ALGEBRAIC GROUPS WITH TORSORS THAT ARE VERSAL FOR ALL AFFINE VARIETIES

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ABSTRACT. Let k be a field and let G be an affine k-algebraic group. Call a G-torsor *weakly versal* for a class of k-schemes $\mathscr{C}$ if it specializes to every G-torsor over a scheme in $\mathscr{C}$. A recent result of the first author, Reichstein and Williams says that for any $d \geq 0$, there exists a G-torsor over a finite type k-scheme that is weakly versal for finite type affine k-schemes of dimension at most d. The first author also observed that if G is unipotent, then G admits a torsor over a finite type k-scheme that is weakly versal for *all* affine k-schemes, and that the converse holds if $\operatorname{char} k = 0$. In this work, we extend this to all fields, showing that G is unipotent if and only if it admits a G-torsor over a quasi-compact base that is weakly versal for all finite type regular affine k-schemes. Our proof is characteristic-free and it also gives rise to a quantitative statement: If G is a non-unipotent subgroup of $\mathbf{GL}_n$, then a G-torsor over a quasi-projective k-scheme of dimension d is *not* weakly versal for finite type regular affine k-schemes of dimension $n(d+1)+2$. This means in particular that every such G admits a nontrivial torsor over a regular affine $(n+2)$-dimensional variety. When G contains a nontrivial torus, we show that nontrivial torsors already exist over 3-dimensional smooth affine varieties (even when G is *special*), and this is optimal in general.

In the course of the proof, we show that for every $m, \ell \in \mathbb{N} \cup \{0\}$ with $\ell \neq 1$, there exists a smooth affine k-scheme X carrying an ℓ-torsion line bundle that cannot be generated by m global sections. We moreover study the minimal possible dimension of such an X and show that it is m, $m+1$ or $m+2$.

1. INTRODUCTION

Let k be a field and let G be an affine k-algebraic group. Recall that a G-torsor $E \to X$, where X is a k-scheme, is said to be *weakly versal* if $E \to X$ specializes to every G-torsor over any k-field $E' \to \operatorname{Spec} K$. When the joint image of all such specializations $\operatorname{Spec} K \to X$ is dense in X, the torsor $E \to X$ is called *versal*; see [9], for instance. Versal torsors have many applications, notably to study of essential dimension and the theory of cohomological invariants, e.g., see [21] and [36]. In particular, the essential dimension of G, denoted $\operatorname{ed}_k(G)$, is equal to the minimal possible dimension of a k-variety that is the base of a versal G-torsor.

Given a class of k-schemes $\mathscr{C}$, let us say that a G-torsor $E \to X$ is *weakly versal for* $\mathscr{C}$ if every G-torsor over a scheme in $\mathscr{C}$ is a specialization of $E \to X$. This definition first appeared in [13, §5], where it was shown that if k is infinite, then for every $d \in \mathbb{N}$, there exists a G-torsor over a finite type k-scheme that is weakly versal for the class of finite type affine k-schemes of dimension at most d. Special examples of this phenomena were observed earlier, e.g., [30] and [2, §5.1]. Weak versality for general classes of k-schemes, as well as stronger variants of this property, were studied systematically by the first author in [11], and we refer to introduction of

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this source for a more detailed history. By [11, Thm. 8.2], for every $d \in \mathbb{N}$ and every k-algebraic subgroup G of $\mathbf{GL}_n$, there is a G-torsor over a quasi-projective k-scheme of dimension $nd + n^2 - \dim G$ that is *strongly* versal for the class of affine noetherian k-schemes of dimension at most d.

Perhaps more surprisingly, it was observed in [11, Thm. 10.1] that if G is unipotent, then there are G-torsors over quasi-projective k-schemes that are weakly versal for *all* affine k-schemes.[1] The special case where $p := \operatorname{char} k > 0$ and G is a finite constant p-group goes back to Saltman [30]; see also [15]. The first author also showed that the converse holds in characteristic 0, that is, a k-algebraic group admitting G-torsor over a finite type k-scheme that is versal for all affine k-schemes is unipotent [11, Thm. 10.6]. They asked whether this also holds in positive characteristic. Our first main result answers this question on the positive.

Theorem 1.1. *Let G be an affine algebraic group over a field k. Then the following conditions are equivalent:*

(a) there exists a G-torsor over a quasi-compact k-scheme that is weakly versal for all finite type regular affine k-schemes;
(b) there exists a G-torsor over a smooth quasi-projective k-scheme that is weakly versal for all affine k-schemes;
(c) G is unipotent.

Our proof is very different from the argument in [11]. It works in any characteristic, applies to torsors over a quasi-compact base (rather than of finite type over k), and it also yields a quantitative variant for G-torsors over quasi-projective k-schemes, which is our second main result.

Theorem 1.2. *Let k be a field, and let G be a non-unipotent k-algebraic subgroup of $\mathbf{GL}_n$. Let $E \to X$ be a G-torsor such that X is quasi-projective over a k-field and of dimension d. Then $E \to X$ is* not *weakly versal for the class of finite type regular affine k-schemes of dimension at most $n(d+1)+2$.*

Both theorems admit mild improvements when some assumptions are imposed on G or k; see Section 6. For example, if G is connected or k is perfect, then we may replace "finite type regular" in condition (a) of Theorem 1.1 and in Theorem 1.2 with "smooth". Furthermore, if G contains a nontrivial torus, then we can assert in Theorem 1.2 that $E \to X$ is not versal for smooth affine k-schemes of dimension $\leq n(d+1)+1$. The strongest statements we can make about particular G and k are given in Theorems 6.1 and 6.2.

Theorem 1.2 implies that every non-unipotent k-algebraic subgroup of $\mathbf{GL}_n$ admits a nontrivial G-torsor over a finite type regular affine k-scheme of dimension $\leq n+2$; mild improvements of this are given in Corollaries 6.5 and 6.6. However, we can do significantly better when G contains a nontrivial torus.

Theorem 1.3. *Let G be an affine k-algebraic group containing a nontrivial torus (resp. central torus). Then there exists a smooth affine k-scheme X of dimension ≤ 3 (resp. 1) carrying a nontrivial G-torsor. When $\operatorname{char} k = 0$ and $\operatorname{trdeg}_{\mathbb{Q}} k$ is infinite, we can take X to be of dimension ≤ 2.*

Theorem 1.3 is the best result we can hope for with general k and G. Indeed, we show that any $\mathbf{SL}_n$-torsor ($n > 1$) over a noetherian affine scheme of dimension smaller than 2 is trivial (Example 7.3), and moreover, every $\mathbf{SL}_n$-torsor over a smooth affine $\overline{\mathbb{F}}_p$-scheme of dimension smaller then 3 is trivial (Corollary 7.5). The characteristic-0 analogue of the latter, i.e., that every $\mathbf{SL}_n$-torsor over a smooth

[1] Note that in characteristic 0, every G-torsor over an affine scheme is trivial when G is unipotent, but in positive characteristic, unipotent groups may have nontrivial torsors.

affine $\overline{\mathbb{Q}}$-scheme of dimension ≤ 2 is trivial, is equivalent to the Bloch–Beilinson conjecture for surfaces (Remark 7.6).

The "local" counterpart of Theorem 1.3, i.e., the problem of finding nontrivial torsors over k-fields, was already considered by Reichstein and Youssin [29]. To put their result in context, recall that Serre made two conjectures about the triviality of torsors over fields of small cohomological dimension [34], [35, III.§3]. Serre's Conjecture I — now a theorem of Steinberg [40] — states that if G is a connected algebraic group over a field K of cohomological dimension ≤ 1, then every G-torsor over K is trivial. Serre's Conjecture II states that if G is a semisimple[2] simply connected algebraic group over a perfect field K of cohomological dimension ≤ 2, then every G-torsor over K is trivial; this conjecture is known in many cases [18]. Serre further called a k-algebraic group G *special* if every G-torsor over a k-field is trivial [33]. The main result of [29], which we state here only in characteristic 0 for simplicity, informally says that Serre's conjectures I and II are optimal. Formally, suppose that G is an algebraic group over an algebraically closed field k (with $\operatorname{char} k = 0$). Then:

(i) If G is not connected, then G has nontrivial torsors over any k-field of transcendence degree 1.
(ii) If $G/\operatorname{rad}(G)$ is not semisimple simply connected, then G has nontrivial torsors over any k-field of transcendence degree 2.
(iii) If G is not special, then G has nontrivial torsors over any k-field of transcendence degree 3.

By spreading out, these statements imply a result similar to Theorem 1.3, but which applies only to non-special groups. In addition to addressing the case of special groups containing a nontrivial torus, Theorem 1.3 improves the "global" consequences of (i)–(iii) when G is semisimple simply connected and $\operatorname{trdeg}_{\mathbb{Q}} k$ is infinite by showing that there exist nontrivial G-torsors over 2-*dimensional* smooth affine k-schemes. (There are also additional improvements in positive characteristic.) On the other hand, the global analogue of (i) gives a better result than Theorem 1.3 for non-connected (smooth) algebraic groups.

We return to the case where G is an affine algebraic group over some field k. Another way to look at Theorem 1.2 is via the following corollary:

Corollary 1.4. *Let G be a k-algebraic subgroup of $\mathbf{GL}_n$, and let $m_d(G)$ be the smallest $m \in \mathbb{N} \cup \{0\}$ for which there is a G-torsor over an m-dimensional quasi-projective k-scheme that is weakly versal for finite type affine k-schemes of dimension at most d. If G is not unipotent, then*

$$\left\lfloor \frac{d-2}{n} \right\rfloor \leq m_d(G) \leq nd + n^2 - \dim G.$$

Otherwise,

$$m_d(G) \leq \frac{n(n-1)}{2} - \dim G.$$

Proof. The lower bound follows from Theorem 1.2. The upper bounds follow from [11, Thms. 8.2, 10.1] □

[2]We follow the convention that reductive groups, and in particular semisimple groups, are connected.

When $d = 0$, the integer $m_d(G)$ is just the essential dimension of G,[3] so the invariants $\{m_d(G)\}_{d\geq 0}$ may be regarded as "higher" analogues of the essential dimension. Corollary 1.4 tells us that the sequence $\{m_d(G)\}_{d\geq 0}$ is either bounded — which is the case precisely when G is unipotent — or it has linear growth rate.

We prove Theorems 1.1 and 1.2 by establishing three results.

First, we show in Theorem 3.6 that an affine k-algebraic group G is not unipotent if and only if $G_{\overline{k}}$ contains a copy of $\boldsymbol{\mu}_\ell$ (the group of ℓ-th roots of unity) for some $\ell > 1$; this was known if G is smooth and connected [7, Ch. IV, Cor. 11.5(2)]. We also show that a general connected k-algebraic group (possibly not affine or not smooth) that is not unipotent has a nontrivial multiplicative algebraic subgroup.

Next, we show in Theorem 4.4 that if G is linear and contains a copy of $\boldsymbol{\mu}_\ell$, and if there is a G-torsor over a quasi-compact scheme X that is weakly versal for a class $\mathscr{C}$ of affine k-schemes, then there is a uniform bound on the number of generators of every ℓ-torsion line bundle over a scheme in $\mathscr{C}$. This upper bound is explicit if X is quasi-projective over a k-field.

Finally, we show in Theorem 5.1 that for every $m, \ell \in \mathbb{N} \cup \{0\}$ with $\ell \neq 1$, there exists a smooth affine k-scheme X admitting an ℓ-torsion line bundle that cannot be generated by m global sections; such examples were known for $\ell = 2$ if $\operatorname{char} k \neq 2$ [37, Prop. 7.13, Prop. 6.3]. More generally, we study what is the minimal possible dimension of X (also in the non-torsion case $\ell = 0$) and show that it is m, $m + 1$ or $m + 2$. A key results which we establish along the way is the following criterion (Lemma 5.4): If the m-th power of the 1-st Chern class of line bundle L is nonzero, then L cannot be generated by m global sections.

Putting Theorems 3.6, 4.4 and 5.1 together tells us (after a little work) that if G is not unipotent, then no G-torsor over a quasi-compact base $E \to X$ is weakly versal for all finite type regular affine k-schemes. This gives Theorem 1.1. When X is quasi-projective, we can moreover point to a particular G-torsor $E' \to X'$ that is not a specialization of $E \to X$ and thus derive the quantitative Theorem 1.2.

Theorem 1.3 is not a corollary of Theorem 1.2, but its proof follows a simpler version of the above strategy. Loosely speaking, we make use of the torus contained in G rather than the copy of $\boldsymbol{\mu}_\ell$.

The paper is organized as follows. Section 2 recalls necessary facts about algebraic groups, torsors and vector bundles. Sections 3 and 4 are dedicated to proving Theorems 3.6 and 4.4, respectively. In Section 5, we give many examples of both torsion and non-torsion line bundles over smooth affine schemes requiring many global sections to generate, thus proving Theorem 5.1. These results are used in Section 6 to prove Theorems 1.1 and 1.2. Finally, Theorem 1.3 is proved in Section 7.

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2. Preliminaries

Throughout this work, k denotes a field and $\overline{k}$ is an algebraic closure of k. If not indicated otherwise, all schemes and group schemes are over k, morphisms are k-morphisms, and products of schemes are taken over $\operatorname{Spec} k$. By a k-variety we

[3] This is true despite the fact that Corollary 1.4 considers only torsors over quasi-projective k-schemes. Indeed, we clearly have $\operatorname{ed}_k(G) \leq m_0(G)$. Furthermore, if $p : E \to X$ is a versal G-torsor, then there is a versal G-torsor $E' \to X'$ with $\dim X' = \dim X$ and X' affine (and in particular quasi-projective) — simply take a dense open affine $X' \subseteq X$ and $E' = p^{-1}(X')$. Thus, $m_0(G)$ is smaller than $\min\{\dim X \mid E \to X \text{ is a versal } G\text{-torsor}\}$, which is $\operatorname{ed}_k(G)$.

mean an integral k-scheme of finite type. The n-dimensional affine and projective spaces over k are denoted $\mathbb{A}^n$ and $\mathbb{P}^n$, respectively.

A k-ring means a commutative (unital) k-algebra. If A is a k-ring and X is a k-scheme, we write X_A or $X \times_k A$ for $X \times_{\operatorname{Spec} k} \operatorname{Spec} A$ viewed as an A-scheme. For example, $\mathbb{P}^n_A$ is the n-dimensional projective space over $\operatorname{Spec} A$. Similar notation applies to k-morphisms.

An algebraic group over k means a finite type group scheme over k (possibly non-smooth). Recall that a k-algebraic group G is affine if and only if it is linear. The identity connected component of G is denoted G° and its unipotent radical is $\operatorname{rad}_u(G)$. As usual, $\mathbf{G_a}$ is the additive group of k, $\mathbf{G_m}$ is the multiplicative group of k, $\boldsymbol{\mu}_\ell$ is the k-algebraic group of ℓ-th roots of unity, $\mathbf{GL}_n$ is the k-algebraic group of invertible $n \times n$ matrices and $\mathbf{SL}_n$ is the k-algebraic group of $n \times n$ matrices of determinant 1. When $p = \operatorname{char} k > 0$, we also write $\boldsymbol{\alpha}_{p^r}$ for the subgroup of $\mathbf{G_a}$ cut by the equation $x^{p^r} = 0$. In accordance with our earlier conventions, given a k-field K, we write $\mathbf{G}_{\mathbf{a},K}$ for the K-algebraic group $\mathbf{G_a} \times_k K$, and likewise for $\mathbf{G_m}$, $\boldsymbol{\mu}_\ell$, etcetera.

We shall sometimes view k-schemes as sheaves on the site of all k-schemes with the fppf topology, denoted $(\mathsf{Sch}/k)_{\mathrm{fppf}}$. In particular, k-morphisms $f : X \to Y$ will sometimes be defined by specifying their action on sections, namely, by specifying $f_S : X(S) \to Y(S)$ for every scheme S. When G is a k-group scheme acting on a k-scheme X, we will write X/G to denote the sheafification of the presheaf $S \mapsto X(S)/G(S)$ on $(\mathsf{Sch}/k)_{\mathrm{fppf}}$. We shall treat X/G as a k-scheme when it is represented by one. For example, if G is a subgroup of a k-algebraic group H, then H/G is known to be a k-scheme, which is moreover quasi-projective when H is affine [22, Thm. 5.28, §7e, proof of Thm. 7.18].

Let G be a k-group scheme. Our conventions regarding G-torsors are the same as in [11, §1]. That is, a G-torsor over a k-scheme X consists of a sheaf E on $(\mathsf{Sch}/k)_{\mathrm{fppf}}$ together with a morphism $E \to X$ and a *right* G-action $E \times G \to E$ subject to the requirement that there is an fppf covering $X' \to X$ for which $X' \times_X E \cong X' \times_k G$ as schemes over X' carrying a G-action. Thus, G-torsors over X are classified by the cohomology pointed set $\mathrm{H}^1_{\mathrm{fppf}}(X, G)$. In general, not all G-torsors are represented by a k-scheme, but this is true if G is affine [23, III, Thm. 4.3].

Let $E \to X$ be a G-torsor and let $f : Y \to X$ be a morphism. Then the first projection $Y \times_X E \to Y$ is also a G-torsor; we write f^*E for $Y \times_X E$. We say that a G-torsor $E \to X$ *specializes* to another G-torsor $E' \to X'$ if there is a k-morphism $f : X' \to X$ such that $E' \cong f^*E$ as G-torsors over Y. This is equivalent to the existence of a G-equivariant map $\hat{f} : E' \to E$.

Given a morphism of k-group schemes $\varphi : G \to H$ and a G-torsor $p : E \to X$, one can form the H-torsor

$$\varphi_*(p) : E \times^G H \to X.$$

Here, $E \times^G H$ is the quotient of $E \times_k H$ by the equivalence relation

$$(e, g, h) \mapsto ((eg, h), (e, \varphi(g)h)) : E \times G \times H \to (E \times H) \times (E \times H),$$

and $\varphi_*(p)$ is given by $\varphi_*(p)[e, h] = p(e)$ on sections.

A vector bundle over a k-scheme X is a locally free $\mathcal{O}_X$-module V. In this case, the functor $(f : Y \to X) \mapsto \Gamma(Y, f^*V)$ from X-schemes to sets is represented by a scheme which we also denote by V. (Here, as usual, $f^*V = f^{-1}V \otimes_{f^{-1}\mathcal{O}_X} \mathcal{O}_Y$.)

Recall that there is a bijection between isomorphism classes of rank-n vector bundles on X and isomorphism classes of $\mathbf{GL}_n$-torsors over X. Moreover, this equivalence is compatible with base-change. Briefly, given a rank-n vector bundle V, we can choose a Zariski covering $\{U_i \to X\}_{i \in I}$ and isomorphisms $\{\varphi_i : \mathcal{O}^n_{U_i} \to$

$V_{U_i}\}_{i\in I}$. Setting $U_{ij} = U_i \cap U_j$, this data gives rise to a $\mathbf{GL}_n$-valued 1-cocycle on X, namely $\alpha = \{\varphi_j^{-1}|_{U_{ij}} \circ \varphi_i|_{U_{ij}}\}_{i,j\in I}$, and it defines the $\mathbf{GL}_n$-torsor corresponding to V (as well as its cohomology class in $\mathrm{H}^1_{\mathrm{fppf}}(X, \mathbf{GL}_n)$).

A line bundle L over X is a rank-1 vector bundle over X. We say that L is *ℓ-torsion* ($\ell \in \mathbb{N} \cup \{0\}$) if $L^{\otimes \ell} := L \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} L$ (ℓ times) is isomorphic to $\mathcal{O}_X$. Note that every line bundle is 0-torsion. By the previous paragraph, there is a one-to-one correspondence between isomorphism classes of line bundles over X, isomorphism classes of $\mathbf{G_m}$-torsors over X and $\mathrm{H}^1_{\mathrm{fppf}}(X, \mathbf{G_m})$. For $\ell > 0$, the Kummer short exact sequence $\boldsymbol{\mu}_\ell \hookrightarrow \mathbf{G_m} \xrightarrow{\ell} \mathbf{G_m}$ of sheaves on $(\mathsf{Sch}/k)_{\mathrm{fppf}}$ gives rise to a long cohomology exact sequence

$$\cdots \to \mathrm{H}^1_{\mathrm{fppf}}(X, \boldsymbol{\mu}_\ell) \to \mathrm{H}^1_{\mathrm{fppf}}(X, \mathbf{G_m}) \xrightarrow{\ell} \mathrm{H}^1_{\mathrm{fppf}}(X, \mathbf{G_m}) \to \dots,$$

from which we see that every $\boldsymbol{\mu}_\ell$-torsor gives rise to an ℓ-torsion line bundle, and every ℓ-torsion line bundle arises from some $\boldsymbol{\mu}_\ell$-torsor (that not unique in general). Explicitly, the line bundle arising from a $\boldsymbol{\mu}_\ell$-torsor $T \to X$ corresponds to the $\mathbf{G_m}$-torsor $T \times^{\boldsymbol{\mu}_\ell} \mathbf{G_m} \to X$.

3. A Characterization of Non-Unipotent Groups

Recall the following important result: Among *smooth* connected linear algebraic groups over a field k, unipotent groups are characterized by the property that they do not contain a nontrivial multiplicative subgroup. When k is algebraically closed, this is [7, Ch. IV, Cor. 11.5(2)], and over general fields it follows from Grothendieck's theorem on the existence of geometrically maximal tori [8, Exp. XIV, Th. 1.1]. Our goal in this section is to prove an analogous result for all connected algebraic groups over general fields, and arbitrary algebraic groups over algebraically closed fields. The result for connected affine groups over algebraically closed fields is contained in SGA [8, Exp. XVII, Prop. 4.3.1, (i) $\iff$ (iv)]. In fact, the same proof works over perfect fields. The purpose of this section is to prove the same result without affineness or perfectness hypotheses.

We begin with the following lemma.

Lemma 3.1. *Over a field k, an extension of a nontrivial finite connected multiplicative algebraic group M by a finite connected unipotent k-algebraic group I contains a nontrivial multiplicative k-algebraic group.*

Proof. We may assume that $\mathrm{char}(k) = p > 0$ (otherwise M is trivial), and that M is p-torsion [22, Cor. 11.30]. By [8, Exp. XVII, Prop. 4.2.1,(i) $\iff$ (iii)], we may also assume that $U = \boldsymbol{\alpha}_p^r$. The proof is now essentially identical to that given in [22, Cor. 15.33], using [22, Prop. 15.31]. $\square$

The key case, when G is finite connected, is contained in the following lemma.

Lemma 3.2. *Let k be a field of characteristic $p > 0$. A finite connected k-group scheme I is unipotent if and only if it does not contain a nontrivial multiplicative subgroup.*

Proof. The "only if" direction is trivial, so we concentrate on the converse. First assume that I is commutative. The connected-étale sequence of $\widehat{I}$, the Cartier dual of I, then shows that $\widehat{I}$ must be connected (since I does not contain a multiplicative group), hence I is a commutative infinitesimal k-group with infinitesimal dual. This implies that it is unipotent. (In fact, for this it is not hard to show that it is enough to have infinitesimal dual.) Indeed, by filtering we may suppose that I and its dual are both killed by Frobenius, and Dieudonné theory then implies that, at least over the perfection of k, I is isomorphic to a power of α_p, which then implies that it is unipotent; see [8, Exp. XVII, Def. 1.1].

We now prove the result in general by induction on the order of I. Note first that if one has an exact sequence

$$1 \longrightarrow I' \longrightarrow I \longrightarrow I'' \longrightarrow 1$$

with $I' \trianglelefteq I$ proper and nontrivial, then by induction I' is unipotent. If I'' contained a nontrivial multiplicative subgroup scheme, then so would I by Lemma 3.1. Therefore, I'' does not contain such a copy, so is also unipotent by induction, hence so is I. We are therefore free to assume that I does not admit a normal nontrivial proper k-subgroup. In particular, we may assume that the Frobenius morphism of I is trivial. That is, I is a height one algebraic group.

Thanks to the equivalence between height one k-algebraic groups and p-Lie algebras [8, Exp. VII$_A$, Thm. 8.1.2], and the fact that we have already handled the commutative case, the problem translates into the following p-Lie algebra question: If $\mathfrak{g}$ is a nonabelian p-Lie algebra admitting no proper nonzero p-ideals (that is, ideals preserved by the p operation), then $\mathfrak{g}$ contains a nontrivial abelian p-Lie subalgebra with surjective p operation. Indeed, this follows from the fact that multiplicative groups have surjective p operation (just check over an algebraic closure where it reduces to the case of $\boldsymbol{\mu}_n$) while unipotent commutative groups of height one have nilpotent p operation (as all such groups over an algebraic closure admit a filtration by either $\boldsymbol{\alpha}_p$ or $\mathbb{Z}/p\mathbb{Z}$), plus the fact that all commutative finite k-groups may be filtered by multiplicative and unipotent ones.

First assume that $\mathrm{ad}(X)$ is nilpotent for all $X \in \mathfrak{g}$. Then Engel's theorem implies that $\mathfrak{g}$ is a nilpotent Lie algebra. In particular, it has nontrivial center. Since $\mathrm{ad}(X^{[p]}) = \mathrm{ad}(X)^p$, the center is preserved by the p operation. It follows from our assumption that $\mathfrak{g}$ has no proper nonzero ideals that $\mathfrak{g}$ is abelian, contrary to assumption.

Now assume that not all $\mathrm{ad}(X)$ are nilpotent. Let $X \in \mathfrak{g}$ be such that $\mathrm{ad}(X)$ is not nilpotent. If $f(T) \in k[T]$ is the minimal polynomial of $\mathrm{ad}(X)$, then f is not a power of T. Furthermore, for some $r \geq 0$, $f(T) = g(T^{p^r})$ with $g \in k[T]$ separable. Because $\mathrm{ad}(X^{[p^r]}) = \mathrm{ad}(X)^{p^r}$, g is the minimal polynomial of $\mathrm{ad}(X^{[p^r]})$. Therefore, replacing X by $X^{[p^r]}$, we may assume that $\mathrm{ad}(X)$ is non-nilpotent and that its minimal polynomial is separable over k. It follows that X admits a Jordan-Chevalley decomposition $X = X_s + X_n$ with $\mathrm{ad}(X_s)$ geometrically semisimple, and $\mathrm{ad}(X_n)$ nilpotent. Replacing X by X_s, therefore, we obtain nonzero $X \in \mathfrak{g}$ such that $\mathrm{ad}(X)$ is geometrically semisimple.

Consider the nonzero subspace $\mathfrak{h} := \langle X, X^{[p]}, X^{[p^2]}, \dots \rangle \subset \mathfrak{g}$. We claim that $\mathfrak{h}$ is abelian. Indeed, this follows from the fact that, for any $Y \in \mathfrak{g}$, $\mathrm{ad}(Y^{[p^n]}) = \mathrm{ad}(Y)^{p^n}$, hence $Y^{[p^n]}$ commutes with Y. This equation also shows that $\mathrm{ad}(Y) \in \mathfrak{gl}(\mathfrak{g})$ is geometrically semisimple for all $Y \in \mathfrak{h}$. Because $\mathfrak{h}$ is abelian, one has $(Y+Z)^{[p]} = Y^{[p]} + Z^{[p]}$ for any $Y, Z \in \mathfrak{h}$. Thus $\mathfrak{h} \subset \mathfrak{g}$ is a (nonzero) abelian p-Lie subalgebra of $\mathfrak{g}$ all of whose elements have geometrically semisimple adjoint action on $\mathfrak{g}$, and furthermore this property is preserved upon passage to $\overline{k}$.

We must show that the p operation on $\mathfrak{h}$ is surjective, and it suffices to show this after extending scalars to $\overline{k}$, where it is equivalent to injectivity, so we may assume that k is algebraically closed. Suppose we are given $Y \in \mathfrak{h}$ such that $Y^{[p]} = 0$. We must show that $Y = 0$. One has $\mathrm{ad}(Y^{[p]}) = \mathrm{ad}(Y)^p = 0 \in \mathfrak{gl}(\mathfrak{g})$. Because $\mathrm{ad}(Y)$ is (geometrically) semisimple, it follows that $\mathrm{ad}(Y) = 0$. That is, Y is central in $\mathfrak{g}$. Since $Y^{[p]} = 0$, it follows that $\langle Y \rangle$ is a p-ideal of $\mathfrak{g}$, a contradiction because we assumed that $\mathfrak{g}$ is nonabelian with no proper nonzero p-ideals. This contradiction shows that the p operation on $\mathfrak{h}$ is injective, and completes the proof of the lemma. $\square$

Lemma 3.3. *Let k be a perfect field. Then any extension E of a multiplicative étale k-algebraic group M by a connected unipotent k-algebraic group U splits.*

Proof. By [8, Exp. XVII, Prop. 4.3.1(i) $\Longleftrightarrow$ (iii)], we may assume that U is a power of either $\mathbf{G}_a$ or $\boldsymbol{\alpha}_p$. The former case follows from [22, Prop. 15.31], while the latter follows from [8, Exp. XVII, Lem. 1.6]. $\square$

The following two lemmas will be required in order to deal with groups that are not affine. The non-affine case will not be needed in the next sections.

Lemma 3.4. *If G is a connected algebraic group over a field k, then there is an exact sequence of k-algebraic groups*

$$1 \longrightarrow H \longrightarrow G \longrightarrow A \longrightarrow 1$$

with H affine and connected and A an abelian variety.

Proof. By [8, Exp. $\mathrm{VII_A}$, Prop. 8.3], there is an infinitesimal normal k-algebraic group $I \trianglelefteq G$ such that G/I is smooth, so we are free to assume that G is smooth. If k is perfect, then we are done by Chevalley's Theorem. Otherwise, $G_{k_{\mathrm{perf}}}$ may be written as an extension of an abelian variety by a linear algebraic k-group, and therefore $G_{k^{p^{-n}}}$ may be so written for some $n \geq 0$. Via the isomorphism $k^{p^{-n}} \xrightarrow{\sim} k$, $\lambda \mapsto \lambda^{p^n}$, $G_{k^{p^{-n}}}$ is identified with $G^{(p^n)}$, the n-fold Frobenius twist of G. Therefore, $G^{(p^n)}$ may be written as an extension of an abelian variety by a linear algebraic group. Since G is smooth, the n-fold Frobenius map $G \to G^{(p^n)}$ is surjective, hence the result for G. $\square$

Lemma 3.5. *Over a field k of characteristic $p > 0$, for any extension E of an abelian variety A by a unipotent k-algebraic group U, there is an integer $n \geq 0$ such that the extension splits upon pullback by the map $[p^n]\colon A \to A$.*

Proof. We may assume that U is either connected or finite étale. In the latter case, we may by filtration assume that U is commutative and p-torsion. In the former case, [8, Exp. XVII, Prop. 4.3.1(i) $\Longleftrightarrow$ (iii)] allows us to assume that U is a power of either $\boldsymbol{\alpha}_p$ or $\mathbf{G_a}$. Thus we may assume that U is either p-torsion finite étale commutative, or else a power of either $\boldsymbol{\alpha}_p$ or $\mathbf{G_a}$. If the extension E is commutative, then because U is p-torsion, $\mathrm{Ext}^1(A, U)$ is killed by $[p]$, hence $[p]_A$ induces the 0 map on $\mathrm{Ext}^1(A, U)$, which proves the lemma. It therefore only remains to prove that E is commutative.

First suppose that U is either finite étale or a power of $\boldsymbol{\alpha}_p$. Then the automorphism functor $\mathrm{Aut}_{U/k}$ of U is an affine k-group scheme, and the conjugation action of A on U is then given by a homomorphism $A \to \mathrm{Aut}_{U/k}$, which must be trivial. Therefore, E is a central extension. The commutator map of E therefore descends to a map $A \times A \to U$, which must be constant because U is finite and $A \times A$ is connected and geometrically reduced. Thus E is commutative.

It remains to show that E is commutative under the assumption that $U = \mathbf{G}_\mathbf{a}^r$ for some $r > 0$. Once again, conjugation induces an action of A on $\mathbf{G}_\mathbf{a}^r$. But we claim that any such action is trivial, and hence E is central. To prove this, we may temporarily extend scalars and thereby assume that k is infinite. Then for any $x \in k^r = \mathbf{G}_\mathbf{a}^r(k)$, the map $a \mapsto a \cdot x$ induces a map $A \to \mathbf{G}_\mathbf{a}^r$. Because $\mathbf{G}_\mathbf{a}^r$ is affine, this map must be constant, hence A acts trivially on x. Because $\mathbf{G}_\mathbf{a}^r(k)$ is Zariski dense in $\mathbf{G}_\mathbf{a}^r$, it follows that the action of A on $\mathbf{G}_\mathbf{a}^r$ is trivial and E is central, as claimed. Once again, therefore, the commutator map of E descends to a map $A \times A \to \mathbf{G}_{\mathbf{a}}^r$ which must be constant because the target is affine. Thus E is commutative, and the proof is complete. □

We now prove the main result of this section.

Theorem 3.6. *Let k be a field and G a k-algebraic group.*

(i) If G is connected, then G is unipotent if and only if it does not contain a nontrivial multiplicative k-algebraic group.

(ii) If k is algebraically closed, then G is unipotent if and only if it does not contain a nontrivial multiplicative k-algebraic group.

Proof. The "only if" direction is immediate, so we concentrate on the "if" direction. First we prove (i). Let G be a connected k-algebraic group that has no nontrivial multiplicative subgroup. We must show that G is unipotent. Let us first give the proof under the additional assumption that G is affine. By [8, Exp. VII$_{\mathrm{A}}$, Prop. 8.3], there is an infinitesimal algebraic subgroup $I \trianglelefteq G$ such that $\overline{G} := G/I$ is smooth (and connected), and I is unipotent by Lemma 3.2. If $\overline{G}$ contains a nontrivial k-torus, then Lemma 3.1 implies that G contains a nontrivial multiplicative subgroup. Therefore, $\overline{G}$ does not contain a nontrivial torus, hence is unipotent by [7, Ch. IV, Cor. 11.5(2)] and [8, Exp. XIV, Thm. 1.1], which completes the proof in the affine case.

In the general (not necessarily affine) case, Lemma 3.4 furnishes an exact sequence

$$1 \longrightarrow H \longrightarrow G \longrightarrow A \longrightarrow 1$$

with H connected and affine, and A an abelian variety. By the already-treated affine case, H is unipotent. Assume for the sake of contradiction that $A \neq 0$. First suppose that $\operatorname{char} k = 0$, or even just that k is perfect. For any $n > 1$ not divisible by $\operatorname{char} k$, $A[n]$ is a nontrivial étale multiplicative algebraic group [39, Tag 03RP], hence Lemma 3.3 implies that G contains such an algebraic group, in violation of our assumption.

Now suppose that $\operatorname{char} k = p > 0$. By Lemma 3.5, G contains a copy of A. If A were nontrivial, it would follow that G contains a nontrivial multiplicative subgroup — for instance, $A[n]$ for any $n > 1$ not divisible by $\operatorname{char} k$. It follows that $A = 0$.

For (ii), let G be a k-algebraic group not containing a nontrivial multiplicative subgroup. By (i), G^0 is unipotent, and we need only show that the finite constant k-algebraic group $E := G/G^0$ is unipotent – that is, $E(k)$ is a p-group. If not, then $E(k)$ admits a nontrivial n-torsion element for some n not divisible by p. This element then defines a nontrivial multiplicative étale k-subgroup of E, so G contains such a group by Lemma 3.3. This violates our assumption about G, hence completes the proof. □

Remark 3.7. The assumption of connectedness in Theorem 3.6(i) is necessary beyond the algebraically closed case. Indeed, for k imperfect of characteristic p, for instance, [22, Cor. 15.33] implies that there are non-split extensions of $\boldsymbol{\mu}_\ell$ by $\boldsymbol{\alpha}_p$ for primes $\ell \neq p$. Such an extension is not unipotent, but it cannot contain a nontrivial multiplicative subgroup, as this would yield a splitting of the extension.

4. Reduction to a Statement About Torsion Line Bundles

Let k be a field and let G be a linear algebraic group over k. The purpose of this section is to show that if a G-torsor $E \to X$ is weakly versal for a class of affine k-schemes $\mathscr{C}$ and G contains a copy of $\boldsymbol{\mu}_\ell$ (resp. $\mathbf{G}_{\mathbf{m}}$), then there is a uniform upper bound on the number of generators of an ℓ-torsion (resp. any) line-bundle over a scheme in $\mathscr{C}$. This is stated formally in Theorems 4.4 and 4.5, which make use of the following invariant.

Definition 4.1. *Let V be a vector bundle of rank n over a scheme X. An open covering $\mathcal{U}$ of X is said to split V if $V_U \cong \mathcal{O}_U^n$ for all $U \in \mathcal{U}$. The smallest possible cardinality of a covering which splits V is called the* splitting number *of V and denoted*

$$\operatorname{spl}(V).$$

Observe that $\operatorname{spl}(V)$ is finite if X is quasi-compact. Furthermore, for every morphism $f : Y \to X$, we have $\operatorname{spl}(f^*V) \leq \operatorname{spl}(V)$.

Proposition 4.2. *Let X be noetherian scheme such that every finite collection of closed points of X is contained in an open affine subscheme of X, e.g., a quasi-projective scheme over a field. Let V be a rank-n vector bundle over X. Then* $\operatorname{spl}(V) \leq \dim X + 1$.

Proof. It is enough to show that there is a sequence $\{U_i\}_{i=1}^{\dim X+1}$ of open affine subsets of X such that $V_{U_r} \cong \mathcal{O}_{U_r}^n$ and $\dim(X - \bigcup_{i=1}^r U_i) \leq \dim X - r$ for all $r \in \{0, 1, \dots, \dim X + 1\}$. Indeed, if this holds, then $\{U_i\}_{i=1}^{\dim X+1}$ is a covering of X which splits V. We prove the existence of the sequence by induction. Let $r \in \{1, \dots, \dim X + 1\}$ and suppose that $U_1, \dots, U_{r-1}$ were constructed. Put $Y = X - \bigcup_{i=1}^{r-1} U_i$ and let $y_1, \dots, y_t$ be closed points of Y meeting every irreducible component of Y. These points are also closed in X, so there is an open affine $U \subseteq X$ with $y_1, \dots, y_t \in U$. Since U is affine, the localization of V_U at $\{y_1, \dots, y_t\}$ is free. We may therefore shrink U to assume that $V_U \cong \mathcal{O}_U^n$. Now take U_r to be U. We have $V_{U_r} \cong \mathcal{O}_{U_r}^n$ by construction, and since U_r meets every irreducible component of Y, we also have $\dim(X - \bigcup_{i=1}^r U_i) = \dim(Y - U_r) \leq \dim Y - 1 \leq \dim X - (r-1) - 1 = \dim X - r$. □

Let X be a scheme, let $\mathcal{M}$ be an $\mathcal{O}_X$-module and let $S \subseteq \Gamma(X, \mathcal{M})$. As usual, we say that S generates $\mathcal{M}$ if the $\mathcal{O}_X$-module map

$$(\alpha_s)_{s\in S} \mapsto \sum_{s\in S} \alpha_s s : \bigoplus_S \mathcal{O}_X \to \mathcal{M}$$

is surjective. The $\mathcal{O}_X$-module $\mathcal{M}$ is said to be *globally generated* if there is some $S \subseteq \Gamma(X, \mathcal{M})$ generating $\mathcal{M}$, and in this case we write

$$\operatorname{gen}(\mathcal{M})$$

for the minimal possible cardinality of such S. This number is always finite if X is affine and $\mathcal{M}$ is quasicoherent of finite type.

Proposition 4.3. *Let X be an affine scheme and let V be a rank-n vector bundle over X. Then* $\operatorname{gen}(V) \leq n \operatorname{spl}(V)$.

Proof. Write $r = \operatorname{spl}(V)$. Then X has an open covering $\{U_1, \dots, U_r\}$ splitting V. Write $X = \operatorname{Spec} A$. Then there are ideals $I_1, \dots, I_r \trianglelefteq A$ such that

$$U_i = \{\mathfrak{p} \in \operatorname{Spec} A \,:\, I_i \not\subseteq \mathfrak{p}\}$$

for every $i \in \{1, \dots, r\}$. Since $X = \bigcup_{i=1}^r U_i$, we have $\sum_{i=1}^r I_i = A$. Thus, we can choose $a_i \in I_i$ for every i such that $\sum_{i=1}^r a_i = 1_A$. Let W_i be the principal open affine corresponding to a_i, namely,

$$W_i = \{\mathfrak{p} \in \operatorname{Spec} A \,:\, a_i \notin \mathfrak{p}\}.$$

Then $X = \bigcup_{i=1}^r W_i$ and $W_i \subseteq U_i$ for all i.

Let $i \in \{1, \dots, r\}$. Since $V_{U_i} \cong \mathcal{O}_{U_i}^n$, we also have $V_{W_i} \cong \mathcal{O}_{W_i}^n$. Thus, there exist $f_1^{(i)}, \dots, f_n^{(i)} \in \Gamma(W_i, V)$ which generate V_{W_i}. By construction, $\Gamma(W_i, V)$ is the localization of the A-module $\Gamma(X, V)$ at the multiplicative set $\{1, a_i, a_i^2, \dots\}$, so there are $g_1^{(i)},\dots,g_n^{(i)} \in \Gamma(X,V)$ and $k\in\mathbb{N}$ such that $f_j^{(i)} = a_i^{-k}g_j^{(i)}$ for all $j\in\{1,\dots,n\}$.

Finally, put $S=\{g_j^{(i)}\,|\,i\in\{1,\dots,r\}, j\in\{1,\dots,n\}\}$. Then every $x\in X$ admits an open neighborhood where S generates V. By Nakayama's Lemma, this means that S generates V, and it follows that $\mathrm{gen}(V)\leq |S|\leq nr$. □

Theorem 4.4. *Let k be a field, let G be an affine k-algebraic group containing a copy of $\boldsymbol{\mu}_\ell$ for some prime number ℓ, and let $\rho : G\to \mathbf{GL}_n$ be a monomorphism of algebraic groups. Let $E\to X$ be a G-torsor, and let V denote the rank-n vector bundle corresponding to the $\mathbf{GL}_n$-torsor $E\times^G \mathbf{GL}_n$.*

(i) Suppose that Y is an affine k-scheme, $M\to Y$ is an $\boldsymbol{\mu}_\ell$-torsor, and L is the ℓ-torsion line-bundle on Y induced by M. If $E\to X$ specializes to $M\times^{\boldsymbol{\mu}_\ell} G\to Y$, then

$$\mathrm{gen}(L)\leq n\,\mathrm{spl}(V).$$

(ii) If $E\to X$ is weakly versal for a class of affine k-schemes $\mathscr{C}$, then for every $Y\in\mathscr{C}$ and every ℓ-torsion line bundle L on Y, we have $\mathrm{gen}(L)\leq n\,\mathrm{spl}(V)$.

Proof. (ii) is an immediate consequence of (i), because every ℓ-torsion line bundle is induced by a $\boldsymbol{\mu}_\ell$-torsor. We turn to prove (i).

By assumption, there is a k-group monomorphism $\gamma:\boldsymbol{\mu}_\ell\to G$. Write $\varphi=\rho\circ\gamma : \boldsymbol{\mu}_\ell\to\mathbf{GL}_n$. Since $\boldsymbol{\mu}_\ell$ is diagonalizable, we can post-compose ρ with $\mathrm{Int}(g)\circ\rho$ for some $g\in\mathrm{GL}_n(k)$ to assume that φ is given section-wise by

$$\varphi(x)=\begin{bmatrix} x^{a_1} & & \\ & \ddots & \\ & & x^{a_n}\end{bmatrix}$$

for some $a_1,\dots,a_n\in\{0,\dots,p-1\}$. Since φ is a monomorphism, there is some $m\in\{1,\dots,n\}$ with $a_m\neq 0$. Let $b\in\mathbb{Z}$ represent an inverse of a_m in the ring $\mathbb{Z}/p\mathbb{Z}$. By replacing γ with $\gamma\circ[x\mapsto x^b]$, we may assume that $a_m=1$.

Put $F=M\times^{\boldsymbol{\mu}_\ell} G$, $T=F\times^G\mathbf{GL}_n = M\times^{\boldsymbol{\mu}_\ell}\mathbf{GL}_n$ and let W be the rank-n vector bundle on Y which corresponds to T. Since $E\to X$ specializes to $F\to Y$, there is a k-morphism $f:Y\to X$ such that $f^*E\cong F$ as G-torsors. This means that $f^*(E\times^G\mathbf{GL}_n)\cong f^*E\times^G\mathbf{GL}_n\cong F\times^G\mathbf{GL}_n=T$, so $f^*V\cong W$. As a result, $\mathrm{spl}(W)\leq\mathrm{spl}(V)$. Now, since Y is affine, we may apply Proposition 4.3 to W and get

$$\mathrm{gen}(W)\leq n\,\mathrm{spl}(V).$$

To finish, observe that since $T=M\times^{\boldsymbol{\mu}_\ell}\mathbf{GL}_n$, we have

$$W\cong L^{\otimes a_1}\oplus\cdots\oplus L^{\otimes a_n};$$

this can be seen by inspecting the cocycles representing the classes in $\mathrm{H}^1_{\mathrm{fppf}}(Y,\mathbf{GL}_n)$ corresponding to both vector bundles. It follows that $L=L^{\otimes a_m}$ is an epimorphic image of W, and therefore $\mathrm{gen}(L)\leq\mathrm{gen}(W)\leq n\,\mathrm{spl}(V)$. □

When $\boldsymbol{\mu}_\ell$ is replaced with $\mathbf{G_m}$, we have a similar result which applies to all line bundles. However, in most cases, the upper bound on $\mathrm{gen}(L)$ is much larger than the one in Theorem 4.4.

Theorem 4.5. *Let k be a field, let G be an affine k-algebraic group containing a copy of $\mathbf{G_m}$ and let $\rho : G\to\mathbf{GL}_n$ be a monomorphism of algebraic groups. Denote by $\{x\mapsto x^{a_i}\}_{i=1}^n$ the characters of $\mathbf{G_m}$ occurring in the representation $\mathbf{G_m}\to G\to\mathbf{GL}_n$, and let r denote* $\min|S|$ *as S ranges over all nonempty subsets of $\{a_1,\dots,a_n\}$ with* $\gcd(S)=1$*. Let $E\to X$ be a G-torsor and let V denote the rank-n vector bundle corresponding to the $\mathbf{GL}_n$-torsor $E\times^G\mathbf{GL}_n$.*

(i) *Suppose that Y is an affine k-scheme, $M \to Y$ is an $\mathbf{G_m}$-torsor and L is the line-bundle on Y corresponding to M. If $E \to X$ specializes to $M \times^{\mathbf{G_m}} G \to Y$, then*

$$\mathrm{gen}(L) \le (n\,\mathrm{spl}(V))^r.$$

(ii) *If $E \to X$ is weakly versal for a class of affine k-schemes $\mathscr{C}$, then for every $Y \in \mathscr{C}$ and every line bundle L on Y, we have* $\mathrm{gen}(L) \le (n\,\mathrm{spl}(V))^r$.

Proof. Again, it is enough to prove (i). Let φ denote the composition $\mathbf{G_m} \to G \xrightarrow{\rho} \mathbf{GL}_n$. As in the proof of Theorem 4.4, we may replace ρ by $\mathrm{Int}(g) \circ \rho$ for some $g \in \mathrm{GL}_n(k)$ to assume that φ is given by

$$\varphi(x) = \mathrm{diag}(x^{a_1}, \dots, x^{a_n}).$$

By proceeding as the proof of Theorem 4.4(i), we find that $\mathrm{gen}(W) \le n\,\mathrm{spl}(V)$ for $W = L^{\otimes a_1} \oplus \cdots \oplus L^{\otimes a_n}$. (Here, $L^{\otimes a}$ means $(L^\vee)^{\otimes(-a)}$ if $a < 0$.) This means that $\mathrm{gen}(L^{\otimes a_i}) \le n\,\mathrm{spl}(V)$ for all i. Since φ is a monomorphism, $\gcd(a_1, \dots, a_n) = 1$. Choose a subset $S \subseteq \{a_1, \dots, a_n\}$ of cardinality r such that $\gcd(S) = 1$, say $S = \{a_{i_1}, \dots, a_{i_r}\}$. Then there are $b_1, \dots, b_r \in \mathbb{Z}$ such that $\sum_{j=1}^r a_{i_j} b_j = 1$. Then $L \cong \bigotimes_{j=1}^r (L^{\otimes a_{i_j}})^{\otimes b_j}$. By a lemma of Swan [6, Lem. 3.4], $\mathrm{gen}((L^{\otimes a_{i_j}})^{\otimes b_j}) \le \mathrm{gen}(L^{\otimes a_{i_j}}) \le n\,\mathrm{spl}(V)$, so $\mathrm{gen}(L) \le \prod_{i=1}^r \mathrm{gen}((L^{\otimes a_{i_j}})^{\otimes b_j}) \le (n\,\mathrm{spl}(V))^r$. □

5. Torsion Line Bundles Requiring Many Sections to Generate

Let k be a field. The purpose of this section is to show that for every $m \in \mathbb{N}$ and $1 < \ell \in \mathbb{N}$, there is an ℓ-torsion line bundle over a smooth affine k-scheme X that requires at least m global sections to generate. This will be coupled with Theorem 4.4 in the next section in order to prove Theorem 1.1.

For any $m \in \mathbb{N}$, the existence of affine schemes of finite type over $\mathbb{R}$ admitting 2-torsion line bundles which cannot be generated by m elements goes back at least as far as Swan [41, Thm. 4], who attributes the example to Chase. Similar 2-torsion examples over affine varieties over any field of characteristic not 2 were given by Shukla and Williams [37, Props. 7.13, 6.3].

Since the dimension of the base scheme X would affect the quantitative version of Theorem 1.1, i.e. Theorem 1.2, we would like $\dim X$ to be as small as possible. To that end, given $\ell, m \in \mathbb{N} \cup \{0\}$ with $\ell \ne 1$, it is convenient to define

$$g_k(\ell, m)$$

to be the smallest $d \in \mathbb{N} \cup \{0\}$ for which there exists a d-dimensional smooth affine k-scheme X carrying an ℓ-torsion line bundle L satisfying $\mathrm{gen}(L) > m$. Note that $g_k(0, m)$ is the minimal dimension of a smooth affine k-scheme carrying a (possibly non-torsion) line bundle which cannot be generated by m elements. By the end of this section, we will show:

Theorem 5.1. *Let k be a field and let ℓ, m be non-negative integers with $\ell \ne 1$.*

(i) $g_k(\ell, 0) = 0$ *and* $g_k(\ell, 1) = 1$.
(ii) $g_k(\ell, m) \in \{m, m+1, m+2\}$.
(iii) *If ℓ is divisible by a prime number greater than m (e.g. $\ell = 0$), then* $g_k(\ell, m) \in \{m, m+1\}$.
(iv) *If k is algebraically closed, $m > 1$ and $\ell > 0$, then $g_k(\ell, m) \ge m + 1$. Equality holds if ℓ is moreover divisible by a prime number greater than m.*
(v) *If $\mathrm{char}\, k = 0$ and $\mathrm{trdeg}_{\mathbb{Q}} k$ is infinite, then $g_k(0, m) = m$.*
(vi) $g_{\overline{\mathbb{F}}_p}(0, m) = m + 1$ *for every prime $p > 0$ and $m \ge 2$.*

We expect that part (v) also holds in positive characteristic, see Remark 5.16. The characteristic-0 counterpart of (vi), i.e., the statement $g_{\overline{\mathbb{Q}}}(0,m) = m+1$, is related to the Bloch–Beilinson conjecture, and in particular follows from it (Remark 5.18). We do not know if there are k, ℓ, m with $g_k(\ell,m) = m+2$; this is related to conjectures of Paulsen and Griffiths–Harris (Remark 5.14).

In general, for $\ell > 1$, the precise value of $g_k(\ell,m)$ is sensitive to arithmetic properties of the field k. This demonstrated in Example 5.9 and Corollary 5.11.

We begin by recalling a result of Förster [16] (see [12] for a more modern treatment): Every rank-n projective module over a noetherian ring of Krull dimension m is generated by $m+n$ elements. This implies:

Lemma 5.2. *For every field k and $\ell, m \in \mathbb{N}\cup\{0\}$ with $\ell \neq 1$, we have $g_k(\ell,m) \geq m$.*

In what follows, given a smooth (irreducible) k-variety X, we denote by $\mathrm{CH}^i X$ the i-th Chow group of X; see [17, §8.3]. The Chow ring of X is $\mathrm{CH}^* X = \bigoplus_{i\geq 0} \mathrm{CH}^i X$. Recall that a flat morphism $f : X \to Y$ between smooth k-varieties induces a ring homomorphism $f^* : \mathrm{CH}^* Y \to \mathrm{CH}^* X$. If E is a vector bundle over X, then the i-th Chern class of E is denoted $c_i(E) \in \mathrm{CH}^i X$. The total Chern class of E is $c(E) = \sum_{i=0}^{\operatorname{rank} E} c_i(E) \in \mathrm{CH}^* X$; see [17, §3.2] for details.

Proposition 5.3 (Murthy [26, Cor. 3.16])**.** *Suppose that k is algebraically closed, let X be a smooth affine k-variety of dimension m and let L be a line bundle over X. Then $\mathrm{gen}(L) = m$ if and only if $c_1(L)^m = 0$.*

It turns out that the "only if" part of Murthy's result holds for any smooth scheme over any field; this will be our main tool for bounding $g_k(\ell,m)$ from above.

Lemma 5.4. *Let X be a smooth k-variety, let L be a line bundle over X and let $m \in \mathbb{N}$. Suppose that $c_1(L)^m \neq 0$ in $\mathrm{CH}^m X$. Then $\mathrm{gen}(L) > m$.*

Proof. For the sake of contradiction, suppose that L can be generated by m global sections. Then there is an exact sequence of vector bundles on X,

$$0 \to E \to \mathcal{O}_X^{\oplus m} \to L \to 0.$$

By tensoring with $L^\vee$, the dual of L, we get an exact sequence

$$0 \to E' \to (L^\vee)^{\oplus m} \to \mathcal{O}_X \to 0,$$

where $E' = E \otimes L^\vee$. Thus,

$$c(L^\vee)^m = c((L^\vee)^{\oplus m}) = c(E')c(\mathcal{O}_X) = c(E').$$

Since $\operatorname{rank} E' = m-1$, it follows that

$$(-1)^m c_1(L)^m = c_1(L^\vee)^m = c_m(E') = 0$$

in $\mathrm{CH}^m X$. But this contradicts our assumptions. □

In light of Proposition 5.3 and Lemma 5.4, it is convenient to define

$$G_k(\ell,m)$$

to be the minimum possible dimension of a smooth affine k-scheme X admitting an ℓ-torsion line bundle L satisfying $c_1(L)^m \neq 0$. It is clear that $G_k(\ell,m) \geq m$, and Proposition 5.3 and Lemma 5.4 imply:

Corollary 5.5. *For every field k and integers $\ell, m \geq 0$ with $\ell \neq 1$, we have*

$$g_k(\ell,m) \leq G_k(\ell,m).$$

Moreover, when k is algebraically closed, $g_k(\ell,m) = m$ if and only if $G_k(\ell,m) = m$.

We can use the second assertion of Corollary 5.5 and another result of Murthy to get a better lower bound on $g_k(\ell, m)$.

Proposition 5.6. *If k is algebraically closed, $\ell > 0$ and $m \geq 2$, then $G_k(\ell, m) \geq m+1$. Consequently, $g_k(\ell, m) \geq m+1$.*

Proof. Let L be an ℓ-torsion line bundle over a smooth affine m-dimensional k-scheme X. Murthy [26, Thms. 2.11, 2.14] showed that $\mathrm{CH}^m X$ is torsion-free. Since $c_1(L)$ is ℓ-torsion, we must have $c_1(L)^m = 0$. □

In the next results, we establish upper bounds on $G_k(\ell, m)$ by exhibiting smooth affine k-varieties X carrying an ℓ-torsion line bundle L such that $c_1(L)^m \neq 0$. We will use the following proposition to produce such examples. Recall that the index of a k-scheme X, denoted $\operatorname{ind} X$, is the greatest common divisor of $\{\dim_k K \mid K \text{ is a finite } k\text{-field with } X(K) \neq \emptyset\}$.

Proposition 5.7. *Let $m, \ell \in \mathbb{N}$ and let Z be an integral hypersurface in $\mathbb{P}^m$ having degree d and index n. Let $X = \mathbb{P}^m - Z$, let $L = \mathcal{O}_{\mathbb{P}^m}(1)|_X$, let X' be the complement of the zero section in the total space of $\mathcal{O}_{\mathbb{P}^m}(\ell)|_X$, let $p : X' \to X$ be its structure morphism and let $L' = p^*L$. Then:*

(i) X is affine, $L^{\otimes d} \cong \mathcal{O}_X$ and the order of $c_1(L)^m$ in the (additive) group $\mathrm{CH}^m X$ is n. Consequently, $c_1(L)^m \neq 0$ if $n > 1$.

(ii) X' is affine, $L'^{\otimes \ell} \cong \mathcal{O}_{X'}$ and the order of $c_1(L')^m$ in $\mathrm{CH}^m X'$ is $\gcd(n, \ell)$. Consequently, $c_1(L')^m \neq 0$ if $\gcd(n, \ell) > 1$.

Proof. (i) The scheme X is affine because it is the complement of an ample divisor in $\mathbb{P}^m$.

Recall [17, §8.4] that $\mathrm{CH}^* \mathbb{P}^m = \mathbb{Z}[h \mid h^{m+1} = 0]$, where h is the class of a hyperplane in $\mathbb{P}^m$ and lives in degree 1. Write $A_i(Z)$ for the group of i-cycles on Z modulo rational equivalence (we have $A_i(Z) = \mathrm{CH}^{m-1-i} Z$ when Z is smooth). By [17, Prop. 1.8], there is an exact sequence

$$A_i(Z) \to \mathrm{CH}^{m-i} \mathbb{P}^m \to \mathrm{CH}^{m-i} X \to 0 \tag{1}$$

in which the first map is pushforward along $Z \to \mathbb{P}^m$ and the second map is pullback along $X \to \mathbb{P}^m$. Let A'_i denote the image of $A_i(Z)$ in $\mathrm{CH}^{m-i} \mathbb{P}^m = \mathbb{Z}h^{m-i}$. We may and shall identify $\mathrm{CH}^{m-i} X$ with $\mathbb{Z}h^{m-i}/A'_i$ using (1). For $i = 0$, this gives

$$\mathrm{CH}^m X = \mathbb{Z}h^m/A'_0 = \mathbb{Z}h^m/\mathbb{Z}(\operatorname{ind} Z)h^m = \mathbb{Z}h^m/\mathbb{Z}nh^m,$$

while for $i = m-1$, this gives

$$\operatorname{Pic} X \cong \mathrm{CH}^1 X = \mathbb{Z}h/A'_{m-1} = \mathbb{Z}h/\mathbb{Z}[Z] = \mathbb{Z}h/\mathbb{Z}dh.$$

Thus, L is d-torsion. Since $c_1(\mathcal{O}_{\mathbb{P}^m}(1)) = h$, we have

$$c_1(L)^m = h^m|_X = h^m + \mathbb{Z}nh,$$

which means that $c_1(L)^m$ has order n in $\mathrm{CH}^m X$.

(ii) The scheme X' is affine because the morphism $X' \to X$ is affine and X is affine. We continue to use the notation from the proof of (i).

By [17, Ex. 2.6.3], for every i, there is an exact sequence

$$\mathrm{CH}^{i-1} X \xrightarrow{c_1(\mathcal{O}(\ell)|_X)} \mathrm{CH}^i X \xrightarrow{p^*} \mathrm{CH}^i X' \to 0.$$

Since $c_1(\mathcal{O}(\ell)|_X) = \ell h + \mathbb{Z}dh$, this allows us to identify $\mathrm{CH}^i X'$ with

$$\mathrm{CH}^i X/(\ell h)\, \mathrm{CH}^{i-1} X = \mathbb{Z}h^i/(A'_{m-i} + \mathbb{Z}\ell h^i).$$

and p^* with the quotient map. In particular, $\operatorname{Pic} X' = \mathrm{CH}^1 X'$ is ℓ-torsion, hence L is ℓ-torsion. Furthermore,

$$\mathrm{CH}^m X' = \mathbb{Z}h^m/\langle nh^m, \ell h^m \rangle = \mathbb{Z}h^m/\mathbb{Z}\gcd(n, \ell)h^m.$$

Now, by (i),

$$c_1(L')^m = p^*(c_1(L)^m) = p^*(h^m + \mathbb{Z}nh^m) = h^m + \mathbb{Z}\gcd(n,\ell)h^m$$

and it follows that $c_1(L')^m$ has order $\gcd(n,\ell)$ in $\mathrm{CH}^m X'$. □

Remark 5.8. In an earlier version of this text, we gave a different construction of ℓ-torsion ($\ell > 1$) line bundles L with $c_1(L)^m \neq 0$. However, we found out that complements of high-index hypersurfaces in projective spaces give rise to examples of smaller dimension.

In our original construction, we took X to be the total space of the line bundle $\mathcal{O}(-\ell)$ over $\mathbb{P}^m$ minus the zero section, and L to be the pullback of $\mathcal{O}(-1)$ along $X \to \mathbb{P}^m$. We then replaced X and L with their pullback along the standard *Jouanolou device* of $\mathbb{P}^m$ [19] to make X affine. The resulting ℓ-torsion line is induced by the $\boldsymbol{\mu}_\ell$-torsor $\mathbf{GL}_{m+1}/\left[\begin{smallmatrix}\mathbf{GL}_m & \\ & 1\end{smallmatrix}\right] \to \mathbf{GL}_{m+1}/\left[\begin{smallmatrix}\mathbf{GL}_m & \\ & \boldsymbol{\mu}_\ell\end{smallmatrix}\right]$. We omit the details as they will not be needed here; the proof that L is ℓ-torsion and $c_1(L)^m \neq 0$ is similar to the proof of Proposition 5.7(ii). (When passing to the Jouanolou device, apply [39, Tag 0GUB].)

Example 5.9. Recall that the u-invariant of k, denoted $u(k)$, is the supremum of all the integers n for which k admits an n-dimensional nonsingular anisotropic quadratic form. We claim that

$$g_k(2,m) = G_k(2,m) = m$$

whenever $1 \leq m < u(k)$. In particular, if $u(k) = \infty$, e.g., if k is real, then $g_k(2,m) = G_k(2,m) = m$ for all m. To see this, let $q : k^{m+1} \to k$ be an anisotropic nonsingular quadratic form, let Z be the quadric hypersurface $q = 0$ in $\mathbb{P}^m$ and let $X = \mathbb{P}^m - Z$. Since q is nonsingular and anisotropic, Z is integral, and by Springer's Theorem [31, II, Thm. 5.3], $\operatorname{ind} Z = 2$. Proposition 5.7(i) now tells us that the line bundle $\mathcal{O}_{\mathbb{P}^m}(1)|_X$ is 2-torsion and satisfies $c_1(L)^m \neq 0$. As a result, $G_k(2,m) \leq m$, and it follows that $g_k(2,m) = G_k(2,m) = m$.

In the special case $k = \mathbb{R}$ and $q(x) = \sum_{i=1}^{m+1} x_i^2$, the scheme X is isomorphic to quotient of the m-sphere by the antipodal map, and we recover the example in Swan [41, Thm. 4].

In order to apply Proposition 5.7, we need to find integral hypersurfaces in $\mathbb{P}^m$ having index greater than 1. Norm forms of finite field extensions give rise to such examples.

Lemma 5.10. *Let K/k be a finite separable field extension of degree n and let $a \in K$ be an element such that $K = k[a]$. For every $m \in \{1, \dots, n-1\}$ set $V_m = \operatorname{span}\{1, a, \dots, a^m\}$, and let Z_m be the hypersurface in $\mathbb{P}(V_m) \cong \mathbb{P}^m$ cut by the equation $\operatorname{Nr}_{K/k}(x_0 + x_1 a + \dots + x_m a^m) = 0$. Then Z_m is integral. Moreover, if $[K : k]$ is a power of a prime number ℓ, then $\ell \mid \operatorname{ind} Z_m$.*

Proof. The fact that Z_m is integral follows from [14, Thm. 1] and its proof.

Suppose now that $[K : k] = \ell^r$ for a prime number ℓ and $r \in \mathbb{N}$. We need to show that $\ell \mid \operatorname{ind} Z_m$. Let L be a finite k-field with $Z_m(L) \neq \emptyset$. Then the norm $N_{L \otimes_k K/L} : L \otimes_k K \to L$ has a non-trivial zero. This means that $L \otimes_k K$ is not a field. This can happen on if $\ell^r = [K : k]$ and $[L : k]$ are not coprime, so ℓ must divide $[L : k]$. □

Corollary 5.11. *Let ℓ be a prime number.*

(i) If k admits a separable field extension of degree ℓ, then $G_k(\ell, m) = m$ for all $m \in \{1, \dots, \ell - 1\}$.

(ii) If k admits a separable field extension of degree ℓ^r for some $r \in \mathbb{N}$, then $G_k(\ell, m) \leq m + 1$ for all $m \in \{1, \dots, \ell^r - 1\}$.

Proof. (i) Let K/k be a finite separable field extension of degree ℓ. Choose $a \in K$ with $K = k[a]$, which exists by the Primitive Element Theorem, and define Z_m as in Lemma 5.10. Then Z_m is an integral hypersurface in $\mathbb{P}^m$ having degree and index ℓ. Proposition 5.7(i) now provides us with a line bundle bundle L over $\mathbb{P}^m - Z_m$ such that $c_1(L)^m \neq 0$, so $G_k(\ell, m) \leq m$.

(ii) This similar to (i), but we use Proposition 5.7(ii) instead of part (i) of that proposition. □

Corollary 5.11 cannot be applied to algebraically closed fields. This is remedied by the following proposition at the cost of increasing the upper bound on $G_k(\ell, m)$ by 1.

Proposition 5.12. *Let k be a field and $\ell, m \in \mathbb{N} \cup \{0\}$ with $\ell \neq 1$. Then $G_k(\ell, m) \leq G_{k(x)}(\ell, m) + 1$.*

Proof. This is shown by *spreading out* from $\operatorname{Spec} k(x)$ to $\operatorname{Spec} k[x]$. What we need is summarized neatly in [28, Thms. 3.2.1, 6.4.3].

In detail, let X be a smooth affine $k(x)$-scheme of dimension $G_{k(x)}(\ell, m)$ carrying an ℓ-torsion line bundle L with $c_1(L)^m \neq 0$. Put $S = \operatorname{Spec} k[x]$ and let $\xi = \operatorname{Spec} k(x)$ be the generic point of S. By [28, Thm. 3.2.1], there is a dense open subscheme U of S and a U-scheme of finite presentation Y such that $Y_\xi = X$. Moreover, by shrinking U if necessary, we may further assume that $Y \to U$ is smooth and U is affine. Thus, Y is a smooth affine k-scheme. By [28, Thm. 6.4.3] (applied to $G = \mathbf{G_m}$), we can shrink U even further to guarantee that there exists an ℓ-torsion line bundle M on Y such that the pullback of M along $\xi \to S$ is L. The map $X = Y_\xi \to Y$ induces a ring homomorphism $\mathrm{CH}^* Y \to \mathrm{CH}^* X$ which is compatible with Chern classes [10, §52F]. Thus, the image of $c_1(M)^m$ in $\mathrm{CH}^* X$ is $c_1(L)^m$, which is nonzero. We conclude that $G_k(\ell, m) \leq \dim Y = \dim X + 1 = G_{k(x)}(\ell, m) + 1$. □

Corollary 5.13. *Let k be a field and let $m, \ell \in \mathbb{N} \cup \{0\}$ with $\ell \neq 1$. Then $G_k(\ell, m) \leq m + 2$. When ℓ has a prime divisor greater than m (this holds when $\ell = 0$), we moreover have $G_k(\ell, m) \leq m + 1$.*

Proof. Let $K = k(t)$. By Proposition 5.12, it is enough to show that $G_K(\ell, m) = m$ when ℓ has a prime divisor $p > m$, and $G_k(\ell, m) \leq m + 1$ in general. Observe that K has a separable field extension of degree n for every $n \in \mathbb{N}$, e.g., $K[x \mid x^n - tx - t = 0]$. Letting n range on the powers of some prime divisor p of ℓ and applying Corollary 5.11(ii) gives $G_K(\ell, m) \leq G_K(p, m) \leq m + 1$. If ℓ has a prime divisor p greater than m, then applying Corollary 5.11(i) to that divisor gives $G_K(\ell, m) \leq G_K(p, m) = m$. □

Remark 5.14. Suppose that $k = \mathbb{C}$. In [27], Paulsen conjectured the following:

(P) Let $m, d \in \mathbb{N}$ with $d \geq 2m$ and let Y be a very general hypersurface in $\mathbb{P}^{m+1}$. Then the degree of every positive-dimensional closed subvariety Z of Y is divisible by d.

The case $m = 3$ is a famous conjecture of Griffiths–Harris. We shall consider conjecture (P) for a general field k.

Fix k and m. If Paulsen's conjecture holds for every d sufficiently large, then $G_k(\ell, m) \leq m + 1$ for every $\ell \in \mathbb{N} - \{1\}$. Indeed, choose $r \in \mathbb{N}$ coprime to ℓ such that (P) holds with $d = \ell r$, let Y be as in the conjecture and let $X = \mathbb{P}^{m+1} - Y$. With notation as in the proof of Proposition 5.7, the conjecture says that for every $0 \leq i < m$, the image of the pushforward map $\mathrm{CH}^i Y \to \mathrm{CH}^{i+1} \mathbb{P}^{m+1} = \mathbb{Z} h^{i+1}$ is $\mathbb{Z}^{i+1} d h^{i+1}$. The cokernel of this map is $\mathrm{CH}^{i+1} X$, so it can be identified with $\mathbb{Z} h^{i+1} / \mathbb{Z} d h^{i+1}$ whenever $0 \leq i < m$. Now put $L = \mathcal{O}_{\mathbb{P}^{m+1}}(r)|_X$. Then $c_1(L) = rh + dh\mathbb{Z}$ while $c_1(L)^m = r^m h^m + \mathbb{Z} d h^m$. Since $\ell r = d$ and $\gcd(\ell, r) = 1$, this means that L is ℓ-torsion and $c_1(L)^m \neq 0$. As X is affine of dimension $m+1$, we conclude that $G_k(\ell, m) \leq m+1$.

When $k = \mathbb{C}$, Paulsen [27, Thm. 3, Prop. 7] showed that for every m there is a subset $S \subseteq \mathbb{N}$ of positive density such that (P) holds for every $d \in S$. The set S consists of numbers that are coprime to $m!$, and together with the argument of the last paragraph, it can be used to show that $G_{\mathbb{C}}(\ell, m) \leq m+1$ if all the prime factors of ℓ are greater than m. However, this is already included in Corollary 5.13.

The next results give bounds on $G_k(0,m)$. They rely on deeper results of Bloch, Mohan Kumar, Murthy, Szpiro, Roitman and Roy appearing in [25], [6], [26].

Proposition 5.15 ([6, §5])**.** *For every $m \in \mathbb{N}$, there is $t \in \mathbb{N}$ such that for every field k with $\operatorname{char} k = 0$ and $\operatorname{trdeg}_{\mathbb{Q}} k \geq t$, we have $G_k(0,m) = m$.*

Proof. Suppose first that k is algebraically closed and uncountable. Let $C_1, \dots, C_m$ be elliptic curves over k and let X be a dense open affine of $C_1 \times \cdots \times C_m$. By the first paragraph of the proof of [6, Thm. 5.8], there is a line bundle L on X such that $c_1(L)^m \neq 0$, so $G_k(0,m) = m$. (We remark that [6, Thm. 5.8] is stated for the case $k = \mathbb{C}$, but the proof works for any uncountable algebraically closed field of characteristic 0.)

Suppose now that k is uncountable. Let X' and L' be the X and L constructed in the last paragraph for the field $\overline{k}$. As in the proof of Proposition 5.12, there is a finite extension k_1 of k such that X' and L' descend to k_1, i.e., there is a smooth affine k_1-scheme X and a line bundle L over X such that $X \times_{k_1} \overline{k} = X'$ and L' is the pullback of L along $X' \to X$. The image of $c_1(L)^m$ under $\mathrm{CH}^m X \to \mathrm{CH}^m X'$ is $c_1(L')^m \neq 0$, so we must have $c_1(L)^m \neq 0$. As X is a smooth affine k-scheme of dimension m, it follows that $G_k(0,m) = m$.

Finally, suppose that k is countable. Let X' and L' be the X and L constructed for the field $\mathbb{C}$. Again, there is a finitely generated $\mathbb{Q}$-field k_0, such that X' descends to a smooth affine k_0-scheme X_0 and L' descends to a line bundle L_0 over X_0. We take $t := \operatorname{trdeg}_{\mathbb{Q}} k_0$. If $\operatorname{trdeg}_{\mathbb{Q}} k \geq t = \operatorname{trdeg}_{\mathbb{Q}} k_0$, then there is a finite extension k_1 of k that is isomorphic to a field lying between k_0 and $\mathbb{C}$. Identifying k_1 with that field, put $X = X_0 \times_{k_0} k_1$ (viewed as a k-scheme) and let L to be the pullback of L_0 along $X \to X_0$. As before, $c_1(L)^m \neq 0$ (because its image in $\mathrm{CH}^m X'$ is $c_1(L')^m \neq 0$) and $G_k(0,m) = m$. □

Remark 5.16. The example in [6, Thm. 5.8] relies on a criterion of Roitman [6, Lem. 5.2] for showing that the group of 0-cycles on a complete k-variety Y is *infinite dimensional* in the sense that there is no closed subvariety $Z \subseteq Y$ such that $A_0(Z) \to A_0(Y)$ is surjective. This criterion, originally proved for $k = \mathbb{C}$, requires k that is uncountable algebraically closed of characteristic 0.

V. Srinivas pointed out to us that Bloch gave an analogue of Roitman's result for uncountable fields of positive characteristic using ℓ-adic cohomology [5, Thm. 1A.6, Ex. 1A.7] (see also [4, Prop. 1]). Moreover, it seems likely that with some work, this result could be applied to the example in [6, Thm. 5.8]. We therefore expect that Proposition 5.15 also holds in positive characteristic.

Proposition 5.17. *Let $m \in \mathbb{N} - \{1\}$ and let p be a prime number. Then $G_{\overline{\mathbb{F}}_p}(0,m) = m+1$.*

Proof. Since $G_{\overline{\mathbb{F}}_p}(0,m) \leq m+1$ (Corollary 5.13), it is enough to prove that if $X = \operatorname{Spec} A$ is a smooth affine $\overline{\mathbb{F}}_p$-scheme of dimension m, then $\mathrm{CH}^m X = 0$. By [25, Thm. 3.6], every closed point of X is a complete intersection, and by [26, Cor. 3.4, Thm. 2.14ii)], this means that $\mathrm{CH}^m X \cong F^m K_0(A) = 0$. □

Remark 5.18. The characteristic-0 counterpart of Proposition 5.17, i.e., whether $G_{\overline{\mathbb{Q}}}(0,m) = m+1$ for all $m \geq 2$, is related to the Bloch–Beilinson conjecture. Indeed, in some formulations, e.g. [20, p. 157] or [38, Cnj. 7.4, Cnj. 7.7], the conjecture may be stated as follows:

(BB) For every smooth affine $\overline{\mathbb{Q}}$-variety X of dimension $m \geq 2$, we have $\mathrm{CH}^m X = 0$.

If this holds, then we must have $G_{\overline{\mathbb{Q}}}(0,m) \geq m+1$, and then equality holds by Corollary 5.13.

We conclude our discussion of $G_k(\ell,m)$ with the following theorem, which implies Theorem 5.1.

Theorem 5.19. *Let k be a field and let ℓ, m be non-negative integers with $\ell \neq 1$.*

(i) $G_k(\ell,0) = 0$ *and* $G_k(\ell,1) = 1$.
(ii) $G_k(\ell,m) \in \{m, m+1, m+2\}$.
(iii) If ℓ is divisible by a prime number greater than m (e.g. $\ell = 0$), then $G_k(\ell,m) \in \{m, m+1\}$.
(iv) If k is algebraically closed, $m > 1$ and $\ell > 0$, then $G_k(\ell,m) \geq m+1$. Equality holds if ℓ is moreover divisible by a prime number greater than m.
(v) If $\operatorname{char} k = 0$ *and* $\operatorname{trdeg}_{\mathbb{Q}} k$ *is infinite, then* $G_k(0,m) = m$.
(vi) $G_{\overline{\mathbb{F}}_p}(0,m) = m+1$ *for every prime $p > 0$ and $m \geq 2$.*

Proof. (ii), (iii), (iv), (v), (vi) follow from Corollary 5.13 and Propositions 5.6, 5.15, 5.17. As for (i), that $G_k(\ell,0) = 0$ is clear — take L to be the trivial line bundle on $\operatorname{Spec} k$. To show that $G_k(\ell,1) = 1$, note first that there is a finite separable k-field k_1 and an ordinary elliptic curve C_1 over k_1. (Indeed, all elliptic curves are ordinary in characteristic 0, and if $p = \operatorname{char} k > 0$, then any elliptic curve with j-invariant not lying in $\mathbb{F}_{p^2}$ is ordinary.) Since C_1 is ordinary, there is a finite separable k_1-field k_2 and $P \in C_1(k_2)$ of order ℓ [39, Tag 03RP]. Let $C_2 = C_1 \times_{k_1} k_2$ and let O be the trivial point of C_2. Then $X := C_2 - \{O\}$ is affine and smooth over k, and $[P] \in \mathrm{CH}^1 X \cong \operatorname{Pic} X$ is the 1-st Chern class of a nontrivial ℓ-torsion line bundle L over X. □

Proof of Theorem 5.1. Thanks Corollary 5.5 and Lemma 5.2, this follows from Theorem 5.19. □

6. Main Results

We finally prove Theorems 1.1 and 1.2. We will use the invariant $g_k(\ell,m)$ defined in Section 5. Recall from Theorem 5.1 that $g_k(\ell,m) \in \{m, m+1, m+2\}$ and $g_k(0,m) \in \{m, m+1\}$.

We derive Theorems 1.1 and 1.2 from the following theorem.

Theorem 6.1. *Let k be a field, let G be a closed subgroup of $\mathbf{GL}_n$ ($n \in \mathbb{N}$), let $E \to X$ be a G-torsor with X a quasi-compact k-scheme and let V be the vector bundle over X corresponding to the $\mathbf{GL}_n$-torsor $E \times^G \mathbf{GL}_n \to X$. Let m be a non-negative integer such that one of the following hold:*

(1) $m \geq n \operatorname{spl} V$, *or*
(2) $m \geq n(\dim X + 1)$ *and X is quasi-projective over a field.*

Suppose further that there is a k-field K and $\ell \in \mathbb{N} - \{1\}$ such that $\boldsymbol{\mu}_{\ell,K}$ embeds in G_K. Then there is a smooth affine K-scheme X' of dimension $g_K(\ell,m)$ carrying a G-torsor $E' \to X'$ that is not a specialization of $E \to X$.

Proof. Assumption (2) implies (1) by Proposition 4.2. By the definition of $g := g_K(\ell,m)$, there is a g-dimensional smooth affine K-scheme X' carrying an ℓ-torsion line bundle L' such that $\mathrm{gen}(L') > m$. Let $T' \to X'$ be a $\boldsymbol{\mu}_{\ell,K}$-torsor giving rise to L' and let $E' = T' \times^{\boldsymbol{\mu}_{\ell,K}} G_K$. Note that we may view $E' \to X'$ both as a G_K-torsor and a G-torsor. If it were the case that $E \to X$ specializes to $E' \to X'$ viewed as a G-torsor, then $E_K \to X_K$ would specialize to $E' \to X'$ viewed as a G_K-torsor. Theorem 4.4(i) would then tell us that $\mathrm{gen}(L') \le n \operatorname{spl} V_K \le n \operatorname{spl} V$. But that is absurd because L' was chosen so that $\mathrm{gen}(L') > m \ge n \operatorname{spl} V$. We therefore conclude that $E \to X$ does not specialize to $E' \to X'$. $\square$

We also have the following variant of Theorem 6.1 which gives a slightly better upper bound on $\dim X'$ when G_K contains a copy of $\mathbf{G}_{\mathbf{m},K}$.

Theorem 6.2. *Let k, n, G, $E \to X$, V and m be as in Theorem 6.1. Suppose further that there is a k-field K such that $\mathbf{G}_{\mathbf{m},K}$ embeds in G_K and the characters of $\mathbf{G}_{\mathbf{m},K}$ occurring in the representation $\mathbf{G}_{\mathbf{m},K} \to G_K \to \mathbf{GL}_{n,K}$ include the identity character or $x \mapsto x^{-1}$. Then there is a smooth affine K-scheme X' of dimension $g_K(0,m)$ carrying a G-torsor $E' \to X'$ that is not a specialization of $E \to X$.*

Proof. This is similar to the proof of Theorem 6.1, except $T' \to X'$ is taken to be a $\mathbf{G}_{\mathbf{m},K}$-torsor and one uses Theorem 4.5 instead of Theorem 4.4. $\square$

We can now prove Theorems 1.1 and 1.2 as well as a few improvements.

Proof of Theorem 1.1. The implication (b) $\Longrightarrow$ (a) is clear and (c) $\Longrightarrow$ (b) is a special case of [11, Thm. 10.1]. It remains to show that (a) $\Longrightarrow$ (c), that is, we need to show that if X is a quasi-compact k-scheme and $E \to X$ is a G-torsor that is weakly versal for all finite type regular affine k-schemes, then G is unipotent.

For the sake of contradiction, suppose that G is not unipotent. By Theorem 3.6, $G_{\overline{k}}$ contains a copy of $\boldsymbol{\mu}_{\ell,\overline{k}}$ for some $\ell > 1$. This means that there is a finite-dimensional k-field K such that G_K contains a copy of $\boldsymbol{\mu}_{\ell,K}$. Theorem 6.1 now gives us a G-torsor $E' \to X'$, where X' is a smooth affine K-scheme, which is not a specialization of $E \to X$. Since X' is regular affine and also of finite type over k, this contradicts our assumption on $E \to X$. We conclude that G must be unipotent. $\square$

Theorem 6.3. *Let k be a field and G be an affine k-algebraic group. Suppose that G is connected, or k is perfect. Then the conditions of Theorem 1.1 are also equivalent to:*

(a$^+$) there exists a G-torsor over a quasi-compact k-scheme that is weakly versal for all smooth *affine k-schemes.*

Proof. As in the proof of Theorem 1.1, it is enough to prove that (a$^+$) implies that G is unipotent. Suppose that (a$^+$) holds but G is not unipotent. Then, as in the proof of Theorem 1.1, given a finite-dimensional k-field K such that G_K contains a copy of $\boldsymbol{\mu}_{\ell,K}$ with $\ell > 1$, we could use Theorem 6.1 to get a smooth affine K-scheme X' and a G-torsor $E' \to X'$ that is not a specialization of $E \to X$. We will reach the desired contradiction by showing that K can be taken to be separable over k, and thus X' is smooth over k.

When k is perfect, we could simply take K to be the k-field from the proof Theorem 1.1. If G is connected, then Theorem 3.6(i) tells us that G contains a nontrivial multiplicative group M. In this case, take K to be a finite separable k-field such that M_K is diagonalizable. $\square$

Theorem 1.2 is part (i) of the following theorem.

Theorem 6.4. *Let G be a non-unipotent k-algebraic subgroup of $\mathbf{GL}_n$, let X be a d-dimensional k-scheme that is quasi-projective over a field and let $E \to X$ be a G-torsor. Then:*

(i) *$E \to X$ is not weakly versal for finite type regular affine k-schemes of dimension $\leq n(d+1)+2$.*

(ii) *If k is perfect or G is connected, then $E \to X$ is not weakly versal for smooth affine k-schemes of dimension $\leq n(d+1)+2$.*

(iii) *If G contains a nontrivial torus, then $E \to X$ is not weakly versal for smooth affine k-schemes of dimension $\leq n(d+1)+1$.*

(iv) *If $\operatorname{char} k = 0$, $\operatorname{trdeg}_{\mathbb{Q}} k$ is infinite and $G_{\overline{k}}$ contains a copy of $\mathbf{G}_{\mathbf{m},\overline{k}}$ such that characters of the representation $\mathbf{G}_{\mathbf{m},\overline{k}} \to G_{\overline{k}} \to \mathbf{GL}_{n,\overline{k}}$ include the identity character or $x \mapsto x^{-1}$, then $E \to X$ is not weakly versal for smooth affine k-schemes of dimension $\leq n(d+1)$.*

Proof. (i) By Theorem 3.6(ii), there is a a finite dimensional k-field K such that G_K contains a copy of $\boldsymbol{\mu}_{\ell,K}$ for some $\ell > 1$. Put $m = n(d+1)$ and let $E' \to X'$ be the G-torsor guaranteed in Theorem 6.1. Then $E \to X$ does not specialize to $E' \to X'$. Since $\dim X' = g_K(\ell, m) \leq m+2$, we conclude that $E \to X$ is not weakly versal for finite type regular affine k-schemes of dimension $\leq n(d+1)+2$.

(ii) As in the proof of Theorem 6.3, there is a separable finite-dimensional k-field K such that G_K contains a copy of $\boldsymbol{\mu}_{\ell,K}$ for some $\ell > 1$. Proceed as in (i) using this K. The scheme X' is smooth over k because it is smooth over K and K is smooth over k.

(iii) Let T be a nontrivial subtorus of G and let K be a finite separable k-field splitting T. Then G_K contains a copy $\mathbf{G}_{\mathbf{m},K}$. Let ℓ be a prime number greater than $m := n(d+1)$. Then G_K contains a copy of of $\boldsymbol{\mu}_{\ell,K}$. Proceed as in (i), but note that $g_K(\ell, m) \leq m+1$ by Theorem 5.1(iii).

(iv) The assumption on G implies that there is a finite separable k-field K such that G_K contains a copy of $\mathbf{G}_{\mathbf{m},K}$ and the characters of the representation $\mathbf{G}_{\mathbf{m},K} \to G_K \to \mathbf{GL}_{n,K}$ include $x \mapsto x$ or $x \mapsto x^{-1}$. We now argue as in (i) using Theorem 6.2 instead of Theorem 6.1. Theorem 5.1(v) tells us that $g_K(0,m) = m$. □

An immediate application of Theorem 6.4 is the following.

Corollary 6.5. *Let G be a non-unipotent k-algebraic subgroup of $\mathbf{GL}_n$. Then there exists a finite type regular affine k-scheme of dimension $n+2$ carrying a non-trivial G-torsor $E \to X$. If k is perfect or G is connected, then X can be taken to be smooth as well.*

Proof. Apply Theorem 6.4 to the G-torsor $G \to \operatorname{Spec} k$. □

As we noted in the introduction, this result can be significantly improved in terms of dimension when G contains a nontrivial torus; this will be shown in the next section. Another way to strengthen Corollary 6.5 is the following.

Corollary 6.6. *Let H be a k-algebraic subgroup of $\mathbf{GL}_n$ and let G be a non-unipotent k-algebraic subgroup of H. Then there exists a G-torsor $E \to X$, with X a finite type regular affine k-scheme such that the H-torsor $E \times^G H \to X$ is nontrivial. Furthermore, X can be taken so that $\dim X \leq n(\dim H - \dim G + 1) + 2$.*

The corollary is obtained by applying Theorem 6.4 to the G-torsor $H \to H/G$, so improvements analogous to parts (ii), (iii), (iv) of Theorem 6.4 hold. We leave it to the reader to work them out explicitly.

Proof. Let X be a k-scheme. It is well-known that there is a short exact sequence of pointed sets

$$(H/G)(X) \xrightarrow{\alpha} \mathrm{H}^1_{\mathrm{fppf}}(X, G) \xrightarrow{\beta} \mathrm{H}^1_{\mathrm{fppf}}(X, H)$$

in which α takes an X-point $p : X \to H/G$ to the cohomology class of the G-torsor obtained by base-changing $H \to H/G$ along p, and β takes the cohomology class of a G-torsor $E \to X$ to the cohomology class of $E \times^G H \to X$. From the sequence we see that every H-torsor $E \to X$ such that $E \times^G H \to X$ is trivial is a specialization of the H-torsor $H \to H/G$. Thus, for any G-torsor $E \to X$ that is a not a specialization of $H \to H/G$, the extension $E \times^G H \to X$ is a nontrivial H-torsor. With this observation at hand, the corollary follows by applying Theorem 6.4 to the G-torsor $H \to H/G$. (Note that H/G is quasi-projective.) □

7. Non-Trivial Torsors

We finish with using results from Sections 4 and 5 to prove Theorem 1.3. This will improve Corollary 6.6 for algebraic groups containing a nontrivial torus.

We will derive Theorem 1.3 from a slightly more general result. To phrase it, we introduce a variation of the invariant $G_k(\ell, m)$ considered in Section 5. For every $m \in \mathbb{N} \cup \{0\}$, let

$$\tilde{G}_k(m)$$

denote the smallest $d \in \mathbb{N} \cup \{0\}$ such that for every $n \in \mathbb{N}$, there is a smooth affine d-dimensional k-scheme and a line bundle L such that $n \cdot c_1(L)^m \neq 0$ in $\mathrm{CH}^m X$.

Theorem 7.1. *Let G be an affine k-algebraic group containing a nontrivial torus (resp. central torus). Then there exists a smooth affine k-scheme of dimension at most $\tilde{G}_k(2)$ (resp. $\tilde{G}_k(1)$) carrying a non-trivial G-torsor.*

Proof of Theorem 1.3 using Theorem 7.1. It is enough to show that $\tilde{G}_k(m) \leq m + 1$, $\tilde{G}_k(1) \leq 1$ and, provided that $\operatorname{char} k = 0$ and $\operatorname{trdeg}_{\mathbb{Q}} k$ is infinite, $\tilde{G}_k(m) \leq m$.

For the first inequality, given $n \in \mathbb{N}$, choose some prime number ℓ larger than m and n. By Theorem 5.19(iii), there is a smooth affine k-scheme X with $\dim X \leq m+1$ and an ℓ-torsion line bundle L such that $c_1(L)^m \neq 0$. Since $c_1(L)$ is ℓ-torsion and $\ell \nmid n$, we must have $n \cdot c_1(L)^m \neq 0$. Thus, $\tilde{G}_k(m) \leq m + 1$.

The second inequality is shown similarly, using Theorem 5.19(i).

For the last inequality, write $K = \overline{k}$. We may assume that $m > 1$ as we already showed that $\tilde{G}_k(1) \leq 1$. By Theorem 5.19(v), there is a smooth affine K-scheme X with $\dim X = m$ and a line bundle L such that $c_1(L)^m \neq 0$. By a result of Murthy [26, Thms. 2.11, 2.14], $\mathrm{CH}^m X$ is torsion-free (here we need $m > 1$ and K to be algebraically closed). Thus, $n \cdot c_1(L)^m \neq 0$ for all $n \in \mathbb{N}$. The K-scheme X and the line bundle L descend to some finite-dimensional k-field, which is separable over k because $\operatorname{char} k = 0$, hence $\tilde{G}_k(m) \leq m$. □

We turn to prove Theorem 7.1. We first prove the following lemma.

Lemma 7.2. *Let $\sigma_1, \sigma_2 : \mathbb{Z}^n \to \mathbb{Z}$ be the first and second symmetric functions. Then for every $x \in \mathbb{Z}^n - \{0\}$, either $\sigma_1(x) \neq 0$ or $\sigma_2(x) \neq 0$.*

Proof. Write $x = (x_1, \dots, x_n)$. We have $\sigma_1(x)^2 - 2\sigma_2(x) = \sigma_1(x_1^2, \dots, x_n^2) \neq 0$, so at least one of $\sigma_1(x)$, $\sigma_2(x)$ must be nonzero. □

Proof of Theorem 7.1. Let T be a nontrivial torus contained in G (resp. the center of G). It is harmless to base-change from k to a finite separable extension splitting T. We may therefore assume that G (resp. its center) contains a copy of $\mathbf{G_m}$.

Choose a morphism $G \to \mathbf{GL}_n$ such that the composition $\varphi : \mathbf{G_m} \to G \to \mathbf{GL}_n$ is nontrivial, e.g., a faithful representation of G. It is enough to show that there is a smooth affine k-scheme X with $\dim X \leq \tilde{G}_k(2)$ (resp. $\dim X \leq \tilde{G}_k(1)$) and a $\mathbf{G_m}$-torsor E such that $E \times^{\mathbf{G_m}} \mathbf{GL}_n$ is a nontrivial $\mathbf{GL}_n$-torsor. Indeed, since $E \times^{\mathbf{G_m}} \mathbf{GL}_n = (E \times^{\mathbf{G_m}} G) \times^G \mathbf{GL}_n$, this would force $E \times^{\mathbf{G_m}} G \to X$ to be a nontrivial G-torsor.

Let $\{x \mapsto x^{a_i}\}_{i=1}^n$ be the characters of the representation $\varphi : \mathbf{G_m} \to \mathbf{GL}_n$ (with multiplicities). We observed in the proof of Theorem 4.5 that if L is the line bundle corresponding to E, then the vector bundle corresponding to $E \times^{\mathbf{G_m}} \mathbf{GL}_n$ is $V := L^{\otimes a_1} \oplus \cdots \oplus L^{\otimes a_n}$. Thus, the total Chern class of V is $c(V) = c(L^{\otimes a_1}) \cdots c(L^{\otimes a_n}) = \prod_{i=1}^n (1 + a_i c_1(L))$. Writing $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, this means that

$$c_i(V) = \sigma_i(a) c_1(L)^i$$

for all $i \in \{1, \ldots, n\}$, where σ_i is the i-th symmetric function. Since $a \neq 0$, Lemma 7.2 tells us that $\sigma_m(a) \neq 0$ for some $m \in \{1, 2\}$.

We now choose X to be a $\tilde{G}_k(m)$-dimensional smooth affine k-scheme carrying a line bundle L with $\sigma_m(a) \cdot c_1(L)^m \neq 0$. By what we just showed, $c_m(V) = \sigma_m(a) \cdot c_1(L)^m \neq 0$ and it follows that $E \times^{\mathbf{G_m}} \mathbf{GL}_n \to X$ is a nontrivial $\mathbf{GL}_n$-torsor. As $\tilde{G}_k(m) \leq \tilde{G}_k(2)$ (because $m \in \{1, 2\}$), this proves the theorem when T is not assumed to be central in G.

When T is contained in the center of G, we choose $\varphi : G \to \mathbf{GL}_n$ in such a way that $\mathbf{G_m}$ is also mapped into the center of $\mathbf{GL}_n$. (Such representations exist, e.g., start with a faithful representation V of G, chose a nontrivial character $\chi : \mathbf{G_m} \to \mathbf{G_m}$ occurring in $\mathbf{G_m} \to G \to \mathbf{GL}(V)$ and take the eigenspace V_χ.) Then there is $a \in \mathbb{Z} - \{0\}$ such that $a_1 = \cdots = a_n = a$. This means that $\sigma_1(a) = na \neq 0$, so we can take $m = 1$ and get $\dim X \leq \tilde{G}_k(1)$. □

We finish with showing that there are algebraic groups for which Theorem 1.3 is the best possible result in terms of dimension.

Example 7.3. Consider the algebraic group $\mathbf{SL}_n$ $(n > 1)$. By Theorem 1.3, there is a smooth affine k-scheme X with $\dim X \leq 3$ carrying a nontrivial $\mathbf{SL}_n$-torsor. Provided that $\operatorname{char} k = 0$ and $\operatorname{trdeg}_{\mathbb{Q}} k$ is infinite, we can even take X with $\dim X = 2$.

On the other hand, every $\mathbf{SL}_n$-torsor over an affine noetherian scheme of dimension smaller than 2 is trivial. Indeed, it is well-known that there is a bijection between isomorphism classes of $\mathbf{SL}_n$-torsors over X and rank-n vector bundles X with trivial determinant, in which the trivial torsor corresponds to $\mathcal{O}_X^n$. (This can be seen, for instance, by inspecting the long fppf cohomology exact sequence associated to the short exact sequence $\mathbf{SL}_n \hookrightarrow \mathbf{GL}_n \xrightarrow{\det} \mathbf{G_m}$.) Let E be an $\mathbf{SL}_n$-torsor over a noetherian k-scheme X with $\dim X < 2$ and let V be the corresponding vector bundle with $\det(V) \cong \mathcal{O}_X$. By a theorem of Bass–Serre [3, Thm. 8.2] (see also [32, Thm. 1]), our assumptions on X imply that $V \cong \mathcal{O}_X^{n-1} \oplus L$ for some line bundle L, so $\det(V) \cong L$. Since we also have $\det(V) \cong \mathcal{O}_X$, it follows that $V \cong \mathcal{O}_X^n$ and $E \to X$ is trivial.

For algebraically closed fields, we can make Example 7.3 more precise and even extend it to semisimple groups of type G_2. The latter case is essentially due to Asok, Hoyois and Wendt [1].

Proposition 7.4. *Let k be an algebraically closed field, let X be a smooth irreducible affine k-scheme with $\dim X < 3$, let $n > 1$ and let G denote the semisimple k-algebraic group of type G_2. Then the following are equivalent:*

(a) $\mathrm{CH}^2 X = 0$*;*
(b) every $\mathbf{SL}_n$-torsor over X is trivial;
(c) every G-torsor over X is trivial.

Proof. We start with showing that (a) $\iff$ (c). The field $k(X)$ has cohomoligcal dimension at most 2, and thus every G-torsor over X is trivial over the generic point [35, Apx. 2.3.3]. Thus, by [1, Thm. 1], $\mathrm{CH}^2 X$ is in bijection with $\mathrm{H}^1_{\mathrm{fppf}}(X, G)$ and the equivalence is immediate.

We proceed with showing that (a) $\iff$ (b). Suppose first that (a) holds. Let E be an $\mathbf{SL}_n$-torsor over X and let V be its corresponding vector bundle as in

Example 7.3. We have seen that $E \to X$ is trivial when $\dim X < 2$, so assume $\dim X = 2$. By the Bass–Serre Theorem [32, Thm. 1], $V \cong V' \oplus \mathcal{O}_X^{n-2}$ for some rank-2 vector bundle V'. Since $\mathrm{CH}^2 X = 0$, we have $c_2(V') = 0$. By a theorem of Murthy [26, Thm. 3.8], this means that there is a line bundle L such that $V' \cong L \oplus \mathcal{O}_X$ and as in Example 7.3, we get that $E \to X$ is trivial.

Suppose now that (b) holds. If $\dim X < 2$, then $\mathrm{CH}^2 X = 0$, so assume $\dim X = 2$. Let $c \in \mathrm{CH}^2 X$. As explained in [24, §1, a)], there is a rank-2 vector bundle V' with $c_1(V') = 0$ and $c_2(V') = c$. Since $c_1(V') = c_1(\det(V')) = 0$, it follows that $\det(V') \cong \mathcal{O}_X$. Let $E \to X$ be the $\mathbf{SL}_n$-torsor corresponding to $V := V' \oplus \mathcal{O}_X^{n-2}$. By (c), E is trivial, so V is free, and it follows that $c = c_2(V') = c_2(V) = 0$. □

Corollary 7.5. *Let p be a prime number and let G be a semisimple simply connected $\overline{\mathbb{F}}_p$-algebraic group of type A_n or G_2. Then every G-torsor over a smooth affine $\overline{\mathbb{F}}_p$-scheme X of dimension < 3 is trivial.*

Proof. We observed in the proof of Proposition 5.17 that $\mathrm{CH}^2 X = 0$, so this follows from Proposition 7.4. □

Remark 7.6. Let G be a semisimple simply connected $\overline{\mathbb{Q}}$-algebraic group of type A_n or G_2. By Proposition 7.4, the statement that every G-torsor over a smooth affine 2-dimensional $\overline{\mathbb{Q}}$-scheme is trivial is equivalent to the Bloch–Beilinson conjecture for surfaces [20, p. 157, (b)].

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