{\mathrm{Fl}(V)})$. The case $m \ge 2$ follows by induction, like this. Consider the exact sequence

$$0 \longrightarrow \mathcal{V}_m^{(r)\vee} \otimes \mathcal{V}_{m-1}^{(r)} \longrightarrow \mathrm{End}(\mathcal{V}_m^{(r)}) \longrightarrow \mathcal{V}_m^{(r)\vee} \otimes \mathcal{L}_m^{(r)} \longrightarrow 0.$$

Using item (2), it remains to prove vanishing of $R^1F_*(\mathcal{V}_m^{(r)\vee} \otimes \mathcal{V}_{m-1}^{(r)})$.
To do so, consider the natural extension

$$0 \longrightarrow \mathcal{L}_m^{-p^r} \otimes \mathcal{V}_{m-1}^{(r)} \longrightarrow \mathcal{V}_m^{(r)\vee} \otimes \mathcal{V}_{m-1}^{(r)} \longrightarrow \mathrm{End}(\mathcal{V}_{m-1}^{(r)}) \longrightarrow 0.$$

Using the induction hypothesis, it remains to prove vanishing of the induced arrow

$$h : R^1F_*(\mathcal{L}_m^{-p^r} \otimes \mathcal{V}_{m-1}^{(r)}) \longrightarrow R^1F_*(\mathcal{V}_m^{(r)\vee} \otimes \mathcal{V}_{m-1}^{(r)}).$$

Using item (3), the left side is a free $\mathcal{O}_S$-module of rank one, whose generator $\mathcal{E}_{m-1,m}^{(r)}$ is tautologically killed by h. The proof is over. □

The next Lemma is a rigorous formulation of the

Motto. A "sufficiently general" extension of line bundles has no non-trivial automorphism, and its class generates the cohomology group in which it dwells.

LEMMA 9.6. *Assume that* $n = \dim(V) = 3$, *and that* S *is an* $\mathbb{F}_p$*-scheme. Consider the composite*

$$f : \mathrm{Fl}(V) = \mathrm{Fl}(1, 2, V) \xrightarrow{f_1} \mathrm{Fl}(2, V) = \mathbb{P}(V) \xrightarrow{f_2} S.$$

Then

$$f_*(\mathcal{O}(p^r, -p^r, 0)) = 0,$$

and

$$R^1f_*(\mathcal{O}(p^r, -p^r, 0)) = \mathcal{O}_S,$$

with generator given by the class of the tautological extension

$$\mathcal{E}_{1,2}^{(r)} : 0 \longrightarrow \mathcal{L}_1^{(r)} \longrightarrow \mathcal{V}_2^{(r)} \longrightarrow \mathcal{L}_2^{(r)} \longrightarrow 0.$$

Proof. The usual computation of cohomology of $\mathbb{P}^1$-bundles, applied to f_1, gives

$$(f_1)_*(\mathcal{O}(p^r, -p^r, 0)) = 0,$$

and

$$R^1(f_1)_*(\mathcal{O}(p^r, -p^r, 0)) = \Gamma^{2p^r-2}(\mathcal{V}_2) \otimes \mathrm{Det}(\mathcal{V}_2)^{1-p^r}.$$

Indeed, the arrow f_1 is a $\mathbb{P}^1$-bundle, the projective bundle of $\mathcal{V}_2$, and

$$\mathcal{O}(p^r, -p^r, 0) = \mathcal{O}(-2p^r) \otimes \mathrm{Det}(\mathcal{V}_2)^{\otimes p^r},$$

where $\mathcal{O}(1)$ denotes the usual twisting sheaf of a $\mathbb{P}^1$-bundle. Over $\mathrm{Fl}(V)$, the vector bundle $\Gamma^{2p^r-2}(\mathcal{V}_2) \otimes \mathrm{Det}(\mathcal{V}_2)^{1-p^r}$ has a natural filtration, with graded pieces the line bundles $\mathcal{O}(-a, a, 0)$, where $1 - p^r \leq a \leq p^r - 1$. These have $f_*(.) = 0$, except when $a = 0$, for then $f_*(\mathcal{O}_{\mathrm{Fl}(V)}) = \mathcal{O}_S$. Using (a tiny portion of) Leray's spectral sequence for $f = f_2 \circ f_1$, we get an injective arrow of coherent $\mathcal{O}_S$-modules

$$R^1f_*(\mathcal{O}(p^r, -p^r, 0)) \xrightarrow{\sim} (f_2)_*(\Gamma^{2p^r-2}(\mathcal{V}_2) \otimes \mathrm{Det}(\mathcal{V}_2)^{1-p^r}) \hookrightarrow \mathcal{O}_S.$$

It admits the natural arrow

$$\begin{array}{ccc} \mathcal{O}_S & \longrightarrow & R^1f_*(\mathcal{O}(p^r, -p^r, 0)) \\ 1 & \longmapsto & < \mathcal{E}_{1,2}^{(r)} > \end{array},$$

as a splitting, which is hence an isomorphism. □

Remark 9.7. Assumptions and notation being those of Lemma 9.6, one can show that $R^1f_*(\mathcal{O}(a, -a, 0)) = 0$, for every integer $a \geq 0$, which is not a p-th power.

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