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# SPLITTING FAMILIES IN GALOIS COHOMOLOGY

by

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**Abstract.** — Let  $k$  be a field, with absolute Galois group  $\Gamma$ . Let  $A/k$  be a finite étale group scheme of multiplicative type, i.e. a finite discrete  $\Gamma$ -module. Let  $n \geq 2$  be an integer, and let  $x \in H^n(k, A)$  be a cohomology class. We show that there exists a countable set  $I$ , and a family  $(X_i)_{i \in I}$  of (smooth, geometrically integral)  $k$ -varieties, such that the following holds: for any field extension  $\ell/k$ , the restriction of  $x$  vanishes in  $H^n(\ell, A)$  if and only if (at least) one of the  $X_i$ 's has an  $\ell$ -point. In addition, we show that the  $X_i$ 's can be made into an ind-variety. In the case  $n = 2$ , we note that one variety is enough.

**Résumé (Familles de déploiement en cohomologie galoisienne)**

Soit  $k$  un corps, de groupe de Galois absolu  $\Gamma$ . Soit  $A/k$  un schéma en groupes fini étale de type multiplicatif, i.e. un  $\Gamma$ -module fini discret. Soit  $n \geq 2$  un entier, et  $x \in H^n(k, A)$  une classe de cohomologie. On montre qu'il existe un ensemble dénombrable  $I$ , et une famille  $(X_i)_{i \in I}$  de  $k$ -variétés (lisses, géométriquement intègres) telles que : pour toute extension de corps  $\ell/k$ , la restriction de  $x$  s'annule dans  $H^n(\ell, A)$  si et seulement si (au moins) une des  $X_i$  a un  $\ell$ -point. De plus, on montre qu'on peut choisir les  $X_i$  pour qu'elles forment une ind-variété. Dans le cas  $n = 2$ , on remarque qu'une seule variété suffit.

## Introduction

Let  $k$  be a field, and let  $p$  be a prime number, which is invertible in  $k$ . The notion of a norm variety was introduced in the study of the Bloch-Kato conjecture. It is

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a key tool in the proof provided by Rost, Suslin and Voevodsky. The norm variety  $X(s)$  of a pure symbol

$$s = (x_1) \cup (x_2) \cup \dots \cup (x_n) \in H^n(k, \mu_p^{\otimes n}),$$

where the  $x_i$ 's are elements of  $k^\times$ , was constructed by Rost (cf. [6] or [3]). The terminology 'norm variety' reflects that it is defined through an inductive process involving the norm of finite field extensions of degree  $p$ . It has the remarkable property that, if  $\ell/k$  is a field extension, then the restriction of  $s$  vanishes in  $H^n(\ell, \mu_p^{\otimes n})$  if and only if the  $\ell$ -variety  $X(s)_\ell$  has a 0-cycle of degree prime-to- $p$ . It enjoys nice geometric features, which we will not mention here. For  $n \geq 3$ , norm varieties are, to the knowledge of the authors of this paper, known to exist for pure symbols only.

In this paper, we shall be interested in the following closely related problem. Let  $A/k$  be a finite étale group scheme of multiplicative type, that is to say, a finite discrete  $\Gamma$ -module. Consider a class  $x \in H^n(k, A)$ . Does there exist a countable family of smooth  $k$ -varieties  $(X_i)_{i \in I}$ , such that, for every field extension  $\ell/k$ , the presence of a  $\ell$ -point in (at least) one of the  $X_i$ 's is equivalent to the vanishing of  $x$  in  $H^n(\ell, A)$ ? If such a family exist, can it always be endowed with the structure of an ind-variety?

We provide answers to those questions. The main results of the paper are the following:

**Theorem 0.1 (Corollary 4.2 and Corollary 5.8).** — *Let  $A/k$  be a finite étale group scheme of multiplicative type and let  $\alpha \in H^n(k, A)$ , where  $n \geq 2$  is an integer.*

- *There exists a countable family  $(X_i)_{i \in I}$  of smooth geometrically integral  $k$ -varieties, such that for any field extension  $\ell/k$  with  $\ell$  infinite,  $\alpha$  vanishes in  $H^n(\ell, A)$  if and only if  $X_i(\ell) \neq \emptyset$  for some  $i$ . In addition, there is such a family  $(X_i)$  which is an ind-variety.*
- *If  $n = 2$ , the family  $(X_i)$  can be replaced by a single smooth geometrically integral  $k$ -variety.*

Note that our main "non-formal" tool, as often (always?) in this context, is Hilbert's Theorem 90.

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## 1. Notation and definitions.

In this paper,  $k$  is a field, with a given separable closure  $k^s$ . We denote by  $\Gamma := \text{Gal}(k^s/k)$  the absolute Galois group. The letters  $d$  and  $n$  denote two positive integers. We assume  $d$  to be invertible in  $k$ .

We denote by  $\mathcal{M}_d$  the Abelian category of finite  $\mathbf{Z}/d\mathbf{Z}$ -modules, and by  $\mathcal{M}_{\Gamma,d}$  that of finite and discrete  $\Gamma$ -modules of  $d$ -torsion. The latter is equivalent to the category of finite  $k$ -group schemes of multiplicative type, killed by  $d$ . We denote this category by  $\mathcal{M}_{k,d}$ . When no confusion can arise, we will identify these categories without further notice. We have an obvious forgetful functor  $\mathcal{M}_{\Gamma,d} \rightarrow \mathcal{M}_d$ .

**1.1. Groups and cohomology.** — Let  $G$  be a linear algebraic  $k$ -group; that is, an affine  $k$ -group scheme of finite type. We denote by  $H^1(k, G)$  the set of isomorphism classes of  $G$ -torsors, for the fppf topology. It coincides with the usual Galois cohomology set if  $G$  is smooth. Let  $\varphi : H \rightarrow G$  be a morphism of linear algebraic  $k$ -groups. It induces, for every field extension  $\ell/k$ , a natural map  $H^1(\ell, H) \rightarrow H^1(\ell, G)$ , which we denote by  $\varphi_{\ell,*}$ .

**1.2. Yoneda Extensions.** — Let  $\mathcal{A}$  be an Abelian category. For all  $n \geq 0$ ,  $A, B \in \mathcal{A}$ , we denote by  $\mathbf{YExt}_{\mathcal{A}}^n(A, B)$  (or  $\mathbf{YExt}^n(A, B)$ ) the (additive) category of Yoneda  $n$ -extensions of  $B$  by  $A$ , and by  $\mathbf{YExt}_{\mathcal{A}}^n(A, B)$  (or  $\mathbf{YExt}^n(A, B)$ ) the Abelian group of Yoneda equivalence classes in  $\mathbf{YExt}^n(A, B)$ . Recall (see [5], section 2) that an object of  $\mathbf{YExt}_{\mathcal{A}}^n(A, B)$  is an exact sequence

$$\mathcal{E} = (0 \rightarrow B \xrightarrow{f_0} E_1 \xrightarrow{f_1} \cdots \rightarrow E_{n-1} \xrightarrow{f_{n-1}} E_n \xrightarrow{f_n} A \rightarrow 0)$$

of objects in  $\mathcal{A}$ , and morphisms between two such  $n$ -extensions of  $A$  by  $B$  are morphisms of complexes for which the induced morphism from  $A$  (resp.  $B$ ) to itself is the identity map.

Recall also that one says that two  $n$ -extensions  $\mathcal{E}_1$  and  $\mathcal{E}_2$  in  $\mathbf{YExt}_{\mathcal{A}}^n(A, B)$  are equivalent if there exists a third extension  $\mathcal{E}$  in  $\mathbf{YExt}_{\mathcal{A}}^n(A, B)$  and morphisms of  $n$ -extensions  $\mathcal{E}_1 \leftarrow \mathcal{E} \rightarrow \mathcal{E}_2$ . In our setting, this indeed defines an equivalence relation between objects of  $\mathbf{YExt}_{\mathcal{A}}^n(A, B)$  (see for instance [5], end of section 2).

**Remark 1.1.** — The groups  $\mathbf{YExt}_{\mathcal{A}}^n(A, B)$  can also be defined as  $\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(A, B[n])$ , where  $\mathcal{D}(\mathcal{A})$  denotes the derived category of  $\mathcal{A}$ .

Given  $A, B \in \mathcal{M}_d$ , we put  $\mathbf{YExt}_d^n(A, B) := \mathbf{YExt}_{\mathcal{M}_d}^n(A, B)$ . Given  $A, B \in \mathcal{M}_{k,d}$ , we put  $\mathbf{YExt}_{k,d}^n(A, B) := \mathbf{YExt}_{\mathcal{M}_{k,d}}^n(A, B)$ .

**Remark 1.2.** — Let  $A$  be a finite discrete  $\Gamma$ -module. Let  $d$  be the exponent of  $A$ . The group  $\mathbf{YExt}_{k,d}^n(\mathbf{Z}/d\mathbf{Z}, A)$  coincide with the usual Ext-group defined via injective resolutions (see [9], Ch. III, section 3), and we have a natural isomorphism  $\mathbf{YExt}_{k,d}^0(\mathbf{Z}/d\mathbf{Z}, A) \xrightarrow{\sim} A^\Gamma$ , therefore there is a canonical isomorphism

$$\mathbf{YExt}_{k,d}^n(\mathbf{Z}/d\mathbf{Z}, A) \xrightarrow{\sim} H^n(\Gamma, A)$$

where  $H^n(\Gamma, A)$  denotes the usual  $n$ -th cohomology group.

**Remark 1.3.** — Let  $\ell/k$  be any field extension. For  $A, B \in \mathcal{M}_{k,d}$ , we have a restriction map

$$\mathrm{Res}_{\ell/k} : \mathbf{YExt}_{k,d}^n(A, B) \longrightarrow \mathbf{YExt}_{\ell,d}^n(A, B).$$

**1.3. Lifting triangles.** — Let  $\varphi : H \rightarrow G$  be a morphism of linear  $k$ -algebraic groups. A lifting triangle (relative to  $\varphi$ ) is a commutative triangle

$$T : \begin{array}{ccc} Q & \xrightarrow{f} & P \\ & \searrow^{H_X} & \downarrow^{G_X} \\ & & X, \end{array}$$

where  $X$  is a  $k$ -scheme,  $Q \rightarrow X$  (resp.  $P \rightarrow X$ ) is an  $H_X$ -torsor (resp. a  $G_X$ -torsor), and where  $f$  is an  $H$ -equivariant morphism (formula on the functors of points:  $f(h.x) = \varphi(h).f(x)$ ).

Note that such a diagram is equivalent to the data of an isomorphism between the  $G_X$ -torsors  $P$  and  $\varphi_*(Q)$ .

The  $k$ -scheme  $X$  is called the *base* of the lifting triangle  $T$ .

We have an obvious notion of isomorphism of lifting triangles.

Moreover, if  $\eta : Y \rightarrow X$  is a morphism of  $k$ -schemes, we can form the pullback  $\eta^*(T)$ ; it is a lifting triangle, over the base  $Y$ .

**1.4. Lifting varieties.** — Let  $\varphi : H \rightarrow G$  be a morphism of linear  $k$ -algebraic groups. Let  $P \rightarrow \text{Spec}(k)$  be a torsor under the group  $G$ .

A geometrically integral  $k$ -variety  $X$  will be called a *lifting variety* (for the pair  $(\varphi, P)$ ) if it fits into a lifting triangle  $\mathcal{T}$ :

$$\mathcal{T} : \begin{array}{ccc} Q & \xrightarrow{F} & P \times_k X \\ & \searrow^{H_X} & \downarrow^{G_X} \\ & & X, \end{array}$$

such that the following holds:

For every field extension  $\ell/k$ , with  $\ell$  infinite, and for every lifting triangle  $t$ :

$$t : \begin{array}{ccc} Q & \xrightarrow{f} & P \times_k \ell \\ & \searrow^{H_\ell} & \downarrow^{G_\ell} \\ & & \text{Spec}(\ell), \end{array}$$

the set of  $\ell$ -rational points  $x : \text{Spec}(\ell) \rightarrow X$  such that the pullback  $\mathcal{T}_x := x^*(\mathcal{T})$  is isomorphic to  $t$  (as a lifting triangle over  $\text{Spec}(\ell)$ ) is Zariski-dense in  $X$ , hence non-empty.

In particular, the variety  $X$  has an  $\ell$ -point if and only if the class of the  $G$ -torsor  $P$  in  $H^1(\ell, G)$  is in the image of the map  $\varphi_{\ell,*} : H^1(\ell, H) \rightarrow H^1(\ell, G)$ .

**1.5. Splitting families.** — Let  $A, B$  be objects of  $\mathcal{M}_{k,d}$ . Pick a class  $x \in \mathbf{YExt}_{k,d}^n(A, B)$ .

A countable set  $(X_i)_{i \in I}$  of (smooth, geometrically integral)  $k$ -varieties will be called a *splitting family* for  $x$  if the following holds:

For every field extension  $\ell/k$ , with  $\ell$  infinite,  $\text{Res}_{\ell/k}(x)$  vanishes in  $\mathbf{YExt}_{\ell, d}^n(A, B)$  if and only if (at least) one of the  $\ell$ -varieties  $X_i$  possesses a  $\ell$ -point.

Whenever a splitting family exists, it is natural to ask whether it can be made into an ind-variety. By this, we mean here that  $I = \mathbb{N}$  and that, for each  $i \geq 0$ , we are given a closed embedding of  $k$ -varieties  $X_i \rightarrow X_{i+1}$ .

## 2. Existence of lifting varieties.

This section contains the non-formal ingredient of this paper, which may have an interest on its own.

Let  $\varphi : H \rightarrow G$  be a morphism of linear  $k$ -algebraic groups; that is, of affine  $k$ -group schemes of finite type.

Let  $P \rightarrow \text{Spec}(k)$  be a torsor under the group  $G$ .

The aim of this section is to construct a lifting variety for  $(\varphi, P)$ . Equivalently, we will build a "nice"  $k$ -variety  $X$  that is a versal object for  $H$ -torsors that lift the  $G$ -torsor  $P$ , in the sense explained in the previous paragraph.

In particular, recall that  $X(\ell) \neq \emptyset$  if and only if  $[P_\ell]$  lifts to  $H^1(\ell, H)$ , for every field extension  $\ell/k$ , with  $\ell$  infinite.

To construct such an  $X$ , we mimic the usual construction of versal torsors (see for instance [7], section I.5). We just have to push it slightly further.

Following for instance [8], Remark 1.4, there exists a finite dimensional  $k$ -vector space  $V$  endowed with a generically free linear action of  $H$ . Furthermore, there exists a dense open subset  $V_0 \subset \mathbf{A}(V)$ , stable under the action of  $H$ , and such that the geometric quotient

$$V_0 \longrightarrow V_0/H$$

exists, and is an  $H$ -torsor, which we denote by  $\mathcal{Q}$ .

Form the quotient

$$X_{\varphi, P} := (P \times_k V_0)/H,$$

where  $H$  acts on  $P$  via  $\varphi$ , and on  $V_0$  in the natural way. Projecting onto  $V_0$  induces a morphism

$$\pi : X_{\varphi, P} \longrightarrow V_0/H,$$

which can also be described as the twist of  $P$  by the  $H$ -torsor  $\mathcal{Q}$ , over the base  $V_0/H$ .

Note that  $X_{\varphi, P}$  depends on the choice of  $V$  (up to stable birational equivalence).

If we denote by  $\mathcal{Q}'$  the pullback via  $\pi$  of the  $H$ -torsor  $\mathcal{Q}$ , there is a natural lifting triangle  $\mathcal{T}_{\varphi,P}$ :

$$\begin{array}{ccc} \mathcal{Q}' & \longrightarrow & X_{\varphi,P} \times_k P \\ & \searrow H & \downarrow G \\ & & X_{\varphi,P} \end{array}$$

Its existence is explained by the following key fact. If  $Y := V_0/H$ , then for any  $Y$ -scheme  $S$ , a point

$$s \in X_{\varphi,P}(S) = \mathrm{Hom}_{Y\text{-sch}}(S, X_{\varphi,P})$$

is exactly the same as an  $H$ -equivariant morphism between  $\mathcal{Q} \times_Y S$  and  $X_{\varphi,P} \times_k S$ , over the base  $S$  (see for instance [2], théorème III.1.6.(ii)), i.e. it is the same as a lifting triangle relative to  $\varphi$  over the base  $S$ , i.e. an isomorphism of  $G$ -torsors between  $\varphi_* \mathcal{Q} \times_Y S$  and  $P \times_k S$ . We shall refer to this property as the universal property of  $X_{\varphi,P}$ .

**Proposition 2.1.** — *The  $k$ -variety  $X_{\varphi,P}$  is a lifting variety for the pair  $(\varphi, P)$ .*

*In particular,  $X_{\varphi,P}(\ell) \neq \emptyset$  if and only if  $[P_\ell]$  lifts to  $H^1(\ell, H)$ .*

*Proof.* — Let  $\ell/k$  be a field extension with  $\ell$  infinite. Let

$$\begin{array}{ccc} t : Q & \xrightarrow{f} & P \times_k \ell \\ & \searrow H_\ell & \downarrow G_\ell \\ & & \mathrm{Spec}(\ell) \end{array}$$

be a lifting triangle, over  $\ell$ . Let  $p : V_0 \rightarrow V_0/H$  denote the quotient map. By construction, given a point  $x \in (V_0/H)(\ell) = \mathrm{Hom}_{k\text{-sch}}(\mathrm{Spec}(\ell), V_0/H)$ , the  $\mathrm{Spec}(\ell)$ -torsor  $x^*(\mathcal{Q})$  is isomorphic to  $Q$  if and only if  $x$  is in the image of  ${}^Q V_0(\ell)$  by the twisted map  ${}^Q p : {}^Q V_0 := (Q \times V_0)/H \rightarrow V_0/H$ . By Hilbert's Theorem 90 (for  $\mathbf{GL}_k(V_0)$ ), the  $\ell$ -variety  ${}^Q V_0$  is isomorphic to an affine space over  $\ell$ , hence its  $\ell$ -points are Zariski dense. Since the map  $p$ , hence  ${}^Q p$ , is dominant, the set of  $\ell$ -rational points  $x \in (V_0/H)(\ell)$  such that  $x^*(\mathcal{Q})$  is isomorphic to  $Q$  (as  $H$ -torsors over  $\ell$ ) is Zariski-dense.

Let  $x$  be such a point. Then the lifting triangle  $t$  corresponds to an isomorphism of  $G$ -torsors between  $\varphi_*(Q)$  and  $P$ , over the base  $\mathrm{Spec}(\ell)$ . Since  $Q$  is isomorphic to  $x^*(\mathcal{Q})$ , the universal property of  $X_{\varphi,P}$  implies that the lifting triangle  $t$  is isomorphic to the fiber of  $\mathcal{T}_{\varphi,P}$  at an  $\ell$ -rational point of  $X_{\varphi,P}$ . This finishes the proof.  $\square$

**Lemma 2.2.** — *The  $k$ -variety  $X_{\varphi,P}$  is smooth and geometrically unirational if  $\varphi : H \rightarrow G$  is surjective, or if  $G$  is smooth and connected.*

*Proof.* — To prove this, we can assume that  $k = \bar{k}$ , in which case the torsor  $P$  is trivial. Then  $X_{\varphi,P} = (G \times V_0)/H$ . If  $G$  is smooth and connected, then it is  $k$ -rational. Hence  $G \times V_0$  is smooth, connected and  $k$ -rational as well. The quotient morphism

$$G \times V_0 \longrightarrow X_{\varphi,P}$$

is an  $H$ -torsor, and smoothness and geometrical unirationality of its total space implies that of its base.

Now, assume that  $\varphi$  is surjective. Denoting by  $K$  its kernel, we see that  $X_{\varphi,P} = V_0/K$ , which implies the result.  $\square$

### 3. Triviality of Yoneda extensions in Abelian categories.

Let  $\mathcal{A}$  be an Abelian category.

The following lemma is well-known.

**Lemma 3.1.** — *Let  $\mathcal{E} = (0 \rightarrow B \xrightarrow{f_0} E_1 \xrightarrow{f_1} \dots \rightarrow E_{n-1} \xrightarrow{f_{n-1}} E_n \xrightarrow{f_n} A \rightarrow 0)$  be an object in  $\mathbf{YExt}^n(A, B)$ , and let  $e$  denote its class in  $\mathbf{YExt}^n(A, B)$ .*

*Then  $e = 0$  in  $\mathbf{YExt}^n(A, B)$  if and only if there exists  $\mathcal{F}$  in  $\mathbf{YExt}^{n-1}(E_n, B)$  and a morphism of complexes  $\phi : \mathcal{E} \rightarrow \mathcal{F}$  inducing the identity on  $B$  and  $E_n$ , i.e. a commutative diagram (with exact rows)*

$$(3.1) \quad \begin{array}{ccccccccccc} 0 & \longrightarrow & B & \xrightarrow{f_0} & E_1 & \xrightarrow{f_1} & \dots & \longrightarrow & E_{n-1} & \xrightarrow{f_{n-1}} & E_n & \xrightarrow{f_n} & A & \longrightarrow & 0 \\ & & \downarrow \text{id} & & \downarrow \phi_1 & & & & \downarrow \phi_{n-1} & & \downarrow \text{id} & & & & \\ 0 & \longrightarrow & B & \longrightarrow & F_1 & \xrightarrow{g_1} & \dots & \longrightarrow & F_{n-1} & \xrightarrow{g_{n-1}} & E_n & \longrightarrow & 0 & & \end{array} .$$

*Proof.* — By [5], section 2 (see also [1], section 7.5, Theorem 1, in the case of categories of modules),  $e = 0$  if and only if there exists a commutative diagram

$$(3.2) \quad \begin{array}{ccccccccccc} 0 & \longrightarrow & B & \longrightarrow & E_1 & \longrightarrow & \dots & \longrightarrow & E_{n-1} & \longrightarrow & E_n & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow = & & \\ 0 & \longrightarrow & B & \longrightarrow & G_1 & \longrightarrow & \dots & \longrightarrow & G_{n-1} & \longrightarrow & G_n & \longrightarrow & A & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & & & \uparrow & & \uparrow & & \uparrow = & & \\ 0 & \longrightarrow & B & \xrightarrow{\text{id}} & B & \longrightarrow & 0 \dots 0 & \longrightarrow & 0 & \longrightarrow & A & \xrightarrow{\text{id}} & A & \longrightarrow & 0. \end{array}$$

Assume  $e = 0$ . In the previous diagram, let  $K' := \text{Ker}(G_n \rightarrow A)$ . Since we are given a splitting  $s$  of  $G_n \rightarrow A$ , there is a natural map  $E_n \rightarrow K'$  defined via the retraction of  $K' \rightarrow G_n$  associated to  $s$ . Define  $\mathcal{F}$  to be the pull-back of the exact sequence

$$0 \rightarrow B \rightarrow G_1 \rightarrow \dots \rightarrow G_{n-1} \rightarrow K' \rightarrow 0$$

by the aforementioned morphism  $E_n \rightarrow K'$ . It is now clear that  $\mathcal{F}$  satisfies the statement of the lemma.

To prove the converse, assume the existence of  $\mathcal{F}$  and  $\phi$  as in the lemma. Define  $F_i := G_i$  for all  $i \leq n-1$ , and  $G_n := E_n \oplus A$ . Consider the maps  $h_i := g_i$  for  $i \leq n-2$ , and let  $h_{n-1} := g_{n-1} \oplus 0 : G_{n-1} \rightarrow G_n = E_n \oplus A$  and  $h_n : G_n = E_n \oplus A \rightarrow A$  be the natural projection. Then the morphism  $\phi$  together with the map  $\text{id} \oplus f_n : E_n \rightarrow G_n = E_n \oplus A$  defines a commutative diagram of the shape (3.2), hence  $e = 0$ .  $\square$

**Definition 3.2.** — Given  $\mathcal{E} \in \mathbf{YExt}^n(A, B)$  as in Lemma 3.1, a  $\mathcal{E}$ -diagram is a pair  $(\mathcal{F}, \phi)$ , where  $\mathcal{F} \in \mathbf{YExt}^{n-1}(E_n, B)$  and  $\phi : \mathcal{E} \rightarrow \mathcal{F}$  is a morphism of complexes inducing the identity on  $B$  and  $E_n$  (see diagram (3.1)). Such a diagram is called injective if  $\phi_i$  is a monomorphism for all  $i$ .

We denote by  $\mathbf{Diag}(\mathcal{E})$  (or  $\mathbf{Diag}_{\mathcal{A}}(\mathcal{E})$ ) the category of  $\mathcal{E}$ -diagrams, where a morphism between  $(\mathcal{F}, \phi)$  and  $(\mathcal{F}', \phi')$  is a morphism between the commutative diagrams associated (as in Lemma 3.1) to both  $\mathcal{E}$ -diagrams, and by  $\mathbf{Diag}(\mathcal{E})$  the set of isomorphism classes in  $\mathbf{Diag}(\mathcal{E})$ .

Note that, given  $\mathcal{D} = (\mathcal{F}, \phi) \in \mathbf{Diag}(\mathcal{E})$ , there is a natural group homomorphism  $\mathrm{Aut}(\mathcal{D}) \rightarrow \mathrm{Aut}(\mathcal{E})$ .

**Example 3.3.** — Consider the particular case when  $\mathcal{A}$  is the category  $\mathcal{M}_{k,d}$ . Recall the obvious functor  $\mathcal{M}_{k,d} \rightarrow \mathcal{M}_d$ .

Then an object  $\mathcal{E}$  of the category  $\mathbf{YExt}_{k,d}^n(A, B)$  is exactly the same as an object  $\mathcal{E}'$  in  $\mathbf{YExt}_d^n(A, B)$  together with a (continuous) group homomorphism  $p : \Gamma \rightarrow \mathrm{Aut}(\mathcal{E}')$ .

Moreover, a  $\mathcal{E}$ -diagram  $\mathcal{D}$  in the category  $\mathcal{M}_{k,d}$  is the same as a  $\mathcal{E}'$ -diagram  $\mathcal{D}'$  in the category  $\mathcal{M}_d$  together with a homomorphism  $q : \Gamma \rightarrow \mathrm{Aut}(\mathcal{D}')$  lifting  $p$ .

Note that in this context, the groups  $\mathrm{Aut}(\mathcal{D}')$  and  $\mathrm{Aut}(\mathcal{E}')$  are finite.

#### 4. Splitting varieties for 2-extensions

In this section, we restrict to the special case of  $\mathrm{YExt}_k^2(A, B)$  and  $H^2(k, A)$ , and we construct splitting varieties.

**Theorem 4.1.** — *Let  $A, B$  be a finite  $d$ -torsion  $\Gamma$ -modules and  $e \in \mathrm{YExt}_{k,d}^2(A, B)$ .*

*Assume  $A$  or  $B$  is free as a  $\mathbf{F}$ -module.*

*Then, there exists a smooth geometrically integral  $k$ -variety  $X$  which is a splitting variety for  $e$ .*

*Proof.* — Let  $\mathcal{E} = (0 \rightarrow B \rightarrow E_1 \rightarrow E_2 \rightarrow A \rightarrow 0)$  be a 2-extension of  $d$ -torsion  $\Gamma$ -modules representing  $e$ . Using Pontryagin duality  $\mathrm{Hom}(\cdot, \mathbf{F})$ , one can assume  $B$  is free. Lemma 5.1 below implies that one can also assume that  $E_2$  is free as a  $\mathbf{F}$ -module.

A  $\mathcal{E}$ -diagram in  $\mathcal{M}_d$  is a commutative diagram with exact lines in the category of finite  $d$ -torsion abelian groups:

$$(4.1) \quad \begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & E_1 & \longrightarrow & E_2 & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow \sim \mathrm{id} & & \downarrow \phi_1 & & \downarrow \sim \mathrm{id} & & & & \\ 0 & \longrightarrow & B & \longrightarrow & F_1 & \longrightarrow & E_2 & \longrightarrow & 0 & & . \end{array}$$

In particular, in such a diagram,  $F_1$  is free as a  $\mathbf{F}$ -module. Therefore Lemma 5.6 below implies that there is a unique such diagram, say  $\mathcal{D}$ , up to isomorphism.

The 2-extension  $\mathcal{E}$  defines a group homomorphism  $p : \Gamma \rightarrow \mathrm{Aut}(\mathcal{E}) := \mathrm{Aut}_{\mathcal{M}_d}(\mathcal{E})$  (see Example 3.3), so that  $p$  corresponds to a  $\mathrm{Spec}(k)$ -torsor  $P_{\mathcal{E}}$  under  $\mathrm{Aut}(\mathcal{E})$ .

Then Example 3.3 relates the triviality of the class  $e$  to the existence of a lifting of the torsor  $P_{\mathcal{E}}$  to the group  $\mathrm{Aut}(\mathcal{D})$ .

Let  $X$  be the lifting variety  $X_{\varphi, P_{\mathcal{E}}}$  for the natural morphism of finite groups  $\varphi : \text{Aut}(\mathcal{D}) \rightarrow \text{Aut}(\mathcal{E})$ , where those groups are considered as constant algebraic  $k$ -groups.

Then Example 3.3 and Proposition 2.1 imply that  $X$  is a splitting variety for  $e$ .  $\square$

**Corollary 4.2.** — *Let  $A$  be a finite  $d$ -torsion  $\Gamma$ -modules and  $\alpha \in H^2(k, A)$ .*

*Then, there exists a smooth geometrically integral  $k$ -variety  $X$  which is a splitting variety for  $\alpha$ .*

**Remark 4.3.** — This corollary recovers a result of Krashen (see [4], Theorem 6.3).

*Proof.* — By Remark 1.2, we have a canonical isomorphism  $\text{YExt}_{k,d}^2(\mathbf{F}, A) \xrightarrow{\sim} H^2(k, A)$ , hence the corollary is a direct consequence of Theorem 4.1.  $\square$

**Remark 4.4.** — Let  $R/k$  be a central simple algebra, of index  $d$ . Assume that  $d$  is invertible in  $k$ . In the previous corollary, take  $A$  to be  $\mu_d$ , the group of  $d$ -th roots of unity, and take  $\alpha \in H^2(k, A) = H^2(k, \mu_d)$  to be the Brauer class of  $R$ . Let  $X$  be a splitting variety, as constructed in the proof of Theorem 4.1. It would be interesting to decide whether  $X$  is stably birational to the Severi-Brauer variety  $\text{SB}(R)$ .

## 5. Splitting families for $n$ -extensions ( $n \geq 3$ )

In this section, we prove the main theorem of the paper (see Theorem 5.7 below).

Let  $d \geq 2$  and  $\mathbf{F} := \mathbf{Z}/d\mathbf{Z}$ .

Let  $n \geq 2$  and let  $A, B$  be objects of  $\mathcal{M}_{k,d}$ .

Fix a class  $e \in \text{YExt}_{k,d}^n(A, B)$ .

**Lemma 5.1.** — *There exists a representative  $\mathcal{E} = (0 \rightarrow B \xrightarrow{f_0} E_1 \xrightarrow{f_1} \dots \rightarrow E_{n-1} \xrightarrow{f_{n-1}} E_n \xrightarrow{f_n} A \rightarrow 0)$  of  $e$  in  $\text{YExt}_{k,d}^n(A, B)$  such that  $E_n$  is free.*

*Proof.* —

$$\begin{array}{ccccccccccccccc} 0 & \longrightarrow & B & \xrightarrow{g_0} & F_1 & \xrightarrow{g_1} & \dots & \longrightarrow & F_{n-2} & \xrightarrow{g_{n-2}} & F'_{n-1} & \xrightarrow{f_{n-1}} & E_n & \xrightarrow{f_n} & A & \longrightarrow & 0 \\ & & \downarrow = & & \downarrow = & & & & \downarrow = & & \downarrow & & \downarrow & & \downarrow = & & \\ 0 & \longrightarrow & B & \xrightarrow{g_0} & F_1 & \xrightarrow{g_1} & \dots & \longrightarrow & F_{n-2} & \xrightarrow{g_{n-2}} & F_{n-1} & \xrightarrow{g_{n-1}} & F_n & \xrightarrow{g_n} & A & \longrightarrow & 0, \end{array}$$

which proves the lemma.  $\square$

**Remark 5.2.** — Repeating the construction of the proof of Lemma 5.1, one can even assume that  $E_2, \dots, E_n$  are free as  $\mathbf{F}$ -modules.

We now fix once and for all a  $n$ -extension

$$\mathcal{E} = (0 \rightarrow B \xrightarrow{f_0} E_1 \xrightarrow{f_1} \dots \rightarrow E_{n-1} \xrightarrow{f_{n-1}} E_n \xrightarrow{f_n} A \rightarrow 0)$$

in  $\text{YExt}_{k,d}^n(A, B)$  representing  $e$  such that  $E_n$  is free as a  $\mathbf{F}$ -module.

Let us define the notion of a free  $n$ -extension.

**Definition 5.3.** — An object

$$\mathcal{L} = (0 \rightarrow D \rightarrow L_1 \rightarrow \cdots \rightarrow L_{n-1} \rightarrow L_n \rightarrow C \rightarrow 0)$$

in  $\mathbf{YExt}_{k,d}^n(C, D)$  is said to be free if  $L_i$  is free as a  $\mathbf{F}$ -module, for all  $1 \leq i \leq n$ .

**Lemma 5.4.** — Assume that  $B$  is free as a  $\mathbf{F}$ -module. Then the class  $e$  is trivial in  $\mathbf{YExt}_{k,d}^n(A, B)$  if and only if there exists an injective  $\mathcal{E}$ -diagram  $\phi : \mathcal{E} \rightarrow \mathcal{L}$ , where  $\mathcal{L} \in \mathbf{YExt}_{k,d}^{n-1}(E_n, B)$  is free.

*Proof.* — The existence of such a diagram implies the triviality of  $e$ , by Lemma 3.1.

Let us now prove the converse. Assume  $e = 0$ . Then by Lemma 3.1, there exists a  $\mathcal{E}$ -diagram  $\varphi : \mathcal{E} \rightarrow \mathcal{G}$  of the following shape:

$$(5.1) \quad \begin{array}{ccccccccccc} 0 & \longrightarrow & B & \xrightarrow{f_0} & E_1 & \xrightarrow{f_1} & \cdots & \longrightarrow & E_{n-1} & \xrightarrow{f_{n-1}} & E_n & \xrightarrow{f_n} & A & \longrightarrow & 0 \\ & & \downarrow = & & \downarrow \varphi_1 & & & & \downarrow \varphi_{n-1} & & \downarrow = & & & & \\ 0 & \longrightarrow & B & \xrightarrow{h_0} & G_1 & \xrightarrow{h_1} & \cdots & \longrightarrow & G_{n-1} & \xrightarrow{h_{n-1}} & E_n & \longrightarrow & 0 & & \end{array},$$

We now prove that we can replace the diagram  $\varphi : \mathcal{E} \rightarrow \mathcal{G}$  by another one  $\psi : \mathcal{E} \rightarrow \mathcal{F}$  that is injective. We construct  $\mathcal{F}$  and  $\psi$  by modifying  $\mathcal{G}$  and  $\varphi$  as follows: let  $F_1 := G_1 \oplus E_2$ ,  $F_{n-1} := G_{n-1} \oplus E_{n-1}$ , and for all  $2 \leq i \leq n-2$ ,  $F_i := G_i \oplus E_i \oplus E_{i+1}$ , with the following morphisms:

- $g_0 : B \rightarrow F_1 = G_1 \oplus E_2$  is  $h_0 \oplus 0$ ,
- $g_1 : F_1 = G_1 \oplus E_2 \rightarrow G_2 \oplus E_2 \oplus E_3 = F_2$  is given by the matrix  $\begin{pmatrix} h_1 & 0 \\ 0 & \text{id} \\ 0 & 0 \end{pmatrix}$ ,
- for  $2 \leq i \leq n-3$ ,  $g_i : F_i = G_i \oplus E_i \oplus E_{i+1} \rightarrow G_{i+1} \oplus E_{i+1} \oplus E_{i+2} = F_{i+1}$  is given by the matrix  $\begin{pmatrix} h_i & 0 & 0 \\ 0 & 0 & \text{id} \\ 0 & 0 & 0 \end{pmatrix}$ ,
- $g_{n-2} : F_{n-2} = G_{n-2} \oplus E_{n-2} \oplus E_{n-1} \rightarrow G_{n-1} \oplus E_{n-1} = F_{n-1}$  is given by the matrix  $\begin{pmatrix} h_{n-2} & 0 & 0 \\ 0 & 0 & \text{id} \end{pmatrix}$ ,
- $g_{n-1} : F_{n-1} = G_{n-1} \oplus E_{n-1} \rightarrow E_n$  is given by the matrix  $\begin{pmatrix} h_{n-1} & 0 \end{pmatrix}$ .

The maps  $\psi_i : E_i \rightarrow F_i$  are given by:

- $\psi_1 := \varphi_1 \oplus f_1 : E_1 \rightarrow F_1 = G_1 \oplus E_2$ ,
- for  $2 \leq i \leq n-2$ ,  $\psi_i := \varphi_i \oplus \text{id} \oplus f_i : E_i \rightarrow F_i = G_i \oplus E_i \oplus E_{i+1}$ ,
- $\psi_{n-1} := \varphi_{n-1} \oplus \text{id} : E_{n-1} \rightarrow F_{n-1} = G_{n-1} \oplus E_{n-1}$ .

Then one checks that  $\mathcal{F}$  is a  $(n-1)$ -extension of  $E_n$  by  $B$ , and that  $\psi : \mathcal{E} \rightarrow \mathcal{F}$  is an injective  $\mathcal{E}$ -diagram.

It is now sufficient to prove the existence of an injective morphism  $\psi' : \mathcal{F} \rightarrow \mathcal{L}$  in  $\mathbf{YExt}_{k,d}^{n-1}(E_n, B)$ , with  $\mathcal{L}$  free.

The  $\Gamma$ -module  $F_1$  can be embedded in a finite  $\Gamma$ -module  $L_1$  that is free as a  $\mathbf{F}$ -module. Then we have a natural commutative diagram of exact sequences:

$$\begin{array}{ccccccccccccccccccc} 0 & \longrightarrow & B & \xrightarrow{g_0} & F_1 & \xrightarrow{g_1} & F_2 & \xrightarrow{g_2} & F_3 & \xrightarrow{g_3} & \dots & \longrightarrow & F_{n-1} & \xrightarrow{g_{n-1}} & E_n & \longrightarrow & 0 \\ & & \downarrow = & & \downarrow \psi'_1 & & \downarrow \widetilde{\psi}_2 & & \downarrow = & & & & \downarrow = & & \downarrow = & & \\ 0 & \longrightarrow & B & \xrightarrow{m_0} & L_1 & \xrightarrow{g'_1} & F'_2 & \xrightarrow{g'_2} & F_3 & \xrightarrow{g_3} & \dots & \longrightarrow & F_{n-1} & \xrightarrow{g_{n-1}} & E_n & \longrightarrow & 0 \end{array},$$

where the second square (i.e. the  $\Gamma$ -module  $F'_2$ , and the maps  $\widetilde{\psi}_2, g'_1, g'_2$ ) is defined as the pushout of  $g_1$  and  $\psi'_1$ . In particular,  $\psi'_1$  and  $\widetilde{\psi}_2$  are injective and  $L_1$  is free. An easy induction (starting by embedding  $F'_2$  into a  $\Gamma$ -module that is free as a  $\mathbf{F}$ -module) proves that there exists a commutative diagram

$$\begin{array}{ccccccccccccccccccc} 0 & \longrightarrow & B & \xrightarrow{g_0} & F_1 & \xrightarrow{g_1} & F_2 & \xrightarrow{g_2} & \dots & \longrightarrow & F_{n-1} & \xrightarrow{g_{n-1}} & E_n & \longrightarrow & 0 \\ & & \downarrow = & & \downarrow \psi'_1 & & \downarrow \psi'_2 & & & & \downarrow \psi'_{n-1} & & \downarrow = & & \\ 0 & \longrightarrow & B & \xrightarrow{m_0} & L_1 & \xrightarrow{m_1} & L_2 & \xrightarrow{m_2} & \dots & \longrightarrow & L_{n-1} & \xrightarrow{m_{n-1}} & E_n & \longrightarrow & 0 \end{array},$$

with all vertical maps injective, and  $L_1, \dots, L_{n-2}$  free as  $\mathbf{F}$ -modules. Since  $E_n$  is free as a  $\mathbf{F}$ -module (see Lemma 5.1), then  $L_{n-1}$  is also free as a  $\mathbf{F}$ -module, which concludes the proof.  $\square$

For any non-negative integers  $a, b, m$ , define  $\mathcal{E}_{n-1}(a, b, m)$  to be the following  $(n-1)$ -extension of free  $\mathbf{F}$ -modules:

$$\mathcal{E}(a, b, m) := (0 \rightarrow F_0 \xrightarrow{g_0} F_1 \xrightarrow{g_1} \dots \rightarrow F_{n-2} \xrightarrow{g_{n-2}} F_{n-1} \xrightarrow{g_{n-1}} F_n \rightarrow 0),$$

where  $F_0 := \mathbf{F}^b$ ,  $F_1 := \mathbf{F}^b \oplus \mathbf{F}^m$ ,  $F_2 = \dots = F_{n-2} = \mathbf{F}^m \oplus \mathbf{F}^m$ ,  $F_{n-1} = \mathbf{F}^m \oplus \mathbf{F}^a$ ,  $F_n = \mathbf{F}^a$ , and  $g_0(x) := (x, 0)$ ,  $g_i(x, y) = (y, 0)$  for  $1 \leq i \leq n-2$  and  $g_{n-1}(x, y) := y$ .

**Lemma 5.5.** — *Assume  $B$  is free as a  $\mathbf{F}$ -module. Let  $\mathcal{L} = (0 \rightarrow B \xrightarrow{m_0} L_1 \xrightarrow{m_1} \dots \rightarrow L_{n-1} \xrightarrow{m_{n-1}} E_n \rightarrow 0)$  be an object in  $\mathbf{YExt}_{k,d}^{n-1}(E_n, B)$  that is free. Let  $a$  (resp.  $b$ ) be the rank of  $E_n$  (resp.  $B$ ).*

*Then there exist an integer  $m$ , a Galois action on  $\mathcal{E}_{n-1}(a, b, m)$  and an injective morphism  $\phi : \mathcal{L} \rightarrow \mathcal{E}_{n-1}(a, b, m)$  in  $\mathbf{YExt}_{k,d}^{n-1}(E_n, B)$ .*

*Proof.* — Choose  $m$  large enough such that  $m$  is greater than or equal to the rank of  $L_i$ , for all  $i$ .

Splitting the  $(n-1)$ -extension into short exact sequences, the statement reduces to two facts :

- given any free  $\mathbf{F}$ -module  $L$  with a  $\Gamma$ -action and any integer  $s$  greater than or equal to the rank of  $L$ , there exists a decomposition  $\mathbf{F}^s = L \oplus L'$  and therefore a  $\Gamma$ -action on  $\mathbf{F}^s = L \oplus L'$  such that  $\Gamma$  acts trivially on  $L'$ , with a  $\Gamma$ -equivariant embedding of  $L$  into  $\mathbf{F}^s$ .

- given a diagram of short exact sequences of free  $\mathbf{F}$ -modules (the second one being the obvious one)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_1 & \longrightarrow & L_2 & \longrightarrow & L_3 & \longrightarrow & 0 \\ & & \downarrow & & & & \downarrow & & \\ 0 & \longrightarrow & \mathbf{F}^r & \xrightarrow{i} & \mathbf{F}^r \oplus \mathbf{F}^s & \xrightarrow{p} & \mathbf{F}^s & \longrightarrow & 0 \end{array}$$

where the vertical maps are injective,  $L_1, L_2, L_3, \mathbf{F}^r$  and  $\mathbf{F}^s$  are endowed with a  $\Gamma$ -action such that the arrows are  $\Gamma$ -equivariant, and assuming there is a decomposition  $\mathbf{F}^s = L_3 \oplus L'$  such that the action on  $L'$  is trivial, there exists a  $\Gamma$ -action on  $\mathbf{F}^r \oplus \mathbf{F}^s$  and an embedding  $L_2 \rightarrow \mathbf{F}^r \oplus \mathbf{F}^s$  making the previous diagram a commutative diagram of  $\Gamma$ -modules.

Indeed, the choice of a section of the first line and the action of  $\Gamma$  on  $L_2$  define a map  $\rho : \Gamma \rightarrow \text{Hom}(L_3, L_1)$  satisfying a cocycle condition

$$\rho(\sigma\tau)(x) = {}^\sigma\rho(\tau)(x) + \rho(\sigma)(\tau x).$$

In order to prove the aforementioned fact, one needs to extend  $\rho$  to a map  $\tilde{\rho} : \Gamma \rightarrow \text{Hom}(\mathbf{F}^s, \mathbf{F}^r)$  satisfying a similar condition. One easily checks that the maps  $\tilde{\rho}(\gamma) : \mathbf{F}^s = L_3 \oplus L' \rightarrow \mathbf{F}^r$  defined by  $\tilde{\rho}(\gamma)(x, y) := \rho(\gamma)(x)$  do satisfy this condition.  $\square$

**Lemma 5.6.** — *Assume  $B$  is free as a  $\mathbf{F}$ -module. Let  $\phi, \psi : \mathcal{E} \rightarrow \mathcal{F}$  be two injective  $\mathcal{E}$ -diagrams in the category  $\mathcal{M}_d$ , such that  $\mathcal{F}$  is free.*

*Then there exists an automorphism  $\varepsilon : \mathcal{F} \rightarrow \mathcal{F}$  in  $\mathbf{YExt}_d^{n-1}(E_n, B)$  such that  $\psi = \varepsilon \circ \phi$ .*

*Proof.* — As in the proof of Lemma 5.5, splitting the  $n$ -extension  $\mathcal{E}$  into short exact sequences reduces the statement to the following facts:

- given two embeddings of  $\mathbf{F}$ -modules  $\phi : E \rightarrow F$  and  $\psi : E \rightarrow F$  with  $F$  free, there exists an automorphism  $\varepsilon$  of  $F$  such that  $\psi = \varepsilon \circ \phi$ . Indeed, one only needs to choose one basis of  $F$  adapted to each embedding.
- given two diagrams of short exact sequences of  $\mathbf{F}$ -modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & 0 \\ & & \downarrow \phi_1, \psi_1 & & \downarrow \phi_2, \psi_2 & & \downarrow \phi_3, \psi_3 & & \\ 0 & \longrightarrow & F_1 & \xrightarrow{i} & F_2 & \xrightarrow{p} & F_3 & \longrightarrow & 0 \end{array}$$

where the vertical maps are injective and the  $F_i$  are free, and given  $\varepsilon_1 \in \text{Aut}(F_1)$  and  $\varepsilon_3 \in \text{Aut}(F_3)$  such that  $\psi_i = \varepsilon_i \circ \phi_i$ , there exists  $\varepsilon_2 \in \text{Aut}(F_2)$  such that  $\psi_2 = \varepsilon_2 \circ \phi_2$  and  $(\varepsilon_i)_{1 \leq i \leq 3}$  is an automorphism of the bottom exact sequence.

Indeed, the modules  $F_i$  being free, one can first fix a section  $F_2 = F_1 \oplus F_3$ . Then  $\phi_2$  and  $\psi_2$  induce morphisms  $\phi_{2,1}, \psi_{2,1} \in \text{Hom}(A_2, F_1)$  and  $\phi_{2,3}, \psi_{2,3} \in \text{Hom}(A_2, F_3)$ . Then the existence of  $\varepsilon_2$  is equivalent to the existence of  $\varepsilon \in \text{Hom}(F_3, F_1)$  such that  $\psi_{2,1} = \varepsilon_1 \circ \phi_{2,1} + \varepsilon \circ \phi_{2,3}$ . Such an  $\varepsilon$  exists since the map  $\phi_3^* : \text{Hom}(F_3, F_1) \rightarrow \text{Hom}(A_3, F_1)$  is onto (because  $F_1$  and  $F_3$  are free and  $\phi_3$  is injective).  $\square$

The following statement is the main result of this section:

**Theorem 5.7.** — *Let  $n \geq 3$  and let  $A, B$  be objects of  $\mathcal{M}_{k,d}$  and assume  $A$  or  $B$  is free in  $\mathcal{M}_d$ . Pick a class  $e \in \mathrm{YExt}_{k,d}^n(A, B)$ .*

*Then, there exists a smooth geometrically integral ind-variety  $(X_i)_{i \in \mathbf{N}}$ , which is a splitting family for  $e$ .*

Before proving this theorem, we state explicitly the following consequence:

**Corollary 5.8.** — *Let  $n \geq 3$ , let  $A$  be a finite  $\Gamma$ -module and let  $\alpha \in H^n(k, A)$ .*

*Then there exists a smooth geometrically integral ind-variety  $(X_i)_{i \in \mathbf{N}}$ , which is a splitting family for  $\alpha$ .*

*Proof.* — Combine the previous theorem and remark 1.2. □

We now focus on the proof of the main theorem.

*Proof of Theorem 5.7.* — Using Pontryagin duality  $\mathrm{Hom}(\cdot, \mathbf{F})$ , one can assume that  $B$  is free.

Let  $\mathcal{E}$  be a  $n$ -extension representing  $e$ , as given by Lemma 5.1. Let  $a$  (resp.  $b$ ) be the rank of  $E_n$  (resp.  $B$ ) as a free  $\mathbf{F}$ -module.

The  $n$ -extension  $\mathcal{E}$  defines a group homomorphism  $p : \Gamma \rightarrow \mathrm{Aut}(\mathcal{E}) := \mathrm{Aut}_{\mathcal{M}_d}(\mathcal{E})$  (see Example 3.3), so that  $p$  corresponds to a  $\mathrm{Spec}(k)$ -torsor  $P_{\mathcal{E}}$  under the constant  $k$ -group  $\mathrm{Aut}(\mathcal{E})$ .

Then Example 3.3 relates the triviality of the class  $e$  to the existence of a  $\mathcal{E}$ -diagram  $\mathcal{D}$  in the category  $\mathcal{M}_d$  together with a lifting of the torsor  $P_{\mathcal{E}}$  to the (constant) group  $\mathrm{Aut}_{\mathcal{M}_d}(\mathcal{D})$ .

Lemmas 5.4 and 5.5 ensure that in order to construct the splitting varieties, it is sufficient to consider only injective diagrams (of  $\mathbf{F}$ -modules)  $\phi : \mathcal{E} \rightarrow \mathcal{E}_{n-1}(a, b, m)$ , for some  $m \in \mathbf{N}$ , i.e. diagrams of the following shape (the aforementioned lemmas essentially say that such diagrams are cofinal in the category of diagrams):

$$(5.2) \quad \begin{array}{ccccccccccc} 0 & \longrightarrow & B & \xrightarrow{f_0} & E_1 & \xrightarrow{f_1} & \dots & \longrightarrow & E_{n-1} & \xrightarrow{f_{n-1}} & E_n & \xrightarrow{f_n} & A & \longrightarrow & 0 \\ & & \downarrow \sim \phi_0 & & \downarrow \phi_1 & & & & \downarrow \phi_{n-1} & & \downarrow \sim \phi_n & & & & \\ 0 & \longrightarrow & F_0 & \xrightarrow{g_0} & F_1 & \xrightarrow{g_1} & \dots & \longrightarrow & F_{n-1} & \xrightarrow{g_{n-1}} & F_n & \longrightarrow & 0 & & \end{array},$$

where all  $\phi_i$  are injective.

In addition, Lemma 5.6 implies that one only needs to consider one such diagram for each  $m$  (since such diagrams with the same  $m$  are equivalent up to an automorphism of  $\mathcal{E}_{n-1}(a, b, m)$ ).

Therefore, let us fix, for some  $m_0 \in \mathbf{N}$  (sufficiently large), one diagram  $\mathcal{D}_{m_0}$  of the shape (5.2) in the category of  $\mathbf{F}$ -modules: such a diagram  $\mathcal{D}_{m_0}$  exists in  $\mathcal{M}_d$  because  $\mathcal{E}$  is trivial as a  $n$ -extension of  $\mathbf{F}$ -modules, since  $B$  is free. Define now diagrams  $\mathcal{D}_m$ , for  $m \geq m_0$ , in a compatible way: the diagram  $\mathcal{D}_m$  is obtained from the diagram  $\mathcal{D}_{m_0}$  by composing the morphism  $\phi_{m_0} : \mathcal{E} \rightarrow \mathcal{E}(a, b, m_0)$  with the natural (injective) morphism  $\mathcal{E}(a, b, m_0) \rightarrow \mathcal{E}(a, b, m)$ .

We have thus defined a direct system of diagrams  $\mathcal{D}_m$ . For all  $m$ , let  $X_m$  denote the  $k$ -variety  $X_{\text{Aut}(\mathcal{D}_m) \rightarrow \text{Aut}(\mathcal{E}), P_{\mathcal{E}}}$  defined in Proposition 2.1. By functoriality of the construction of these varieties and by the natural (injective) group homomorphisms  $\text{Aut}(\mathcal{D}_m) \rightarrow \text{Aut}(\mathcal{D}_{m+1})$ , we get a direct system of  $k$ -varieties  $X_m$ .

In addition, Lemma 5.6 implies that the morphisms  $\text{Aut}(\mathcal{D}_m) \rightarrow \text{Aut}(\mathcal{E})$  are surjective, hence the varieties  $X_m$  are smooth and geometrically unirational.

By construction,  $(X_m)_{m \in \mathbf{N}}$  is a splitting family for  $e$ , which concludes the proof.  $\square$

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