Smooth profinite groups, II: the Uplifting Theorem

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Abstract. Let p be a prime. In this paper, we investigate the existence of liftings of mod p representations of a profinite group, to mod p^2 representations. As a concrete application of general results, we get the following. Let F be a field, with separable closure F_s, and let $d \geq 1$ an integer. Then, every Galois representation

$$\rho_1 : \mathrm{Gal}(F_s/F) \longrightarrow \mathbf{GL}_d(\mathbb{Z}/p)$$

lifts to

$$\rho_2 : \mathrm{Gal}(F_s/F) \longrightarrow \mathbf{GL}_d(\mathbb{Z}/p^2).$$

This is a vast improvement on previously known results- see the Introduction for details. To achieve it, we work in the framework of cyclotomic pairs and of smooth profinite groups, developped in [6], and prove a much deeper result. Namely, complete flags of mod p semi-linear representations of a $(1,1)$-smooth profinite group lift, step by step, modulo p^2. This is Theorem 14.1, which we refer to as the Uplifting Theorem. It solves our initial lifting question, in an optimal way. It can be viewed as a non-commutative generalisation of mod p^2 Kummer theory- beyond the usual case of Galois groups of fields, and to higher dimensions.

Contents

1. Introduction.

Let G be a profinite group. Let p be a prime. In this paper, we deal with the following task.

(T): Axiomatize properties of G, ensuring the existence of liftings of its mod p representations (equipped with a complete filtration), to their mod p^2 counterparts. As a concrete result, we obtain:

Theorem. *(Theorem 14.2).*
Let F be a field, with separable closure F_s/F. Let $d \geq 1$ be an integer.
Denote by $\mathbf{G}$ one of the following groups: $\mathbf{GL}_d$, or its Borel subgroup $\mathbf{B}_d$.
Then, the natural map

$$\mathrm{Hom}(\mathrm{Gal}(F_s/F), \mathbf{G}(\mathbb{Z}/p^2)) \longrightarrow \mathrm{Hom}(\mathrm{Gal}(F_s/F), \mathbf{G}(\mathbb{Z}/p))$$

is surjective. Here $\mathrm{Hom}(.,.)$ *stands for continuous arrows of profinite groups.*
Restatement for $\mathbf{G} = \mathbf{GL}_d$: mod p Galois representations of F lift mod p^2.

Remarks 1.1.

- For $d = 1$, the statement follows from the existence of the Teichmüller (multiplicative) section $\mathbb{F}_p^\times \longrightarrow (\mathbb{Z}/p^2)^\times$, given by $(p-1)$-th roots of unity.
- For $d = 2$, the statement goes back to Serre. In the literature, its first occurrence is in [20], Theorem 1- stated under the extra assumption that $F = L$ is a number field, but provided with a proof that actually works for all fields.
- When $d = 2$ and $F = \mathbb{Q}$, and under mild assumptions, mod p Galois representations with coefficients in a finite field k lift to representations over $\mathbf{W}(k)$, by [26].
- The recent texts [5] and [21] contain lifting results in dimensions $d = 3$ and $d = 4$, under extra assumptions.
- When F is a p-adic local field, mod p Galois representations of F actually lift to representations over $\mathbb{Z}_p$. This is proved in the recent work [10].
- For $d \geq 3$ arbitrary, the statement is brand new. Note that the case of $\mathbf{GL}_d$ follows from that of $\mathbf{B}_d$, by a classical restriction/corestriction argument.

- Denote by $\mathbf{U}_d \subset \mathbf{B}_d$ the group of strictly upper triangular matrices. When p is odd, we construct in section 15 a field F, containing $\mathbb{C}$, such that the arrow
$$\mathrm{Hom}(\mathrm{Gal}(F_s/F), \mathbf{U}_3(\mathbb{Z}/p^2)) \longrightarrow \mathrm{Hom}(\mathrm{Gal}(F_s/F), \mathbf{U}_3(\mathbb{F}_p))$$
is not surjective. In other words, in the case of fields, the generic mod p Heisenberg representation does not lift mod p^2.

The most evolved result of this paper is Theorem 14.1 -"The Uplifting Theorem". It is much more general than the previous Theorem, and completes task (T). Roughly speaking, it states that complete flags of mod p semi-linear representations of a $(1,1)$-smooth profinite group lift, step by step, modulo p^2. To get the liftability of Galois representations from the Uplifting Theorem, one then has to apply most of the major results of [6].

Our initial motivation, for considering (T), is the following question.

(Q): Is there a proof of the (surjectivity part of the) Norm Residue Isomorphism Theorem of Rost, Suslin, Voevodsky and Weibel, that just uses Kummer theory?

We will not establish here the link between (T) and (Q). This is done in [7], where we apply Theorem 14.1 to provide a positive answer to (Q) (see [7, Theorem 5.1] for a precise statement).
A common feature of (T) and (Q) is the need for a refined axiomatization of Kummer theory, with coefficients in $\mu_{p^{1+e}}$. Here $e \in \mathbb{N}_{\geq 1} \cup \{\infty\}$ is a number, called *depth*. When $e = \infty$, it is understood that
$$\mu_{p^\infty} = \varprojlim \mu_{p^r},$$
the usual Tate module.
Fundations of this refined axiomatization are laid in the recent work [6]. To do so, three new notions are introduced:

(1) (n,e)-cyclotomic pairs,
(2) (n,e)-cyclothymic profinite groups,
(3) (n,e)-smooth profinite groups.

From now on, in the present paper, we focus on the case $n = e = 1$.

The first notion then applies to a pair $(G, \mathbb{Z}/p^2(1))$, consisting of a profinite group G, and a free $\mathbb{Z}/p^2$-module of rank one equipped with a continuous action of G. Note that, up to isomorphism, $\mathbb{Z}/p^2(1)$ is simply given by a continuous character
$$G \longrightarrow (\mathbb{Z}/p^2)^\times,$$
playing the role of the usual cyclotomic character. The two other notions are much more flexible: they just depends on p, and on a profinite group G. An important fact is that they are equivalent: a $(1,1)$-cyclothymic profinite group is the same thing as a $(1,1)$-smooth profinite group ([6], Theorem 11.4). Precisions are provided in section 8.

We list the main tools used to prove the Uplifting Theorem (Theorem 14.1).

- The framework of cyclotomic pairs, and of smooth profinite groups, as introduced in [6].

- G-linearized Witt vector bundles, over a G-scheme S of characteristic p, and complete flags of these. See [8], [6] and section 3.5 for definitions.
- Extensions of vector bundles over flag schemes of vector bundles, and their splitting schemes- see section 5.3. They will be used to create liftings of embedded Witt line bundles, as explained in section 11.2.
- Good filtrations, on quasi-coherent modules over a scheme. Their purpose is to facilitate dévissage arguments, to prove vanishing results- see section 4.

To understand these tools, and how they are used, familiarity with [8] and [6] is advisable.

Section 13 is completely independent. It is an attempt to investigate subtleties of splitting schemes of tautological extensions of vector bundles. We hope that it is useful for the future.

Section 14 contains the statement of the main result: Theorem 14.1. We provide its proof in section 16.

A geometric application is given in section 17: locally for the Zariski topology, complete flags of $\mathbb{F}_p$-étale local systems lift to complete flags of $\mathbb{Z}/p^2$-étale local systems.

Section 18 treats the case of Galois representations- the Theorem in this Introduction.

In the Appendix, we introduce the notions of smooth closure and of cyclotomic closure. They apply, respectively, to a profinite group G, and to a pair $(G, \mathbb{Z}/p^2(1))$. From an arithmetic point of view, they are avatars of the (Galois group of the) separable closure of a field– with one major difference: they are perfectly functorial. From a geometric point of view, they can be thought of as "resolution of singularities" of G and of $(G, \mathbb{Z}/p^2(1))$, respectively. It is worth remembering that the smooth closure makes the Uplifting Theorem 14.1 usable, to study modular (mod p) representations of arbitrary profinite groups. We will not venture to do so in this paper.

2. Notation, conventions and fundamental concepts.

The notation $A := E$ means that A is *defined as* the expression E.
The letter p denotes a prime number.
Oddly enough, the parity of p plays no role. Even better: p can be arbitrary! ;)
The letter G denotes a profinite group. The reader will be notified, when assumptions are made on G- e.g. being finite, being a pro-p-group, or being $(1,1)$-smooth.

2.1. Witt vectors, lifting Frobenius and lifting line bundles.
For an integer $r \geq 1$, we denote by $\mathbf{W}_r$ the p-typical Witt vectors of length r, seen as scheme of commutative rings, defined over $\mathbb{Z}$. To begin with, it is sufficient to know that $\mathbf{W}_r$ is an endofunctor of the category of commutative rings, such that $\mathbf{W}_r(\mathbb{F}_p) = \mathbb{Z}/p^r\mathbb{Z}$. By glueing, it extends to an endofunctor of the category of schemes. If S is a scheme, there are closed immersions $\mathbf{W}_r(S) \longrightarrow \mathbf{W}_{r+1}(S)$, which are nilpotent if S has characteristic p. For details on this contruction (and much more), we refer to [2].
The functor $\mathbf{W}_r$ enjoys two elementary crucial features, which will be ceaselessly exploited in the sequel.

(1) Let S be a scheme of characteristic p. As would any endomorphism of S, the (absolute) Frobenius $\mathrm{Frob} : S \longrightarrow S$ lifts to an endomorphism $\mathbf{W}_r(S) \longrightarrow \mathbf{W}_r(S)$. It is the (absolute) Frobenius of $\mathbf{W}_r(S)$, still denoted by Frob.
(2) Let S be a scheme, and let L be a line bundle over S. Then, for every $r \geq 2$, L functorially extends (lifts) to a line bundle over $\mathbf{W}_r(S)$- its Teichmüller lift. We denote it by $\mathbf{W}_r(L)$. An introduction to the Teichmüller lift of line bundles can be found in [8]. This is a fundamental elementary construction, obtained by applying the formalism of torsors, to the Teichmüller section

$$\mathbf{W}_1^\times = \mathbb{G}_m \longrightarrow \mathbf{W}_r^\times.$$

2.2. Greenberg transfer.
Let S be a scheme of characteristic p. Let $r \geq 2$ be an integer (in this paper, $r = 2$ almost everywhere). We shall need the Greenberg transfer $R_{\mathbf{W}_r/\mathbf{W}_1}$.
It is a functor, from the category of $\mathbf{W}_r(S)$-schemes, to that of S-schemes.
It is comparable to Weil restriction of scalars, transposed to the p-adic context.
Let $T \longrightarrow \mathbf{W}_r(S)$ be a scheme, over $\mathbf{W}_r(S)$. Let $X \longrightarrow S$ be a scheme over S. Then, on the level of functors of points, we have a functorial bijection

$$R_{\mathbf{W}_r/\mathbf{W}_1}(T)(X) \simeq T(\mathbf{W}_r(X)).$$

We will use Greenberg's structure theorem, in specific situations where we provide concrete constructions. For details on the general setting, see [17], or [1] for a recent revisit.

3. G-equivariant constructions.

In this section, we provide definitions of algebro-geometric objects, and of their G-equivariant counterparts. Notation-wise, if $\mathcal{T}$ is a type of algebro-geometric objects (e.g. a scheme, a vector bundle over a scheme, or a $\mathbf{W}_r$-bundle over a scheme), we often denote by $G\mathcal{T}$ the "G-equivariant" version of $\mathcal{T}$. It consists of objects of type $\mathcal{T}$, endowed with the extra datum of an action of G. We may ask that this action satisfies some technical assumption. Let's get to details.

3.1. G-ACTIONS.
All actions of the profinite group G are continuous, in the following strong sense: a given action (on some algebro-geometric structure) occurs through a finite quotient $G^0 := G/G_0$, with G_0 a normal open subgroup of G. In other words, when we consider a G-action a, we always assume that its kernel $\mathrm{Ker}(a) \subset G$ is open.

3.2. G-SCHEMES.
Following [6], all schemes are quasi-compact.
In practice, all schemes S needed to prove the main result of this paper (Theorem 14.1) occur as

$$(*) : S \xrightarrow{f_2} S_2 \xrightarrow{f_1} S_1,$$

where S_1 is affine, f_1 is projective and f_2 is affine.
A G-scheme is a scheme S equipped with an action of G, such that S has a covering by G-invariant open affines. This condition is guaranteed if, for instance, S is quasi-projective over a field, or more generally if every finite set of points of S is contained in a common open affine. This is the case for schemes $(*)$ as above.

3.3. (G, S)-MODULES.
By $(\mathbb{F}_p, G)$-module, we mean a finite dimensional $\mathbb{F}_p$-vector space equipped with an action of G. Assuming G finite, this is a finite module over the group algebra $\mathbb{F}_p[G]$. In general, let A be a commutative ring equipped with an action of a finite group G. We denote by $A[G]$ the corresponding skew group algebra. It is the free A-module with basis e_g, $g \in G$, with multiplication given by the formula

$$(ae_g).(be_h) = ag(b)e_{gh}.$$

If G acts trivially on A, it is the usual group algebra. By "$A[G]^f$-module", we mean an $A[G]$-module, which is finite locally free as an A-module.
More generally, if S is a G-scheme, we can consider the category of quasi-coherent $\mathcal{O}_S$-modules, equipped with a semi-linear action of G. These will be called $(G, \mathcal{O}_S)$-modules, or (G, S)-modules. A (G, S)-module, which is finite locally free as an $\mathcal{O}_S$-module, will be called a G-vector bundle over S, or (G, S)-bundle.

3.4. $G\mathbf{W}_r$-BUNDLES.
Let $r \geq 1$ be an integer. A $(G, \mathcal{O}_{\mathbf{W}_r(S)})$-module is then the same thing as a $(G, \mathbf{W}_r(\mathcal{O}_S))$-module- see [6, §5]. It is called a $G\mathbf{W}_r$-module over S. If it is locally free as a $\mathbf{W}_r(\mathcal{O}_S)$-module, it will be called a G-linearized Witt vector bundle of height r over S, or simply $G\mathbf{W}_r$-bundle over S.
If G is finite, and if $S = \mathrm{Spec}(A)$ is an affine $(\mathbb{F}_p, G)$-scheme, a $G\mathbf{W}_r$-bundle over S is the same thing as a $\mathbf{W}_r(A)[G]^f$-module.

3.5. FLAGS OF (WITT) VECTOR BUNDLES.

The notation $V_{i,r}$ stands for a $\mathbf{W}_r$-bundle of dimension i, over an $\mathbb{F}_p$-scheme S. For a given $\mathbf{W}_r$-bundle $V_{i,r}$, and for an integer $1 \leq s \leq r$, we denote by

$$V_{i,s} := V_{i,r} \otimes_{\mathbf{W}_r} \mathbf{W}_s$$

its reduction to a $\mathbf{W}_s$-bundle. If

$$\nabla_r : 0 = V_{0,r} \subset V_{1,r} \subset \ldots \subset V_{d,r}$$

is a complete flag of $\mathbf{W}_r$-bundles (see Section 9), we use the notation

$$L_{i,r} := V_{i,r}/V_{i-1,r}$$

for the $\mathbf{W}_r$-line bundles forming its graded pieces.
When used, the notation L_i stands for $L_{i,1}$. For relative integers $a_1, \ldots, a_d$, set

$$\mathcal{O}(a_1, a_2, \ldots, a_d) := L_{1,1}^{\otimes a_1} \otimes_{\mathcal{O}_S} L_{2,1}^{\otimes a_2} \otimes_{\mathcal{O}_S} \ldots \otimes_{\mathcal{O}_S} L_{d,1}^{\otimes a_d};$$

these are usual line bundles over S. This notation extends as follows. If M is a quasi-coherent S-module, set

$$M(a_1, a_2, \ldots, a_d) := M \otimes_{\mathcal{O}_S} \mathcal{O}(a_1, a_2, \ldots, a_d).$$

Suppose that

$$S = \mathbf{Fl}(V_1) \xrightarrow{F} T$$

is a complete flag scheme (see section 9.2) where V_1 is a vector bundle over a scheme T. We then use curly letters $\mathcal{V}_{i,1}$ to denote the i-th piece of the tautological flag

$$\nabla_{gen,1} : 0 = \mathcal{V}_{0,1} \subset \mathcal{V}_{1,1} \subset \ldots \subset \mathcal{V}_{d,1} := F^*(V_1),$$

over S. Denote by

$$\mathcal{L}_{i,1} = \mathcal{V}_{i,1}/\mathcal{V}_{i-1,1}$$

its graded pieces; these are line bundles.
We will also consider the G-equivariant versions of these objects.

3.6. The notation $*, +, ++, -, --$.
In this text, we use the notation $*$ for a relative integer, or more generally an object, whose name it is superfluous to mention. For instance, on the complete flag scheme of a 4-dimensional vector bundle, $\mathcal{O}(a, *, b, *)$ stands for a line bundle of the shape $\mathcal{O}(a, x, b, y)$, where the relative integers x, y do not matter. Similarly, the notation $+$ (resp. $++$, $-$, $--$) stands for a non-negative (resp. positive, non-positive, negative) integer. For instance, $\mathcal{O}(*, ++, -, --)$ denotes a line bundle of the shape $\mathcal{O}(a_1, a_2, a_3, a_4)$, when it suffices to retain the informations $a_2 > 0$, $a_3 \leq 0$ and $a_4 < 0$. The notation $*$ obviously extends to other contexts. For instance,

$$0 \longrightarrow A \longrightarrow * \longrightarrow * \longrightarrow B \longrightarrow 0$$

denotes a 2-extension of B by A (in an abelian category), whose middle terms need not be specified.

3.7. Frobenius functoriality.
Let $V = V_r$ be a $\mathbf{W}_r$-bundle, over an $\mathbb{F}_p$-scheme S. Let $m \geq 1$ be an integer. Denote by

$$\mathrm{Frob}^m : S \longrightarrow S$$

the m-th iterate of the (absolute) Frobenius morphism of S.
Write $(\mathrm{Frob}^m)^*(V)$, or preferably $V^{(m)}$, for the pullback of V, with respect to Frob^m. It is a $\mathbf{W}_r$-bundle.
Write $(\mathrm{Frob}^m)_*(V)$ for the pushforward of V, with respect to Frob^m. It is a quasi-coherent $\mathbf{W}_r(\mathcal{O}_S)$-module. If $r = 1$ and if S is regular, then

$$\mathrm{Frob} : S \longrightarrow S$$

is finite and flat, so that $(\mathrm{Frob}^m)_*(V)$ is still a vector bundle. However, in the present text, there is no need to worry about such issues.
Assume that $s > r$ is an integer, and that W_s is a lift of $V^{(m)}$, to a $\mathbf{W}_s$-bundle.

There may not exist a lift V_s of V to an $\mathbf{W}_s$-bundle, such that $W_s \simeq V_s^{(m)}$.
That being said, it is convenient to denote the lift W_s by $V_s^{[m]}$, and to put

$$V_s^{[m+n]} := W_s^{(n)},$$

for all $n \geq 0$. In other words, the notation $V_s^{[m]}$ stands for a lift of $V_r^{(m)}$, which does not need to occur as the Frobenius twist of a lift of V_r.
Here again, notation may apply to G-equivariant objects.

3.8. Extensions and operations on them.
Let A, B be $G\mathbf{W}_r$-modules, over an $(\mathbb{F}_p, G)$-scheme S.
We denote by $\mathbf{Ext}^n_{G,r}(S, B, A)$, or simply by $\mathbf{Ext}^n_{G,r}(B, A)$ if the dependence on S is clear, the category of n-extensions of $G\mathbf{W}_r$-modules over S. Such an extension is denoted as

$$\mathcal{E} : 0 \longrightarrow A \longrightarrow * \longrightarrow \ldots \longrightarrow * \longrightarrow B \longrightarrow 0.$$

We shall always assume that $\mathcal{E}$ is *schematic*, i.e. that it arises from an extension $\mathbf{E}$ of *schemes* in $G\mathbf{W}_r$-modules over S– as described in [6], 4.3, item (3). It will always be clear, from which $\mathbf{E}$ the extension $\mathcal{E}$ arises. In fact, we could have worked with schemes in $G\mathbf{W}_r$-modules, instead of $G\mathbf{W}_r$-modules. We chose not to do so, as it would have resulted in introducing unnecessarily mesmerizing new objects. Our only motivation, for restricting to schematic $\mathcal{E}$'s, is to allow changing the base by arbitrary morphisms of $(\mathbb{F}_p, G)$-schemes– see below.
The extension $\mathcal{E}$ is said to be *trivial*, if it is trivial in the Yoneda (or, equivalently, derived category) setting- see, for instance, [6] or [9]. Equivalence classes of n-extensions are denoted by $\mathrm{Ext}^n_{G,r}(S, B, A)$, or simply by $\mathrm{Ext}^n_{G,r}(B, A)$ if generating no confusion. We use the notation

$$H^n_{G,r}(S, A) := \mathrm{Ext}^n_{G,r}(S, \mathbf{W}_r(\mathcal{O}_S), A),$$

or simply $H^n_{G,r}(A)$. This group does in fact not depend on r, as it is given by cohomology of G-equivariant sheaves of abelian groups on S. To see why, use the local-to-global spectral sequence (see section 3.10) to reduce to the case $G = 1$. It then follows from the fact that, for $s \geq r$, $\mathbf{W}_r(S)$ and $\mathbf{W}_s(S)$ agree as topological spaces. Thus, we will denote $H^n_{G,r}(A)$ simply by $H^n_G(A)$.

Remarks 3.1.

(1) When $G = 1$, what precedes boils down to cohomology groups $\mathrm{Ext}^n(.,.)$ and $H^n(.)$, for quasi-coherent $\mathbf{W}_r(S)$-modules.
(2) When $S = \mathrm{Spec}(\mathbb{F}_p)$ and $r = 1$, what precedes boils down to cohomology of G, with coefficients in p^r-torsion discrete G-modules: $\mathrm{Ext}^n_G(.,.)$ and $H^n(G, .)$.
(3) Let $s > r$ be an integer. Regarding A and B as $G\mathbf{W}_s$-modules over S, we can consider $\mathrm{Ext}^n_{G,s}(B, A)$. There is a natural arrow

$$\mathrm{Ext}^n_{G,r}(B, A) \longrightarrow \mathrm{Ext}^n_{G,s}(B, A).$$

It fails to be surjective in general. It is injective when $n = 1$, but it can fail to be injective when $n \geq 2$.
(4) Assume that A and B are $G\mathbf{W}_r$-bundles, and that S is quasi-projective over a field (or more generally, over an affine scheme). Then, each class $c \in \mathrm{Ext}^n_{G,r}(B, A)$ is represented by a Yoneda extension

$$\mathcal{E} : 0 \longrightarrow A \longrightarrow * \longrightarrow \ldots \longrightarrow * \longrightarrow B \longrightarrow 0,$$

where the $*$'s are $G\mathbf{W}_r$-bundles as well. Equivalently, using dimension shifting, there exists an extension of $G\mathbf{W}_r$-bundles

$$0 \longrightarrow A \xrightarrow{\iota} A' \longrightarrow A'' \longrightarrow 0,$$

such that

$$\iota_*(c) = 0 \in \mathrm{Ext}^n_{G,r}(B, A').$$

If $G = 1$, this is a classical result, using an ample invertible sheaf on S. The case G arbitrary is then an exercise, using induction from open subgroups of G, and Shapiro's lemma.

Extensions are subject to four functorial operations, which we briefly recall. For a more detailled exposition, see [6], section 4.1. Let

$$\mathcal{E} : 0 \longrightarrow A \longrightarrow * \longrightarrow \ldots \longrightarrow * \longrightarrow B \longrightarrow 0$$

be an n-extension of $G\mathbf{W}_r$-modules over S.

- Pushforward. If $f : A \longrightarrow A'$ is an arrow in $\{G\mathbf{W}_r - \mathrm{Mod}\}$, the pushforward
$$f_*(\mathcal{E}) : 0 \longrightarrow A' \longrightarrow * \longrightarrow \ldots \longrightarrow * \longrightarrow B \longrightarrow 0$$
is defined in the usual fashion.
- Pullback. If $g : B' \longrightarrow B$ is an arrow in $\{G\mathbf{W}_r - \mathrm{Mod}\}$, the pullback
$$g^*(\mathcal{E}) : 0 \longrightarrow A \longrightarrow * \longrightarrow \ldots \longrightarrow * \longrightarrow B' \longrightarrow 0$$
is defined in the usual fashion.
- Change of the base. Let $F : S' \longrightarrow S$ be a morphism of G-schemes. Following [6], 4.3, item (3), using that $\mathcal{E}$ is schematic, we can form the change of the base
$$F^*(\mathcal{E}) : 0 \longrightarrow F^*(A) \longrightarrow * \longrightarrow \ldots \longrightarrow * \longrightarrow F^*(B) \longrightarrow 0.$$
It is an object of $\mathbf{Ext}^n_{G,r}(S', F^*(B), F^*(A))$. Note that, if $\mathcal{E}$ is an extension of $G\mathbf{W}_r$-bundles, and the schematic structure considered is the natural one, then $F^*(\mathcal{E})$ is the usual change of the base, for extensions of vector bundles.
We sometimes use the terminology "pullback" also for "change of the base".
- Baer sum. If
$$\mathcal{E}_1 : 0 \longrightarrow A \longrightarrow * \longrightarrow \ldots \longrightarrow * \longrightarrow B \longrightarrow 0$$
and
$$\mathcal{E}_2 : 0 \longrightarrow A \longrightarrow * \longrightarrow \ldots \longrightarrow * \longrightarrow B \longrightarrow 0$$
are two extensions (objects of $\mathbf{Ext}^n_{G,r}(B, A)$), we can form their Baer sum
$$\mathcal{E}_1 + \mathcal{E}_2 : 0 \longrightarrow A \longrightarrow * \longrightarrow \ldots \longrightarrow * \longrightarrow B \longrightarrow 0.$$

Note that pushforwards and pullbacks commute, in the sense that there is a natural isomorphism

$$f_*(g^*(\mathcal{E})) \xrightarrow{\sim} g^*(f_*(\mathcal{E})).$$

Similarly, pushforward, pullback and change of the base commute to Baer sum. Slightly abusing notation, for

$$\mathcal{F} \in \mathrm{Ext}^n_{G,r}(\mathrm{Frob}^*(B), \mathrm{Frob}^*(A)),$$

we use $f_*(\mathcal{F})$ to denote $\mathrm{Frob}(f)_*(\mathcal{F})$; similarly for pullbacks.

3.9. Geometric triviality. Let A and B be $G\mathbf{W}_r$-bundles over S.
Say that A is geometrically trivial iff $A \simeq \mathbf{W}_r(\mathcal{O}_S)^d$ is trivial as a $\mathbf{W}_r$-bundle. Accordingly, a 1-extension of $G\mathbf{W}_r$-bundles is geometrically split (or trivial) if it splits, as an extension of $\mathbf{W}_r$-bundles. For $n \geq 1$, define

$$\mathrm{ext}^n_r(A,B) := \mathrm{Ker}(\mathrm{Ext}^n_{G,r}(A,B) \longrightarrow \mathrm{Ext}^n_r(A,B)),$$

and

$$h^n(A) := \mathrm{Ker}(H^n_G(A) \longrightarrow H^n(A)).$$

Elements of the group $\mathrm{ext}^n_r(A,B)$ will be called *geometrically trivial* cohomology classes. For $n \geq 2$, note that this notion is much weaker, than that of *strongly geometrically trivial* n-extensions, introduced in [6]. In the present text, we shall only need geometrically trivial cohomology classes, in degrees $n = 1$ and $n = 2$.
In general, “something geometrically trivial over S” means ”something over S, equipped with an action of G, which becomes trivial when the action of G is forgotten”.

Remark 3.2. This definition is inspired by classical ones. For instance, in the theory of Chow groups of (projective homogeneous) varieties X over a field F, the groups

$$\mathrm{ch}^n(X) := \mathrm{Ker}(\mathrm{CH}^n(X) \longrightarrow \mathrm{CH}^n(\overline{X}))$$

are a major topic of investigation. So does the algebraic Brauer group

$$\mathrm{Br}_a(X) := \mathrm{Ker}(\mathrm{Br}(X) \longrightarrow \mathrm{Br}(\overline{X})).$$

Here $\overline{X}$ stands for the fiber product $X \times_F \overline{F}$, where $\overline{F}/F$ is an algebraic closure of F.

3.10. The “local-to-global” spectral sequence. Let S be an $(\mathbb{F}_p, G)$-scheme. Let A, B be $G\mathbf{W}_r$-modules over S. There is the usual local-to-global spectral sequence

$$H^i(G, \mathrm{Ext}^j_r(A,B)) \Rightarrow \mathrm{Ext}^{i+j}_{G,r}(A,B).$$

We shall exclusively use it in low degree, to compute the obstruction to lifting extensions of $G\mathbf{W}_r$-bundles. It then has a tangible interpretation, as follows. Let

$$\mathcal{E}_r : 0 \longrightarrow V_r \longrightarrow * \longrightarrow W_r \longrightarrow 0$$

be an extension of $G\mathbf{W}_r$-bundles over S. Let V_{r+1} (resp. W_{r+1}) be a given lift of V_r (resp. W_r) to a $G\mathbf{W}_{r+1}$-bundle. We would like to lift $\mathcal{E}_r$, to an extension of $G\mathbf{W}_{r+1}$-bundles

$$\mathcal{E}_{r+1} : 0 \longrightarrow V_{r+1} \longrightarrow * \longrightarrow W_{r+1} \longrightarrow 0.$$

The obstruction to do so is a class

$$c \in \mathrm{Ext}^2_{G,1}(W_1^{(r)}, V_1^{(r)}).$$

It belongs to $\mathrm{ext}^2_1(W_1^{(r)}, V_1^{(r)})$ if, and only if, $\mathcal{E}_r$ lifts to an extension of $\mathbf{W}_{r+1}$-bundles (dismissing the action of G). If this is the case, the edge map

$$\mathrm{ext}^2_1(W_1^{(r)}, V_1^{(r)}) \longrightarrow H^1(G, \mathrm{Ext}^1_1(W_1^{(r)}, V_1^{(r)}))$$

kills c, if and only if $\mathcal{E}_r$ lifts to an extension of $\mathbf{W}_{r+1}$-bundles

$$\mathcal{F}_{r+1} : 0 \longrightarrow V_{r+1} \longrightarrow * \longrightarrow W_{r+1} \longrightarrow 0,$$

whose cohomology class is G-invariant. More accurately, for all $g \in G$, there should exist an isomorphism

$$\phi_g : g^*(\mathcal{F}_{r+1}) \simeq \mathcal{F}_{r+1},$$

lifting the isomorphism $g^*(\mathcal{E}_r) \simeq \mathcal{E}_r$ arising from the G-structure on $\mathcal{E}_r$. If this is the case, the obstruction to endowing $\mathcal{F}_{r+1}$ with a semi-linear action of G, and thus turning it into an extension of $G\mathbf{W}_r$-bundles, belongs to

$$H^2(G, \mathrm{Hom}(W_1^{(r)}, V_1^{(r)})).$$

It vanishes if and only if the ϕ_g's can be chosen to satisfy the usual compatibility (cocycle) condition. Writing down computational details is an instructive exercise.

Remark 3.3. What precedes is a technical variation on a simple theme. Consider some algebro-geometric strucure V, over a G-base S. To G-linearize V ("to make it G-equivariant"), a necessary condition is that V be G-invariant, up to isomorphism. If this holds, one may then proceed to search for a G-structure on V. Note that, if G is a free profinite group (e.g. $G = \hat{\mathbb{Z}}$), there is no difference between "G-invariant" and "G-linearizable". Thus, we get the equivalence of the following assertions.

(1) The structure V is G-invariant.
(2) For every homomorphism $\phi : \hat{\mathbb{Z}} \longrightarrow G$, the structure V is $\hat{\mathbb{Z}}$-linearizable via ϕ.

3.11. Recollections on induction and restriction.

Definition 3.4 (Restriction). *Let $H \subset G$ be an open subgroup. Let S be a G-scheme. We denote by $\mathrm{Res}_H^G(S)$ the scheme S, viewed as an H-scheme.*
When this creates no confusion, we may denote the H-scheme $\mathrm{Res}_H^G(S)$ simply by S.

Definition 3.5 (Induction). *Let $H \subset G$ be an open subgroup. Let S be an H-scheme. We define the induced G-scheme*

$$\mathrm{Ind}_H^G(S) := \mathrm{Hom}_H(G, S)$$

as follows. On the level of functors of points, it consists of functions

$$f : G \longrightarrow S$$

such that

$$f(hg) = hf(g),$$

for all $h \in H$ and $g \in G$. Its G-scheme structure is given by

$$(x.f)(g) = f(gx),$$

for $x, g \in G$ and $f \in \mathrm{Ind}_H^G(S)$.
Induction is functorial in the H-scheme S.
The functor Ind_H^G is right adjoint to the forgetful functor

$$\mathrm{Res}_H^G : \{G - \mathrm{Sch}\} \longrightarrow \{H - \mathrm{Sch}\}.$$

If S is a G-scheme, we denote by

$$\Delta : S \longrightarrow \mathrm{Ind}_H^G(\mathrm{Res}_H^G(S)),$$

$$s \mapsto (g \mapsto gs).$$

the adjunction morphism.

Definition 3.6 (Induction, relative version).
Let S be a G-scheme. Let $H \subset G$ be an open subgroup.
Let

$$f : T \longrightarrow \mathrm{Res}_H^G(S)$$

be an H-scheme, over $\mathrm{Res}_H^G(S)$.
We define $\mathrm{Ind}_H^G(T, f)$, a G-scheme over S, as the fibered product

$$\begin{tikzcd} \mathrm{Ind}_H^G(T,f) \arrow[rr] \arrow[d] & & S \arrow[d, "\Delta"] \\ \mathrm{Ind}_H^G(T) \arrow[r, "\mathrm{Ind}_H^G(f)"] & \mathrm{Ind}_H^G(\mathrm{Res}_H^G(S)). & \end{tikzcd}$$

We shall denote it simply by

$$\mathrm{Ind}_H^G(f) \longrightarrow S,$$

if there is no risk of confusing it with the arrow $\mathrm{Ind}_H^G(f)$.

The functor

$$\mathrm{Ind}_H^G : \{H - \mathrm{Sch}/S\} \longrightarrow \{G - \mathrm{Sch}/S\}$$

is right adjoint to the forgetful functor

$$\mathrm{Res}_H^G : \{G - \mathrm{Sch}/S\} \longrightarrow \{H - \mathrm{Sch}/S\}.$$

Remark 3.7. Denote by n the index of H in G. Choosing a system

$$\{g_1, \ldots, g_n\} \subset G$$

of representatives of cosets in $H\backslash G$ yields a (noncanonical) isomorphism

$$\mathrm{Ind}_H^G(T) \simeq \underbrace{T \times_{\mathbb{Z}} T \times_{\mathbb{Z}} \ldots \times_{\mathbb{Z}} T}_{n \text{ times}}.$$

Similarly, we get an isomorphism

$$\mathrm{Ind}_H^G(f) \simeq \underbrace{T \times_f T \times_f \ldots \times_f T}_{n \text{ times}}.$$

In case f is the restriction of a (G, S)-scheme $T \longrightarrow S$, there is a canonical isomorphism

$$\mathrm{Ind}_H^G(f) \simeq T^{G/H},$$

where the right-hand side denotes a product of copies of T, fibered over S, and indexed by the set G/H.

3.12. Cohomological detox. This paper follows the guideline “effectiveness of a proof matters as much as its result”. It is consistent with minimizing the use of cohomological devices of degree ≥ 2. I do, however, respect these- for without knowing them, I would not have written a single line of this text.

4. Good filtrations.

Filtrations are a main tool in this text. They are, in many cases, just geometric filtrations: they need not respect G-actions.
Filtrations encountered in this text are natural. They enjoy nice properties, which facilitate dévissage arguments. They will be refered to as *good* filtrations.
We give a motivated definition.
Let $F : S \longrightarrow T$ be a morphism of schemes. Let M be a quasi-coherent S-module. As usual in algebraic geometry, and for various reasons, we will need to show the vanishing of $R^iF_*(M)$. In the current context, derived functors are taken with respect to the Zariski topology, and $i = 0$ or 1. The price to pay, for working in such a delightfully light cohomological setting, is that the module M is often messy. Rather than being a low-dimensional object (e.g. a line bundle), it will often be a direct limit of complicated vector bundles. Such modules M typically arise when considering compositions of splitting schemes of torsors under vector bundles. The reader may take a glimpse at section 16, where this is particularly obvious. For proving the required vanishing of $R^iF_*(M)$, we use dévissage arguments, relatively to filtrations of the following shape.

Definition 4.1. *(Good filtrations, well-filtered morphisms)*
Let S be a scheme, and let M be a quasi-coherent S-module.
A good filtration on M is the data of

- *A well-ordered set J. In practice, J is often a subset of $\mathbb{N}^n$, with the usual lexicographic order. In all cases, we use 0 to denote its least element.*
- *An increasing filtration $(M_j)_{j\in J}$ of M, by quasi-coherent sub-modules, whose graded pieces*

$$F_j := M_j/\sum_{i<j} M_i$$

are vector bundles, for all $j \in J$. In practice, they are often line bundles.

Let $g : S' \longrightarrow S$ be an affine morphism. Let $(M_j)_{j\in J}$ be a good filtration of the quasi-coherent S-module $g_*(\mathcal{O}_{S'})$, with first step

$$M_0 = \mathcal{O}_S \subset g_*(\mathcal{O}_{S'}).$$

We then say that g is well-filtered, w.r.t. the good filtration $(M_j)_{j\in J}$.

By dévissage on the good filtration, to prove the vanishing of $R^iF_*(M)$, it suffices to prove that of $R^iF_*(F_j)$, for all $j \in J$. This follows from the fact that J is well-ordered. Details are left to the reader.

We now list a few tools, of which we shall make an intensive use. We then conclude with examples of good filtrations.

4.1. Tensor product of good filtrations.

Let M, M' be quasi-coherent modules over a scheme S, respectively equipped with good filtrations $(M_j)_{j\in J}$ and $(M'_{j'})_{j'\in J'}$. Put

$$M_{j,j'} := \sum_{(i,i')\leq(j,j')} M_i \otimes M'_{i'}.$$

Then, $(M_{j,j'})_{(j,j')\in J\times J'}$ is a good filtration of $M\otimes M'$, for the lexicographic order on $J\times J'$. Its graded pieces are the vector bundles

$$F_{j,j'} := (F_j \otimes F_{j'})_{(j,j')\in J\times J'}.$$

This extends to tensor products of three or more good filtrations of quasi-coherent modules. Note that the tensor product of good filtrations depends on a chosen order of the factors.

4.2. Composition of well-filtered morphisms.
Let

$$g_1 : S_2 \longrightarrow S_1$$

and

$$g_2 : S_3 \longrightarrow S_2$$

be affine morphisms of schemes.
For $i=1,2$, assume that g_i is well-filtered w.r.t. the filtration $(M_{i,j_i})_{j_i\in J_i}$, whose graded pieces we denote by F_{i,j_i}.
Put

$$J := J_2 \times J_1,$$

equipped with the lexicographic order:
for $j=(j_2,j_1)$ and $j'=(j_2',j_1')$, we have $j<j'$ iff $j_2<j_2'$ or ($j_2=j_2'$ and $j_1<j_1'$).
Consider the composite

$$g := g_1 \circ g_2 : S_3 \longrightarrow S_1.$$

We would like g to be well-filtered in a natural way, using J to label the steps of the filtration. To ensure the existence of such a filtration, we assume the existence of the following extra data.

(D): For all $j_2\in J_2$, a vector bundle V_{1,j_2} on S_1 is given, together with an isomorphism

$$\phi_{j_2} : g_1^*(V_{1,j_2}) \xrightarrow{\sim} F_{2,j_2}.$$

We have $V_{1,0}=\mathcal{O}_{S_1}$, and $\phi_0=\mathrm{Id}$.

Thus, the graded pieces of the filtration $(M_{2,j_2})_{j_2\in J_2}$, which are vector bundles over S_2, should be defined over S_1. In general, this holds (trivially) only for $F_{2,0}=\mathcal{O}_{S_2}$. Note that it is important, in (D), to specify the isomorphisms ϕ_{j_2}.
For $j=(j_2,j_1)\in J$, put

$$F_j := V_{1,j_2} \otimes_{\mathcal{O}_{S_1}} F_{1,j_1}.$$

Via ϕ_{j_2}, using the projection formula, the quasi-coherent S_1-module

$$(g_1)_*(F_{2,j_2}) \simeq V_{1,j_2} \otimes (g_1)_*(\mathcal{O}_{S_2})$$

is then naturally well-filtered, w.r.t. the filtration

$$(\Phi_{j_2,j_1})_{j_1\in J_1} := (V_{1,j_2} \otimes M_{1,j_1})_{j_1\in J_1},$$

having $(F_{j_2,j_1})_{j_1\in J_1}$ as graded pieces.
For $j=(j_2,j_1)\in J$, we then denote by

$$M_j \subset (g_1)_*(M_{2,j_2}) \subset g_*(\mathcal{O}_{S_3})$$

the inverse image of Φ_{j_2,j_1}, under the quotient

$$(g_1)_*(M_{2,j_2}) \longrightarrow (g_1)_*(F_{2,j_2}).$$

Lemma 4.2. *The data of* $(M_j)_{j\in J}$ *is a good filtration of* $g_*(\mathcal{O}_{S_3})$*, with* $M_0 = \mathcal{O}_{S_1}$.

Proof. Exercise. □

Whenever two composable well-filtered morphisms g_1 and g_2 are given, together with the data (D), we will thus consider $g := g_1 \circ g_2$ as a well-filtered morphism, in the way described above.
This construction clearly extends to compositions of three or more well-filtered morphisms.

Remark 4.3. Assume that $T \longrightarrow S_1$ and $g_1 : S_2 \longrightarrow S_1$ are two well-filtered (affine) morphisms. Denote by g_2 the projection

$$S_3 := S_2 \times_{S_1} T \longrightarrow S_2;$$

it is naturally a well-filtered morphism. The preceding construction then applies, and can be recovered from that of the tensor product of good filtrations.

4.3. Classical results in coherent sheaf cohomology. Let V be a vector bundle over a scheme S. Let $F : \mathbf{Fl}(V) \longrightarrow S$ be the structure morphism of its complete flag scheme. Let L be a line bundle over $\mathbf{Fl}(V)$.
Proposition 9.4 then gives a simple condition for the vanishing of $F_*(L)$. With the help of Proposition 9.5, it can be used to actually compute $R^1F_*(L)$.

4.4. Examples of good filtrations.

Example 4.4. (A typical good filtration, on symmetric powers.)
Let S be an $\mathbb{F}_p$-scheme. Let V be a vector bundle over S, of rank $d \geq 3$. Denote by $F : \mathbf{Fl}(V) \longrightarrow S$ its complete flag scheme, and by

$$\nabla_{gen} : 0 \subset \mathcal{V}_1 \subset \ldots \subset \mathcal{V}_d = F^*(V)$$

its tautological complete flag. Pick a positive integer n. Consider the vector bundle (over $\mathbf{Fl}(V)$)

$$\mathcal{W} := \mathrm{Sym}^n(\mathcal{V}_d)$$

Then, $\mathcal{W}$ has a natural good filtration $(M_j)_{j\in J}$. Here

$$J := \{(a_1, \ldots, a_d) \in \mathbb{N}^d, \ \sum a_i = n\}$$

is endowed with the lexicographic order,

$$M_{(a_1,\ldots,a_d)} := \mathrm{Span}(v_1 \ldots v_d, \ v_i \in \mathcal{V}_{b_i}, \ (b_1, \ldots, b_d) \leq (a_1, \ldots, a_d)).$$

We have $F_{(a_1,\ldots,a_d)} = \mathcal{O}(a_1, \ldots, a_d)$.

Example 4.5. Splitting schemes are archetypes of well-filtered affine morphisms.
Splitting schemes of extensions of vector bundles are introduced later, in Proposition 5.1, where we emphasize that they are well-filtered morphisms. Using section 4.2, it is even true that splitting schemes of extensions of $\mathbf{W}_r$-bundles are well-filtered- see Proposition 5.2.

5. $G\mathbf{W}_r$-affine spaces and splitting schemes.

5.1. Affine spaces of vector bundles.

Let S be a scheme, and let V be a vector bundle over S. We denote by

$$\mathbb{A}(V) := \mathrm{Spec}(\mathrm{Sym}(V^\vee)) \longrightarrow S$$

the affine space of V. Its functor of points is given by

$$\mathbb{A}(V)(T) = H^0(T, V \otimes_{\mathcal{O}_S} \mathcal{O}_T),$$

for every $T \longrightarrow S$.
Clearly, if V is a G-vector bundle over a G-scheme S, then $\mathbb{A}(V)$ is a (G,S)-scheme. It is also the *scheme* in $G\mathbf{W}_1$-modules over S, naturally attached to V.

5.2. $G\mathbf{W}_r$-affine spaces.

Let S be a $(G, \mathbb{F}_p)$-scheme. We can reinterpret G-equivariant torsors under $G\mathbf{W}_r$-vector bundles over S, as G-*affine spaces* over S. This point of view is analoguous to a well-known fact in real geometry, taught to our second-year students. Namely, a set with a simply transitive action of $\mathbb{R}^n$ is a *real affine space*, in the sense of barycentric geometry. The general notion of a G-affine space over S is discussed in [6], from which we follow conventions. In particular, a G-affine space X over $\mathbf{W}_r(S)$ will be called a $G\mathbf{W}_r$-affine space over S. In this text, we will always assume that $\overrightarrow{X}$, the $G\mathbf{W}_r$-module of translations of X, is a $G\mathbf{W}_r$-bundle.

5.3. Affine spaces, as splitting schemes of extensions of vector bundles.

The following notion was introduced in [6, §4], to which we refer for details.

Definition 5.1. *Let V be a G-vector bundle over a G-scheme S. Let X be a (G,V)-torsor over S- that is to say, a G-affine space directed by V.*
Then, X is represented by a G-scheme, affine over S.

If X corresponds to an extension (of G-vector bundles over S)

$$\mathcal{E} : 0 \longrightarrow V \xrightarrow{i} E \xrightarrow{\pi} \mathcal{O}_S \longrightarrow 0,$$

then this (G,S)-scheme is the scheme of sections of π, which we denote by

$$g : \mathbb{S}(\mathcal{E}) \longrightarrow S.$$

It is a G-affine subspace of $\mathbb{A}(E)$, having $\mathbb{A}(V)$ as its space of translations.
As such, it is the Spec *of the (G,S)-algebra*

$$\varinjlim(\mathrm{Sym}^n(E^\vee)),$$

where the limit is taken with respect to the injections of the exact sequences

$$0 \longrightarrow \mathrm{Sym}^n(E^\vee) \xrightarrow{\times\pi^\vee} \mathrm{Sym}^{n+1}(E^\vee) \xrightarrow{\mathrm{Sym}^{n+1}(i^\vee)} \mathrm{Sym}^{n+1}(V^\vee) \longrightarrow 0,$$

which are the symmetric powers of the dual extension

$$\mathcal{E}^\vee : 0 \longrightarrow \mathcal{O}_S \xrightarrow{\pi^\vee} E^\vee \xrightarrow{i^\vee} V^\vee \longrightarrow 0.$$

This description yields a natural (G-equivariant) filtration on the quasi-coherent $\mathcal{O}_S$-module $g_*(\mathcal{O}_{\mathbb{S}(\mathcal{E})})$, by the sub-vector bundles $\mathrm{Sym}^n(E^\vee)$*. It is indexed by the well-ordered set $\mathbb{N}$. Its n-th graded piece is the vector bundle* $\mathrm{Sym}^n(V^\vee)$*. It is a* *good filtration, in the sense of section 4. Hence, splitting schemes are well-filtered in a natural way.*

Let S be an $(\mathbb{F}_p, G)$-scheme.
The Greenberg transfer allows to consider splitting schemes of extensions of $G\mathbf{W}_r$-bundles over S, as (G, S)-schemes. How to do so is explained in the next Proposition, borrowed from [6, Proposition 5.5], to which we refer for a proof.

Proposition 5.2. *Let*

$$\mathcal{E}_r : 0 \longrightarrow V_r \longrightarrow E_r \longrightarrow \mathbf{W}_r(\mathcal{O}_S) \longrightarrow 0$$

be an extension of $G\mathbf{W}_r$-bundles, over an $(\mathbb{F}_p, G)$-scheme S. For $1 \leq i \leq r$, denote by

$$\mathcal{E}_i : 0 \longrightarrow V_i \longrightarrow E_i \longrightarrow \mathbf{W}_i(\mathcal{O}_S) \longrightarrow 0$$

its reduction, to an extension of $G\mathbf{W}_i$-bundles over S.
Viewing $\mathcal{E}_r$ as an extension of vector bundles over $\mathbf{W}_r(S)$, denote by

$$\mathbb{S}_r(\mathcal{E}_r) \longrightarrow \mathbf{W}_r(S)$$

its splitting scheme. Form its Greenberg transfer

$$g_r : \mathbb{S}(\mathcal{E}_r) := R_{\mathbf{W}_r/\mathbf{W}_1}(\mathbb{S}_r(\mathcal{E}_r)) \longrightarrow S.$$

We also refer to g_r as the splitting scheme of $\mathcal{E}_r$. It is naturally presented as a composite

$$\mathbb{S}(\mathcal{E}_r) = X_r \xrightarrow{h_r} X_{r-1} \xrightarrow{h_{r-1}} \ldots \xrightarrow{h_2} X_1 \xrightarrow{g_1} S.$$

The morphism g_1 is the splitting scheme of $\mathcal{E}_1$.
The h_i's are defined inductively, as follows. Denote by

$$0 \longrightarrow \mathrm{Frob}_*^i(V_{r-i}^{(i)}) \xrightarrow{j_i} V_r \xrightarrow{\rho_i} V_i \longrightarrow 0$$

the natural extension of $G\mathbf{W}_r$-modules on S, where ρ_i is the reduction arrow.
Over X_i, there exists a natural extension of $G\mathbf{W}_{r-i}$- bundles

$$\mathcal{F}_{i,r-i} : 0 \longrightarrow V_{r-i}^{(i)} \longrightarrow * \longrightarrow \mathbf{W}_{r-i}(\mathcal{O}_S) \longrightarrow 0,$$

such that $\mathcal{E}_r$, over X_i, becomes canonically isomorphic to $(j_i)_*(\mathcal{F}_{i,r-i})$.
Define the arrow $h_{i+1} : X_{i+1} \longrightarrow X_i$ as the splitting scheme

$$\mathbb{S}(\mathcal{F}_{i,1}) \longrightarrow X_i$$

of the mod p reduction of $\mathcal{F}_{i,r-i}$,

$$\mathcal{F}_{i,1} : 0 \longrightarrow V_1^{(i)} \longrightarrow * \longrightarrow \mathcal{O}_S \longrightarrow 0.$$

The arrow g_r is well-filtered in a natural (G-equivariant) way, indexed by $(a_r, \ldots, a_1) \in \mathbb{N}^r$, well-ordered by the lexicographic order. The graded pieces associated to the filtration of $(g_r)_*(\mathcal{O}_{\mathbb{S}(\mathcal{E}_r)})$ are the vector bundles

$$\mathrm{Sym}^{a_1}(V_1^{\vee}) \otimes \mathrm{Sym}^{a_2}(V_1^{(1)\vee}) \otimes \ldots \otimes \mathrm{Sym}^{a_r}(V_1^{(r-1)\vee}).$$

This follows from Proposition 5.1, combined to the composition process 4.2.

6. Permutation embedded flags of $G\mathbf{W}_r$-bundles.

6.1. Permutation $G\mathbf{W}_r$- bundles.

Definition 6.1. *Let S be an $(\mathbb{F}_p, G)$-scheme. Let X be a finite G-set. To fix ideas, we write the G-action on S on the left, and the G-action on X on the right. Let $(L_x)_{x\in X}$ be a collection of line bundles over S, together with isomorphisms of line bundles over S*

$$\phi_{x,g} : L_{xg} \xrightarrow{\sim} (g.)^*(L_x),$$

one for each $g \in G$ and $x \in X$, satisfying the cocycle condition

$$\phi_{x,gh} = (h.)^*(\phi_{x,g}) \circ \phi_{xg,h}.$$

We then say that the $G\mathbf{W}_r$-bundle (over S)

$$\bigoplus_{x\in X} \mathbf{W}_r(L_x)$$

is a permutation $G\mathbf{W}_r$-bundle, with $(L_x)_{x\in X}$ as a basis.
A $G\mathbf{W}_r$-bundle, which is isomorphic to a permutation $G\mathbf{W}_r$-bundle relative to some basis, will simply be called a permutation $G\mathbf{W}_r$-bundle.
In other words, a permutation $G\mathbf{W}_r$-bundle is a direct sum of $G\mathbf{W}_r$-bundles which are induced, from finitely many open subgroups $H_i \subset G$, from line $H_i\mathbf{W}_r$-bundles.

Clearly, Teichmüller lifts of G-line bundles over S are permutation $G\mathbf{W}_r$-bundles. Morphisms between permutation $G\mathbf{W}_r$-bundles always lift, as follows.

Lemma 6.2. *Let S be an $(\mathbb{F}_p, G)$-scheme.*

1) Let V_r be a permutation $G\mathbf{W}_r$-bundle over S. Then, V_r admits a system of compatible liftings, to permutation $G\mathbf{W}_s$-bundle over S, $s \geq r$.
2) Let V_r, W_r be two permutation $G\mathbf{W}_r$-bundles over S. Consider a G-equivariant arrow

$$f_r : V_r \longrightarrow W_r.$$

Then, f_r admits a system of compatible liftings, to G-equivariant arrows

$$f_s : V_s \longrightarrow W_s,$$

for all $s \geq r$.

Proof. The case $r = 1$ follows from functoriality of the Teichmüller lift of line bundles over S, $L \mapsto \mathbf{W}_r(L)$. The general case is by induction on r. $\square$

7. Embedded complete flags.

In this section, we introduce key tools, for proving the Uplifting Theorem.

Let $r \geq 1$ be an integer. Let S be an $\mathbb{F}_p$-scheme. The following concept is essential.

Definition 7.1. *(Complete flags of $\mathbf{W}_r$-bundles.)*
A complete flag of $\mathbf{W}_r$-bundles over S, of rank $d \geq 1$, is the data of a filtration of $\mathbf{W}_r$-bundles over S

$$\nabla = \nabla_{d,r} : 0 = V_{0,r} \subset V_{1,r} \subset \ldots \subset V_{d,r},$$

whose graded pieces $L_{i,r} = V_{i,r}/V_{i-1,r}$ are $\mathbf{W}_r$-line bundles.
If S is an $(\mathbb{F}_p, G)$-scheme, a G-linearized complete flag of $\mathbf{W}_r$-bundles over S is called a complete flag of $G\mathbf{W}_r$-bundles.

In order to make our future lifting problems representable by schemes, we need an "embedded" version of these complete flags.

Definition 7.2. *(Embedded complete flag of $\mathbf{W}_r$-bundles.)*
An embedded complete flag of $\mathbf{W}_r$-bundles over S is the data of a complete flag of $\mathbf{W}_r$-bundles over S

$$\nabla = \nabla_{d,r} : 0 = V_{0,r} \subset V_{1,r} \subset \ldots \subset V_{d,r},$$

together with an embedding of $\mathbf{W}_r$-bundles over S

$$V_{d,r} \subset V_{D,r},$$

where $V_{D,r}$ is a $\mathbf{W}_r$-bundle over S, of rank $D \geq d$.
By "embedding", it is understood that $V_{D,r}/V_{d,r}$ is a $\mathbf{W}_r$-bundle.

Definition 7.3. *(Permutation embedded complete flag of $G\mathbf{W}_r$-bundles.)*
Let S be an $(\mathbb{F}_p, G)$-scheme. A permutation embedded complete flag of $G\mathbf{W}_r$-bundles over S is the data of a complete flag of $G\mathbf{W}_r$-bundles over S

$$\nabla = \nabla_{d,r} : 0 = V_{0,r} \subset V_{1,r} \subset \ldots \subset V_{d,r},$$

together with an embedding of $G\mathbf{W}_r$-bundles over S

$$V_{d,r} \subset V_{D,r},$$

where $V_{D,r}$ is a permutation $G\mathbf{W}_r$-bundle over S.

7.1. Operations on embedded complete flags.

Let

$$\nabla = \nabla_{d,r} : 0 = V_{0,r} \subset V_{1,r} \subset \ldots \subset V_{d,r} \subset V_{D,r}$$

be an embedded complete flag of $\mathbf{W}_r$-bundles, over an $\mathbb{F}_p$-scheme S.

7.1.1. *Truncation.* For any $1 \leq d' \leq d$, the truncation

$$t_{d'}(\nabla) := 0 = V_{0,r} \subset V_{1,r} \subset \ldots \subset V_{d',r} \subset V_{D,r}$$

is an embedded complete flag of $\mathbf{W}_r$-bundles, of rank d'.

7.1.2. *Reduction.* For any $1 \leq r' \leq r$, the reduction

$$\rho_{r'}(\nabla) := 0 = V_{0,r'} \subset V_{1,r'} \subset \ldots \subset V_{d,r'} \subset V_{D,r'}$$

is an embedded complete flag of $\mathbf{W}_{r'}$-bundles, of rank d.

8. Recollections on cyclotomic pairs and on smooth profinite groups.

The notions of cyclotomic pair and of smooth profinite group used here are introduced in [6]. They depend on a pair of integers (n, e), where $n \geq 1$ is the degree of the cohomology groups involved, and where $e \geq 1$ is the depth. In the present work, we retain from [6] just what we need. In particular, we focus on the case $n = e = 1$, as mentionned in the Introduction.

Definition 8.1. *($(1,1)$-smooth profinite group; see* [6, Definition 6.7]*)*
A profinite group G is said to be $(1,1)$-smooth if the following lifting property holds. Let A be a perfect $\mathbb{F}_p$-algebra equipped with an action of G (factoring through an open subgroup). Let L_1 be a locally free A-module of rank one, equipped with a semi-linear action of G. Let $c \in H^1(G, L_1)$ be a cohomology class. Then, there exists a lift of L_1, to a $(\mathbf{W}_2(A), G)$-module $L_2[c]$, locally free of rank one as a $\mathbf{W}_2(A)$-module (and depending on c), such that c belongs to the image of the natural map

$$H^1(G, L_2[c]) \longrightarrow H^1(G, L_1).$$

As explained in [6, §11], $(1,1)$-smoothness is equivalent to a simple fact: the liftability of one-dimensional mod p G-affine spaces, to mod p^2 G-affine spaces.

Definition 8.2. *($(1,1)$-cyclotomic pair; see* [6, §6]*)*
Let G be a profinite group, and let $\mathbb{Z}/p^2(1)$ be a free $\mathbb{Z}/p^2$-module of rank one, equipped with a continuous action of G. We say that the pair $(G, \mathbb{Z}/p^2(1))$ is $(1,1)$-cyclotomic if the following lifting property holds.
For all open subgroups $H \subset G$, the natural map

$$H^1(H, \mathbb{Z}/p^2(1)) \longrightarrow H^1(H, \mathbb{Z}/p(1))$$

is surjective.

We shall need the following result.

Theorem 8.3. *(see* [6]*, Theorem A)*
Let $(G, \mathbb{Z}/p^2(1))$ be a $(1,1)$-cyclotomic pair. Let A be a perfect $(\mathbb{F}_p, G)$-algebra. Let L_1 be an invertible A-module, equipped with a semi-linear action of G. Then, for all open subgroups $H \subset G$, the natural map

$$H^1(H, \mathbf{W}_2(L_1)(1)) \longrightarrow H^1(H, L_1(1))$$

is surjective. In particular, G is $(1,1)$-smooth.

9. Flag schemes and their cohomology.

Definition 9.1 (Subbundles).
Let S be a scheme. Let V be a vector bundle over S. A sub-vector bundle $W \subset V$ (or subbundle) is a coherent sub-S-module $W \subset V$, such that V/W is a vector bundle.
(Hence, W is automatically a vector bundle.)

Definition 9.2 (Flag schemes).
Let S be a scheme. Let V be a vector bundle over S, of rank $D \geq 1$. Let

$$1 \leq n_1 < \ldots < n_s \leq D$$

be a strictly increasing sequence of integers. We denote by

$$F(= F_{n_1,\ldots,n_s}) : \mathbf{Fl}(n_1, \ldots, n_s, V) \longrightarrow X$$

the scheme of flags of subbundles of V, of dimensions $n_1, \ldots, n_s$. Its universal property is the following : for any morphism $T \stackrel{t}{\longrightarrow} S$, the set $\mathbf{Fl}(n_1, \ldots, n_s, V)(T)$ consists of flags of subbundles over T

$$(0 = Z_0 \subset Z_1 \subset \ldots \subset Z_s \subset Z_D := t^*(V)),$$

with $\dim(Z_i) = n_i$ for all $i = 1, \ldots, s$.
We use the notation

$$0 = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \ldots \subset \mathcal{V}_s \subset \mathcal{V}_D = F^*(V)$$

for the tautological filtration (flag) on the vector bundle $F^*(V)$.

We denote the flag scheme $\mathbf{Fl}(1, 2, \ldots, d, V)$, for $d \in \{1, \ldots, D\}$, by

$$F(= F_{d,V}) : \mathbf{Fl}(d, V) \longrightarrow S.$$

If $d = D$, we denote $\mathbf{Fl}(d, V)$ simply by $\mathbf{Fl}(V)$; it is the scheme of complete flags of the vector bundle V. We then denote by

$$\mathcal{L}_i := \mathcal{V}_i / \mathcal{V}_{i-1}$$

the associated quotient line bundles, for $i = 1, \ldots, D$.
Following the notation introduced earlier, for an arbitrary sequence of relative integers $a_1, \ldots, a_D$, we put

$$\mathcal{O}(a_1, \ldots, a_D) := \mathcal{L}_1^{\otimes a_1} \otimes \ldots \otimes \mathcal{L}_d^{\otimes a_D};$$

it is a line bundle on $\mathbf{Fl}(V)$.

Remark 9.3. The scheme $\mathbf{Fl}(V)$ can be naturally constructed as a composite of projective bundles; see [14, §3.2].

The following Proposition will serve us well.

Proposition 9.4. *Denote by $F : \mathbf{Fl}(V) \longrightarrow S$ the scheme of complete flags of a vector bundle V over S, of rank D.*
Let $a = (a_1, \ldots, a_D) \in \mathbb{Z}^D$ be a sequence of D relative integers.
If a is not an increasing sequence, then

$$F_*(\mathcal{O}(a_1, \ldots, a_D)) = 0.$$

Proof. See [3, Proposition 1.4.5]. □

Let $F : \mathbf{Fl}(V) \longrightarrow S$ be the complete flag scheme of a vector bundle over S, of rank D. It is an exercise to show that the tautological flag

$$\nabla_{gen} : 0 = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \ldots \subset \mathcal{V}_{D-1} \subset \mathcal{V}_D = F^*(V)$$

is not split. More precisely, for all integers $1 \leq m < n \leq D$ the natural extension

$$Nat : 0 \longrightarrow \mathcal{V}_m \longrightarrow \mathcal{V}_n \longrightarrow \mathcal{V}_n / \mathcal{V}_m \longrightarrow 0$$

is not split. If S has characteristic p, neither are its Frobenius pullbacks $Nat^{(r)}$. In the sequel, we will need to split (part of) this tautological flag, using splitting schemes.

We now recall fairly classical cohomological material, focusing on cohomology groups of degree ≤ 1.

Proposition 9.5. *(Cohomology of $\mathbb{P}^{d-1}$-bundles).*
Let V be a vector bundle of rank $d \geq 2$, over a scheme S. Denote by

$$F : \mathbb{P}(V) \longrightarrow S$$

its projective bundle, and by $\mathcal{O}(1)$ its twisting sheaf. Let $n > 0$ be an integer.

1) We have $F_*(\mathcal{O}(-n)) = 0$.
2) If $d \geq 3$, we have $R^1F_*(\mathcal{O}(-n)) = 0$.
3) If $d = 2$, we have $R^1F_*(\mathcal{O}(-1)) = 0$, and

$$R^1F_*(\mathcal{O}(-n)) = \Gamma^{n-2}(V^\vee) \otimes \mathrm{Det}(V)^\vee$$

for $n \geq 2$, where Γ^* denotes divided powers, dual to symmetric powers Sym^*.

The same computations hold over the complete flag scheme $\mathbf{Fl}(V)$, instead of the projective bundle $\mathbb{P}(V)$. Just replace $\mathcal{O}(-n)$ by $\mathcal{O}(0, \ldots, 0, -n)$, or dually $\mathcal{O}(n, 0, \ldots, 0)$.
When $d = 2$, we get from 3) a natural good filtration for the vector bundle $F^*(R^1F_*(\mathcal{O}(n, 0)))$, having as graded pieces line bundles of the shape $\mathcal{O}(n_1, n_2)$, with n_1, n_2 positive, and $n_1 + n_2 = n$.

Proof. Stacks project ([31] 30.8) does it well. □

10. S-polynomial functors.

The purpose of this section is to give a meaning to "a polynomial functor" applied to a vector bundle V over a scheme. We do this adopting the trendy language of stacks. This section is rather informal, and intended to be inspiring. Note that the only polynomial functors used later in this text are symmetric functors-defined at the end of this section.

Denote by S a scheme, not necessarily of characteristic p.

Definition 10.1. *We define $\mathbf{Vect}_S$ to be the category whose objects are pairs*

$$(T \xrightarrow{f} S, V)$$

where f is a morphism of schemes, and where V is a vector bundle over T (of arbitrary constant rank). Morphisms

$$(T \xrightarrow{f} S, V) \longrightarrow (T' \xrightarrow{f'} S, V')$$

exist if and only if $f = f'$, in which case they are morphisms of $\mathcal{O}_T$-Modules $V \longrightarrow V'$.

The category $\mathbf{Vect}_S$ is fibered over Sch_S, through the forgetful functor and pullbacks of vector bundles, giving rise to the stack of vector bundles over S (with respect, say, to the fpqc topology).

The next Definition, for polynomial functors, mimics that of polynomial laws (see [27]), in a categorical context.

Definition 10.2. *(S-polynomial functors.)*
Let n be a positive integer. An S-polynomial functor, homogeneous of degree $n \geq 0$, is an endofunctor Φ of the stack $\mathbf{Vect}_S$, together with the data, for every

$$(T \xrightarrow{f} S, V) \in \mathbf{Vect}_S$$

and every line bundle L over T, of a functorial isomorphism

$$\phi_{f,V,L} : \Phi(L \otimes V) \xrightarrow{\sim} L^{\otimes n} \otimes \Phi(V)$$

of vector bundles over T.

Remark 10.3. If $S = \mathrm{Spec}(k)$ with k a field, the preceding Definition is most likely equivalent to the usual definition of a polynomial functor- see, for instance, [13].

Examples 10.4. The n-th symmetric power

$$V \mapsto \mathrm{Sym}^n(V)$$

and n-th exterior power

$$V \mapsto \Lambda^n(V)$$

both define S-polynomial functors, homogeneous of degree n.

The n-th divided power

$$V \mapsto \Gamma^n(V) := \mathrm{Sym}^n(V^\vee)^\vee$$

also defines such a functor. (Warning: this formula for divided powers applies to vector bundles only.)

If S has characteristic p, the r-th Frobenius pullback

$$V \mapsto V^{(r)} = (\mathrm{Frob}^r)^*(V)$$

defines an S-polynomial functor, homogeneous of degree p^r.

Remark 10.5. Note that Γ^n, Sym^n and Λ^n are $\mathbb{Z}$-polynomial functors: they are defined over $S = \mathrm{Spec}(\mathbb{Z})$.

Definition 10.6. *Let Φ be an S-polynomial functor, homogeneous of degree n. Then, the association*

$$V \mapsto \Phi(V^\vee)^\vee$$

defines another polynomial functor $\Phi^\vee$, dual to Φ.

We shall say that Φ is self-dual if Φ and $\Phi^\vee$ are isomorphic.

Exercise 10.7. Exterior powers Λ^n are self-dual over any base. Symmetric powers Sym^n are self-dual over S if and only $n!$ is everywhere invertible on S.

Remark 10.8. (Frobenius and Verschiebung.)
Let S be a scheme of characteristic p. Let V be a vector bundle over S. There is a natural Frobenius arrow

$$\mathrm{Frob}_V : \mathrm{Frob}^*(V) \longrightarrow \mathrm{Sym}^p(V),$$

$$v \otimes 1 \mapsto v^p,$$

identifying $\mathrm{Frob}^*(V)$ to a subbundle of $\mathrm{Sym}^p(V)$. It is an isomorphism if, and only if, V is a line bundle. It gives rise to a natural transformation of $\mathbb{F}_p$-polynomial functors

$$\mathrm{Frob}^*(.) \xrightarrow{\mathrm{Frob}} \mathrm{Sym}^p(.).$$

The Verschiebung occurs as its dual:

$$\Gamma^p(.) \stackrel{\mathrm{Ver}}{\longrightarrow} \mathrm{Frob}^*(.).$$

Applied to V, it is given by the formula

$$\Gamma^p(V) \stackrel{\mathrm{Ver}}{\longrightarrow} \mathrm{Frob}^*(V),$$

$$[v]_p \mapsto v \otimes 1.$$

Note that Frob and Ver actually apply to *all* quasi-coherent modules, not just to vector bundles.

10.1. Symmetric functors. These are the polynomial functors that we shall use in practice.

Definition 10.9. *(Symmetric functors.)*
A symmetric functor over $\mathbb{F}_p$ is an $\mathbb{F}_p$-polynomial functor of the shape

$$\Phi : V \mapsto \bigotimes_{i=1}^{n} \mathrm{Sym}^{a_i}(V^{(r_i)}),$$

where n, the a_i's and the r_i's are non-negative integers, with $a_i p^{r_i} > 0$ for all i. It is thus homogeneous, of degree $\sum_{i=1}^n a_i p^{r_i}$.
It is called pure if $n = 1$. If $n \geq 2$, it is said to be composite.

11. Lifting extensions of $\mathbf{W}_r$-bundles...

In this section, we investigate how to lift extensions of $\mathbf{W}_r$-bundles. For simplicity, we restrict to extensions with kernel a line bundle- which is all we shall need later. The general case can be dealt with along the same lines.
This question is of crucial importance in the proof of the Uplifting Theorem 14.1. It is clearly equivalent to the problem of lifting line subbundles of Witt vector bundles- and to its dual counterpart, lifting quotient line bundles of Witt vector bundles.

Let S be a scheme over $\mathbb{F}_p$, let L be a line bundle over S, let $r \geq 1$ be an integer, and let

$$\mathcal{E}_r : 0 \longrightarrow L_r \stackrel{i_r}{\longrightarrow} V_r \stackrel{\pi_r}{\longrightarrow} Q_r \longrightarrow 0$$

be an exact sequence of $\mathbf{W}_r$-bundles over S. Let $s \geq 1$ be an integer. Let V_{r+s} be a given lift of V_r, to a $\mathbf{W}_{r+s}$-bundle on S. We want to perform a change of the base

$$\mathbf{S}_{r+s} \longrightarrow S,$$

which solves the moduli problem of lifting $\mathcal{E}_r$, to an extension of $\mathbf{W}_{r+s}$-bundles

$$\mathcal{E}_{r+s} : 0 \longrightarrow L_{r+s} \stackrel{i_{r+s}}{\longrightarrow} V_{r+s} \stackrel{\pi_{r+s}}{\longrightarrow} Q_{r+s} \longrightarrow 0.$$

Note that, if we can lift i_r to a homomorphism of $\mathbf{W}_{r+s}$-bundles

$$i_{r+s} : L_{r+s} \longrightarrow V_{r+s},$$

this lift is then automatically an embedding. Indeed, by Nakayama's lemma, its dual arrow $i_{r+s}^\vee : V_{r+s}^\vee \longrightarrow L_{r+s}^\vee$ is surjective. Then, simply set Q_{r+s} to be the cokernel of i_{r+s}.
There are two ways to deal with our lifting problem: either L_{r+s} is prescribed in advance, or it is not. We now get to details.

11.1. ...PRESCRIBING THE LIFT OF THEIR KERNEL... In this section, we prescribe the $\mathbf{W}_{r+s}$-bundle L_{r+s}. In other terms, we are given the data of $\mathcal{E}_r$, L_{r+s} and V_{r+s}, and our goal is to define the morphism $\mathbf{S}_{r+s} \longrightarrow S$, which parametrizes liftings $\mathcal{E}_{r+s}$ of $\mathcal{E}_r$. For instance, if L_r is the Teichmüller lift of its mod p reduction ($L_r = \mathbf{W}_r(L_1)$), we can require $L_{r+s} = \mathbf{W}_{r+s}(L_1)$.
Consider the natural exact sequence of $\mathbf{W}_{r+s}$-modules over S

$$\mathcal{F}_{r+s} : 0 \longrightarrow \mathrm{Frob}_*^r((\mathrm{Frob}^r)^*(V_s \otimes L_s^{-1})) \longrightarrow V_{r+s} \otimes L_{r+s}^{-1} \xrightarrow{\rho} V_r \otimes L_r^{-1} \longrightarrow 0,$$

where

$$V_s := V_r \otimes_{\mathbf{W}_r} \mathbf{W}_s$$

denotes the reduction of V_r to a $\mathbf{W}_s$-bundle on S, and where ρ is given by reduction. The arrow i_r is given by a global section

$$s_r \in H^0(S, V_r \otimes L_r^{-1}) = \mathrm{Hom}_{\mathbf{W}_r(\mathcal{O}_S)-\mathrm{Mod}}(\mathbf{W}_r(\mathcal{O}_S), V_r \otimes L_r^{-1}).$$

Lifting it to $i_{r+s} : L_{r+s} \longrightarrow V_{r+s}$ amounts to lifting s_r, via ρ, to a global section

$$s_{r+s} \in H^0(S, V_{r+s} \otimes L_{r+s}^{-1}) = \mathrm{Hom}_{\mathbf{W}_{r+s}(\mathcal{O}_S)-\mathrm{Mod}}(\mathbf{W}_{r+s}(\mathcal{O}_S), V_{r+s} \otimes L_{r+s}^{-1}).$$

The space of such liftings is naturally equipped with the structure of a torsor, under the $\mathbf{W}_s$-bundle $(V_s \otimes L_s^{-1})^{(r)}$. Using adjunction between Frob_* and Frob^*, it is given by the extension $s_r^*(\mathcal{F}_{r+s})$, corresponding to an extension of $\mathbf{W}_s$-bundles over S

$$\mathcal{G}_{r+s} : 0 \longrightarrow (\mathrm{Frob}^r)^*(V_s \otimes L_s^{-1}) \longrightarrow * \longrightarrow \mathcal{O}_S \longrightarrow 0.$$

Denote by

$$\mathbf{S}_{r+s}(= \mathbf{S}_{r+s}(i_r, V_{r+s}, L_{r+s})) := \mathbb{S}(\mathcal{G}_{r+s}) \xrightarrow{g} S$$

its splitting scheme; see 5.2. Over $\mathbf{S}_{r+s}$, $\mathcal{G}_{r+s}$ acquires a canonical section, giving rise to a lifting i_{r+s} of i_r. The affine morphism

$$g : \mathbf{S}_{r+s} \longrightarrow S$$

is then, indeed, the universal change of the base, parametrizing the desired liftings of $\mathcal{E}_r$.
We have just proved a useful representability statement, as follows.

DEFINITION 11.1. *Let $0 \longrightarrow L_r \xrightarrow{i_r} V_r \longrightarrow Q_r \longrightarrow 0$ be an extension of $\mathbf{W}_r$-bundles over an $\mathbb{F}_p$-scheme S. Assume given a lift V_{r+s}(resp. L_{r+s}) of V_r(resp. L_r) to a $\mathbf{W}_{r+s}$-bundle over S. Consider the functor*

$$\Psi_{\mathcal{E}_r, V_{r+s}, L_{r+s}} : \begin{array}{ccc} Sch/S & \longrightarrow & Sets \\ (T \xrightarrow{t} S) & \longmapsto & \{\mathcal{E}_{r+s}\} \end{array},$$

from the category of S-schemes to that of sets, defined as follows. It sends t to the set of liftings of $t^*(\mathcal{E}_r)$ to an extension

$$\mathcal{E}_{r+s} : 0 \longrightarrow t^*(L_{r+s}) \longrightarrow t^*(V_{r+s}) \longrightarrow * \longrightarrow 0$$

of $\mathbf{W}_{r+s}$-bundles over T.

PROPOSITION 11.2. *The functor $\Psi_{\mathcal{E}_r, V_{r+s}, L_{r+s}}$ is represented by the affine morphism $g : \mathbf{S}_{r+s} \longrightarrow S$ constructed above.*

Proof. As we have seen above, over a given $T \xrightarrow{t} S$, the data of a lifting of $t^*(\mathcal{E}_r)$ is equivalent to that of a lifting of $t^*(i_r) : t^*(L_r) \longrightarrow t^*(V_r)$; that is to say, to a splitting of the extension $t^*(s_r^*(\mathcal{F}_{r+s}))$. The claim follows from the universal property of splitting schemes. $\square$

Remark 11.3. Assume that $s=1$. Then, by Proposition 5.1, the affine morphism g is well-filtered, in a natural way. The graded pieces of the associated good filtration of $g_*(\mathcal{O}_{\mathbf{S}_{r+1}})/\mathcal{O}_S$ are of the shape

$$\mathrm{Sym}^n((\mathrm{Frob}^r)^*(V_1^\vee\otimes L_1))=\mathrm{Sym}^n((\mathrm{Frob}^r)^*(V_1^\vee))\otimes L_1^{np^r}.$$

Hence, setting $a:=np^r$,

$$\Phi_a(.):=\mathrm{Sym}^n(\mathrm{Frob}^r(.))$$

is a pure symmetric functor, homogenenous of degree $a\geq 1$, which describes these graded pieces.
The situation for s arbitrary is similar, replacing Φ_a by a composite symmetric functor, and using Proposition 5.2.

11.2. ...OR WITHOUT CONSTRAINT. In this section, we describe the space of liftings $\mathcal{E}_{r+s}$ of $\mathcal{E}_r$, without prescribing L_{r+s} in advance. This space is the fiber of the reduction morphism

$$R_{\mathbf{W}_{r+s}/\mathbf{W}_1}(\mathbb{P}_{\mathbf{W}_{r+s}(S)}(V_{r+s}^\vee))\longrightarrow R_{\mathbf{W}_r/\mathbf{W}_1}(\mathbb{P}_{\mathbf{W}_r(S)}(V_r^\vee)),$$

over the S-point given by i_r. Recall that $R_{\mathbf{W}_r/\mathbf{W}_1}$ denotes Greenberg's functor, from the category of $\mathbf{W}_r(S)$-schemes to that of S-schemes. This fiber can be described using Greenberg's Structure Theorem- see, for instance, [1]. We give a self-contained exposition, in our particular context.
Assume first that S is *affine*, in which case :

- all extensions of $\mathbf{W}_r$-bundles over S are split, and
- every $\mathbf{W}_r$-line bundle over S is (non-canonically) isomorphic to the Teichmüller lift of its mod p reduction.

Both assertions follows from the vanishing of coherent cohomology over S. Actually, the second one just uses $H^1(S,\mathcal{O}_S)=0$.
Our moduli problem can then be reformulated as follows.
Fix an isomorphism $L_r\simeq\mathbf{W}_r(L_1)$. Let V_{r+s} be a given lift of V_r, to a $\mathbf{W}_{r+s}$-bundle on S. Parametrize equivalence classes of liftings of

$$\mathbf{W}_r(L_1)\xrightarrow{i_r}V_r,$$

to

$$i_{r+s}:\mathbf{W}_{r+s}(L_1)\xrightarrow{i_{r+s}}V_{r+s},$$

where two liftings i_{r+s} and i'_{r+s} are identified if there exists an automorphism

$$\sigma\in\mathrm{Ker}(\mathrm{Aut}(\mathbf{W}_{r+s}(L_1)\longrightarrow\mathrm{Aut}(\mathbf{W}_r(L_1))=\mathrm{Ker}(\mathbf{W}_{r+s}^\times\longrightarrow\mathbf{W}_r^\times),$$

such that

$$i'_{r+s}=i_{r+s}\circ\sigma.$$

Remark 11.4. The expression

$$\mathrm{Ker}(\mathbf{W}_{r+s}^\times\longrightarrow\mathbf{W}_r^\times)$$

defines a linear algebraic group over $\mathbb{Z}$. Its mod p reduction is a split unipotent linear algebraic group over $\mathbb{F}_p$. If $p\neq 2$, or if $r\geq 2$, it is isomorphic to $(\mathbf{W}_s,+)$, via the p-adic logarithm. See [8, Remark 2.4].

Keeping the notation of the previous paragraph, these equivalence classes form a torsor under the vector bundle $(Q_s \otimes L_s^\vee)^{(r)}$. Concretely, it is given by the extension of $\mathbf{W}_s$-bundles $(\pi_s)_*(\mathcal{G}_{r+s}) = (\pi_s)_*(s_r^*(\mathcal{F}_{r+s}))$, reading as

$$\mathcal{G}_{r+s} : 0 \longrightarrow (Q_s \otimes L_s^\vee)^{(r)} \longrightarrow * \longrightarrow \mathbf{W}_s(O_S) \longrightarrow 0.$$

This extension is canonical: it does not depend on the choice of the isomorphism $L_r \xrightarrow{\sim} \mathbf{W}_r(L_1)$. By glueing, it is defined over an arbitrary S. Denote by

$$\mathbf{S}'_{r+s} (= \mathbf{S}'_{r+s}(i_r, V_{r+s})) := \mathbb{S}(\mathcal{G}_{r+s}) \xrightarrow{g'} S$$

its splitting scheme. We have proved the following useful representability statement.

Definition 11.5. *Let*

$$\mathcal{E}_r : 0 \longrightarrow L_r \xrightarrow{i_r} V_r \longrightarrow Q_r \longrightarrow 0$$

be an extension of $\mathbf{W}_r$-bundles over an $\mathbb{F}_p$-scheme S. Assume given a lift V_{r+s} of V_r to a $\mathbf{W}_{r+s}$-bundle over S. Consider the functor

$$\begin{array}{cccc} \Phi_{\mathcal{E}_r, V_{r+s}} : & Sch/S & \longrightarrow & Sets \\ & (T \xrightarrow{t} S) & \longmapsto & \{\mathcal{E}_{r+s}\}, \end{array}$$

from the category of S-schemes to that of sets, defined as follows. It sends t to the set of liftings of $t^*(\mathcal{E}_r)$ to an extension

$$\mathcal{E}_{r+s} : 0 \longrightarrow * \longrightarrow t^*(V_{r+s}) \longrightarrow * \longrightarrow 0$$

of $\mathbf{W}_{r+s}$-bundles over T.

Proposition 11.6. *The functor $\Phi_{\mathcal{E}_r, V_{r+s}}$ is represented by the affine morphism $g' : \mathbf{S}'_{r+s} \longrightarrow S$ constructed above.*

Remark 11.7. In the present situation, the analogue of Remark 11.3 goes as follows. The morphism g' is well-filtered in a natural way. The graded pieces of the associated good filtration of $g'_*(\mathcal{O}_{\mathbf{S}'_{r+1}})/\mathcal{O}_S$ are of the shape

$$\Phi'_a((V_1/L_1)^\vee \otimes L_1),$$

where $a \geq 1$ is an integer, and where Φ'_a is a symmetric functor, homogenenous of degree a, which is pure if $s = 1$.

Exercise 11.8. Give a precise meaning to the following motto:
"Splitting creates functions which are *dual* to those created for lifting."
Of course, splitting is a much stronger operation than lifting!

11.3. The equivariant case. Assume now that S is an $(\mathbb{F}_p, G)$-scheme, that $\mathcal{E}_r$ is an exact sequence of $G\mathbf{W}_r$-bundles over S, and that V_{r+s} is a $G\mathbf{W}_{r+s}$-bundle over S. In the situation of subsection 11.1, assume also that L_{r+s} is a $G\mathbf{W}_{r+s}$-line bundle over S. By functoriality of the constructions above, we get the following. The schemes $\mathbf{S}_{r+s}$ and $\mathbf{S}'_{r+s}$ are $(\mathbb{F}_p, G)$-schemes in a natural way, for which the arrows g and g' are G-equivariant. Furthermore, their G-equivariant sections correspond to liftings $\mathcal{E}_{r+s}$ of $\mathcal{E}_r$, that are extensions of $G\mathbf{W}_{r+s}$-bundles.

12. Glueing extensions of $G\mathbf{W}_2$-bundles.

Let S be an $(\mathbb{F}_p, G)$-scheme. Let M_2 be a $G\mathbf{W}_2$-bundle over S. Denote by

$$M_1 = \rho(M_2)$$

its mod p reduction, where ρ stands for *reduction*. Recall that we have an exact sequence of $G\mathbf{W}_2$-modules over S

$$0 \longrightarrow \mathrm{Frob}_*(M_1^{(1)}) \xrightarrow{j} M_2 \xrightarrow{\rho} M_1 \longrightarrow 0.$$

In the rest of the proof, to simplify notation, we will not write Frobenius pushforwards. By section 3.8, we know that

$$\mathrm{Ext}^i_{G,2}(S, \mathbf{W}_2(\mathcal{O}_S), *) = H^i_G(S, *),$$

for all $i \geq 0$, and for all $G\mathbf{W}_2$-modules $*$.
Assume given two extensions of $G\mathbf{W}_2$-bundles over S,

$$\mathcal{E}_{d,2} : 0 \longrightarrow V_{d-1,2} \xrightarrow{i_{d,2}} V_{d,2} \xrightarrow{q_{d,2}} L_{d,2} \longrightarrow 0$$

and

$$\mathcal{P}_{d+1,2} : 0 \longrightarrow L_{d,2} \longrightarrow P_{d,d+1,2} \longrightarrow L_{d+1,2} \longrightarrow 0,$$

where $V_{i,2}$ (resp. $L_{i,2}$, $P_{d,d+1,2}$) is of dimension i (resp. 1, 2).

Definition 12.1. *(Glueing)*
A glueing of $\mathcal{E}_{d,2}$ and $\mathcal{P}_{d+1,2}$ is a pair $(\mathcal{E}_{d+1,2}, \phi_2)$, consisting of an extension of $G\mathbf{W}_2$-bundles over S

$$\mathcal{E}_{d+1,2} : 0 \longrightarrow V_{d,2} \xrightarrow{i_{d+1,2}} V_{d+1,2} \xrightarrow{q_{d+1,2}} L_{d+1,2} \longrightarrow 0,$$

and an isomorphism of extensions of $G\mathbf{W}_2$-bundles over S

$$\phi_2 : (q_{d,2})_*(\mathcal{E}_{d+1,2}) \xrightarrow{\sim} \mathcal{P}_{d+1,2}.$$

Isomorphisms of glueings are defined in the obvious way.
This definition clearly extends, to glueing extensions of $G\mathbf{W}_r$-bundles over S for any $r \geq 1$.

We know that the obstruction to glueing $\mathcal{E}_{d,2}$ and $\mathcal{P}_{d+1,2}$ is the cup-product

$$\mathcal{C}_2 := \mathcal{E}_{d,2} \cup \mathcal{P}_{d+1,2} : 0 \longrightarrow V_{d-1,2} \longrightarrow V_{d,2} \longrightarrow P_{d,d+1,2} \longrightarrow L_{d+1,2} \longrightarrow 0.$$

It is a 2-extension of $G\mathbf{W}_2$-bundles over S, whose class

$$c_2 \in \mathrm{Ext}^2_{G,2}(S, L_{d+1,2}, V_{d-1,2}) = H^2_G(S, L^\vee_{d+1,2} \otimes V_{d-1,2})$$

vanishes if, and only if, our two extensions can be glued.

Definition 12.2. *(Lifting a glueing)*
Let $(\mathcal{E}_{d+1,1}, \phi_1)$ be a glueing of $\mathcal{E}_{d,1} := \rho(\mathcal{E}_{d,2})$ and $\mathcal{P}_{d+1,1} := \rho(\mathcal{P}_{d+1,2})$, as extensions of $G\mathbf{W}_1$-bundles.
A lifting of the glueing $(\mathcal{E}_{d+1,1}, \phi_1)$, is the data of a glueing $(\mathcal{E}_{d+1,2}, \phi_2)$ of $\mathcal{E}_{d,2}$ and $\mathcal{P}_{d+1,2}$, together with an isomorphism

$$\rho(\mathcal{E}_{d+1,2}, \phi_2) \xrightarrow{\sim} (\mathcal{E}_{d+1,1}, \phi_1),$$

as glueings of $\mathcal{E}_{d,1}$ and $\mathcal{P}_{d+1,1}$.

In the remaining part of this section, we are going to show that the obstruction to lifting a glueing, is a 2-extension of G-bundles

$$\mathcal{C}_1 : 0 \longrightarrow V_{d-1,1}^{(1)} \longrightarrow * \longrightarrow * \longrightarrow L_{d+1,2}^{(1)} \longrightarrow 0,$$

whose class

$$c_1 \in \mathrm{Ext}^2_{G,1}(S, L_{d+1,1}^{(1)}, V_{d-1,1}^{(1)}) = H^2_G(S, (L_{d+1,1}^{\vee} \otimes V_{d-1,1})^{(1)})$$

satisfies

$$c_2 = j_*(c_1) \in H^2_G(S, L_{d+1,2}^{\vee} \otimes V_{d-1,2}).$$

To simplify, we first assume that $L_{d+1,2} = \mathbf{W}_2(\mathcal{O}_S)$ is trivial.
Consider the natural surjection of $G\mathbf{W}_2$-modules

$$V_{d,1} \oplus L_{d,2} \xrightarrow{q} L_{d,1} \longrightarrow 0,$$

given by

$$q(v,l) := q_{d,1}(v) - \rho(l).$$

It fits into an extension of $G\mathbf{W}_2$-modules

$$\mathcal{Q} : 0 \longrightarrow N \xrightarrow{\iota} V_{d,1} \oplus L_{d,2} \xrightarrow{q} L_{d,1} \longrightarrow 0,$$

whose kernel N is presented as

$$\mathcal{N} : 0 \longrightarrow V_{d-1,1}^{(1)} \longrightarrow V_{d,2} \xrightarrow{s} N \longrightarrow 0,$$

where

$$s(v) := (\rho(v), q_{d,2}(v)) \in N \subset V_{d,1} \oplus L_{d,2},$$

and where the injection is given by the natural inclusions

$$V_{d-1,1}^{(1)} \subset V_{d,1}^{(1)} \subset V_{d,2}.$$

Consider the Baer sum

$$\mathcal{F} := (\rho^*(\mathcal{E}_{d+1,1}) + \mathcal{P}_{d+1,2}) : 0 \longrightarrow V_{d,1} \oplus L_{d,2} \longrightarrow * \longrightarrow L_{d+1,2} \longrightarrow 0,$$

where $\rho : L_{d+1,2} \longrightarrow L_{d+1,1}$ is the reduction. It is an extension of $G\mathbf{W}_2$-modules. Since $L_{d+1,2} = \mathbf{W}_2(\mathcal{O}_S)$, we can associate to $\mathcal{F}$ the G-affine space of its sections. It is a (G,S)-torsor X, under the $G\mathbf{W}_2$-module $V_{d,1} \oplus L_{d,2}$ (see [6], section 4- especially Lemma 4.12). We now work with (G,S)-torsors, instead of extensions. The data of the patching $(\mathcal{E}_{d+1,1}, \phi_1)$ yields a natural trivialization of $q_*(X)$, which is a (G,S)-torsor under $L_{d,1}$. Using the extension $\mathcal{Q}$, we get a (G,S)-torsor Y, under N, together with a natural isomorphism $X \xrightarrow{\sim} \iota_*(Y)$. Lifting the glueing $(\mathcal{E}_{d+1,1}, \phi_1)$ is then equivalent to lifting Y, to a (G,S)-torsor under $V_{d,2}$ via s. Using the connecting map associated to $\mathcal{N}$, the obstruction to this liftability is an element

$$c_1 \in H^2_G(S, V_{d-1,1}^{(1)}).$$

An important fact is that

$$c_2 = j_*(c_1) \in H^2_G(S, V_{d-1,2}).$$

Thus, we have constructed a natural reduction of c_2, to a mod p cohomology class c_1. If S is quasi-projective over a field, by Remark 4 of 3.1, c_1 is represented by an extension of G-*bundles* over S

$$\mathcal{C}_1 : 0 \longrightarrow V_{d-1,1}^{(1)} \longrightarrow * \longrightarrow * \longrightarrow \mathcal{O}_S \longrightarrow 0.$$

In the general case, where $L_{d+1,2}$ is not assumed to be trivial, we apply

$$. \otimes L_{d+1,2}^{\vee},$$

tensor product of $G\mathbf{W}_2$-modules. For instance, we get

$$\mathcal{E}_{d+1,2} \otimes L_{d+1,2}^{\vee} : 0 \longrightarrow V_{d,2} \otimes L_{d+1,2}^{\vee} \longrightarrow V_{d+1,2} \otimes L_{d+1,2}^{\vee} \longrightarrow \mathbf{W}_2(\mathcal{O}_S) \longrightarrow 0.$$

Replacing all $G\mathbf{W}_2$-modules M by $M \otimes L_{d+1,2}^{\vee}$, we are sent back to the case where $L_{d+1,2} = \mathbf{W}_2(\mathcal{O}_S)$. We thus get an element

$$c_1 \in H_G^2(S, (V_{d-1,1} \otimes L_{d+1,1}^{\vee})^{(1)}),$$

such that

$$c_2 = j_*(c_1) \in H_G^2(S, V_{d-1,2} \otimes L_{d+1,2}^{\vee}).$$

If S is quasi-projective over a field, by Remark 4 of 3.1, c_1 is represented by an extension of G-*bundles* over S

$$\mathcal{C}_1 : 0 \longrightarrow V_{d-1,1}^{(1)} \longrightarrow * \longrightarrow * \longrightarrow L_{d+1,1}^{(1)} \longrightarrow 0.$$

13. Computations of extensions, over complete flag schemes.

This section contains results of interest, among which the unicity of the tautological section.

None of them are used in the proof of the Uplifting Theorem, so that this section is logically independent from the main objective of this paper. We advise the reader to skip it, hoping it is useful for future considerations.

13.1. Unicity of the tautological section. Let's start with a "toy example"-just to use a bandied around expression;)

Let V be a vector bundle of rank 3, over a base scheme S. Consider its complete flag scheme $f : \mathbf{F} := \mathbf{Fl}(V) \longrightarrow S$. Denote by

$$Nat_{1,2} : 0 \longrightarrow \mathcal{L}_1 \longrightarrow \mathcal{V}_2 \xrightarrow{\pi} \mathcal{L}_2 \longrightarrow 0$$

the tautological extension, of vector bundles on $\mathbf{F}$. Denote by

$$g : \mathbf{S}_{1,2} := \mathbb{S}(Nat_{1,2}) \longrightarrow \mathbf{F}$$

its splitting scheme, and by $s := f \circ g$ the structure morphism of the S-scheme $\mathbf{S}_{1,2}$. Over $\mathbf{S}_{1,2}$, the extension $Nat_{1,2}$ acquires a canonical (tautological) section $\sigma : \mathcal{L}_2 \longrightarrow \mathcal{V}_2$. View it as an element of $H^0(\mathbf{S}_{1,2}, \mathcal{L}_2^{\otimes -1} \otimes \mathcal{V}_2)$. It is reasonable to expect that σ is the *only* section of $Nat_{1,2}$, over $\mathbf{S}_{1,2}$. This is indeed the case, thanks to the next Lemma.

Lemma 13.1. *The following is true.*
1) The natural inclusion

$$\mathcal{O}_S \longrightarrow s_*(\mathcal{O}_{\mathbf{S}_{1,2}})$$

is an isomorphism.
2) For all $n \geq 1$, we have

$$s_*(\mathcal{O}(n, -n, 0)) = 0.$$

3) The natural arrow

$$\mathcal{O}_S \longrightarrow s_*(\mathcal{L}_2^{\otimes -1} \otimes \mathcal{V}_2),$$
$$1 \mapsto \sigma$$

is an isomorphism.

Proof. By Proposition 5.1, we know that the quasi-coherent $\mathcal{O}_{\mathbf{F}}$-module $g_*(\mathcal{O}_{\mathbf{S}_{1,2}})/\mathcal{O}_{\mathbf{F}}$ has a good filtration by the subbundles $\mathrm{Sym}^a(\mathcal{V}_2)\otimes\mathcal{L}_1^{\otimes -a}$, with $\mathcal{O}(-a,a,0)$ as associated graded pieces, $a\geq 1$. These have $f_*(.)=0$, by Proposition 9.4. Since $\mathcal{O}_S=f_*(\mathcal{O}_{\mathbf{F}})$, Claim 1) follows. We prove 2), by dévissage using the preceding good filtration. We have to check the vanishing of $f_*(\mathrm{Sym}^a(\mathcal{V}_2)(n-a,-n,0))$, for all $a\geq 0$. If $a\leq n$, then $n-a>-n$, and we can apply Proposition 9.4 and the projection formula. For $a>n$, proceed as follows. Consider the natural extension $\mathrm{Sym}^a(Nat_{1,2})(n-a,n,0)$. It is an extension of vector bundles over $\mathbf{F}$, reading as

$$0\longrightarrow \mathrm{Sym}^{a-1}(\mathcal{V}_2)(n-a+1,-n,0)\longrightarrow \mathrm{Sym}^a(\mathcal{V}_2)(n-a,-n,0)\longrightarrow \mathcal{O}(n-a,-n+a,0)\longrightarrow 0.$$

Since $-n+a>0$, we have $s_*(\mathcal{O}(n-a,-n+a,0))=0$ by Proposition 9.4. Applying s_* to this exact sequence, we conclude by induction on a.
We prove the third part. Let f be a section of $s_*(\mathcal{L}_2^{\otimes -1}\otimes\mathcal{V}_2)$. The composite $\pi\circ f$ is a section of $s_*(\mathcal{L}_2^{\otimes -1}\otimes\mathcal{L}_2)=\mathcal{O}_S$ (use 1)). Thus, there exists $\lambda\in\mathcal{O}_S$, such that $\pi\circ(f-\lambda\sigma)$=0. In other words, $f-\lambda\sigma$ belongs to $s_*(\mathcal{O}(1,-1,0))=0$ (use 2)). $\square$
We can adapt this Lemma to a more general setting, as follows.

Proposition 13.2. *Let V be a vector bundle of rank $D\geq 4$, over a base scheme S of characteristic p. Let $1\leq d\leq D-2$ be an integer. Consider its complete flag scheme*

$$F:\mathbf{F}:=\mathbf{Fl}(V)\longrightarrow S.$$

Denote by

$$t:\mathbf{S}_d:=\mathbb{S}(Nat)\longrightarrow \mathbf{F}$$

the splitting scheme of the natural extension

$$Nat: 0\longrightarrow \mathcal{V}_d\longrightarrow\mathcal{V}_{d+1}\longrightarrow\mathcal{L}_{d+1}\longrightarrow 0,$$

over $\mathbf{F}$.
Let $r\geq 0$ and $m\geq 1$ be integers. For $i=1,\dots,m$, denote by

$$\sigma_i:\mathcal{L}_{d+1}\longrightarrow\mathcal{V}_{d+1},$$

the m tautological sections of Nat, over the m-fold product $t^m:\mathbf{S}_d^m\longrightarrow\mathbf{F}$. They arise as pullbacks of the tautological section, by the m projections $\mathbf{S}_d^m\longrightarrow\mathbf{S}_d$. Then, the natural arrow

$$g:\mathcal{O}_S^m\longrightarrow (F\circ t^m)_*(\mathcal{L}_{d+1}^{\vee(r)}\otimes\mathcal{V}_{d+1}^{(r)}),$$
$$e_i\mapsto\sigma_i^{(r)},$$

is an isomorphism.

Proof. We have a natural arrow

$$h:\mathcal{O}_S^{m-1}\longrightarrow (F\circ t^m)_*(\mathcal{L}_{d+1}^{\vee(r)}\otimes\mathcal{V}_{d}^{(r)}),$$
$$e_i\mapsto\sigma_i^{(r)}-\sigma_m^{(r)},$$

$i=1,\dots,m-1$. We first study h. By proposition 5.1, we know that $t_*(\mathcal{O}_{\mathbf{S}_d})$ has a natural good filtration, by vector bundles of the shape $\mathrm{Sym}^a(\mathcal{V}_{d+1}^\vee\otimes\mathcal{L}_{d+1})$, with graded pieces $\mathrm{Sym}^a(\mathcal{V}_d^\vee\otimes\mathcal{L}_{d+1})$, $a\geq 0$. Thus, $t_*^m(\mathcal{O}_{\mathbf{S}_d^m})$ has a natural good filtration, by vector bundles of the shape $\Phi(\mathcal{V}_{d+1}^\vee\otimes\mathcal{L}_{d+1})$, where Φ is a composite symmetric functor. Consider the inclusion (of the component of total degree p^r of this good filtration)

$$\mathcal{W}_{p^r}:=\sum_{\deg(\Phi)=p^r}\Phi(\mathcal{V}_{d+1}^\vee\otimes\mathcal{L}_{d+1})\subset t_*^m(\mathcal{O}_{\mathbf{S}_d^m}).$$

We claim that it induces an isomorphism

$$\iota' : F_*(\mathcal{W}_{p^r} \otimes \mathcal{L}_{d+1}^{\vee(r)} \otimes \mathcal{V}_d^{(r)}) \xrightarrow{\sim} F_*(t_*^m(\mathcal{O}_{\mathbf{S}_d^m}) \otimes \mathcal{L}_{d+1}^{\vee(r)} \otimes \mathcal{V}_d^{(r)}) = t_*^m(\mathcal{L}_{d+1}^{\vee(r)} \otimes \mathcal{V}_d^{(r)}).$$

To see this, it suffices to show that

$$F_*(\Psi(\mathcal{V}_d^\vee \otimes \mathcal{L}_{d+1}) \otimes \mathcal{L}_{d+1}^{\vee(r)} \otimes \mathcal{V}_d^{(r)}) = F_*(\Psi(\mathcal{V}_d^\vee) \otimes \mathcal{V}_d^{(r)} \otimes \mathcal{L}_{d+1}^{a-p^r})$$

vanishes, for all symmetric functors Ψ, of degree $a > p^r$. To do so, consider the factorization

$$F : \mathbf{Fl}(V) \xrightarrow{F_1} \mathbf{Fl}(1, \ldots, d, V) \xrightarrow{F_2} S.$$

By Proposition 9.5, since $d+1 < D-1$, we know that $(F_1)_*(\mathcal{L}_{d+1}^{a-p^r})$ vanishes. Conclude using the projection formula.
We infer that the inclusion $\mathcal{W}_{p^r} \subset t_*^m(\mathcal{O}_{\mathbf{S}_d^m})$ also yields an isomorphism

$$\iota : F_*(\mathcal{W}_{p^r} \otimes \mathcal{L}_{d+1}^{\vee(r)} \otimes \mathcal{V}_{d+1}^{(r)}) \xrightarrow{\sim} F_*(t_*^m(\mathcal{O}_{\mathbf{S}_d^m}) \otimes \mathcal{L}_{d+1}^{\vee(r)} \otimes \mathcal{V}_{d+1}^{(r)}) = t_*^m(\mathcal{L}_{d+1}^{\vee(r)} \otimes \mathcal{V}_{d+1}^{(r)}).$$

Indeed, the tautological sections $\sigma_i^{(r)}$ obviously lie in the image of ι, and we can then use the fact that ι' is an isomorphism to conclude.

It remains to show that the natural arrow

$$g : \mathcal{O}_S^m \longrightarrow F_*(\mathcal{W}_{p^r} \otimes \mathcal{L}_{d+1}^{\vee(r)} \otimes \mathcal{V}_{d+1}^{(r)}),$$

$$e_i \mapsto \sigma_i^{(r)}$$

is an isomorphism. To do this, we use the natural good filtration of $\mathcal{W}_{p^r}$, with graded pieces vector bundles of the shape $\Psi(\mathcal{V}_d^\vee \otimes \mathcal{L}_{d+1})$, where Ψ is a symmetric functor, of degree $a \leq p^r$. Note that $\Psi(.)$ can be written as $\bigotimes_{i=1}^m \mathrm{Sym}^{a_i}(.)$, where $a_1 + \ldots + a_m = a$. If $a < p^r$, then

$$F_*(\Psi(\mathcal{V}_d^\vee \otimes \mathcal{L}_{d+1}) \otimes \mathcal{L}_{d+1}^{\vee(r)} \otimes \mathcal{V}_{d+1}^{(r)}) = F_*(\Psi(\mathcal{V}_d^\vee) \otimes \mathcal{L}_{d+1}^{a-p^r} \otimes \mathcal{V}_{d+1}^{(r)}) = 0.$$

This is clear if $a = 0$ (i.e $\Psi = 1$). If $a \geq 1$, consider the natural exact sequence

$$0 \longrightarrow \Psi(\mathcal{V}_d^\vee) \otimes \mathcal{L}_{d+1}^{a-p^r} \otimes \mathcal{V}_d^{(r)} \longrightarrow \Psi(\mathcal{V}_d^\vee) \otimes \mathcal{L}_{d+1}^{a-p^r} \otimes \mathcal{V}_{d+1}^{(r)} \longrightarrow \Psi(\mathcal{V}_d^\vee) \otimes \mathcal{L}_{d+1}^{a} \longrightarrow 0.$$

Its cokernel has a natural good filtration, with graded pieces degree zero line bundles of the shape $\mathcal{O}(-b_1, \ldots, -b_d, a, 0, \ldots, 0)$, where the non-negative integers b_i satisfy $b_1 + \ldots + b_d = a$. These have have $F_*(.) = 0$ thanks to Proposition 9.4. We are thus reduced to show the vanishing of $F_*(\Psi(\mathcal{V}_d^\vee) \otimes \mathcal{L}_{d+1}^{a-p^r} \otimes \mathcal{V}_d^{(r)})$. The vector bundle $\Psi(\mathcal{V}_d^\vee) \otimes \mathcal{L}_{d+1}^{a-p^r} \otimes \mathcal{V}_d^{(r)}$ has a natural good filtration, with graded pieces degree zero line bundles of the shape $\mathcal{O}(c_1, \ldots, c_d, a-p^r, 0, \ldots, 0)$, where the relative integers c_i satisfy $c_1 + \ldots + c_d = p^r - a > 0$. Thanks to Proposition 9.4, again, these have $F_*(.) = 0$. Conclude by dévissage.
If $a = p^r$, then $F_*(\Psi(\mathcal{V}_d^\vee) \otimes \mathcal{V}_{d+1}^{(r)}) = F_*(\Psi(\mathcal{V}_d^\vee) \otimes \mathcal{V}_d^{(r)})$. If Ψ is composite (meaning that at least two of the a_i's are positive), we can apply Lemma 13.3, to get $F_*(\Psi(\mathcal{V}_d^\vee) \otimes \mathcal{V}_d^{(r)}) = 0$. It now remains to consider the graded pieces where Ψ is pure, i.e. $\Psi(.) = \mathrm{Sym}^{p^r}(.)$. There are m such graded pieces, corresponding to the "trivial" partitions, where one a_i equals p^r, and all other vanish. Applying Lemma 13.3 again, we get $F_*(\Psi(\mathcal{V}_d^\vee) \otimes \mathcal{V}_d^{(r)}) = \mathcal{O}_S$, for each of these graded pieces. Each of these m copies of $\mathcal{O}_S$ corresponds to a direct factor of the source of g- showing that g is, indeed, an isomorphism. $\square$

13.2. A RESULT FOR Ext^0. Let $r \geq 1$ be an integer. For a vector bundle W on an $\mathbb{F}_p$-scheme S, the r-th iterate of the Frobenius

$$\mathrm{Frob}_W^r : W^{(r)} \longrightarrow \mathrm{Sym}^{p^r}(W),$$
$$w \otimes 1 \mapsto w^{p^r}$$

is an injective homomorphism from $W^{(r)}$ to a pure symmetric functor, applied to W. The next Lemma states that, for tautological subquotient bundles, every such homomorphism is collinear to Frob_W^r. It also stipulates that there is no non-zero homomorphism from $W^{(r)}$, to a composite symmetric functor of degree p^r, applied to W.

LEMMA 13.3. *Let V be a vector bundle of rank d, over an $\mathbb{F}_p$-scheme S. Denote by*

$$F : \mathbf{Fl}(V) \longrightarrow S$$

its complete flag scheme. Let m, n be two integers, with $0 \leq m < n-1 \leq d-2$. Put

$$\mathcal{W} := \mathcal{V}_n/\mathcal{V}_m;$$

it is a vector bundle defined over $\mathbf{Fl}(V)$. Let $r \geq 0$ be an integer. Then, the following is true.
1) One has

$$F_*(\mathcal{W}^{(r)\vee} \otimes \mathrm{Sym}^{p^r}(\mathcal{W})) = \mathcal{O}_S,$$

with generator given by the Frobenius

$$\mathrm{Frob}^r : \mathcal{W}^{(r)} \longrightarrow \mathrm{Sym}^{p^r}(\mathcal{W}).$$

1) Let $s \geq 2$ be an integer. Let $a_1, \ldots, a_s$ be positive integers, adding up to p^r. Then, one has

$$H^0(\mathbf{Fl}(V), \mathcal{W}^{(r)\vee} \otimes \mathrm{Sym}^{a_1}(\mathcal{W}) \otimes \ldots \mathrm{Sym}^{a_s}(\mathcal{W})) = 0.$$

Proof. Let's prove 1). Consider the natural exact sequence

$$0 \longrightarrow \mathcal{W}^{(r)\vee} \otimes \mathcal{W}^{(r)} \longrightarrow \mathcal{W}^{(r)\vee} \otimes \mathrm{Sym}^{p^r}(\mathcal{W}) \longrightarrow \mathcal{W}^{(r)\vee} \otimes (\mathrm{Sym}^{p^r}(\mathcal{W})/\mathcal{W}^{(r)}) \longrightarrow 0.$$

Its kernel is $\mathrm{End}(\mathcal{W})^{(r)}$, which has $F_*(.) = \mathcal{O}_S$, with generator given by the identity. Checking this fact is left to the reader, as an exercise. Its cokernel has a natural good filtration, inherited from that of $\mathcal{W}$, with successive quotients degree zero line bundles of the shape $\mathcal{O}(0, \ldots, 0, a_{m+1}, \ldots, a_n, 0, \ldots, 0)$. Here $(a_{m+1}, \ldots, a_n)$ is a non-zero sequence of relative integers, adding up to zero. According to Proposition 9.4, these line bundles have $F_*(.) = 0$ (this uses $n \leq d-1$). By dévissage, the cokernel in question has $F_*(.) = 0$ as well. The claim is proved.
Let's deal with 2). Using the projection formula, we see that

$$H^0(\mathbf{Fl}(V), \mathcal{W}^{(r)\vee} \otimes \mathrm{Sym}^{a_1}(\mathcal{W}) \otimes \ldots \otimes \mathrm{Sym}^{a_s}(\mathcal{W})) =$$
$$H^0(\mathbf{Fl}(V), \mathcal{W}^{(r)\vee} \otimes \mathrm{Sym}^{a_1}(\mathcal{W}) \otimes \ldots \otimes \mathrm{Sym}^{a_{s-1}}(\mathcal{W}) \otimes \mathcal{L}_n^{\otimes a_s}).$$

Using the natural good filtration of $\mathcal{W}^{(r)\vee}$, by dévissage, we reduce to showing the vanishing of

$$H^0(\mathbf{Fl}(V), \mathrm{Sym}^{a_1}(\mathcal{W}) \otimes \ldots \otimes \mathrm{Sym}^{a_{s-1}}(\mathcal{W}) \otimes \mathcal{L}_n^{\otimes a_s} \otimes \mathcal{L}_i^{\otimes -p^r}),$$

for $i = m+1, \ldots, n$. The case $i = n$ is straightfoward, using the projection formula and Proposition 9.4, because $0 > a_s - p^r$. Note that this uses $\dim(\mathcal{W}) = n-m \geq 2$. We now deal with the case $m+1 \leq i \leq n-1$. The vector bundle $\mathrm{Sym}^{a_1}(\mathcal{W}) \otimes \ldots \otimes \mathrm{Sym}^{a_{s-1}}(\mathcal{W})$ has a natural good filtration, with graded pieces line bundles of the shape $\mathcal{O}(0,\ldots,0,b_{m+1},\ldots,b_n,0,\ldots,0)$, where $b_{m+1}+\ldots+b_n = a-a_s$, and $0 \leq b_j \leq a-a_s$ for all j. Thus, the vector bundle

$$\mathrm{Sym}^{a_1}(\mathcal{W}) \otimes \ldots \otimes \mathrm{Sym}^{a_{s-1}}(\mathcal{W}) \otimes \mathcal{L}_i^{\otimes(-p^r)} \otimes \mathcal{L}_n^{\otimes a_s}$$

has a natural good filtration, with graded pieces line bundles of the shape

$$\mathcal{O}(0,\ldots,0,b_{m+1},\ldots,b_{i-1},b_i-p^r,b_{i+1}\ldots,b_n+a_s,0,\ldots,0)$$

where $b_i - p^r < 0$. The sequence

$$(0,\ldots,0,b_{m+1},\ldots,b_{i-1},b_i-p^r,b_{i+1}\ldots,b_n+a_s,0,\ldots,0)$$

is not increasing: it ends by zero, its terms add up to zero, and one of its terms is non-zero. We conclude by dévissage, using Proposition 9.4.

□

13.3. A result for Ext^1.

Lemma 13.4. *Let V be a vector bundle of rank 3, over an $\mathbb{F}_p$-scheme S. Denote by*

$$F : \mathbf{Fl}(V) \xrightarrow{f_1} \mathbb{P}(V) = \mathbf{Fl}(2,V) \xrightarrow{f_2=f} S$$

its complete flag scheme. Recall that we denote by

$$0 \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset \mathcal{V}_3 = F^*(V)$$

the tautological complete flag, over $\mathbf{Fl}(V)$.
Let $b \geq 0$ be an integer. The following is true.

1) There exists a canonical isomorphism of $\mathcal{O}_{\mathbb{P}(V)}$-modules

$$\Gamma^{2b}(\mathcal{V}_2) \otimes \mathrm{Det}^{-b}(\mathcal{V}_2) \xrightarrow{\sim} R^1(f_1)_*(\mathcal{O}(b+1,-b-1,0)).$$

2) If $b+1$ is not a p-th power, we have

$$f_*(\Gamma^{2b}(\mathcal{V}_2) \otimes \mathrm{Det}^{-b}(\mathcal{V}_2)) = 0.$$

3) Assume that $b = p^s - 1$. Then, there exists a canonical isomorphism

$$f_*(\Gamma^{2b}(\mathcal{V}_2) \otimes \mathrm{Det}^{-b}(\mathcal{V}_2)) \simeq \mathcal{O}_S.$$

Proof. Point 1) is a reformulation of point 3) of Proposition 9.5, applied to the 2-dimensional vector bundle $\mathcal{V}_2$ over $\mathbb{P}(V)$, and to $n = 2b+2$. Note that f_1 is a $\mathbb{P}^1$-bundle (the projective bundle of $\mathcal{V}_2$).
It remains to prove 2) and 3).
The vector bundle $\Gamma^{2b}(\mathcal{V}_2) \otimes \mathrm{Det}^{-b}(\mathcal{V}_2)$ has a good filtration by subbundles, with quotients line bundles of the shape $\mathcal{O}(i,-i,0)$, $i = -b,-b+1\ldots,b-1,b$. These have $F_*(.) = 0$, except for $i = 0$. By dévissage, we get a canonical embedding

$$\alpha : f_*(\Gamma^{2b}(\mathcal{V}_2) \otimes \mathrm{Det}^{-b}(\mathcal{V}_2)) \longrightarrow \mathcal{O}_S,$$

which is a first step towards proving 2) and 3).
Assertions 2) and 3) can be checked Zariski-locally on S, using a gluing argument for 3). We thus reduce to the case where $V = \mathcal{O}_S^3$ is trivial. If desired, we can further reduce to $S = \mathrm{Spec}(A)$, with A an $\mathbb{F}_p$-algebra of finite-type. Consider the structure morphism $g : S \longrightarrow \mathrm{Spec}(\mathbb{F}_p)$. If we can prove the statements when $S = \mathrm{Spec}(F_p)$, then we can pull everything back through g by proper base change, and get the same statements over S.
Next, we give a proof that 2) holds, working for $S = \mathrm{Spec}(k)$, with k any field of

characteristic p.

The exact sequence of vector bundles

$$0 \longrightarrow \mathcal{L}_1 \longrightarrow \mathcal{V}_2 \xrightarrow{\pi} \mathcal{L}_2 \longrightarrow 0$$

induces a natural surjection

$$\Gamma^{2b}(\mathcal{V}_2) \longrightarrow \Gamma^b(\mathcal{V}_2)(0,b,0),$$

$$[v]_{2b} \mapsto [v]_b \otimes \pi(v)^b.$$

Twisting it by $\mathrm{Det}^{-b}(\mathcal{V}_2) = \mathcal{O}(-b,-b,0)$, we get a surjection of vector bundles over $\mathbf{Fl}(V)$

$$\phi : \Gamma^{2b}(\mathcal{V}_2) \otimes \mathrm{Det}^{-b}(\mathcal{V}_2) \longrightarrow \Gamma^b(\mathcal{V}_2)(-b,0,0),$$

Denote by $\mathcal{K}_b$ its kernel; it is a vector bundle of rank b over $\mathbb{P}(V)$. It has a natural good filtration by subbundles, having as successive quotients the line bundles $\mathcal{O}(i,-i,0)$, for $i = 1,\dots,b$. These have $F_*(.) = 0$, so that $F_*(\mathcal{K}_b) = 0$ by dévissage. Taking $F_*(\phi)$ thus yields an injection

$$\iota : f_*(\Gamma^{2b}(\mathcal{V}_2) \otimes \mathrm{Det}^{-b}(\mathcal{V}_2)) \longrightarrow f_*(\Gamma^b(\mathcal{V}_2) \otimes \mathrm{Sym}^b(\mathcal{V}_2) \otimes \mathrm{Det}^{-b}(\mathcal{V}_2)),$$

given on sections by the formula

$$[v]_{2b} \otimes \delta^{-b} \mapsto [v]_b \otimes v^b \otimes \delta^{-b}.$$

Using a good filtration argument analogous to those used before, we prove that the k-vector space $f_*(\Gamma^b(\mathcal{V}_2) \otimes \mathrm{Sym}^b(\mathcal{V}_2) \otimes \mathrm{Det}^{-b}(\mathcal{V}_2))$ has dimension at most one. Consider the perfect duality pairing

$$\Delta = <.,.>: \Gamma^b(\mathcal{V}_2) \times \mathrm{Sym}^b(\mathcal{V}_2) \longrightarrow \mathrm{Det}^b(\mathcal{V}_2),$$

$$< [v]_b, w_1 w_2 \dots w_b > = (v \wedge w_1)(v \wedge w_2) \dots \otimes (v \wedge w_b).$$

It gives a non-zero vector inside $f_*(\Gamma^b(\mathcal{V}_2) \otimes \mathrm{Sym}^b(\mathcal{V}_2) \otimes \mathrm{Det}^{-b}(\mathcal{V}_2))$, which we still denote by Δ. The vector space $f_*(\Gamma^b(\mathcal{V}_2) \otimes \mathrm{Sym}^b(\mathcal{V}_2) \otimes \mathrm{Det}^{-b}(\mathcal{V}_2))$ is therefore one-dimensional, directed by Δ. Arguing by contradiction, assume that the k-vector space $f_*(\Gamma^{2b}(\mathcal{V}_2) \otimes \mathrm{Det}^{-b}(\mathcal{V}_2))$ is one dimensional too, with generator u. By what precedes, $\iota(u)$ would then be a non-zero multiple of Δ. Rescaling, we can assume $\iota(u) = \Delta$. Considering a fiber V_2 of $\mathcal{V}_2$, at a k-rational point of $\mathbb{P}(V)$, we would then get the following. There exists

$$u = \sum_1^r a_i [u_i]_{2b} \in \Gamma_k^{2b}(V_2),$$

where $a_i \in k$, and $u_i \in V$, such that the perfect duality pairing

$$<.,.>: \Gamma_k^b(V_2) \times \mathrm{Sym}_k^b(V_2) \longrightarrow \mathrm{Det}^b(V_2)$$

is expressed as

$$< [v]_b, w_1 w_2 \dots w_b > = \sum_1^r a_i (v \wedge u_i)^b (w_1 \wedge u_i)(w_2 \wedge u_i) \dots (w_b \wedge u_i).$$

Write the base-p expansion

$$b = a_0 + a_1 p + \dots + a_s p^s.$$

One see that the expression on the right factors, in the first variable v, through the natural surjective k-linear map

$$\theta : \Gamma^b(V_2) \longrightarrow \bigotimes_0^s \Gamma^{a_i}(V_2^{(i)}),$$

$$[v]_b \mapsto [v]_{a_0} \otimes [v^{(1)}]_{a_1} \otimes \ldots \otimes [v^{(s)}]_{a_s}.$$

Indeed, this factorisation is given by

$$(.,.) : \bigotimes_{i=0}^{s} \Gamma^{a_i}(V_2^{(i)}) \times \mathrm{Sym}_k^b(V_2) \longrightarrow \mathrm{Det}^b(V_2),$$

$$([v_0]_{a_0}[v_1]_{a_1 p} \ldots [v_s]_{a_s p^s}, w_1 w_2 \ldots w_b) :=$$

$$\sum_1^r a_i (v_0 \wedge u_i)^{a_0} (v_1 \wedge u_i)^{a_1 p} \ldots (v_s \wedge u_i)^{a_s p^s} (w_1 \wedge u_i)(w_2 \wedge u_i) \ldots (w_b \wedge u_i).$$

Since $< .,. >$ is perfect, the surjection θ has to be an isomorphism. Equating dimensions of the source and target of θ yields

$$b + 1 = (a_0 + 1)(a_1 + 1) \ldots (a_s + 1),$$

implying $a_i = p - 1$ for all i, so that $b = p^s - 1$.
Let's just sketch the proof of statement 3), which is much easier because it is constructive. Consider the extension

$$0 \longrightarrow \mathcal{L}_1^{(s)} \longrightarrow \mathcal{V}_2^{(s)} \longrightarrow \mathcal{L}_2^{(s)} \longrightarrow 0,$$

of vector bundles over $\mathbf{Fl}(V)$, defined as the s-th Frobenius pullback of the tautological sequence. Its class yields an injection of $\mathcal{O}_{\mathbb{P}(V)}$-modules

$$\mathcal{O}_{\mathbb{P}(V)} \longrightarrow R^1(f_1)_*(\mathcal{O}(p^s, -p^s, 0)) \xrightarrow{\sim} \Gamma^{2b}(\mathcal{V}_2) \otimes \mathrm{Det}^{-b}(\mathcal{V}_2),$$

where we have used the isomorphism of point 1) of our Lemma. Applying f_*, we get an injection of $\mathcal{O}_S$-modules

$$\beta : f_*(\mathcal{O}_{\mathbb{P}(V)}) \xrightarrow{\sim} \mathcal{O}_S \longrightarrow f_*(\Gamma^{2b}(\mathcal{V}_2) \otimes \mathrm{Det}^{-b}(\mathcal{V}_2)),$$

which is then checked to be the inverse of α. □

Remark 13.5. Using the projection formula, Lemma 13.4 can be generalized to vector bundles of arbitrary rank $D \geq 3$. In particular, we get the following result. Let V be a vector bundle of rank $D \geq 3$, over an $\mathbb{F}_p$-scheme S. Denote by

$$F : \mathbf{Fl}(V) \longrightarrow S$$

its complete flag scheme. Let $1 \leq d \leq D - 1$ be an integer. Let $a \geq 0$ be an integer.

- If a is not a p-th power, then $R^1 F_*(\mathcal{L}_d^a \otimes \mathcal{L}_{d+1}^{-a}) = 0$.
- If $a = p^s$ is a p-th power, then $R^1 F_*(\mathcal{L}_d^{p^s} \otimes \mathcal{L}_{d+1}^{-p^s}) = \mathcal{O}_S$, with canonical generator given by (the s-th Frobenius twist of) the natural extension

$$Nat_{d,d+1} : 0 \longrightarrow \mathcal{L}_d \longrightarrow \mathcal{V}_{d+1}/\mathcal{V}_{d-1} \longrightarrow \mathcal{L}_{d+1} \longrightarrow 0.$$

14. Statement of the Uplifting Theorem.

Let S be an $(\mathbb{F}_p, G)$-scheme. Let L be a G-line bundle over S. Recall that L lifts to a $G\mathbf{W}_2$-bundle over S; namely, its Teichmüller lift $\mathbf{W}_2(L)$.
We come to the main result of this paper. It extends the preceding fact to higher dimensions- under appropriate assumptions.

Theorem 14.1. *(The Uplifting Theorem)*
Let G be a $(1,1)$-smooth profinite group. Let S be a perfect affine $(\mathbb{F}_p, G)$-scheme. Let

$$\nabla_{d+1,1} : 0 \subset V_{1,1} \subset V_{2,1} \subset \ldots \subset V_{d+1,1}$$

be a complete flag of G-vector bundles, of dimension $d+1 \geq 2$ over S.
Assume given a lift of the truncation

$$\nabla_{d,1} := \tau_d(\nabla_{d+1,1}) : 0 \subset V_{1,1} \subset V_{2,1} \subset \ldots \subset V_{d,1},$$

to a complete flag of $G\mathbf{W}_2$- bundles over S

$$\nabla_{d,2} : 0 \subset V_{1,2} \subset V_{2,2} \subset \ldots \subset V_{d,2}.$$

Then, $\nabla_{d,2}$ can be extended, to a lift

$$\nabla_{d+1,2} : 0 \subset V_{1,2} \subset V_{2,2} \subset \ldots \subset V_{d,2} \subset V_{d+1,2}$$

of $\nabla_{d+1,1}$, to a complete flag of $G\mathbf{W}_2$- bundles over S.

We get the following result as a consequence of the Uplifting Theorem.

Theorem 14.2. *(The Uplifting Theorem, weak form)*
Let G be a $(1,1)$-smooth profinite group. Let S be a perfect affine $(\mathbb{F}_p, G)$-scheme. Let

$$\nabla_1 : 0 \subset V_{1,1} \subset V_{2,1} \subset \ldots \subset V_{d,1}$$

be a complete flag of G-vector bundles over S, of dimension $d \geq 1$.
Then, ∇_1 admits a lift, to a complete flag ∇_2 of $G\mathbf{W}_2$- bundles over S.

Here is an equivalent reformulation, in the tongue of embedding problems.
Let A be a perfect $(\mathbb{F}_p, G)$-algebra. Let $d \geq 1$ be an integer. Denote by $\mathbf{B}_d \subset \mathbf{GL}_d$ the Borel subgroup of upper triangular matrices.
Then, the natural arrow

$$H^1(G, \mathbf{B}_d(\mathbf{W}_2(A))) \longrightarrow H^1(G, \mathbf{B}_d(A)),$$

induced by reduction, is surjective.

Proof. Induction on d, using Theorem 14.1. □

Remark 14.3. Note that the statement of the Uplifting Theorem does not provide information about the graded pieces $L_{i,2}$ of ∇_2.
Assume that G is $(1,1)$-cyclotomic, relative to a cyclotomic module $\mathbb{Z}/p^2(1)$. It is then $(1,1)$-smooth by [6, Theorem A]. The Uplifting Theorem thus applies to G. It is then normal to wonder whether we can prescribe

$$L_{i,2} = \mathbf{W}_2(L_{i,1})(-i).$$

The answer is negative in general- see the next section.

Remark 14.4. We can ask whether the Uplifting Theorem extends to depth $e \geq 2$. This could be the subject of future investigation. For many possible applications though, the degree of generality allowed by the Uplifting Theorem, in depth $e = 1$, is arguably sufficient. This belief is materialized in [7], where the Uplifting Theorem is applied, to provide a self-contained proof of the Norm Residue Isomorphism Theorem.

15. Non-liftability of the generic Heisenberg representation.

In the second part of the statement of Theorem 14.2, it is natural to ask whether $\mathbf{B}_d$ can be replaced by its unipotent radical $\mathbf{U}_d$.
In other words, under the assumptions of this Theorem, is the natural arrow

$$H^1(G, \mathbf{U}_d(\mathbf{W}_2(A))) \longrightarrow H^1(G, \mathbf{U}_d(A))$$

surjective?
A positive answer would be an improvement, as surjectivity for coefficients in $\mathbf{U}_d$ implies surjectivity for coefficients in $\mathbf{B}_d$ (exercise for the reader).

In this section, we provide a negative answer to the question above. Precisely, for p odd, we give an example of a field F, containing $\mathbb{C}$, such that

$$H^1(\mathrm{Gal}(F_s/F), \mathbf{U}_3(\mathbb{Z}/p^2)) \longrightarrow H^1(\mathrm{Gal}(F_s/F), \mathbf{U}_3(\mathbb{F}_p))$$

is not surjective, using a result of Karpenko as the key ingredient. In other words: in general, mod p Heisenberg representations fail to lift mod p^2.

Remarks 15.1.

- When $p = 2$, one can show that the preceding arrow is surjective, for any field F. However, replacing $\mathbb{F}_2$ with the finite field $\mathbb{F}_4$, and $\mathbb{Z}/4$ with $\mathbf{W}_2(\mathbb{F}_4)$, surjectivity again fails, in general.
- In the recent work [21], it is proved that the preceding arrow is surjective for p odd, when F is a global field or a non-archimedean local field, under the presence of p^2-th roots of unity. Note that the proof provided for F local, in fact extends to the case of fields F, containing p^2-th roots of unity, and such that the $\mathbb{F}_p$-vector space $H^2(F, \mathbb{F}_p)$ is one-dimensional.

Start with a field F, containing the function field in two variables $\mathbb{C}(x, y)$. Set G to be its absolute Galois group. It is $(1, 1)$-smooth by [6, Theorem A]. For each $n \geq 1$, use $e^{\frac{2\pi i}{n}} \in F$ to identify μ_n to $\mathbb{Z}/n$, as finite G-modules.
Using Kummer theory, we have two classes

$$(x)_p, (y)_p \in H^1(F, \mu_p),$$

respectively associated to extensions of $(\mathbb{F}_p, G)$-modules

$$\mathcal{E}_x : 0 \longrightarrow \mathbb{F}_p = \mu_p \longrightarrow E_x \longrightarrow \mathbb{F}_p \longrightarrow 0$$

and

$$\mathcal{E}_y : 0 \longrightarrow \mathbb{F}_p = \mu_p \longrightarrow E_y \longrightarrow \mathbb{F}_p \longrightarrow 0.$$

These give rise to arrows

$$\rho_x : G \longrightarrow \mathbf{U}_2(\mathbb{F}_p) = \mathbb{F}_p$$

and

$$\rho_y : G \longrightarrow \mathbf{U}_2(\mathbb{F}_p) = \mathbb{F}_p.$$

Definition 15.2. *Assume there exists a complete flag of* $(\mathbb{F}_p, G)$*-modules*

$$\nabla_1 : 0 \subset V_{1,1} \subset V_{2,1} \subset V_{3,1},$$

such that the truncated extension of $(\mathbb{F}_p, G)$*-modules*

$$0 \longrightarrow V_{1,1} \longrightarrow V_{2,1} \longrightarrow V_{2,1}/V_{1,1} \longrightarrow 0$$

is isomorphic to $\mathcal{E}_x$*, and such that the quotient extension*

$$0 \longrightarrow V_{2,1}/V_{1,1} \longrightarrow V_{3,1}/V_{1,1} \longrightarrow V_{3,1}/V_{2,1} \longrightarrow 0$$

is isomorphic to $\mathcal{E}_y$*.*
We then say that $\mathcal{E}_x$ *and* $\mathcal{E}_y$ *glue, to the complete flag* ∇_1*.*

The extensions $\mathcal{E}_x$ and $\mathcal{E}_y$ glue to a ∇_1 as above, if and only if the cup-product

$$a = (x)_p \cup (y)_p \in H^2(F, \mu_p^{\otimes 2}) = H^2(F, \mu_p) = \mathrm{Br}(F)[p]$$

vanishes.
Assume that this is the case, and let ∇_1 be such a gluing. Using the same construction as above, for mod p^2 coefficients, we get the following.
The complete flag ∇_1 lifts to a complete flag of $(\mathbb{Z}/p^2, G)$-bundles

$$\nabla_2 : 0 \subset V_{1,2} \subset V_{2,2} \subset V_{3,2},$$

with trivial graded pieces

$$L_{i,2} = \mathbb{Z}/p^2,$$

if and only if $(x)_p$ and $(y)_p$ respectively lift to classes

$$(X)_{p^2}, (Y)_{p^2} \in H^1(F, \mu_{p^2}),$$

such that

$$(X)_{p^2} \cup (Y)_{p^2} = 0 \in H^2(F, \mu_{p^2}^{\otimes 2}) = H^2(F, \mu_{p^2}) = \mathrm{Br}(K)[p^2].$$

We now show that the field F can be chosen, so that this liftability property fails. Equivalently:

- The extensions $\mathcal{E}_x$ and $\mathcal{E}_y$ glue, to a ∇_1 as above.
- The flag ∇_1 does not admit a lift to a flag of $G\mathbf{W}_2$ bundles ∇_2, with trivial graded pieces.

We can then conclude, that the class of ∇_1 in $H^1(\mathrm{Gal}(F_s/F), \mathbf{U}_3(\mathbb{F}_p))$ cannot be lifted via

$$H^1(\mathrm{Gal}(F_s/F), \mathbf{U}_3(\mathbb{Z}/p^2)) \longrightarrow H^1(\mathrm{Gal}(F_s/F), \mathbf{U}_3(\mathbb{F}_p)),$$

completing the goal of this section.
Start with the generic symbol algebra

$$A = (x, y)_{p^2},$$

over the field $\mathbb{C}(x, y)$. It is a division algebra of degree p^2. Consider the Severi-Brauer variety $\mathrm{SB}(A^{\otimes p})$, and define F to be its function field. By [19], Theorem 2.1, the F-algebra $A \otimes_{\mathbb{C}(x,y)} F$ is an indecomposable division algebra of degree p^2 and exponent p. Assume that $(x)_p$ and $(y)_p$ lift to classes

$$(X)_{p^2}, (Y)_{p^2} \in H^1(F, \mu_{p^2}),$$

such that $(X)_{p^2} \cup (Y)_{p^2} = 0$. Write

$$(X)_{p^2} = (x)_{p^2} - p(u)_{p^2}$$

and

$$(Y)_{p^2} = (y)_{p^2} - p(v)_{p^2},$$

for $u, v \in F^\times$. Expanding the equality $(X)_{p^2} \cup (Y)_{p^2} = 0$, we get

$$[A] = (u)_p \cup (y)_p + (x)_p \cup (v)_p \in \mathrm{Br}(F).$$

In other words, A decomposes as a tensor product of two symbol algebras of degree p over F- a contradiction.

16. Proof of the Uplifting Theorem.

Remark 16.1. In this proof, we could have worked over perfect $\mathbb{F}_p$-schemes (e.g. the perfection of flag schemes and of splitting schemes). This would have made the proof slightly more readable, by dismissing some Frobenius twists. Meanwhile, it would also have made it less explicit, and would have concealed the possibility of measuring the growth of these Frobenius twists- a goal that I will not pursue.

Let

$$\nabla_{d+1,1} : 0 \subset V_{1,1} \subset V_{2,1} \subset \ldots \subset V_{d+1,1}$$

be a complete flag of G-vector bundles, of arbitrary dimension $d+1 \geq 2$, over $S = \mathrm{Spec}(A)$. We think of it as "a complete flag of semi-linear representations of G over A". Assume given a lift of the truncation

$$\nabla_{d,1} : 0 \subset V_{1,1} \subset V_{2,1} \subset \ldots \subset V_{d,1},$$

to a complete flag of $G\mathbf{W}_2$-bundles

$$\nabla_{d,2} : 0 \subset V_{1,2} \subset V_{2,2} \subset \ldots \subset V_{d,2}.$$

Replacing $\nabla_{d,2}$ by

$$\nabla'_{d,2} := \nabla_{d,2} \otimes L_{1,2}^{-1} \otimes \mathbf{W}_2(L_{1,2}),$$

which is another lift of $\nabla_{d,1}$, we are free to assume $V_{1,2} = \mathbf{W}_2(L_{1,2})$.

Denote by

$$G_0 \subset G$$

the kernel of the action of G on S, on $V_{d+1,1}$ and on $V_{d,2}$. Put

$$G^0 := G/G_0.$$

All G-actions, so far, come from G^0-actions.
Choose an embedding of $A[G^0]$-modules

$$V_{d+1,1} \longrightarrow V_{D,1} := A[G^0]^n,$$

with

$$D := n|G^0| \geq d+2.$$

Note that the existence of such an embedding is equivalent to its dual counterpart: writing $V_{d+1,1}^\vee$ as a quotient of a free $A[G^0]$-module, whose rank can be chosen to be arbitrarily large. Set

$$V_{D,2} := \mathbf{W}_2(A)[G^0]^n.$$

For $r = 1, 2$, note that $V_{D,r}$, seen as a $G\mathbf{W}_r$-bundle over S, is permutation. We can now view $\nabla_{d+1,1}$ as a permutation embedded flag of $G\mathbf{W}_1$-bundles

$$\nabla_{d+1,1} : 0 \subset V_{1,1} \subset V_{2,1} \subset \ldots \subset V_{d+1,1} \subset V_{D,1}.$$

Because $V_{D,1}$ is a projective $A[G^0]$-module, the embedding

$$V_{d,1} \longrightarrow V_{D,1}$$

lifts, to an embedding of $G\mathbf{W}_2$-bundles

$$V_{d,2} \longrightarrow V_{D,2},$$

yielding an embedded flag of $G\mathbf{W}_2$-bundles

$$0 \subset V_{1,2} \subset V_{2,2} \subset \ldots \subset V_{d,2} \subset V_{D,2},$$

which we still denote by $\nabla_{d,2}$.

Lemma 16.2. *We can, and will, assume that $V_{1,2} \subset V_{D,2}$ is the Teichmüller lift $\tau_2(i_{1,D,1})$ of the natural inclusion*

$$(i_{1,D,1} : V_{1,1} \longrightarrow V_{D,1}) \in H^0(G, V_{1,1}^\vee \otimes V_{D,1}),$$

provided by Lemma 6.2.

Proof. Denote by

$$i_{1,D,2} : V_{1,2} \longrightarrow V_{D,2} \in H^0(G, V_{1,2}^\vee \otimes V_{D,2})$$

the inclusion appearing in $\nabla_{d,2}$. The difference $\tau_2(i_{1,D,1}) - i_{1,D,2}$ has trivial mod p reduction. Using the natural extension of $G\mathbf{W}_2$-modules

$$0 \longrightarrow (V_{1,1}^\vee \otimes V_{D,1})^{(1)} \xrightarrow{j} V_{1,2}^\vee \otimes V_{D,2} \xrightarrow{\rho} V_{1,1}^\vee \otimes V_{D,1} \longrightarrow 0,$$

it is hence given by an element

$$\epsilon_{1,D,1} \in H^0(G^0, (V_{1,1}^\vee \otimes V_{D,1})^{(1)}).$$

Since $V_{D,1}$ is a projective $A[G^0]$-module, and since $V_{1,1}$ is locally free as an A-module, $(V_{1,1}^\vee \otimes V_{D,1})^{(1)}$ is a projective $A[G^0]$-module. Thus, the extension of $A[G^0]$-modules

$$0 \longrightarrow ((V_{d,1}/V_{1,1})^\vee \otimes V_{D,1})^{(1)} \longrightarrow (V_{d,1}^\vee \otimes V_{D,1})^{(1)} \longrightarrow (V_{1,1}^\vee \otimes V_{D,1})^{(1)} \longrightarrow 0$$

splits, so that $\epsilon_{1,D,1}$ extends (lifts) to an element

$$\epsilon_{d,D,1} \in H^0(G^0, (V_{d,1}^\vee \otimes V_{D,1})^{(1)}) \subset H^0(G^0, (V_{d,2}^\vee \otimes V_{D,2})).$$

The claim follows, replacing the inclusions

$$i_{j,D,2} : V_{j,2} \longrightarrow V_{D,2}$$

by

$$i_{j,D,2} + (\epsilon_{d,D,1})_{|V_{j,2}}.$$

□

Put $V_1 := V_{D,1}$. Introduce the flag scheme

$$F : \mathbf{F} := \mathbf{Fl}(1, \ldots, d+1, V_1) \longrightarrow S.$$

Denote by

$$\nabla_{gen,d+1,1} : 0 \subset \mathcal{V}_{1,1} \subset \ldots \subset \mathcal{V}_{d,1} \subset \mathcal{V}_{d+1,1} \subset \mathcal{V}_{D,1} := F^*(V_1)$$

the tautological flag. The data of $\nabla_{d+1,1}$, embedded in $V_{D,1}$, naturally corresponds to a G-equivariant arrow $s : S \longrightarrow \mathbf{F}$ (a G-equivariant section of F), together with an isomorphism of G-flags embedded in $V_{D,1}$,

$$\nabla_{d+1,1} \simeq s^*(\nabla_{gen,d+1}).$$

We are now going to perform successive changes of the base, from $\mathbf{F}$ to suitable G-schemes. These parametrize liftings (resp. splittings) of some relevant extensions of vector bundles. Let's get to details.

Denote by

$$\nabla_{gen,d,1} : 0 \subset \mathcal{V}_{1,1} \subset \mathcal{V}_{2,1} \subset \ldots \subset \mathcal{V}_{d,1} \subset \mathcal{V}_{D,1}$$

the truncation of $\nabla_{gen,d+1,1}$.

16.1. Step 1: Geometric splitting of $Nat_{d,d+1,1}$.

Over $\mathbf{F}$, we have a natural extension of G-bundles

$$Nat_{d,d+1,1} : 0 \longrightarrow \mathcal{L}_{d,1} \longrightarrow \mathcal{V}_{d+1,1}/\mathcal{V}_{d-1,1} \longrightarrow \mathcal{L}_{d+1,1} \longrightarrow 0.$$

We produce a G-equivariant change of the base

$$T : \mathbf{T} \longrightarrow \mathbf{F}$$

such that, over $\mathbf{T}$, the extension $Nat_{d,d+1,1}$ splits, as an extension of G_0-bundles. To do so, simply set

$$\mathbf{T} := \mathrm{Ind}_{(G_0,\mathbf{F})-Sch}^{(G,\mathbf{F})-Sch}(\mathbb{S}(Nat_{d,d+1,1})) = \mathbb{S}(Nat_{d,d+1,1})^{G^0} \longrightarrow \mathbf{L}_{d+1},$$

where the product is fibered over $\mathbf{F}$.

Lemma 16.3. *The following holds.*

(1) *The G-equivariant sections of $\mathbf{T} \longrightarrow \mathbf{L}_{d+1}$ parametrize splittings of the extension of $(G_0, \mathbf{W}_1)$-bundles $Nat_{d,d+1,1}$. In particular, over $\mathbf{T}$, the extension of $(G_0, \mathbf{W}_1)$-bundles $Nat_{d,d+1,1}$ splits.*
(2) *The quasi-coherent $\mathcal{O}_\mathbf{F}$-Module $T_*(\mathcal{O}_\mathbf{T})$ has a natural good filtration, with graded pieces vector bundles of the shape*

$$\mathcal{L}_{d,1}^{\otimes -b} \otimes \mathcal{L}_{d+1,1}^{\otimes b},$$

for $b \geq 0$.
(3) *Over S, the natural extension of $(G_0, \mathbf{W}_1)$-bundles*

$$0 \longrightarrow L_{d,1} \longrightarrow V_{d+1,1}/V_{d-1,1} \longrightarrow L_{d+1,1} \longrightarrow 0$$

splits. The data of such a splitting determines a G-equivariant point

$$s_1 : S \longrightarrow \mathbf{T},$$

lifting s (formula: $T \circ s_1 = s$).
In short: s naturally lifts through T, in a G-equivariant fashion.

Proof. Point 1) follows from the universal property of induction, given in 3.11. Point 2) follows from Proposition 5.1. Since G_0 acts trivially on everything, 3) holds simply because S is affine. $\square$

16.2. Step 2: Equivariant lifting of $Nat_{d,d+1,1}$.
Over $\mathbf{T}$, the extension of G-vector bundles

$$Nat_{d,d+1,1} : 0 \longrightarrow \mathcal{L}_{d,1} \longrightarrow \mathcal{V}_{d+1,1}/\mathcal{V}_{d-1,1} \longrightarrow \mathcal{L}_{d+1,1} \longrightarrow 0$$

is geometrically split. Since the profinite group G is $(1,1)$-smooth, we can apply Proposition 11.12 of [6]. There exists $m \geq 0$, and a lift of $\mathcal{L}_{d+1,1}^{(m)}$, to a line $G\mathbf{W}_2$-bundle $\mathcal{L}_{d+1,2}^{[m]}$ over $\mathbf{T}$, such that $Nat_{d,d+1,1}^{(m)}$ lifts, to a (geometrically split) extension of $G\mathbf{W}_2$-bundles over $\mathbf{T}$

$$Nat_{d,d+1,2}^{[m]} : 0 \longrightarrow \mathcal{L}_{d,2}^{(m)} \longrightarrow \mathcal{V}_{d,d+1,2}^{[m]} \longrightarrow \mathcal{L}_{d+1,2}^{[m]} \longrightarrow 0.$$

Note that the $G\mathbf{W}_2$-line bundle $\mathcal{L}_{d+1,2}^{[m]}$ need not be isomorphic to $\mathbf{W}_2(\mathcal{L}_{d+1,1}^{(m)})$.

16.3. Step 3: Equivariant lifting of $\nabla_{gen,1,d}$.
We now produce an equivariant lifting of $\nabla_{gen,1,d}$, over $\mathbf{T}$. We use induction, following the process described in section 11.2.
Using Lemma 6.2, we get that the arrow

$$i_{1,D,1} : \mathcal{V}_{1,1} \longrightarrow \mathcal{V}_{D,1},$$

between permutation $G\mathbf{W}_1$-bundles over $\mathbf{T}$, has a natural lift to

$$i_{1,D,2} := \tau_2(i_{1,D,1}) : \mathcal{V}_{1,2} := \mathbf{W}_2(\mathcal{L}_{1,1}) \longrightarrow \mathcal{V}_{D,2},$$

which is an embedding. To lift the partial G-flag

$$\nabla_{gen,2,1} : \mathcal{V}_{1,1} \subset \mathcal{V}_{2,1} \subset \mathcal{V}_{D,1},$$

a change of the base is needed. Using $i_{1,D,2}$, this problem is equivalent to lifting the G-arrow

$$\mathcal{L}_{2,1} \longrightarrow \mathcal{V}_{D,1}/\mathcal{V}_{1,1},$$

to a G-arrow

$$\mathcal{L}_{2,2} \longrightarrow \mathcal{V}_{D,2}/\mathcal{V}_{1,2},$$

where $\mathcal{L}_{2,2}$ is some lift of $\mathcal{L}_{2,1}$, which we do not prescribe. By section 11.2, we know that the space of such liftings naturally bears the structure of an extension of G-vector bundles over $\mathbf{T}$

$$\mathcal{E}_{2,D} : 0 \longrightarrow (\mathcal{V}_{D,1}/\mathcal{V}_{2,1})^{(1)} \longrightarrow * \longrightarrow \mathcal{L}_{2,1}^{(1)} \longrightarrow 0.$$

Put

$$\mathbf{L}_2 := \mathbb{S}(\mathcal{E}_{2,D}) \longrightarrow \mathbf{T}.$$

Over $\mathbf{L}_2$, the partial G-flag

$$\nabla_{gen,2,1} : \mathcal{V}_{1,1} \subset \mathcal{V}_{2,1} \subset \mathcal{V}_{D,1}$$

then acquires a natural lift, to a flag of $G\mathbf{W}_2$-bundles

$$\nabla_{gen,2,2} : \mathcal{V}_{1,2} \subset \mathcal{V}_{2,2} \subset \mathcal{V}_{D,2},$$

which extends

$$\nabla_{gen,1,2} : 0 \subset \mathcal{V}_{1,2} \subset \mathcal{V}_{D,2}.$$

Iterating this process generates a sequence of G-arrows

$$L : \mathbf{L}_d \longrightarrow \ldots \longrightarrow \mathbf{L}_3 \longrightarrow \mathbf{L}_2 \longrightarrow \mathbf{L}_1 = \mathbf{T},$$

with the following properties.

(1) The arrow $\mathbf{L}_{i+1} \longrightarrow \mathbf{L}_i$ is the splitting scheme of an extension of G-vector bundles over $\mathbf{L}_i$

$$\mathcal{E}_{i+1,D} : 0 \longrightarrow (\mathcal{V}_{D,1}/\mathcal{V}_{i+1,1})^{(1)} \longrightarrow * \longrightarrow \mathcal{L}_{i+1,1}^{(1)} \longrightarrow 0.$$

(2) Over $\mathbf{L}_d$, the embedded flag $\nabla_{gen,d,1}$ lifts to a flag of $G\mathbf{W}_2$-bundles

$$\nabla_{gen,d,2} : 0 \subset \mathcal{V}_{1,2} \subset \mathcal{V}_{2,2} \subset \ldots \subset \mathcal{V}_{d,2} \subset \mathcal{V}_{D,2} := F^*(V_2).$$

(3) The arrow L is well-filtered in a natural way. The corresponding filtration of $L_*(\mathcal{O}_{\mathbf{L}_d})$ is indexed by $(a_d, \ldots, a_2) \in \mathbb{N}^{d-1}$, ordered lexicographically. Its graded pieces are vector bundles of the shape

$$\bigotimes_{i=2}^{d} \Phi_{a_i}((\mathcal{V}_{D,1}/\mathcal{V}_{i,1})^\vee \otimes \mathcal{L}_{i,1}),$$

where

$$\Phi_{a_i}(.) := \mathrm{Sym}^{b_i}(\mathrm{Frob}(.))$$

is a pure symmetric functor, homogenenous of degree $a_i = pb_i$. To see why, apply Remark 11.7 to all splitting schemes $\mathbf{L}_{i+1} \longrightarrow \mathbf{L}_i$, using the composition process of Section 4.2. Note that the only graded piece of degree $a := \sum a_i = 0$ corresponds to $\mathcal{O}_{\mathbf{T}} \subset L_*(\mathcal{O}_{\mathbf{L}_d})$.

(4) The G-arrow L parametrizes liftings of the embedded flag $\nabla_{gen,d,1}$, to a flag of $\mathbf{W}_2$-bundles embedded in $\mathcal{V}_{D,2}$, under the constraint

$$i_{1,D,2} := \tau_2(i_{1,D,1}).$$

In particular, the data of $\nabla_{d,2}$ naturally corresponds to a G-point

$$s_2 : S \longrightarrow \mathbf{L}_d,$$

lifting s_1. Formula: $L \circ s_2 = s_1$.
In short: s_1 naturally lifts through L, in a G-equivariant fashion.

16.4. Step 4: geometric lifting of $\mathcal{L}_{d+1,1} \longrightarrow \mathcal{V}_{D,1}/\mathcal{V}_{d,1}$.

We begin with shrinking G_0: we now denote by $G_0 \subset G$ the intersection of the kernels of the actions of G on S, on $V_{d+1,1}$, on $V_{d,2}$ and on $\mathcal{L}_{d+1,2}^{[m]}$.
In this fourth step, we produce a G-equivariant change of the base

$$\mathbf{L}_{d+1} \longrightarrow \mathbf{L}_d$$

such that, over $\mathbf{L}_{d+1}$, the m-th Frobenius twist of the embedding

$$\mathcal{L}_{d+1,1} \longrightarrow \mathcal{V}_{D,1}/\mathcal{V}_{d,1}$$

lifts to an embedding

$$\mathcal{L}_{d+1,2}^{[m]} \longrightarrow \mathcal{V}_{D,2}^{(m)}/\mathcal{V}_{d,2}^{(m)},$$

enjoying the following properties.

(1) The lifting is an arrow of $(G_0, \mathbf{W}_2)$-bundles (not of $(G, \mathbf{W}_2)$-bundles).
(2) The $G\mathbf{W}_2$-line bundle $\mathcal{L}_{d+1,2}^{[m]}$ is that introduced in Step 2.

To achieve this, we use the process of section 11.2, together with induction from G_0. More precisely, the space of liftings of

$$\mathcal{L}_{d+1,1}^{(m)} \longrightarrow \mathcal{V}_{D,1}^{(m)}/\mathcal{V}_{d,1}^{(m)}$$

to an embedding of $\mathbf{W}_2$-bundles

$$\mathcal{L}_{d+1,2}^{[m]} \longrightarrow \mathcal{V}_{D,2}^{(m)}/\mathcal{V}_{d,2}^{(m)}$$

is governed by a natural extension of vector bundles over $\mathbf{L}_d$

$$\mathcal{E}_{d+1,D} : 0 \longrightarrow (\mathcal{V}_{D,1}/\mathcal{V}_{d,1})^{(m+1)} \longrightarrow * \longrightarrow \mathcal{L}_{d+1,1}^{(m+1)} \longrightarrow 0.$$

Rather than passing to its splitting scheme, we pass to its splitting scheme *induced from* G_0: we set

$$\lambda : \mathbf{L}_{d+1} := \mathrm{Ind}_{G_0}^{G}(\mathbb{S}(\mathcal{E}_{d+1,D})) = \mathbb{S}(\mathcal{E}_{d+1,D})^{G^0} \longrightarrow \mathbf{L}_d$$

(induction and fiber products taken over $\mathbf{L}_d$).
By the universal property of induction, we get that, over $\mathbf{L}_{d+1}$, the embedding

$$\mathcal{L}_{d+1,1}^{(m)} \longrightarrow \mathcal{V}_{D,1}^{(m)}/\mathcal{V}_{d,1}^{(m)}$$

indeed lifts, to an embedding of $G_0\mathbf{W}_2$-bundles

$$\mathcal{L}_{d+1,2}^{[m]} \longrightarrow \mathcal{V}_{D,2}^{(m)}/\mathcal{V}_{d,2}^{(m)}.$$

Denote by

$$\mathcal{V}_{d+1,2}^{[m]} \subset \mathcal{V}_{D,2}^{(m)}$$

the inverse image of $\mathcal{L}_{d+1,2}^{[m]} \subset \mathcal{V}_{D,2}^{(m)}/\mathcal{V}_{d,2}^{(m)}$, under the quotient arrow $\mathcal{V}_{D,2}^{(m)} \longrightarrow \mathcal{V}_{D,2}^{(m)}/\mathcal{V}_{d,2}^{(m)}$.

To sum up, the following holds.

(1) Over $\mathbf{L}_{d+1}$, the m-th Frobenius twist of the tautological embedded flag

$$\nabla_{gen,d+1,1} : 0 \subset \mathcal{V}_{1,1} \subset \mathcal{V}_{2,1} \subset \ldots \subset \mathcal{V}_{d+1,1} \subset \mathcal{V}_{D,1}$$

acquires a lift to a flag

$$\nabla_{gen,d+1,2}^{[m]} := 0 \subset \mathcal{V}_{1,2}^{(m)} \subset \mathcal{V}_{2,2}^{(m)} \subset \ldots \subset \mathcal{V}_{d,2}^{(m)} \subset \mathcal{V}_{d+1,2}^{[m]} \subset \mathcal{V}_{D,2}^{(m)} := F^*(V_2^{(m)}),$$

where:

a) The embeddings $\mathcal{V}_{i,2} \subset \mathcal{V}_{D,2}$, for $i = 1, \ldots, d$, are the embeddings of $G\mathbf{W}_2$-bundles built in Step 3.

b) The embedding $\mathcal{V}_{d+1,2}^{[m]} \subset \mathcal{V}_{D,2}^{(m)}$ is an embedding of $G_0\mathbf{W}_2$-bundles.

c) As a $G_0\mathbf{W}_2$-line bundle, the graded piece $\mathcal{V}_{d+1,2}^{[m]}/\mathcal{V}_{d,2}^{(m)}$ is isomorphic to $\mathcal{L}_{d+1,2}^{[m]}$, built in Step 2.

(2) The quasi-coherent $\mathcal{O}_{\mathbf{L}_d}$-Module $\lambda_*(\mathcal{O}_{\mathbf{L}_{d+1}})$ has a natural good filtration, with graded pieces vector bundles of the shape

$$\Phi_{a_{d+1}}((\mathcal{V}_{D,1}/\mathcal{V}_{d,1})^\vee \otimes \mathcal{L}_{d+1,1}),$$

where $\Phi_{a_{d+1}}(.)$ is a composite symmetric functor, homogenenous of degree $a_{d+1} \geq 0$. Note that, here again, the only graded piece of degree $a_{d+1} = 0$ corresponds to $\mathcal{O}_{\mathbf{L}_d} \subset \lambda_*(\mathcal{O}_{\mathbf{L}_{d+1}})$.

(3) The set of G-equivariant sections of λ parametrizes liftings of the G_0-equivariant embedding

$$\mathcal{L}_{d+1,1}^{(m)} \longrightarrow \mathcal{V}_{D,1}^{(m)}/\mathcal{V}_{d,1}^{(m)},$$

to a G_0-equivariant embedding

$$\mathcal{L}_{d+1,2}^{[m]} \longrightarrow \mathcal{V}_{D,2}^{(m)}/\mathcal{V}_{d,2}^{(m)}.$$

Over S, the embedding

$$L_{d+1,1}^{(m)} \longrightarrow V_{D,1}^{(m)}/V_{d,1}^{(m)}$$

lifts to an embedding of $G_0\mathbf{W}_2$-bundles

$$L_{d+1,2}^{[m]} := s_1^*(\mathcal{L}_{d+1,2}^{[m]}) \longrightarrow V_{D,2}^{(m)}/V_{d,2}^{(m)}.$$

Since G_0 acts trivially on S, $L_{d+1,2}^{[m]}$ and $V_{D,2}^{(m)}/V_{d,2}^{(m)}$, this simply follows from the vanishing of coherent cohomology, over an affine base. The choice of such a lifting naturally determines a G- point

$$s_3 : S \longrightarrow \mathbf{L}_{d+1},$$

lifting s_2. Formula: $\lambda \circ s_3 = s_2$.

In short: s_2 naturally lifts through λ, in a G-equivariant fashion.

16.5. Step 5: a good filtration. Denote by

$$\theta := T \circ L \circ \lambda : \mathbf{L}_{d+1} \longrightarrow \mathbf{L}_d \longrightarrow \mathbf{T} \longrightarrow \mathbf{F}$$

the composite of the G-arrows built in Steps 1,3 and 4.
Using the composition process of section 4.2, the combination of the filtrations numbered (2) in Step 1, (3) in Step 3 and (2) in Step 4 yields the following.
The arrow θ is well-filtered, in a natural way. The associated filtration of $\theta_*(\mathcal{O}_{\mathcal{O}_{\mathbf{L}_{d+1}}})$ is indexed by

$$(a_{d+1}, \dots, a_2, b) \in \mathbb{N}^{d+1},$$

ordered lexicographically. Its graded pieces are vector bundles of the shape

$$(\bigotimes_{i=2}^{d} \Phi_{a_i}((\mathcal{V}_{D,1}/\mathcal{V}_{i,1})^\vee \otimes \mathcal{L}_{i,1})) \otimes \Phi_{a_{d+1}}((\mathcal{V}_{D,1}/\mathcal{V}_{d,1})^\vee \otimes \mathcal{L}_{d+1,1}) \otimes \mathcal{L}_{d,1}^{\otimes -b} \otimes \mathcal{L}_{d+1,1}^{\otimes b},$$

where $\Phi_{a_i}(.)$ is a symmetric functor, homogenenous of degree a_i.
Such a vector bundle itself possesses a natural good filtration by sub-vector bundles, having as graded pieces degree zero line bundles, of the shape

$$\mathcal{O}(0, +, *, \dots, *, -, \dots, -).$$

Reading from the left, the first $-$ symbol occurs as the $(d+2)$-th entry.

16.6. Step 6: a glueing problem.
Over $\mathbf{L}_{d+1}$, we want to glue the extensions of $G\mathbf{W}_2$-bundles

$$Nat_{d,2}^{(m)} : 0 \longrightarrow \mathcal{V}_{d-1,2}^{(m)} \longrightarrow \mathcal{V}_{d,2}^{[m]} \longrightarrow \mathcal{L}_{d,2}^{(m)} \longrightarrow 0$$

and

$$Nat_{d,d+1,2}^{[m]} : 0 \longrightarrow \mathcal{L}_{d,2}^{(m)} \longrightarrow \mathcal{V}_{d+1,2}^{(m)}/\mathcal{V}_{d-1,2}^{(m)} \longrightarrow \mathcal{L}_{d+1,2}^{(m)} \longrightarrow 0.$$

Clearly, this can be done modulo p, using the extension of G-bundles

$$Nat_{d+1,1}^{(m)} : 0 \longrightarrow \mathcal{V}_{d,1}^{(m)} \longrightarrow \mathcal{V}_{d+1,1}^{[m]} \longrightarrow \mathcal{L}_{d+1,1}^{(m)} \longrightarrow 0.$$

Thanks to the process described in Section 12, we get a natural class

$$c \in \mathrm{Ext}^2_{G,1}(\mathbf{L}_{d+1}, \mathcal{L}_{d+1,1}^{(m+1)}, \mathcal{V}_{d-1,1}^{(m+1)}),$$

with the following property.
The vanishing of c is equivalent to the existence of a glueing of $Nat_{d,2}^{(m)}$ and $Nat_{d,d+1,2}^{[m]}$ over $\mathbf{L}_{d+1}$, lifting that given by $Nat_{d+1,1}^{(m)}$.
The class c is geometrically trivial: we have

$$c \in \mathrm{ext}^2_{G,1}(\mathbf{L}_{d+1}, \mathcal{L}_{d+1,1}^{(m+1)}, \mathcal{V}_{d-1,1}^{(m+1)}).$$

To understand why, note that the extension of $G_0\mathbf{W}_2$-bundles

$$0 \longrightarrow \mathcal{V}_{d,2}^{(m)} \longrightarrow \mathcal{V}_{d+1,2}^{[m]} \longrightarrow \mathcal{L}_{d+1,2}^{[m]} \longrightarrow 0,$$

extracted from the flag $\nabla_{gen,d+1,2}^{[m]}$ of Step 4, yields the sought-for glueing– as extensions of $G_0\mathbf{W}_2$-bundles. Thus, the restriction

$$\mathrm{Res}_G^{G_0}(c) \in \mathrm{Ext}^2_{G_0,1}(\mathbf{L}_{d+1}, \mathcal{L}_{d+1,1}^{(m+1)}, \mathcal{V}_{d-1,1}^{(m+1)})$$

dies, proving the claim.

16.7. Step 7: Computations in Ext groups.
To proceed further, our next task is to compute the group

$$\mathrm{ext}^2_{G,1}(\mathbf{L}_{d+1}, \mathcal{L}^{(m+1)}_{d+1,1}, \mathcal{V}^{(m+1)}_{d-1,1}).$$

This is done in Lemma 16.8. The reader may take its statement for granted, and proceed to Step 8.
Consider the exact sequence of G-vector bundles

$$Nat : 0 \longrightarrow \mathcal{V}_{d-1,1} \longrightarrow \mathcal{V}_{D,1} \longrightarrow \mathcal{V}_{D,1}/\mathcal{V}_{d-1,1} \longrightarrow 0.$$

In cohomology, it induces connecting arrows

$$\mathrm{Ext}^i_{G,1}(\mathbf{L}_{d+1}, \mathcal{L}^{(m+1)}_{d+1,1}, (\mathcal{V}_{D,1}/\mathcal{V}_{d-1,1})^{(m+1)}) \longrightarrow \mathrm{Ext}^{i+1}_{G,1}(\mathbf{L}_{d+1}, \mathcal{L}^{(m+1)}_{d+1,1}, \mathcal{V}^{(m+1)}_{d-1,1}),$$

and

$$\mathrm{ext}^i_{G,1}(\mathbf{L}_{d+1}, \mathcal{L}^{(m+1)}_{d+1,1}, (\mathcal{V}_{D,1}/\mathcal{V}_{d-1,1})^{(m+1)}) \longrightarrow \mathrm{ext}^{i+1}_{\mathbf{L}_{d+1}}(\mathcal{L}^{(m+1)}_{d+1,1}, \mathcal{V}^{(m+1)}_{d-1,1}),$$

for all $i \geq 0$.

Lemma 16.4. *The arrow*

$$\mathrm{ext}^1_{G,1}(\mathbf{L}_{d+1}, \mathcal{L}^{(m+1)}_{d+1,1}, (\mathcal{V}_{D,1}/\mathcal{V}_{d-1,1})^{(m+1)}) \longrightarrow \mathrm{ext}^2_{G,1}(\mathbf{L}_{d+1}, \mathcal{L}^{(m+1)}_{d+1,1}, \mathcal{V}^{(m+1)}_{d-1,1})$$

is an isomorphism.

Proof. Chasing in the diagrams induced by $Nat^{(m+1)}$ for $\mathrm{Ext}^i(\mathcal{L}^{(m+1)}_{d+1,1}, .)$'s, it suffices to show that the three groups

$$\mathrm{ext}^2_{G,1}(\mathbf{L}_{d+1}, \mathcal{L}^{(m+1)}_{d+1,1}, \mathcal{V}^{(m+1)}_{D,1}), \mathrm{Ext}^1_{G,1}(\mathbf{L}_{d+1}, \mathcal{L}^{(m+1)}_{d+1,1}, \mathcal{V}^{(m+1)}_{D,1})$$

and

$$\mathrm{Ext}^1_{G,1}(\mathbf{L}_{d+1}, \mathcal{L}^{(m+1)}_{d+1,1}, \mathcal{V}^{(m+1)}_{D,1})$$

vanish. Using the local-to-global spectral sequence, we reduce to proving the vanishing of the groups

$$\mathrm{Ext}^i_1(\mathbf{L}_{d+1}, \mathcal{L}^{(m+1)}_{d+1,1}, \mathcal{V}^{(m+1)}_{D,1}),$$

for $i = 0, 1$. Recall that $V_{D,1} = F^*(V_1)$, where $F : \mathbf{F} \longrightarrow S$ is the structure morphism. Since S is affine, using the projection formula, we further reduce to proving that the (Zariski) cohomology groups

$$H^i(\mathbf{L}_{d+1}, \mathcal{L}^{-p^{m+1}}_{d+1,1}) = \mathrm{Ext}^i_1(\mathbf{L}_{d+1}, \mathcal{L}^{(m+1)}_{d+1,1}, \mathcal{O}_{\mathbf{L}_{d+1}}) = H^i(\mathbf{F}, \theta_*(\mathcal{O}_{\mathbf{L}_{d+1}}) \otimes \mathcal{L}^{-p^{m+1}}_{d+1,1})$$

vanish for $i = 0, 1$. Using the (double) good filtration of Step 5, we see that the quasi-coherent $\mathcal{O}_{\mathbf{F}}$-module $\theta_*(\mathcal{O}_{\mathbf{L}_{d+1}}) \otimes \mathcal{L}^{-p^{m+1}}_{d+1,1}$ has a good filtration, with graded pieces line bundles of total degree $-p^{m+1} < 0$, and of the shape

$$\mathcal{O}(0, +, *, \ldots, *, -, \ldots, -).$$

These have $R^iF_*(.) = 0$, for $i = 0, 1$. Checking this is an exercise, using Propositions 9.4 and 9.5. Conclude by dévissage.

□

Lemma 16.5. *Consider the affine morphism $L \circ \lambda : \mathbf{L}_{d+1} \longrightarrow \mathbf{T}$.*
The natural arrows

$$(L \circ \lambda)_* : \mathrm{Ext}^0_1(\mathbf{T}, \mathcal{L}^{(m+1)}_{d+1,1}, (\mathcal{V}_{d+1,1}/\mathcal{V}_{d-1,1})^{(m+1)}) \longrightarrow \mathrm{Ext}^0_1(\mathbf{L}_{d+1}, \mathcal{L}^{(m+1)}_{d+1,1}, (\mathcal{V}_{d+1,1}/\mathcal{V}_{d-1,1})^{(m+1)})$$

and

$$(L \circ \lambda)_* : \mathrm{Ext}^0_1(\mathbf{T}, \mathcal{L}^{(m+1)}_{d+1,1}, (\mathcal{V}_{D,1}/\mathcal{V}_{d-1,1})^{(m+1)}) \longrightarrow \mathrm{Ext}^0_1(\mathbf{L}_{d+1}, \mathcal{L}^{(m+1)}_{d+1,1}, (\mathcal{V}_{D,1}/\mathcal{V}_{d-1,1})^{(m+1)})$$

are isomorphisms.

Proof. We give the proof for the second arrow. The proof for the first one is similar.
Recall that, for vector bundles A and B over $\mathbf{F}$, we have

$$\mathrm{Ext}^0_1(\mathbf{T}, A, B) = F_*(T_*(A^\vee \otimes B))$$

and

$$\mathrm{Ext}^0_1(\mathbf{L}_{d+1}, A, B) = F_*(\theta_*(A^\vee \otimes B)).$$

Using the exact sequence (of quasi-coherent modules over $\mathbf{F}$)

$$0 \longrightarrow T_*(\mathcal{O}_\mathbf{T}) \longrightarrow \theta_*(\mathcal{O}_{\mathbf{L}_{d+1}}) \longrightarrow \theta_*(\mathcal{O}_{\mathbf{L}_{d+1}})/T_*(\mathcal{O}_\mathbf{T}) \longrightarrow 0,$$

we see that it suffices to show

$$F_*(\mathcal{L}_{d+1,1}^{-p^{m+1}} \otimes (\mathcal{V}_{D,1}/\mathcal{V}_{d-1,1})^{(m+1)} \otimes (\theta_*(\mathcal{O}_{\mathbf{L}_{d+1}})/T_*(\mathcal{O}_\mathbf{T}))) = 0.$$

Using the good filtration of $\theta_*(\mathcal{O}_{\mathbf{L}_{d+1}})$ given in Step 5, we get a good filtration of the quasi-coherent $\mathcal{O}_\mathbf{F}$-module $\theta_*(\mathcal{O}_{\mathbf{L}_{d+1}})/T_*(\mathcal{O}_\mathbf{T})$. Its graded pieces are vector bundles of the shape

$$\mathcal{W} := (\bigotimes_{i=2}^{d} \Phi_{a_i}((\mathcal{V}_{D,1}/\mathcal{V}_{i,1})^\vee \otimes \mathcal{L}_{i,1})) \otimes \Phi_{a_{d+1}}((\mathcal{V}_{D,1}/\mathcal{V}_{d,1})^\vee \otimes \mathcal{L}_{d+1,1}) \otimes \mathcal{L}_{d,1}^{\otimes -b} \otimes \mathcal{L}_{d+1,1}^{\otimes b}.$$

Here $b \geq 0$ is any integer, and the $\Phi_{a_i}(.)$'s are symmetric functors, homogenenous of degrees $a_i \geq 0$, with at least one non-zero a_i. By dévissage, it suffices to prove the vanishing of

$$F_*(\mathcal{L}_{d+1,1}^{-p^{m+1}} \otimes (\mathcal{V}_{D,1}/\mathcal{V}_{d-1,1})^{(m+1)} \otimes \mathcal{W}).$$

To do so, thanks to the projection formula, we may replace

$$F : \mathbf{Fl}(1, \ldots, d+1, V_1) \longrightarrow S$$

by the complete flag scheme

$$F : \mathbf{Fl}(V_1) \longrightarrow S,$$

which we do, in three steps.

(1) At least one of the numbers $a_1, \ldots, a_{d-1}$ does not vanish. Then, the natural good filtration on $\mathcal{W}$ has graded pieces consisting of degree zero line bundles of the shape

$$\mathcal{L} := \mathcal{O}(c_1, \ldots, c_{d-1}, \ldots, c_D),$$

where the relative numbers $c_1, \ldots, c_{d-1}$ do not all vanish, and where the first non-zero of these, reading from the left, is positive. Using the natural good filtration of $\mathcal{L}_{d+1,1}^{-p^{m+1}} \otimes (\mathcal{V}_{D,1}/\mathcal{V}_{d-1,1})^{(m+1)}$, we get that

$$\mathcal{L}_{d+1,1}^{-p^{m+1}} \otimes (\mathcal{V}_{D,1}/\mathcal{V}_{d-1,1})^{(m+1)} \otimes \mathcal{W}$$

possesses the same kind of good filtration. For all of its graded pieces $\mathcal{L}$, we have $F_*(\mathcal{L}) = 0$ by Proposition 9.4. Conclude by (double) dévissage.

It remains to treat the case $a_1 = \ldots = a_{d-1} = 0$. Hence, a_d and a_{d+1} do not both vanish, and we have

$$\mathcal{W} = \Phi((\mathcal{V}_{D,1}/\mathcal{V}_{d,1})^\vee) \otimes \mathcal{L}_{d,1}^{a_d - b} \otimes \mathcal{L}_{d+1,1}^{b + a_{d+1}},$$

where

$$\Phi := \Phi_{a_d} \otimes \Phi_{a_{d+1}}$$

is a symmetric functor, of degree $a_d + a_{d+1} \geq 1$.

(2) We have $b+a_{d+1}-p^{m+1}>0$. Introduce the factorization

$$F:\mathbf{F}=\mathbf{Fl}(V_1)\xrightarrow{F_1}\mathbf{Fl}(1,\ldots,d,V_1)\xrightarrow{F_2}\mathbf{F}.$$

Write

$$\mathcal{L}_{d+1,1}^{-p^{m+1}}\otimes(\mathcal{V}_{D,1}/\mathcal{V}_{d-1,1})^{(m+1)}\otimes\mathcal{W}=\mathcal{L}_{d+1,1}^{b+a_{d+1}-p^{m+1}}\otimes\mathcal{Y},$$

where $\mathcal{Y}$ is a vector bundle, defined over $\mathbf{Fl}(1,\ldots,d,V_1)$. Since $d+1\leq D-1$, using the projection formula and proposition 9.4, we get

$$(F_1)_*(\mathcal{L}_{d+1,1}^{-p^{m+1}}\otimes(\mathcal{V}_{D,1}/\mathcal{V}_{d-1,1})^{(m+1)}\otimes\mathcal{W})=(F_1)_*(\mathcal{L}_{d+1,1}^{b+a_{d+1}-p^{m+1}})\otimes\mathcal{Y}$$
$$=0\otimes\mathcal{Y}=0.$$

Hence

$$F_*(\mathcal{L}_{d+1,1}^{-p^{m+1}}\otimes(\mathcal{V}_{D,1}/\mathcal{V}_{d-1,1})^{(m+1)}\otimes\mathcal{W})=0,$$

and we conclude by dévissage.

(3) We have $b+a_{d+1}-p^{m+1}\leq 0$. The vector bundle $\Phi((\mathcal{V}_{D,1}/\mathcal{V}_{d,1})^\vee)$ has a good filtration, with graded pieces line bundles of the shape

$$\mathcal{O}(0,\ldots,0,-,\ldots,-),$$

where the first of the $-$ symbols occurs as the $(d+1)$-th entry, and one of them at least is $--$. By dévissage, it suffices to prove the vanishing of

$$F_*((\mathcal{V}_{D,1}/\mathcal{V}_{d-1,1})^{(m+1)}(0,\ldots,0,a_d-b,-,-,\ldots,-)),$$

where one at least of the symbols $-$ is $--$. If $a_d-b\geq 0$, then the sequence $(a_d-b,-,-,\ldots,-)$ is not increasing. Using the factorization

$$F:\mathbf{F}=\mathbf{Fl}(V_1)\xrightarrow{F_3}\mathbf{Fl}(1,\ldots,d-1,V_1)\xrightarrow{F_4}\mathbf{F},$$

we conclude using Proposition 9.4. If $a_d-b<0$, we have to show

$$F_*((\mathcal{V}_{D,1}/\mathcal{V}_{d-1,1})^{(m+1)}(0,\ldots,0,--,-,\ldots,-))=0,$$

where the first of the $-$ symbols occurs as the $(d+1)$-th entry, and one of them, at least, is $--$. The vector bundle $(\mathcal{V}_{D,1}/\mathcal{V}_{d-1,1})^{(m+1)}$ is equipped with its natural good filtration, having as graded pieces the line bundles $\mathcal{L}_{i,1}^{p^{m+1}}$, for $d\leq i\leq D$. Thus, the vector bundle

$$(\mathcal{V}_{D,1}/\mathcal{V}_{d-1,1})^{(m+1)}(0,\ldots,0,--,-,\ldots,-)$$

has a natural good filtration, with graded pieces (degree zero) line bundles, of the shape

$$\mathcal{O}(0,\ldots,0,*,\ldots,*),$$

where one at least of the symbols $*$ is negative. These have $F_*(.)=0$, by Proposition 9.4. Conclude by dévissage.

□

Consider the natural extension, of vector bundles over $\mathbf{F}$,

$$Nat:0\longrightarrow\mathcal{V}_{d+1,1}/\mathcal{V}_{d-1,1}\longrightarrow\mathcal{V}_{D,1}/\mathcal{V}_{d-1,1}\longrightarrow\mathcal{V}_{D,1}/\mathcal{V}_{d+1,1}\longrightarrow 0.$$

Lemma 16.6. *The inclusion*

$$\mathrm{Ext}^0_1(\mathbf{T},\mathcal{L}_{d+1,1}^{(m+1)},(\mathcal{V}_{d+1,1}/\mathcal{V}_{d-1,1})^{(m+1)})\longrightarrow\mathrm{Ext}^0_1(\mathbf{T},\mathcal{L}_{d+1,1}^{(m+1)},(\mathcal{V}_{D,1}/\mathcal{V}_{d-1,1})^{(m+1)}),$$

induced by Nat, is an isomorphism.

Proof. Consider the good filtration described in point 2) of Lemma 16.3. Using the exact sequence Nat, and arguing as in the proof of Lemma 16.5 -whose proof is actually much more delicate- we reduce to showing

$$F_*(\mathcal{L}_{d+1,1}^{-p^{m+1}} \otimes (\mathcal{V}_{D,1}/\mathcal{V}_{d+1,1})^{(m+1)} \otimes \mathcal{L}_{d,1}^{-b} \otimes \mathcal{L}_{d+1,1}^{b}) = 0,$$

for all $b \geq 0$. The vector bundle

$$\mathcal{L}_{d+1,1}^{-p^{m+1}} \otimes (\mathcal{V}_{D,1}/\mathcal{V}_{d+1,1})^{(m+1)} \otimes \mathcal{L}_{d,1}^{-b} \otimes \mathcal{L}_{d+1,1}^{b}$$

has a natural good filtration, with degree zero line bundles of the shape

$$\mathcal{O}(0,\ldots,0,-b,b-p^{m+1},*,\ldots,*)$$

as graded pieces– where all but one symbols $*$ vanish. The non-zero $*$ equals p^{m+1}. Such line bundles have $F_*(.) = 0$ by Proposition 9.4. Conclude by dévissage. $\square$

Lemma 16.7. *The natural map*

$$\mathrm{ext}^1_{G,1}(\mathbf{L}_{d+1}, \mathcal{L}_{d+1,1}^{(m+1)}, (\mathcal{V}_{d+1,1}/\mathcal{V}_{d-1,1})^{(m+1)}) \longrightarrow \mathrm{ext}^1_{G,1}(\mathbf{L}_{d+1}, \mathcal{L}_{d+1,1}^{(m+1)}, (\mathcal{V}_{D,1}/\mathcal{V}_{d-1,1})^{(m+1)}),$$

induced by the inclusion $\mathcal{V}_{d+1,1}/\mathcal{V}_{d-1,1} \longrightarrow \mathcal{V}_{D,1}/\mathcal{V}_{d-1,1}$, is an isomorphism.

Proof. We have

$$\mathrm{ext}^1_{G,1}(\mathbf{L}_{d+1},.,.) = H^1(G, \mathrm{Ext}^0_1(\mathbf{L}_{d+1},.,.)).$$

Thus, it suffices to show that the natural injective arrow

$$\mathrm{Ext}^0_1(\mathbf{L}_{d+1}, \mathcal{L}_{d+1,1}^{(m+1)}, (\mathcal{V}_{d+1,1}/\mathcal{V}_{d-1,1})^{(m+1)}) \longrightarrow \mathrm{Ext}^0_1(\mathbf{L}_{d+1}, \mathcal{L}_{d+1,1}^{(m+1)}, (\mathcal{V}_{D,1}/\mathcal{V}_{d-1,1})^{(m+1)})$$

is an isomorphism. This follows from Lemmas 16.5 and 16.6. $\square$

Lemma 16.8. *The natural arrow*

$$\beta : \mathrm{ext}^1_{G,1}(\mathbf{L}_{d+1}, \mathcal{L}_{d+1,1}^{(m+1)}, (\mathcal{V}_{d+1,1}/\mathcal{V}_{d-1,1})^{(m+1)}) \longrightarrow \mathrm{ext}^2_{G,1}(\mathbf{L}_{d+1}, \mathcal{L}_{d+1,1}^{(m+1)}, \mathcal{V}_{d-1,1}^{(m+1)})$$

is an isomorphism.

Proof. Combine Lemmas 16.4 and 16.7.

$\square$

16.8. Step 8: an adjustment.

Recall the class

$$c \in \mathrm{ext}^2_{\mathbf{L}_{d+1}}(\mathcal{L}_{d+1,1}^{(m+1)}, \mathcal{V}_{d-1,1}^{(m+1)}),$$

obtained in Step 6. Thanks to Lemma 16.8, there exists

$$\epsilon \in \mathrm{ext}^1_{\mathbf{L}_{d+1}}(\mathcal{L}_{d+1,1}^{(m+1)}, (\mathcal{V}_{d+1,1}/\mathcal{V}_{d-1,1})^{(m+1)}),$$

such that

$$\beta(\epsilon) = c.$$

Consider the natural extension of $G\mathbf{W}_1$-bundles defined over $\mathbf{F}$,

$$0 \longrightarrow \mathcal{L}_{d+1,1}^{-1} \otimes \mathcal{L}_{d,1} \longrightarrow \mathcal{L}_{d+1,1}^{-1} \otimes (\mathcal{V}_{d+1,1}/\mathcal{V}_{d-1,1}) \xrightarrow{\pi} \mathcal{O}_{\mathbf{F}} \longrightarrow 0.$$

Form the pushforward

$$\pi_*(\epsilon) \in \mathrm{ext}^1_{\mathbf{L}_{d+1}}(\mathcal{O}_{\mathbf{L}_{d+1}}, \mathcal{O}_{\mathbf{L}_{d+1}}) = h^1_{\mathbf{L}_{d+1}}(\mathcal{O}_{\mathbf{L}_{d+1}}).$$

Recall that the space of lifts of $\mathcal{L}^{(m)}_{d+1,1}$, to a $G\mathbf{W}_2$-bundle over $\mathbf{L}_{d+1}$, is pointed by $\mathbf{W}_2(\mathcal{L}^{(m)}_{d+1,1})$, and is equivalent to the space of $(G, \mathcal{O}_{\mathbf{L}_{d+1}})$-torsors (see [8], Proposition 4.3). This gives a meaning to

$$\mathcal{L}^{'[m]}_{d+1,2} := \mathcal{L}^{[m]}_{d+1,2} - \pi_*(\epsilon).$$

Note that $\mathcal{L}^{[m]}_{d+1,2}$ and $\mathcal{L}^{'[m]}_{d+1,2}$, as lifts of $\mathcal{L}^{(m)}_{d+1,1}$, are geometrically isomorphic. The extension ϵ then gives rise to an extension of $G\mathbf{W}_2$-bundles over $\mathbf{L}_{d+1}$

$$Nat^{[m]}_{d,d+1,2} - j_*(\epsilon) : 0 \longrightarrow \mathcal{L}^{(m)}_{d,2} \longrightarrow \mathcal{V}^{'[m]}_{d,d+1,2} \longrightarrow \mathcal{L}^{'[m]}_{d+1,2} \longrightarrow 0,$$

denoted by $Nat^{'[m]}_{d,d+1,2}$. We can now perform the same constructions as in Step 6, replacing $\mathcal{L}^{[m]}_{d+1,2}$ by $\mathcal{L}^{'[m]}_{d+1,2}$, and $Nat^{[m]}_{d,d+1,2}$ by $Nat^{'[m]}_{d,d+1,2}$. Glueing it with $Nat^{(m)}_{d,2}$, in a way that lifts $Nat^{(m)}_{d+1,1}$, is then obstructed by a class

$$c' \in \mathrm{ext}^2_{\mathbf{L}_{d+1}}(\mathcal{L}^{(m+1)}_{d+1,1}, \mathcal{V}^{(m+1)}_{d-1,1}),$$

with

$$c' = c - \beta(\epsilon) = 0.$$

The extensions $Nat^{'[m]}_{d,d+1,2}$ and $Nat^{(m)}_{d,2}$ thus glue, in a way that lifts $Nat^{(m)}_{d+1,1}$, to an extension of $G\mathbf{W}_2$-bundles

$$0 \longrightarrow \mathcal{V}^{(m)}_{d,2} \longrightarrow \mathcal{V}^{'[m]}_{d+1,2} \longrightarrow \mathcal{L}^{'[m]}_{d+1,2} \longrightarrow 0,$$

over $\mathbf{L}_{d+1}$.

16.9. Step 9: done! We have a complete flag of $G\mathbf{W}_2$-bundles over $\mathbf{L}_{d+1}$,

$$\nabla^{[m]}_{gen,d+1,2} : 0 \subset \mathcal{V}^{(m)}_{1,2} \subset \ldots \subset \mathcal{V}^{(m)}_{d,2} \subset \mathcal{V}^{'[m]}_{d+1,2}.$$

It lifts $\nabla^{(m)}_{gen,d+1,1}$, in a way that extends $\nabla^{(m)}_{gen,d,2}$.
Note that it is *not* embedded in $\mathcal{V}^{(m)}_{D,2}$.
Specializing via s_3, we get that

$$\nabla^{[m]}_{d+1,2} := (s_3)^*(\nabla^{[m]}_{gen,d+1,2}),$$

a complete flag of $G\mathbf{W}_2$-bundles over S, lifts $\nabla^{(m)}_{d+1,1}$, in a way that extends $\nabla^{(m)}_{d,2}$.
Since S is perfect, $\nabla_{d+1,1}$ itself lifts, in a way that extends $\nabla_{d,2}$.

The following Corollary generalizes [5, Theorem 6.1], in depth 1.

Corollary 16.9. *(Lifting representations of $(1,1)$-smooth profinite groups.) Let G be a $(1,1)$-smooth profinite group. Let k be a perfect field of characteristic p. Let*

$$\rho_1 : G \longrightarrow \mathbf{GL}_d(k)$$

be a continuous mod p representation of G, of arbitrary dimension d. Then, ρ_1 lifts to a representation

$$\rho_2 : G \longrightarrow \mathbf{GL}_d(\mathbf{W}_2(k)).$$

Proof. Write $V_{d,1} = k^d$, seen as a (k, G)-module via ρ_1. Consider the extension with abelian kernel

$$0 \longrightarrow \mathrm{M}_d(k) \longrightarrow \mathbf{GL}_d(\mathbf{W}_2(k)) \longrightarrow \mathbf{GL}_d(k) \longrightarrow 1.$$

the induced action of $\mathbf{GL}_d(k)$ on the matrix algebra $\mathrm{M}_d(k)$ is given by

$$g.M = \mathrm{Frob}(g) M \mathrm{Frob}(g)^{-1},$$

where

$$\mathrm{Frob} : \mathbf{GL}_d \longrightarrow \mathbf{GL}_d$$

is the Frobenius of $\mathbf{GL}_d$. Using [30], Chapter 5, proposition 41, we get a class

$$c \in H^2(G, \mathrm{End}(V_{d,1})^{(1)}),$$

obstructing the existence of ρ_2. By the usual inflation-restriction argument, to show that it vanishes, it suffices to show that its restriction to $H^2(G_p, \mathrm{End}(V_{d,1})^{(1)})$ vanishes, where $G_p \subset G$ is a pro-p-Sylow subgroup. By [5], Lemma 11.10, G_p is $(1,1)$-smooth as well. In other words, we can assume that $G = G_p$ is a pro-p-group. Then, $V_{d,1}$ possesses a complete G-invariant flag

$$\nabla_1 : 0 \subset V_{1,1} \subset \ldots \subset V_{d,1}.$$

We can then apply the Uplifting Theorem to this flag, with $S = \mathrm{Spec}(k)$. It lifts to a complete flag of $G\mathbf{W}_2$-bundles

$$\nabla_2 : 0 \subset V_{1,2} \subset \ldots \subset V_{d,2}$$

In particular, $V_{d,1}$ lifts modulo p^2. Equivalently, ρ_1 lifts to ρ_2, as desired. $\square$

Exercise 16.10. In the preceding Corollary, remove the perfectness assumption on k, using a Frobenius-splitting argument.
Give a constructive proof, using Section 3 of [5].

17. Lifting $\mathbb{F}_p$-étale local systems, to $\mathbb{Z}/p^2$-étale local systems...

17.1. ...for a semi-local scheme.

Let X be a connected scheme, where p is invertible. Denote by

$$G := \pi_1(X)$$

the étale fundamental group of X. If X is semi-local, it is known by [5], that the pair $(G, \mathbb{Z}_p(1))$ is $(1,\infty)$-cyclotomic. Therefore, G is $(1,1)$-smooth by Theorem A of [6].
Applying the Uplifting Theorem and its corollary, it follows that (completely filtered) $\mathbb{F}_p$-étale local systems on X admit Zariski-local liftings, to (completely filtered) $\mathbb{Z}/p^2$-étale local systems on X.
Equivalently, for all $d \geq 1$, the arrows

$$\mathrm{Hom}(\pi_1(X), \mathbf{B}_d(\mathbb{Z}/p^2)) \longrightarrow \mathrm{Hom}(\pi_1(X), \mathbf{B}_d(\mathbb{F}_p))$$

and

$$\mathrm{Hom}(\pi_1(X), \mathbf{GL}_d(\mathbb{Z}/p^2)) \longrightarrow \mathrm{Hom}(\pi_1(X), \mathbf{GL}_d(\mathbb{F}_p))$$

are surjective.

17.2. ...FOR A SMOOTH CURVE, OVER AN ALGEBRAICALLY CLOSED FIELD.
Let C be a smooth connected curve (not necessarily proper), over an algebraically closed field F. Denote by

$$G := \pi_1(C)$$

its étale fundamental group. By [5], Proposition 4.11, combined to Theorem A of [6], we know that G is $(1,1)$-smooth. Thus, (completely filtered) $\mathbb{F}_p$-étale local systems on C lift to (completely filtered) $\mathbb{Z}/p^2$-étale local systems on C.
Equivalently, for all $d \geq 1$, the arrows

$$\mathrm{Hom}(\pi_1(C), \mathbf{B}_d(\mathbb{Z}/p^2)) \longrightarrow \mathrm{Hom}(\pi_1(X), \mathbf{B}_d(\mathbb{F}_p))$$

and

$$\mathrm{Hom}(\pi_1(C), \mathbf{GL}_d(\mathbb{Z}/p^2)) \longrightarrow \mathrm{Hom}(\pi_1(X), \mathbf{GL}_d(\mathbb{F}_p))$$

are surjective.

Exercise 17.1. (Lifting local systems on curves: an elementary approach)
Assume that C is a smooth *projective* curve C over $F = \mathbb{C}$. Set $G := \pi_1(C)$.
Let

$$\rho_1 : G \longrightarrow \mathbf{GL}_d(\mathbb{F}_p)$$

be a mod p local system on C. We have seen that ρ_1 lifts to

$$\rho_2 : G \longrightarrow \mathbf{GL}_d(\mathbb{Z}/p^2).$$

Show that ρ_1 actually lifts to

$$\rho_\infty : G \longrightarrow \mathbf{GL}_d(\mathbb{Z}_p),$$

using the description of G by generators and relations.
In genus $g = 1$, you have to prove the following. Let $a, b \in \mathbf{GL}_d(\mathbb{F}_p)$ be two commuting invertible matrices. Then, a and b lift, to commuting invertible matrices $A, B \in \mathbf{GL}_d(\mathbb{Z}_p)$.
In genus $g \geq 2$, I do not have a solution.

18. LIFTING MOD p GALOIS REPRESENTATIONS.

Let F be a field. Then, $\mathrm{Spec}(F)$ is a semi-local scheme, so that section 17.1 applies. Translating into the tongue of Galois representations, we get the following.

THEOREM 18.1. *Let*

$$\rho_1 : \mathrm{Gal}(F_s/F) \longrightarrow \mathbf{GL}_d(\mathbb{F}_p)$$

be a continuous Galois representation of G, of arbitrary dimension d.
Then, ρ_1 lifts to

$$\rho_2 : \mathrm{Gal}(F_s/F) \longrightarrow \mathbf{GL}_d(\mathbb{Z}/p^2).$$

Similarly, let

$$\rho_1 : \mathrm{Gal}(F_s/F) \longrightarrow \mathbf{B}_d(\mathbb{F}_p)$$

be a continuous triangular Galois representation of G, of arbitrary dimension d.
Then, ρ_1 lifts to

$$\rho_2 : \mathrm{Gal}(F_s/F) \longrightarrow \mathbf{B}_d(\mathbb{Z}/p^2).$$

Liftability of Galois representations can be formulated in an elementary and joyful fashion, accessible to anyone familiar with group theory:)

THEOREM 18.2. *(Reformulation of the first part of Theorem 18.1).*
Let F be a field. Let E_1/F be a Galois extension of F, whose Galois group

$$\Gamma_1 := \mathrm{Gal}(E_1/F)$$

is a subgroup of a matrix group $\mathbf{GL}_d(\mathbb{F}_p)$.
Then, there exists a field extension E_2/E_1, enjoying the following properties.

- *The extension E_2/F is Galois, and its group*
$$\Gamma_2 := \mathrm{Gal}(E_2/F)$$
is a subgroup of $\mathbf{GL}_d(\mathbb{Z}/p^2)$.
- *The natural surjection $\Gamma_2 \longrightarrow \Gamma_1$, given by Galois correspondence, is induced by the mod p reduction $\mathbf{GL}_d(\mathbb{Z}/p^2) \longrightarrow \mathbf{GL}_d(\mathbb{F}_p)$.*

19. What's next?

The Uplifting Theorem is an extremely fruitful statement. It has many possible applications, especially in algebraic geometry and in modular representation theory. One of these already materialized in [7]: The Smoothness Theorem, providing a new proof the Norm Residue Isomorphism Theorem of Rost, Suslin, Voevodsky and Weibel.
The Uplifting Theorem can also be transposed to other contexts.

20. Appendix: cyclotomic closure and smooth closure.

Let G be a profinite group.
Consider a discrete G-module $\mathbb{Z}/p^2(1)$, which is free of rank one as a $\mathbb{Z}/p^2$-module. We do not assume that the pair $(G, \mathbb{Z}/p^2(1))$ is $(1,1)$-cyclotomic.
Then, there is a canonical cyclotomic closure

$$\Sigma(G, \mathbb{Z}/p^2(1)) \longrightarrow G.$$

It is a surjective homomorphism of profinite groups, whose source is $(1,1)$-cyclotomic w.r.t. to $\mathbb{Z}/p^2(1)$. It can be thought of as a "resolution of singularities" of G, w.r.t. to $\mathbb{Z}/p^2(1)$. It is an important construction, applying to all profinite groups. It can be transposed to other contexts. How to use it is kept for future considerations.

DEFINITION 20.1. *Consider the set of pairs (H, c_h), where $H \subset G$ is an open subgroup, and where $c_h : H \longrightarrow \mathbb{Z}/p(1)$ is a 1-cocycle. Using Shapiro's Lemma, we have a tautological 1-cocycle*

$$C_G : G \longrightarrow \prod_{(H,c_h)} (\mathbb{Z}/p)(1)^{G/H}.$$

Form the (set-theoretic) fibered product

$$\begin{tikzcd}
\sigma(G, \mathbb{Z}/p^2(1)) \arrow[r] \arrow[d] & G \arrow[d, "C_G"] \\
\prod_{(H,c_h)} (\mathbb{Z}/p^2)(1)^{G/H} \arrow[r] & \prod_{(H,c_h)} (\mathbb{Z}/p)(1)^{G/H};
\end{tikzcd}$$

it is naturally a profinite group. Formula for the group law:

$$(g, x)(g', x') := (gg', x + g.x'),$$

for $g \in G$ and $x \in \prod_{(H,c_h)} (\mathbb{Z}/p^2)(1)^{G/H}$.
One can iterate this process.
Set

$$\Sigma(G, \mathbb{Z}/p^2(1)) := \varprojlim_i \sigma^i(G, \mathbb{Z}/p^2(1)).$$

It is the inverse limit of the system

$$\ldots \longrightarrow \sigma(\sigma(G, \mathbb{Z}/p^2(1)), \mathbb{Z}/p^2(1)) \longrightarrow \sigma(G, \mathbb{Z}/p^2(1)) \longrightarrow G.$$

Proposition 20.2. *The pair*

$$(\Sigma(G, \mathbb{Z}/p^2(1)), \mathbb{Z}/p^2(1))$$

is $(1,1)$-cyclotomic. The natural morphism

$$\Sigma(G, \mathbb{Z}/p^2(1)) \longrightarrow G$$

is surjective. It is versal in the category of all continuous morphisms $G' \longrightarrow G$, whose source G' is $(1,1)$-cyclotomic w.r.t. $\mathbb{Z}/p^2(1)$. In other words, for every such morphism $G' \longrightarrow G$, there exists a (non unique) morphism $G' \longrightarrow \Sigma(G, \mathbb{Z}/p^2(1))$, such that the triangle

$$\begin{tikzcd} G' \arrow[d] \arrow[dr] & \\ \Sigma(G, \mathbb{Z}/p^2(1)) \arrow[r] & G \end{tikzcd}$$

commutes.

Proof. We check that $\Sigma(G, \mathbb{Z}/p^2(1))$ is $(1,1)$-cyclotomic w.r.t. $\mathbb{Z}/p^2(1)$. Denote by $\mathcal{G}_i$ the kernel of the natural quotient

$$q_i : \Sigma(G, \mathbb{Z}/p^2(1)) \longrightarrow \sigma^i(G, \mathbb{Z}/p^2(1)).$$

Let $\mathcal{H} \subset \Sigma(G, \mathbb{Z}/p^2(1))$ be an open subgroup. Let $c_h : \mathcal{H} \longrightarrow \mathbb{Z}/p(1)$ be a 1-cocycle. Pick i such that $\mathcal{G}_i \subset \mathcal{H}$, and such that c_h factors through $\mathcal{H} \longrightarrow \mathcal{H}/\mathcal{G}_i$. Denote by $H_i \subset \sigma^i(G, \mathbb{Z}/p^2(1))$ the image of $\mathcal{H}$ under q_i. Then c_h gives rise to a 1-cocycle $x_h : H_i \longrightarrow \mathbb{Z}/p(1)$. By definition of $\sigma(., \mathbb{Z}/p^2(1))$, the composite cocycle

$$H_{i+1} \longrightarrow H_i \xrightarrow{x_h} \mathbb{Z}/p(1)$$

lifts to a 1-cocycle $\tilde{x}_h : H_{i+1} \longrightarrow \mathbb{Z}/p^2(1)$. Thus, c_h itself lifts to $\tilde{c}_h : \mathcal{H} \longrightarrow \mathbb{Z}/p^2(1)$. The fact that $\Sigma(G, \mathbb{Z}/p^2(1)) \longrightarrow G$ is surjective is obvious. That it is versal follows from the definition of a $(1,1)$-cyclotomic pair. $\square$

20.1. The smooth closure. The cyclotomic closure has a "smooth" version, which depends only on G and p.

Definition 20.3. *For a finite G-set X, put*

$$\mathbf{G}_X := \mathbb{G}_a^X \rtimes \mathbb{G}_m,$$

where the semi-direct product is given by the diagonal action

$$\lambda.(t_x)_{x \in X} = (\lambda t_x)_{x \in X}.$$

It is an affine algebraic group, defined over $\mathbb{Z}$. Recall that $\mathbf{G}_{\{*\}}$ is the group of automorphisms of the one-dimensional affine space $\mathbb{A}^1$.
Consider the set of pairs (X, c), where X is a finite G-set, and where $c : G \longrightarrow \mathbf{G}_X(\mathbb{F}_p)$ is a 1-cocycle.

We have a tautological 1*-cocycle*

$$C_G : G \longrightarrow \prod_{(X,c)} \mathbf{G}_X(\mathbb{F}_p).$$

Form the (set-theoretic) fibered product

$$\begin{tikzcd} \sigma(G) \arrow[r] \arrow[d] & G \arrow[d, "C_G"] \\ \prod_{(X,c)} \mathbf{G}_X(\mathbb{Z}/p^2) \arrow[r, "\rho"] & \prod_{(X,c)} \mathbf{G}_X(\mathbb{F}_p), \end{tikzcd}$$

where ρ is induced by the reductions $\mathbf{G}_X(\mathbb{Z}/p^2) \longrightarrow \mathbf{G}_X(\mathbb{F}_p)$.
It is naturally a profinite group. Iterating this process, set

$$\Sigma(G) := \varprojlim_i \sigma^i(G).$$

Proposition 20.4. *The profinite group $\Sigma(G)$ is $(1,1)$-smooth. The natural morphism $\Sigma(G) \longrightarrow G$ is surjective. It is versal in the category of all continuous morphisms $G' \longrightarrow G$, whose source G' is $(1,1)$-smooth. In other words, for every such morphism $G' \longrightarrow G$, there exists a (non unique) morphism $G' \longrightarrow \Sigma(G)$, such that the triangle*

$$\begin{tikzcd} G' \arrow[d] \arrow[dr] & \\ \Sigma(G) \arrow[r] & G \end{tikzcd}$$

commutes.

Proof. Adapt the proof of Proposition 20.1. □

Remark 20.5. Using the smooth closure, Theorem 14.1 can be applied to study modular representations of arbitrary (pro)-finite groups.
This is a worthwhile topic of investigation.

Exercise 20.6. Compute the smooth closure of $\mathbb{Z}/p$. Good luck!

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Index of notation and denomination

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