Multilinear descent theory

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March 2010

1. Notation

1.1. Symmetric groups. Let n be a positive integer. In the symmetric group S_n, we use the standard notation for cycles. For instance, (12) denotes the transposition exchanging 1 and 2, whereas (123) denotes the 3-cycle sending 1 to 2, 2 to 3 and 3 to 1.

1.2. Endomorphisms. Let $(\mathcal{S}, \otimes, 1)$ be a symmetric monoidal category (often denoted by $\mathcal{S}$ in short), and let E be an object of $\mathcal{S}$. For any positive integer n, the symmetric group S_n acts naturally on $E^{\otimes^n}$. For instance, (12) stands for the canonical exchange involution on $E \otimes E$. Let f be an endomorphism of $E \otimes E$. We shall denote by f_1 the endomorphism $\mathrm{Id}_E \otimes f$ of $E \otimes E \otimes E$. Similarly, we denote by f_2 the endomorphism $(12) \circ f_1 \circ (12)$ of $E \otimes E \otimes E$, and by f_3 the endomorphism $f \otimes \mathrm{Id}_E$. This notation obviously extends to the case of an endomorphism g of $E^{\otimes^n}$. It induces $n+1$ endomorphisms g_i of $E^{\otimes^{n+1}}$. For instance, $g_{n+1} = g \otimes \mathrm{Id}_E$.
We denote by f_{12} the endomorphism $f \otimes \mathrm{Id}_{E \otimes E}$ of $E \otimes E \otimes E \otimes E$. Simarly, we have endomorphisms f_{ij} of $E \otimes E \otimes E \otimes E$, for each pair of integers i and j with $1 \leq i < j \leq 4$.
An endomorphism h of E will be called constant if it comes from an endomorphism of 1, i.e. if there exists an endomorphism k of 1 such that $f = k \otimes \mathrm{Id}_E$, via the natural isomorphism $1 \otimes E \simeq E$.

1.3. Projective spaces. Let k be a field, and V a finite-dimensional k-vector space. We denote by $\mathbb{P}_k(V)$ the projective space of lines in V.

2. Why descent theory is a linear theory

In this section, we review some facts about Grothendieck's descent theory and show that, in a sense to be made precise, this theory is linear. We then explain the main motivation of this paper.

Let us briefly recall what descent theory is, in its most simple form. The exposition that follows is not the exact context in which descent theory is generally stated. Meanwhile, it is very close, and the interested reader may check that it implies descent theory for nonzero ring homomorphisms $k \longrightarrow B$, where k is a field.
Let k be a field. Let $\mathcal{V}$ denote the category of k-vector spaces. Fix a nonzero $A \in \mathcal{V}$. Consider the functor

$$\mathcal{V} \longrightarrow \mathcal{V},$$

$$X \mapsto A \otimes X.$$

We want to find a condition for a vector space B to be in the essential image of this functor. Imagine that $B = A \otimes X$, for an $X \in \mathcal{V}$. Then $A \otimes B = A \otimes A \otimes X$ is endowed with a canonical map s_X, the switch, defined by

$$\phi(a_1 \otimes a_2 \otimes x) = (a_2 \otimes a_1 \otimes x).$$

Now, the space $A \otimes A \otimes B$ is naturally endowed with two automorphisms: the switch s_B (exchanging the A factors), and the map $(s_X)_1 = \mathrm{Id}_A \otimes s_X$. Of course, this last map depends on the decomposition $B = A \otimes X$, whereas s_B does not. We instantly compute that $s_B \circ (s_X)_1$ has order three, which amounts to say, since s_X and s_B are involutions, that $(s_X)_1 s_B = s_B (s_X)_1 s_B (s_X)_1$; in other words, that $(s_X)_1 s_B = (s_X)_2 (s_X)_1$. Now define a category $\mathcal{V}_A$ as follows.
Objects of $\mathcal{V}_A$ are couples (B, ϕ), where B is a vector space, and ϕ is an automorphism of $A \otimes B$ such that the relation

$$\phi_1 s_B = \phi_2 \phi_1$$

holds (both sides being automorphisms of $A \otimes A \otimes B$).
A morphism $(B, \phi) \longrightarrow (B', \phi')$ is a linear map $f : B \longrightarrow B'$, such that $\phi' \circ (\mathrm{Id}_A \otimes f) = (\mathrm{Id}_A \otimes f) \circ \phi$.
We have an obvious functor

$$\Psi : \mathcal{V} \longrightarrow \mathcal{V}_A,$$

$$A \mapsto (A \otimes X, s_X).$$

The basic result in descent theory (which contains its main ideas) is that Ψ is an equivalence of categories. This can also be viewed as a generalization of Morita equivalence, since we do not assume A to be finite-dimensional.

In view of this brief exposition, it appears that descent theory is a categorical way of solving the equation

$$B = A \otimes X$$

by endowing B with a descent data, which turns out to be always effective. It is in this sense that descent theory is linear. The main purpose of this paper is to develop a bilinear descent theory. More precisely, we shall consider the equation

$$B \simeq X \otimes Y,$$

with 'unknowns' X and Y, and study which descent data is needed on B in order to solve it. It turns out that, in the category of pointed k-vector spaces over some field (and more generally in pointed symmetric monoidal categories), the descent data that occur are always effective. However, in the category of k-vector spaces, descent data are not always effective. The obstruction to this is the Brauer group of k. Our construction will be used to give a new simple definition of the Brauer group of any symmetric monoidal category. It need not be equal to the Brauer group definied by Vitale in [Vit].

3. Preliminary I: pointed symmetric monoidal categories

Definition 3.1. *Let $\mathcal{S}$ be a symmetric monoidal category. We build another symmetric monoidal category, the pointed symmetric monoidal category $\mathcal{P}(\mathcal{S})$ associated to $\mathcal{S}$, as follows. An object of $\mathcal{P}(\mathcal{S})$ is a morphism $1 \longrightarrow X$ in $\mathcal{S}$, admitting* *a retraction (which is not specified in the data). A morphism from* $1 \longrightarrow X$ *to* $1 \longrightarrow X'$ *is an arrow* $X \longrightarrow X'$ *in* $\mathcal{S}$*, such that the obvious diagram*

$$\begin{tikzcd} 1 \arrow[r] \arrow[d, equal] & X \arrow[d] \\ 1 \arrow[r] & X' \end{tikzcd}$$

commutes. The tensor product of $1 \xrightarrow{f} X$ *and* $1 \xrightarrow{f'} X$ *defined to be* $1 \xrightarrow{f\otimes f'} X \otimes X'$*. The commutativity constraint* $(1 \xrightarrow{f\otimes f'} X \otimes X') \simeq (1 \xrightarrow{f'\otimes f} X' \otimes X)$ *is given by that of* $\mathcal{S}$*.*

It is readily checked that $\mathcal{P}(\mathcal{S})$ is indeed a symmetric monoidal category.
Note that we have an obvious (strong monoidal) forgetful functor

$$F_{\mathcal{S}} : \mathcal{P}(\mathcal{S}) \longrightarrow \mathcal{S}.$$

Definition 3.2. *Let* $\mathcal{S}$ *be a symmetric monoidal category. If the functor* $F_{\mathcal{S}}$ *is an equivalence of categories, we say that* $\mathcal{S}$ *is a pointed symmetric monoidal category.*

Proposition 3.3. *Let* $\mathcal{S}$ *be a symmetric monoidal category. Then* $\mathcal{P}(S)$ *is a pointed symmetric monoidal category. The category* $\mathcal{S}$ *itself is pointed if and only if the following holds: a unit of* $\mathcal{S}$ *is an initial object, and every object of* $\mathcal{S}$ *admits at least one morphism to a unit.*

Proof. Left to the reader. $\square$
The next lemma will be used in the proof of theorem 6.1.

Lemma 3.4. *Let* $\mathcal{S}$ *be a pointed symmetric monoidal category with equalizers, such that, for every object* X *of* $\mathcal{S}$*, the functor* $.\otimes X$ *preserves equalizers. Let* A*,* B*,* A'*,* B' *be objects of* $\mathcal{S}$*. Let* $f_i : A \longrightarrow B$ *and* $f'_i : A' \longrightarrow B'$*,* $i = 1, 2$*, be morphisms. Denote by* X *(resp* X'*) the equalizer of* f_1 *and* f_2 *(resp.* f'_1 *and* f'_2*). Assume the canonical morphism* $i_X : X \longrightarrow A$ *(resp.* $i_{X'} : X' \longrightarrow A'$*) admits a retraction* r_X *(resp.* $r_{X'}$*). Let* Z *be an object of* $\mathcal{S}$*, and* $g : Z \longrightarrow A \otimes A'$ *be a morphism. Then* g *factors through* $X \otimes X' \xrightarrow{i_X \otimes i_{X'}} A \otimes A'$ *if and only if the following two equalities hold:*

i) $(f_1 \otimes \mathrm{Id}_{A'}) \circ g = (f_2 \otimes \mathrm{Id}_{A'}) \circ g$
ii) $(\mathrm{Id}_A \circ f'_1) \circ g = (\mathrm{Id}_A \circ f'_2) \circ g$*.*

Proof. By the hypothesis of the lemma, i) means that g factors through the morphism $X \otimes A' \xrightarrow{i_X \otimes \mathrm{Id}_{A'}} A \otimes A'$. Because r_X is a retraction of i_X, this means that g equals the composite

$$Z \xrightarrow{g} A \otimes A' \xrightarrow{r_X \otimes \mathrm{Id}_{A'}} X \otimes A' \xrightarrow{i_X \otimes \mathrm{Id}_{A'}} A \otimes A'.$$

Condition ii) implies the similar statement for X'. Put $h = (r_X \otimes r_{X'}) \circ g$. We compute:

$$Z \xrightarrow{h} X \otimes X' \xrightarrow{i_X \otimes i_{X'}} A \otimes A'$$

$$= Z \xrightarrow{g} A \otimes A' \xrightarrow{(i_X \circ r_X) \otimes \mathrm{Id}_{A'}} A \otimes A' \xrightarrow{\mathrm{Id}_X \otimes (i_{X'} \circ r_{X'})} A \otimes A' = g.$$

$\square$

4. Preliminary II: some group theory

In this section, W is a free group on one generator $w \in W$. Let G be a group, and $X \subset G$ be subset. Imagine we want to measure what elements of X have in common. On way to do this is to look at relations simultaneously satisfied by all elements of X. Let us be more precise.

Definition 4.1. *Let $G * W$ be the amalgamated sum of G and W. The group of relations of X in G is the group*

$$R(X, G) := \cap_{x \in X} \mathrm{Ker} f_x \subset G * W,$$

where for $x \in X$, $f_x : G * W \longrightarrow G$ is the homomorphism which is the identity on G and which sends w to x.
We shall denote by $S(X, G)$ the quotient $(G * W)/R(X, G)$. We call it the smash group of X (in G). It comes equipped with a canonical element τ_X, which is the class of $w \in (G * W)$ modulo $R(X, G)$. We call it the smash of X.

Examples 4.2. When $X = \varnothing$, we have $S(X, G) = \{e\}$.
When X is a one element set, $S(X, G)$ is canonically isomorphic to G.
When G is abelian, $S(X, G)$ is canonically isomorphic to $G \times (\mathbb{Z}/n\mathbb{Z})$, where n is the smallest positive integer such that the n'th powers of elements of X all coincide, or zero if no such integer exists.

4.1. Smashing in symmetric groups. Let $n > 2$ be an integer, and S_n be the symmetric group on n letters, with the usual notation (cf. section 1).

Lemma 4.3. *The homomorphism*

$$f_{\mathrm{Id}} \times f_{(12)} : S_n * W \longrightarrow S_n \times S_n$$

(notation as in Definition 4.1) induces by passing to the quotient an isomorphism

$$S(\{\mathrm{Id}, (12)\}, S_n) \simeq S_n \times S_n.$$

Proof. We have to show that $f_{\mathrm{Id}} \times f_{(12)}$ is surjective. Let $1 \leq i \leq n$ be an integers. We have $(f_{\mathrm{Id}} \times f_{(12)})((2i)w(2i)) = (\mathrm{Id}, (1i))$. Since the transpositions $(1i)$ generate S_n, and since f_{Id} is obviously surjective, this shows the claim.

□

Proposition 4.4. *The subgroup $R(\{Id, (12)\}, S_3) \subset S_3 * W$ is generated (as a normal subgroup of $S_3 * W$) by the elements w^2,$w(12)w(12)$ and $w(13)w(13)w(23)w(23)$.*

Proof. We will now show that, modulo H, every element of $S_3 * W$ is of the form awb or $wawb$, for $a, b \in S_3$. Since there are at most 18 expressions of each kind (remember that (12) and w commute modulo H), and since $S(\{\mathrm{Id}, (12)\}, S_3) \simeq S_3 \times S_3$ is of cardinality 36, this proves that $H = R(\{\mathrm{Id}, (12)\}, S_3)$. Let $x \in S_3 * W$. Modulo H, x can be written as $a_1 w a_2 w ... a_r$, with $a_i \in S_3$. If $r = 1$, $x = w\mathrm{Id}wa_1$ modulo H. If $r = 2$, there is nothing to prove. Assume $r \geq 3$. Modulo H, w commutes with (12). Hence, one easily sees that we can assume $a_i = \mathrm{Id}, (13)$ or (23) for $i = 1...r-1$. For $\{a, b\} = \{(13), (23)\}$, note that we have $awaw = wbwb$ and $awbw = wawb$ modulo H. Using this, we can assume $a_1 = \mathrm{Id}$, and thus that $r \geq 4$. But the same relations then yield $wa_2wa_3 = bwcw$ for some b,$c \in \{(13), (23)\}$. Hence $x = wa_2wa_3wa_4 \ldots a_r = bwcwwa_4 \ldots a_r = bwca_4 \ldots a_r$ modulo H, so that induction applies. □

5. The category of potential decompositions

Let $(\mathcal{S}, \otimes, 1)$ be a symmetric monoidal category. Let X, Y be two objects of $\mathcal{S}$. Put $E = X \otimes Y$. In the spirit of Grothendieck's descent theory, we want to find some intrinsic data on E which arises from the decomposition of E as the tensor product of X and Y. This is done as follows. Denote by s_X the automorphism of

$$E \otimes E = X \otimes Y \otimes X \otimes Y$$

given by

$$x \otimes y \otimes x' \otimes y' \mapsto x' \otimes y \otimes x \otimes y'.$$

Put $t := (s_X)_3$, an automorphism of $E \otimes E \otimes E$.
Loosely speaking, t is equal to (12) on the X-part of $E \otimes E \otimes E$, and to the identity on the Y-part of $E \otimes E \otimes E$. We thus see -and this is essential- that all words in $R(\{Id, (1,2)\}, S_3)$ (cf. definition 4.1) are equal to the identity when evaluated at t. Thanks to proposition 4.4, it suffices to keep the three relations

$$t^2 = \mathrm{Id},$$

$$(12)t = t(12)$$

and

$$(13)t(13)t = t(23)t(23).$$

We are thus led to defining a new category attached to $\mathcal{S}$.

Definition 5.1. *Let $\mathcal{S}$ be a symmetric monoidal category. We define a category* $\mathrm{PDec}(S)$ *(the category of potential decomposition of objects of $\mathcal{S}$) as follows.*
An object of $\mathrm{PDec}(S)$ *is a pair (E, ϕ), where E is an object of $\mathcal{S}$ and ϕ is an element of* $\mathrm{Aut}(E \otimes E)$ *such that the following equalities hold:*
i) ϕ^2 is a constant automorphism,
ii) $\phi \circ (12) = (12) \circ \phi$,
iii) $\phi_1 \phi_3 = \phi_3 \phi_2$ (as automorphisms of $E \otimes E \otimes E$).

A morphism from (E, ϕ) to (E', ϕ') is a morphism $f : E \longrightarrow E'$ such that $(f \otimes f) \circ \phi = \phi' \circ (f \otimes f)$, up to a constant automorphism.
The identity object $1_{\mathrm{PDec}(\mathcal{S})}$ is equal to $(1 \simeq 1 \otimes 1, s_1)$.
The tensor product of (E, ϕ) and (E', ϕ') is equal to $(E \otimes E', \phi \otimes \phi')$, via the natural identification

$$(E \otimes E) \otimes (E' \otimes E') \simeq (E \otimes E') \otimes (E \otimes E'),$$

$$(e_1 \otimes e_2) \otimes (e_1' \otimes e_2') \mapsto (e_1 \otimes e_1') \otimes (e_2 \otimes e_2').$$

Remark 5.2. Let $(E, \tilde{\phi})$ be an object of $\mathrm{PDec}(\mathcal{S})$. Let i, j, k be three integers such that $\{i, j, k\} = \{1, 2, 3\}$. On $E \otimes E \otimes E$, we have the relations $(jk) \circ \phi_j \circ (jk) = \phi_k$. We thus see that, in the definition of $\mathrm{PDec}(\mathcal{S})$, we can replace iii) by any of the following relations:
$\phi_i \circ \phi_j = \phi_j \circ \phi_k$,
$(jk) \circ \phi_j \circ (jk) \circ \phi_j = \phi_j \circ (ij) \circ \phi_j \circ (ij)$,
$(jk) \circ \phi_j \circ (ij) \circ \phi_j = \phi_j \circ (jk) \circ \phi_j \circ (ij)$.
For instance, the relation $\phi_1 \circ \phi_3 = \phi_3 \circ \phi_2$ is obtained by conjugating ii) by (23) (taking into account that (23) and ϕ_1 commute).

There is a functor

$$\Psi_{\mathcal{S}} : \mathcal{S}^2 \longrightarrow \mathrm{PDec}(S)$$

$$(X, Y) \mapsto (X \otimes Y, s_X).$$

In the spirit of Grothendieck's descent theory, we can wonder whether $\Psi_{\mathcal{S}}$ is an equivalence of categories.
There are several more or less obvious obstructions to this. We list four of them.

Obstruction 1. Let L be an invertible object of $\mathcal{S}$ such that the switch automorphism of $L \otimes L$ is the identity (this always happen, for instance, in the category of finite locally free sheaves on a scheme). Let X and Y be objects of $\mathcal{S}$. One readily checks that $\Psi_{\mathcal{S}}(X \otimes L, Y)$ and $\Psi_{\mathcal{S}}(X, L \otimes Y)$ are canonically isomorphic in $\mathrm{PDec}(\mathcal{S})$. Nevertheless, $(X \otimes L, Y)$ and $(X, L \otimes Y)$ are not isomorphic in $\mathcal{S}^2$ as soon as L is not isomorphic to 1.
Obstruction 2. Let $\lambda \in \mathrm{End}(1)$. If $\lambda \neq \mathrm{Id}_1$, then the two arrows (λ, Id_1) and (Id_1, λ) are different in $\mathcal{S}^2$ but their images under $\Psi_{\mathcal{S}}$ are the same.
Obstruction 3. We present it as an exercise for the reader. Take $\mathcal{S}$ to be the category of finite free modules over the ring $A = \mathbb{C}[\epsilon]$ (the $\mathbb{C}$-algebra of dual numbers). Take X, Y in $\mathcal{S}$ of rank 2, with basis x_1, x_2 and y_1, y_2, respectively. Show that the formula $1 \mapsto \epsilon(x_1 \otimes y_1 + x_2 \otimes y_2)$ defines a morphism $(A, \mathrm{Id}) \longrightarrow (X \otimes Y, c_Y)$ in $\mathrm{PDec}(\mathcal{S})$ and that this morphism does not lie in the image of $\Psi_{\mathcal{S}}$.
Obstruction 4. Take $\mathcal{S}$ to be the category of finite-dimensional vector spaces over a commutative field k. Let A be a central simple k-algebra. Assume first that $A = \mathrm{End}(V) \simeq V \otimes V^*$ for some (finite-dimensional) vector space V. Then $(V \otimes V^*, s_V)$ is an object of $\mathrm{PDec}(\mathcal{S})$. Assume now that A is arbitrary. By a descent argument, we can still define a canonical automorphism ϕ_A on $A \otimes A$ (which agrees with the one we just defined if A is split), such that (A, ϕ_A) is an object of $\mathrm{PDec}(\mathcal{S})$. We will show later that (A, ϕ_A) belongs to the essential image of $\Psi_{\mathcal{S}}$ if and only if A is split.

The fourth one is crucial: it reflects the existence of a Brauer group of $\mathcal{S}$, which we will investigate later.

There is a first trivial case in which Ψ_S is an equivalence of categories: that of sets.

Proposition 5.3. *Let $\mathcal{S}$ be the symmetric monoidal category of sets, with units the singletons and tensor product structure the cartesian product. Then $\Psi_{\mathcal{S}}$ is an equivalence of categories.*

Proof. This is mainly an exercice, which contains nonetheless some of the ideas of the proof of theorem 6.1. We present the proof here as a consequence of this theorem. Let $(E, \phi) \in \mathrm{PDec}(\mathcal{S})$. We first show that, for all $e \in E$, we have $\phi(e,e) = (e,e)$. Take $e \in E$. Then there exists $a \in E$ such that $\phi(e,e) = (a,a)$, since ϕ and (12) commute. Write $\phi(e,a) = (b,c)$. Since $\phi_1\phi_2(e,e,e) = \phi_3\phi_1(e,e,e)$, we have $(a,b,c) = (b,c,a)$, whence $a = b = c$. Thus, $\phi(e,a) = (a,a)$, or equivalently, $\phi(a,a) = (e,a)$. Since ϕ and (12) commute, this implies $e = a$, qed. Now take $e \in E$, and consider E to be pointed by e. In the symmetric monoidal category $\mathcal{S}'$ of pointed sets, the map ϕ induces a map $\phi' : (E \times E, (e,e)) \longrightarrow (E \times E, (e,e))$, hence an object $((E,e), \phi')$ of $\mathrm{PDec}(\mathcal{S}')$. But since $\mathcal{S}'$ is pointed, theorem 6.1 applies: $((E,e),\phi')$ comes from a decomposition $(E,e) \simeq (X,x) \times (Y,y)$ of (E,e) into a cartesian product of two pointed sets. A fortiori, ϕ comes from the decomposition $E \simeq X \times Y$. $\square$
In the next section, we describe a very important case in which $\Psi_{\mathcal{S}}$ is an equivalence of category: the case of pointed symmetric monoidal categories.

6. Effectiveness of bilinear descent data in pointed symmetric monoidal categories

Theorem 6.1. *Let $\mathcal{S}$ be a pointed symmetric monoidal category with equalizers, such that, for every object X of $\mathcal{S}$, the functor $. \otimes X$ preserves equalizers. Then Ψ_S is an equivalence of categories.*

Proof. Denote by Y (resp. X) the equalizer of the two arrows $E \longrightarrow E \otimes E$ given by $u_E \otimes \mathrm{Id}_E$ and $\phi \circ (u_E \otimes \mathrm{Id}_E)$ (resp. $(12) \circ \phi \circ (u_E \otimes \mathrm{Id}_E)$). Denote by $i_X : X \longrightarrow E$ and $i_Y : Y \longrightarrow E$ the canonical inclusions.

Claim 1. The composite

$$1 \otimes E \simeq E \overset{u_E \otimes \mathrm{Id}_E}{\longrightarrow} E \otimes E \overset{\phi}{\longrightarrow} E \otimes E \overset{r \otimes \mathrm{Id}_E}{\longrightarrow} 1 \otimes E \simeq E$$

(resp.

$$1 \otimes E \simeq E \overset{u_E \otimes \mathrm{Id}_E}{\longrightarrow} E \otimes E \overset{\phi}{\longrightarrow} E \otimes E \overset{\mathrm{Id}_E \otimes r}{\longrightarrow} E \otimes 1 \simeq E)$$

factors through Y (resp. X).

Proof of Claim 1. We check the assertion for Y, the proof for X being similar. Consider the composite

$$E \overset{u_E \otimes \mathrm{Id}_E}{\longrightarrow} E \otimes E \overset{\phi}{\longrightarrow} E \otimes E \overset{r \otimes \mathrm{Id}_E}{\longrightarrow} E \overset{u_E \otimes \mathrm{Id}_E}{\longrightarrow} E \otimes E \overset{\phi}{\longrightarrow} E \otimes E.$$

We have to show that it equals

$$E \overset{u_E \otimes \mathrm{Id}_E}{\longrightarrow} E \otimes E \overset{\phi}{\longrightarrow} E \otimes E \overset{r \otimes \mathrm{Id}_E}{\longrightarrow} E \overset{u_E \otimes \mathrm{Id}_E}{\longrightarrow} E \otimes E.$$

We compute:

$$\begin{aligned} & E \overset{u_E \otimes \mathrm{Id}_E}{\longrightarrow} E \otimes E \overset{\phi}{\longrightarrow} E \otimes E \overset{r \otimes \mathrm{Id}_E}{\longrightarrow} E \overset{u_E \otimes \mathrm{Id}_E}{\longrightarrow} E \otimes E \overset{\phi}{\longrightarrow} E \otimes E \\ &= E \overset{u_{E \otimes E} \otimes \mathrm{Id}_E}{\longrightarrow} E \otimes E \otimes E \overset{\phi_1}{\longrightarrow} E \otimes E \otimes E \overset{\mathrm{Id}_E \otimes r \otimes \mathrm{Id}_E}{\longrightarrow} E \otimes E \overset{\phi}{\longrightarrow} E \otimes E \\ &= E \overset{u_{E \otimes E} \otimes \mathrm{Id}_E}{\longrightarrow} E \otimes E \otimes E \overset{\phi_1}{\longrightarrow} E \otimes E \otimes E \overset{\phi_2}{\longrightarrow} E \otimes E \otimes E \overset{\mathrm{Id}_E \otimes r \otimes \mathrm{Id}_E}{\longrightarrow} E \otimes E \\ &= E \overset{u_{E \otimes E} \otimes \mathrm{Id}_E}{\longrightarrow} E \otimes E \otimes E \overset{\phi_3 \circ \phi_1}{\longrightarrow} E \otimes E \otimes E \overset{\mathrm{Id}_E \otimes r \otimes \mathrm{Id}_E}{\longrightarrow} E \otimes E \end{aligned}$$

Since $\phi \circ u_{E \otimes E} = u_{E \otimes E}$ because $\mathcal{S}$ is pointed, this composite equals

$$\begin{aligned} &= E \overset{u_{E \otimes E} \otimes \mathrm{Id}_E}{\longrightarrow} E \otimes E \otimes E \overset{\phi_1}{\longrightarrow} E \otimes E \otimes E \overset{\mathrm{Id}_E \otimes r \otimes \mathrm{Id}_E}{\longrightarrow} E \otimes E \\ &= E \overset{u_E \otimes \mathrm{Id}_E}{\longrightarrow} E \otimes E \overset{\phi}{\longrightarrow} E \otimes E \overset{r \otimes \mathrm{Id}_E}{\longrightarrow} E \overset{u_E \otimes \mathrm{Id}_E}{\longrightarrow} E \otimes E. \end{aligned}$$

This finishes the proof of Claim 1. We thus have two arrows $r_X : E \longrightarrow X$ and $r_Y : E \longrightarrow Y$, which are readily checked to be retractions of i_X and i_Y, respectively. It is readily checked that the association $(E, \phi) \mapsto (X, Y)$ is functorial. It thus defines a functor

$$\Theta_{\mathcal{S}} : \mathrm{PDec}(\mathcal{S}) \longrightarrow \mathcal{S}^2.$$

I claim that $\Theta_{\mathcal{S}}$ is a quasi-inverse of $\Psi_{\mathcal{S}}$. Indeed, let $(E, \phi) \in \mathrm{PDec}(\mathcal{S})$ and let $r : E \longrightarrow 1$ be a morphism. Put $(X, Y) = \Theta_S(E, \phi)$. Recall that we have at our disposal the morphisms i_X, r_X, i_Y and r_Y.

Claim 2. The composite $E \overset{u_E \otimes \mathrm{Id}_E}{\longrightarrow} E \otimes E \overset{\phi}{\longrightarrow} E \otimes E$ factors through $i_X \otimes i_Y : X \otimes Y \longrightarrow E \otimes E$.

Proof of Claim 2. Thanks to lemma 3.4, it suffices to check the following two properties.

i) The composite

$$E \overset{u_E \otimes \mathrm{Id}_E}{\longrightarrow} E \otimes E \overset{\phi}{\longrightarrow} E \otimes E \overset{u_E \otimes \mathrm{Id}_{E \otimes E}}{\longrightarrow} E \otimes E \otimes E$$

equals the composite

$$E \xrightarrow{u_E\otimes \mathrm{Id}_E} E\otimes E \xrightarrow{\phi} E\otimes E \xrightarrow{u_E\otimes \mathrm{Id}_{E\otimes E}} E\otimes E\otimes E \xrightarrow{(12)\circ\phi_3} E\otimes E\otimes E.$$

ii) The composite

$$E \xrightarrow{u_E\otimes \mathrm{Id}_E} E\otimes E \xrightarrow{\phi} E\otimes E \xrightarrow{\mathrm{Id}_E\otimes u_E\otimes \mathrm{Id}_E} E\otimes E\otimes E$$

equals the composite

$$E \xrightarrow{u_E\otimes \mathrm{Id}_E} E\otimes E \xrightarrow{\phi} E\otimes E \xrightarrow{\mathrm{Id}_E\otimes u_E\otimes \mathrm{Id}_E} E\otimes E\otimes E \xrightarrow{\phi_1} E\otimes E\otimes E$$

We check i), ii) being similar. We compute:

$$\begin{aligned}
&E \xrightarrow{u_E\otimes \mathrm{Id}_E} E\otimes E \xrightarrow{\phi} E\otimes E \xrightarrow{u_E\otimes \mathrm{Id}_{E\otimes E}} E\otimes E\otimes E \xrightarrow{(12)\circ\phi_3} E\otimes E\otimes E\\
&= E \xrightarrow{u_{E\otimes E}\otimes \mathrm{Id}_E} E\otimes E\otimes E \xrightarrow{(12)\circ\phi_3\circ\phi_1} E\otimes E\otimes E\\
&= E \xrightarrow{u_{E\otimes E}\otimes \mathrm{Id}_E} E\otimes E\otimes E \xrightarrow{(12)\circ\phi_2\circ\phi_3} E\otimes E\otimes E\\
&= E \xrightarrow{u_{E\otimes E}\otimes \mathrm{Id}_E} E\otimes E\otimes E \xrightarrow{(12)\circ\phi_2} E\otimes E\otimes E
\end{aligned}$$

(since $\phi\circ u_{E\otimes E} = u_{E\otimes E}$)

$$\begin{aligned}
&= E \xrightarrow{u_E\otimes \mathrm{Id}_E} E\otimes E \xrightarrow{\phi} E\otimes E \xrightarrow{\mathrm{Id}_E\otimes u_E\otimes \mathrm{Id}_E} E\otimes E\otimes E \xrightarrow{(12)} E\otimes E\otimes E\\
&= E \xrightarrow{u_E\otimes \mathrm{Id}_E} E\otimes E \xrightarrow{\phi} E\otimes E \xrightarrow{u_E\otimes \mathrm{Id}_E\otimes \mathrm{Id}_E} E\otimes E\otimes E.
\end{aligned}$$

This finishes the proof of Claim 2.

We have thus built a map $f : E \longrightarrow X\otimes Y$, functorial in (E,ϕ). We now show that it is invertible by constructing its inverse explicitly. Let G be the composite $X\otimes Y \xrightarrow{i_X\otimes i_Y} E\otimes E \xrightarrow{\phi} E\otimes E$. Put $g = (r\otimes \mathrm{Id}_E\circ G)$.

Claim 3. We have $G = (u_E\circ \mathrm{Id}_E)g$.

Proof of Claim 3. We compute

$$\begin{aligned}
(u_E\circ \mathrm{Id}_E)g &= X\otimes Y \xrightarrow{i_X\otimes i_Y} E\otimes E \xrightarrow{\phi} E\otimes E \xrightarrow{r\otimes \mathrm{Id}_E} E \xrightarrow{u_E\otimes \mathrm{Id}_E} E\otimes E\\
&= X\otimes Y \xrightarrow{i_X\otimes i_y} E\otimes E \xrightarrow{u_E\otimes \mathrm{Id}_{E\otimes E}} E\otimes E\otimes E \xrightarrow{\phi_1} E\otimes E\otimes E \xrightarrow{\mathrm{Id}_E\otimes r\otimes \mathrm{Id}_E} E\otimes E\\
&= X\otimes Y \longrightarrow E\otimes E \xrightarrow{u_E\otimes \mathrm{Id}_{E\otimes E}} E^{\otimes 3} \xrightarrow{\phi_3\circ(12)} E^{\otimes 3} \xrightarrow{\phi_1} E^{\otimes 3} \xrightarrow{\mathrm{Id}_E\otimes r\otimes \mathrm{Id}_E} E\otimes E
\end{aligned}$$

(by definition of X)

$$\begin{aligned}
&= X\otimes Y \longrightarrow E\otimes E \xrightarrow{u_E\otimes \mathrm{Id}_{E\otimes E}} E^{\otimes 3} \xrightarrow{\phi_1\circ\phi_3\circ(12)} E^{\otimes 3} \xrightarrow{\mathrm{Id}_E\otimes r\otimes \mathrm{Id}_E} E\otimes E\\
&= X\otimes Y \longrightarrow E\otimes E \xrightarrow{\mathrm{Id}_E\otimes u_E\otimes \mathrm{Id}_E} E^{\otimes 3} \xrightarrow{\phi_1\circ\phi_3} E^{\otimes 3} \xrightarrow{\mathrm{Id}_E\otimes r\otimes \mathrm{Id}_E} E\otimes E\\
&= X\otimes Y \longrightarrow E\otimes E \xrightarrow{\mathrm{Id}_E\otimes u_E\otimes \mathrm{Id}_E} E^{\otimes 3} \xrightarrow{\phi_2\circ\phi_1} E^{\otimes 3} \xrightarrow{\mathrm{Id}_E\otimes r\otimes \mathrm{Id}_E} E\otimes E\\
&= X\otimes Y \longrightarrow E\otimes E \xrightarrow{\mathrm{Id}_E\otimes u_E\otimes \mathrm{Id}_E} E^{\otimes 3} \xrightarrow{\phi_2} E^{\otimes 3} \xrightarrow{\mathrm{Id}_E\otimes r\otimes \mathrm{Id}_E} E\otimes E
\end{aligned}$$

(by definition of Y)

$$\begin{aligned}
&= X\otimes Y \longrightarrow E\otimes E \xrightarrow{\phi} E\otimes E\\
&= G.
\end{aligned}$$

This finishes the proof of Claim 3.

In short, we have just proven that G factors through the (split mono) $E \xrightarrow{u_E\otimes \mathrm{Id}_E} E\otimes E$. Hence in particular, g is independent of the choice of r.

Claim 4. The arrows f and g are mutual inverses.

Proof of Claim 4. We compute:

$$g\circ f = E \xrightarrow{u_E\otimes \mathrm{Id}_E} E\otimes E \xrightarrow{\phi} X\otimes Y \longrightarrow E\otimes E \xrightarrow{\phi} E\otimes E \xrightarrow{r\otimes \mathrm{Id}_E} E$$

$$= E \xrightarrow{u_E \otimes \mathrm{Id}_E} E \otimes E \xrightarrow{\phi^2} E \otimes E \xrightarrow{r \otimes \mathrm{Id}_E} E.$$

But we know that ϕ^2 is a constant automorphism of $E \otimes E$, hence the identity since $\mathcal{S}$ is pointed. We thus infer that $g \circ f = \mathrm{Id}_E$. The proof of $f \circ g = \mathrm{Id}_{X \otimes Y}$ is similar, and left to the reader. This finishes the proof of Claim 4.

We now have to show that f (and hence g) are indeed arrows in the category $\mathrm{PDec}(\mathcal{S})$. In other words, we have to show that the square

$$\begin{tikzcd} E \otimes E \arrow[r, "f"] \arrow[d, "\phi"] & X \otimes Y \otimes X \otimes Y \arrow[d, "s_X"] \\ E \otimes E \arrow[r, "f \otimes f"] & X \otimes Y \otimes X \otimes Y \end{tikzcd}$$

commutes. Let us call S this square.

Claim 5. The composite $X \otimes X \xrightarrow{i_X \otimes i_X} E \otimes E \xrightarrow{(12)} E \otimes E$ equals $X \otimes X \xrightarrow{i_X \otimes i_X} E \otimes E \xrightarrow{\phi} E \otimes E$, and the composite $Y \otimes Y \xrightarrow{i_Y \otimes i_Y} E \otimes E$ equals $Y \otimes Y \xrightarrow{i_Y \otimes i_Y} E \otimes E \xrightarrow{\phi} E \otimes E$.

Proof of Claim 5. We prove the first assertion, the second one being similar. We compute:

$$X \otimes X \xrightarrow{i_X \otimes i_X} E \otimes E \xrightarrow{u_{E\otimes E} \otimes \mathrm{Id}_{E \otimes E}} E \otimes E \otimes E \otimes E \xrightarrow{\phi_{34}} E \otimes E \otimes E \otimes E$$

$$= X \otimes X \longrightarrow E \otimes E \xrightarrow{u_{E\otimes E} \otimes \mathrm{Id}_{E \otimes E}} E \otimes E \otimes E \otimes E \xrightarrow{\phi_{34}\phi_{24}\phi_{13}(13)(24)} E \otimes E \otimes E \otimes E$$

by definition of X. Using the computing rules on ϕ, we see that

$$\phi_{34}\phi_{24}\phi_{13} = \phi_{23}\phi_{34}\phi_{13} = \phi_{23}\phi_{14}\phi_{34}.$$

Hence the previous composite equals

$$X \otimes X \longrightarrow E \otimes E \xrightarrow{\mathrm{Id}_{E\otimes E} \otimes u_{E \otimes E}} E \otimes E \otimes E \otimes E \xrightarrow{\phi_{23}\phi_{14}\phi_{34}} E \otimes E \otimes E \otimes E$$

$$= X \otimes X \longrightarrow E \otimes E \xrightarrow{\mathrm{Id}_{E\otimes E} \otimes u_{E \otimes E}} E \otimes E \otimes E \otimes E \xrightarrow{\phi_{23}\phi_{14}} E \otimes E \otimes E \otimes E$$

(since $\phi \circ u_{E \otimes E} = u_{E \otimes E}$)

$$= X \otimes X \longrightarrow E \otimes E \xrightarrow{\mathrm{Id}_{E\otimes E} \otimes u_{E \otimes E}} E \otimes E \otimes E \otimes E \xrightarrow{(23)(14)} E \otimes E \otimes E \otimes E$$

(by definition of X)

$$= X \otimes X \longrightarrow E \otimes E \xrightarrow{u_{E\otimes E} \otimes \mathrm{Id}_{E \otimes E}} E \otimes E \otimes E \otimes E \xrightarrow{(34)} E \otimes E \otimes E \otimes E.$$

We thus have proven that

$$X \otimes X \longrightarrow E \otimes E \xrightarrow{u_{E\otimes E} \otimes \mathrm{Id}_{E \otimes E}} E \otimes E \otimes E \otimes E \xrightarrow{\phi_{34}} E \otimes E \otimes E \otimes E$$

$$= X \otimes X \longrightarrow E \otimes E \xrightarrow{u_{E\otimes E} \otimes \mathrm{Id}_{E \otimes E}} E \otimes E \otimes E \otimes E \xrightarrow{(34)} E \otimes E \otimes E \otimes E,$$

and the assertion follows by composing by $r \otimes r \otimes \mathrm{Id}_{E \otimes E}$. This finishes the proof of claim 5.

We now check that the square S commutes. Consider the map

$$E \otimes E \xrightarrow{\phi} E \otimes E \xrightarrow{f \otimes f} X \otimes Y \otimes X \otimes Y.$$

Composing it with the canonical inclusion $X \otimes Y \otimes X \otimes Y \overset{i_X \otimes i_Y \otimes i_X \otimes i_Y}{\longrightarrow} E^{\otimes^4}$, and using the definition of f, we get the map

$$E \otimes E \overset{u_E \otimes \mathrm{Id}_E \otimes u_E \otimes \mathrm{Id}_E}{\longrightarrow} E^{\otimes^4} \overset{\phi_{12}\phi_{34}\phi_{24}}{\longrightarrow} E^{\otimes^4}.$$

On the other hand, we compute

$$E \otimes E \xrightarrow{f \otimes f} X \otimes Y \otimes X \otimes Y \xrightarrow{s_X} X \otimes Y \otimes X \otimes Y \overset{i_X \otimes i_Y \otimes i_X \otimes i_Y}{\longrightarrow} E^{\otimes^4}$$

$$= E \otimes E \overset{u_E \otimes \mathrm{Id}_E \otimes u_E \otimes \mathrm{Id}_E}{\longrightarrow} E^{\otimes^4} \overset{\phi_{13}\phi_{34}\phi_{12}}{\longrightarrow} E^{\otimes^4}.$$

using the definition of f and Claim 5 (concerning X). What thus remains to be checked is the equality $\phi_{13}\phi_{34}\phi_{12} = \phi_{12}\phi_{34}\phi_{24}$. This is done as follows:

$$\phi_{34}\phi_{12}\phi_{24} = \phi_{34}\phi_{14}\phi_{12} = \phi_{13}\phi_{34}\phi_{12}.$$

This finishes the proof of the construction of a natural isomorphism between $\Psi_S \circ \Theta_S$ and the identity functor of $\mathrm{PDec}(\mathcal{S})$.

It remains to produce, for $(X, Y) \in \mathcal{S}^2$, a natural isomorphism between $\Theta_{\mathcal{S}}(\Psi_S(X, Y)) = \Theta_{\mathcal{S}}(X \otimes Y, s_X)$ and (X, Y) (in the category $\mathcal{S}^2$). This is rather formal. Put $(X', Y') = \Theta_{\mathcal{S}}(X \otimes Y, s_X)$. Then Y' is defined as the equalizer of the arrows

$$X \otimes Y \overset{u_{X \otimes Y} \otimes \mathrm{Id}_{X \otimes Y}}{\longrightarrow} X \otimes Y \otimes X \otimes Y$$

and

$$X \otimes Y \overset{u_{X \otimes Y} \otimes \mathrm{Id}_{X \otimes Y}}{\longrightarrow} X \otimes Y \otimes X \otimes Y \xrightarrow{s_X} X \otimes Y \otimes X \otimes Y.$$

One readily checks that this equilizer is nothing but the arrow $Y \overset{u_X \otimes \mathrm{Id}_Y}{\longrightarrow} X \otimes Y$, hence a canonical isomorphism between Y and Y'. Similarly, we produce a canonical isomorphism between X and X'. Everything is perfectly functorial, so that we get a natural isomorphism between the functors $\Theta_S \circ \Psi_S$ and the identity functor of $\mathcal{S}^2$. $\square$

Definition 6.2. *Let $\mathcal{S}$ be a symmetric monoidal category. Let E be an object of $\mathcal{S}$. We define a category $\mathrm{Dec}'(E)$ as follows. An element of $\mathrm{Dec}'(E)$ is a pair (X, Y) of objects of $\mathcal{S}$, together with an isomorphism $E \xrightarrow{f} X \otimes Y$. A morphism*

$$(E \xrightarrow{f} X \otimes Y) \longrightarrow (E \xrightarrow{f'} X' \otimes Y')$$

is a pair of isomorphisms $g : X \longrightarrow X'$ and $h : Y \longrightarrow Y'$ such that $(g \otimes h) \circ f = f'$. Whenever the isomorphism classes of objects of $\mathrm{Dec}'(E)$ form a set, we denote it by $\mathrm{Dec}(E)$.

Proposition 6.3. *Let $\mathcal{S}$ be a symmetric monoidal category. Assume that $\Psi_{\mathcal{S}}$ is an equivalence of categories. Then $\mathrm{Dec}(E)$ exists. More precisely, the association*

$$\mathrm{Dec}'(E) \longrightarrow \mathrm{PDec}(E),$$

$$(E \xrightarrow{f} X \otimes Y) \mapsto (E, c_Y)$$

(where we identify E and $X \otimes Y$ via f) induces a bijection

$$\mathrm{Dec}(E) \simeq \mathrm{PDec}(E).$$

Proof. Everything is obvious, except perhaps that, if

$$(E \xrightarrow{f} X \otimes Y) \xrightarrow{(g,h)} (E \xrightarrow{f'} X' \otimes Y')$$

is an (iso)morphism in $\mathrm{Dec}'(E)$, then $(E, s_X) = (E, s_{X'})$. In the diagram

$$\begin{tikzcd} E\otimes E \arrow[r, "f\otimes f"] \arrow[d, "\mathrm{Id}"] & X\otimes Y\otimes X\otimes Y \arrow[r, "s_X"] \arrow[d, "g\otimes h\otimes g\otimes h"] & X\otimes Y\otimes X\otimes Y \arrow[r, "f^{-1}\otimes f^{-1}"] \arrow[d, "g\otimes h\otimes g\otimes h"] & E\otimes E \arrow[d, "\mathrm{Id}"] \\ E\otimes E \arrow[r, "f'\otimes f'"] & X'\otimes Y'\otimes X'\otimes Y' \arrow[r, "s_{X'}"] & X'\otimes Y'\otimes X'\otimes Y' \arrow[r, "f'^{-1}\otimes f'^{-1}"] & E\otimes E \end{tikzcd},$$

the three small squares obviously commute, so that the big square commutes, too. $\square$

We now investigate some precise examples.

7. Algebras and Azumaya algebras

Let A be a commutative ring. Denote by $\mathrm{Mod}(A)$ the category of A-modules, by $\mathrm{Mod}_f(A)$ the category of finite locally free A-modules, by $\mathcal{P}(A)$ the category $\mathcal{P}(\mathrm{Mod}(A))$, and by $\mathcal{ALG}(A)$ the category whose objects are A-algebras R such that the canonical map $A \longrightarrow R$ is a split monomorphism of A-modules. All these categories are symmetric monoidal for the tensor product of A-modules.

Recall that an Azumaya algebra over A is an A-algebra R which is finite and locally free as an A-module, and such that the canonical map

$$R\otimes_A R \longrightarrow \mathrm{End}_A(R),$$

$$r\otimes r' \mapsto (x \mapsto rxr')$$

is an isomorphism. Denote by $\mathcal{AZ}(A)$ the symmetric monoidal category of Azumaya algebras over A. It is a full subcategory of $\mathcal{ALG}(A)$. Let $E \in \mathrm{Mod}_f(A)$. Then one easily sees that the functor

$$B \mapsto \mathrm{PDec}_B(E\otimes_A B),$$

from the category of A-algebras to that of sets, is representable by an affine scheme over $\mathrm{Spec}(A)$, which we denote by $\mathrm{PDEC}(E)$.

Proposition 7.1. *The functors $\Psi_{\mathcal{ALG}(A)}$ and $\Psi_{\mathcal{AZ}(A)}$ are equivalences of (symmetric monoidal) categories.*

Proof. Let us first prove the statement for $\Psi_{\mathcal{ALG}(A)}$. There is an obvious (strongly monoidal) functor

$$F : \mathcal{ALG}(A) \longrightarrow \mathcal{P}(A),$$

$$R \mapsto (R, A \overset{a\mapsto a.1}{\longrightarrow} R).$$

Thus, we have a functor $\mathrm{PDec}(F) : \mathrm{PDec}(\mathcal{ALG}(A)) \longrightarrow \mathrm{PDec}(\mathcal{P}(A))$, such that the diagram

$$\begin{tikzcd} \mathcal{ALG}(A)^2 \arrow[r] \arrow[d] & \mathcal{P}(A)^2 \arrow[d] \\ \mathrm{PDec}(\mathcal{ALG}(A)) \arrow[r] & \mathrm{PDec}(\mathcal{P}(A)) \end{tikzcd}$$

commutes. Examining the definition of the quasi-inverse $\Theta_{\mathcal{P}(A)}$ of $\Psi_{\mathcal{P}(A)}$ built in the proof of Theorem 6.1, we see that the composite $\Theta_{\mathcal{P}(A)} \circ \mathrm{PDec}(F)$ factorizes canonically through F^2. Indeed, let (R, ϕ) be in $\mathrm{PDec}(\mathcal{ALG}(A))$. Write $\Theta_{\mathcal{P}(A)}(R, 1, \phi) = ((X, x_0), (Y, y_0))$. One immediately sees that X (resp. Y) is canonically an A-algebra with unit element x_0 (resp. y_0), as it is defined as an

equalizer of two A-algebra morphisms. One readily checks that the canonical isomorphism $X \otimes Y \longrightarrow R$ is then an A-algebra isomorphism. This yields the asserted factorization. In other words, we have built a functor $\Theta_{\mathcal{ALG}(A)}$ which is a quasi-inverse to $\Psi_{\mathcal{ALG}(A)}$.
Now, let $(R, \phi) \in \mathrm{PDec}(\mathcal{AZ}(A))$. Write $\Theta_{\mathcal{ALG}(A)}(R, \phi) = (S, T)$. By construction, S (resp. T) is a direct factor of R as an A-module, hence is a finite and locally free A-module. Since the A-algebra $S \otimes T$ is isomorphic to R, we see that S and T are Azumaya algebras as well. This yields a quasi-inverse for $\Psi_{\mathcal{AZ}(A)}$.

□

Denote by $\mathcal{SB}(A)$ the category of Severi-Brauer schemes over $\mathrm{Spec}(A)$, with morphisms being isomorphisms. It is well-known that this category is equivalent to the (symmetric monoidal) category $\mathcal{AZ}'(A)$ of Azumaya algebras over A, with isomorphisms as morphisms. This endows $\mathcal{SB}(A)$ with a canonical structure of a symmetric monoidal category.

PROPOSITION 7.2. *Let $E \in \mathrm{Mod}_f(A)$. For any $(E, \phi) \in \mathrm{PDec}(E)$, there exists Severi-Brauer schemes X and Y over $\mathrm{Spec}(A)$ and an isomorphism $f : \mathbb{P}(E) \longrightarrow X \otimes Y$, such that $(\mathbb{P}(E \otimes E) = \mathbb{P}(E) \otimes \mathbb{P}(E), s_X)$ (where we identifiy $\mathbb{P}(E)$ and $X \otimes Y$ via f) is equal to $\mathbb{P}(\phi)$.*

Proof. Put $R = \mathrm{End}(E)$; it is an Azumaya algebra. The data of $(E, \phi) \in \mathrm{PDec}(E)$ gives rise to the data of $(R, \psi) \in \mathrm{PDec}(\mathcal{AZ}(A))$ the following way: $\psi \in R \otimes R = \mathrm{End}(E \otimes E)$ is given by conjugation by ψ. One readily checks that this ψ indeed satisfies the required equations. Thanks to proposition 7.1, we get Azumaya algebras S and T, plus an isomorphism $R \simeq S \otimes T$, such that $(R, \psi) = (R, s_S) \in \mathrm{PDec}(\mathcal{AZ}(A))$. Thanks to the equivalence of categories between Severi-Brauer schemes and Azumaya algebras, the assertion follows by taking X (resp. Y) to be the Severi-Brauer scheme corresponding to S (resp. T). □

Remark 7.3. The previous proposition shows that, in the category $\mathrm{Mod}_f(A)$, although a potential decomposition of a module E might not correspond to an effective one, it always comes from a decomposition of $\mathbb{P}(E)$ as the tensor product of two Severi-Brauer schemes.

DEFINITION 7.4. *Let E be an object of $\mathrm{Mod}_f(A)$, of constant dimension e. For any pair of positive integers x and y such that $xy = e$, denote by $\mathrm{PDec}_{x,y}(E)$ the subset of $\mathrm{PDec}(E)$ defined as follows. By proposition 7.2, for any $(E, \phi) \in \mathrm{PDec}(E)$, there exists Severi-Brauer schemes X and Y over $\mathrm{Spec}(A)$ and an isomorphism $f : \mathbb{P}(E) \longrightarrow X \otimes Y$ which is mapped to $(\mathbb{P}(E), \mathbb{P}(\phi))$ under $\Psi_{\mathcal{P}(E)}$. Then (E, ϕ) belongs to $\mathrm{PDec}_{x,y}(E)$ if and only if X (resp. Y) has constant dimension $x - 1$ (resp. $y - 1$).*

DEFINITION 7.5. *Let $E \in \mathrm{Mod}_f(A)$. Then one easily sees that the functor*

$$B \mapsto \mathrm{PDec}_B(E \otimes_A B),$$

from the category of A-algebras to that of sets, is representable by an affine scheme over $\mathrm{Spec}(A)$, which we denote by $\mathrm{PDEC}(E)$. For any pair of positive integers x and y such that $xy = e$, we define an affine scheme $\mathrm{PDEC}_{x,y}(E)$ similarly.

Remark 7.6. If $\mathrm{Spec}(A)$ is connected, then $\mathrm{PDEC}(E)$ is the disjoint union of the $\mathrm{PDEC}_{x,y}(E)$, where x, y range through the positive integers such that $xy = e$.

Remark 7.7. There is a amusing corollary of what we have done so far. Let $n > 0$ be an integer, and let k be a field. Put $E = k^n$, and $X_n = \mathrm{PDEC}(E)$; it is an affine variety over k, defined by rather simple equations (i), ii), iii) of 5.1; one may even prove in this context that i) is implied by ii) and iii)). Remove from X_n its 2 isolated points corresponding to the trivial decompositions $E = 1 \otimes E$ and $E = E \otimes 1$. We obtained a variety X'_n, naturally attached to n, whose connected components correspond exactly to the nontrivial decompositions of n into a product of two positive integers. In particular, X'_n is empty if and only if n is prime. One may wonder whether, given a point of X'_n, there exists a fast algorithm for computing the corresponding decomposition.

For simplicity, assume now that $A = k$ is a field. Let X_0 and Y_0 be finite-dimensional nonzero k-vector spaces of dimensions x and y, respectively. We have an exact sequence of algebraic k-groups

$$1 \longrightarrow \mathbb{G}_{\mathrm{m}} \longrightarrow \mathrm{GL}(X_0) \times_k \mathrm{GL}(Y_0) \longrightarrow \mathrm{GL}(X_0 \otimes Y_0),$$

where the first arrow is given by

$$x \mapsto (x, x^{-1}),$$

and the second by

$$(f, g) \mapsto f \otimes g.$$

Denote by $\mathbb{G}(X_0, Y_0) = \mathrm{GL}(X_0 \otimes Y_0)/((\mathrm{GL}(X_0) \otimes \mathrm{GL}(Y_0))/\mathbb{G}_{\mathrm{m}})$ the cokernel of this exact sequence.

Proposition 7.8. *The variety* $\mathrm{PDEC}_{x,y}(X_0, Y_0)$ *is canonically isomorphic to* $\mathbb{G}(X_0, Y_0)$.

Proof. Consider the morphism

$$F : \mathrm{GL}(X_0 \otimes Y_0) \longrightarrow \mathrm{PDEC}(X_0, Y_0)$$

given, on the level of points in a k-algebra B by

$$f \mapsto (f \otimes f) \circ s_{X_0 \otimes_k B} \circ (f^{-1} \otimes f^{-1}).$$

One immediately sees that this morphism factors through $G(X_0, Y_0)$, yielding a morphism

$$\Phi : G(X_0, Y_0) \longrightarrow \mathrm{PDEC}(X_0, Y_0).$$

I claim that Φ is an isomorphism. It suffices to exhibit its inverse. To that purpose, let B be a k-algebra and $s \in \mathrm{PDEC}_{x,y}(X_0 \otimes_k Y_0)(B)$. By proposition 7.2, there exists Severi-Brauer schemes X and Y over B, of constant dimension $x - 1$ and $y - 1$, respectively, and an isomorphism (of projective schemes over B) $h : \mathbb{P}_B(X_0 \otimes_k Y_0 \otimes_k B) \longrightarrow X \otimes Y$, such that $s = s_x$. Choose a faithfully flat B-algebra C, and isomorphisms $f : X_C \longrightarrow \mathbb{P}(X_0 \otimes_k C)$ (resp. $g : Y_C \longrightarrow \mathbb{P}(Y_0 \otimes_k C)$). Changing C if necessary, we may assume that the automorphism $(f \otimes g) \circ h$ of $\mathbb{P}_C(X_0 \otimes_k Y_0 \otimes_k C)$ is induced by an automorphism $r' \in \mathrm{GL}(X_0 \otimes Y_0)(C)$. The class of r' in $\mathbb{G}(X_0, Y_0)(C)$ is independent of the choice of f and g, hence descends (by classical faithfully flat descent theory) to an element $r \in \mathbb{G}(X_0, Y_0)(B)$. It is readily checked that the associaton $s \mapsto r$ yield the inverse of F. $\square$

Remark 7.9. The interested reader may notice the following fact. Let n, m be two positive integers. The variety $\mathbb{G}(k^n, k^m)$ is stably birational to the classifying space of the k-group $(\mathrm{GL}_n \times_k \mathrm{GL}_m)/\mathbb{G}_{\mathrm{m}}$. One may then prove as an exercise that the stable rationality of this classifying space is equivalent to that of PGL_r, where r is the gcd of n and m. Hence, stable rationality of every connected component of $\mathrm{PDEC}(E)$ for every nonzero E is equivalent to stable rationality of PGL_n for every positive integer n.

8. The Brauer group of a symmetric monoidal category

Let $\mathcal{S}$ be a symmetric monoidal category.

Lemma 8.1. *Let (E,ϕ) be an object of* $\mathrm{PDec}(\mathcal{S})$*. The isomorphism $\phi : E\otimes E \longrightarrow E\otimes E$ induces a canonical isomorphism*

$$(E,(12)\circ\phi)\otimes(E,\phi)\longrightarrow(E\otimes E,s_E)$$

in $\mathrm{PDec}(\mathcal{S})$*.*

Proof. We have to check that the diagram

$$\begin{tikzcd} E\otimes E\otimes E\otimes E \arrow[r, "\phi_{34}\phi_{12}"] \arrow[d, "\phi_{24}\phi_{13}(13)"] & E\otimes E\otimes E\otimes E \arrow[d, "(13)"] \\ E\otimes E\otimes E\otimes E \arrow[r, "\phi_{34}\phi_{12}"] & E\otimes E\otimes E\otimes E \end{tikzcd}$$

commutes. We compute:

$$(13)\phi_{34}\phi_{12}=\phi_{14}(13)\phi_{12}=\phi_{14}\phi_{23}(13),$$

so that we have to show that

$$\phi_{34}\phi_{12}\phi_{24}\phi_{13}=\phi_{14}\phi_{23}$$

. This is done as follows:

$$\phi_{34}\phi_{12}\phi_{24}\phi_{13}=\phi_{34}\phi_{24}\phi_{14}\phi_{13}=\phi_{34}\phi_{24}\phi_{34}\phi_{14}=\phi_{34}\phi_{34}\phi_{23}\phi_{14}=\phi_{14}\phi_{23},$$

up to a constant automorphism.

□

Definition 8.2. *Let $\mathcal{S}$ be a symmetric monoidal category.*
i) An object of $\mathrm{PDec}(\mathcal{S})$ *is called trivial if it is in the essential image of $\Psi_{\mathcal{S}}$.*
ii) Two objects E, E' of $\mathrm{PDec}(\mathcal{S})$ *are said to be stably isomorphic (we then write $E\sim E'$) if there exists trivial objects T and T' such that $E\otimes T$ is isomorphic (in* $\mathrm{PDec}(\mathcal{S})$*) to $E'\otimes T'$.*
iii) If the isomorphism classes of objects of $\mathrm{PDec}(\mathcal{S})$ *modulo $\sim$ form a set, we denote it by* $\mathrm{Br}(\mathcal{S})$*. It is a group with neutral element the class of $(1\otimes 1,s_1)$, for the group structure induced by the tensor product in* $\mathrm{PDec}(\mathcal{S})$*. More precisely, lemma 8.1 implies that the inverse of (E,ϕ) is $(E,(12)\phi)$. The group* $\mathrm{Br}(\mathcal{S})$ *is called the Brauer group of $\mathcal{S}$.*

We now study some examples. Let A be a (commutative) base ring, and $\mathrm{Mod}'_f(A)$ be the symmetric monoidal category of locally free A-modules of finite and everywhere nonzero rank.

Proposition 8.3. *There is a canonical isomorphism between the usual Brauer-Azumaya group* $\mathrm{Br}(A)$ *and* $\mathrm{Br}(\mathrm{Mod}'_f(A))$*.*

Proof. Let $(E,\phi)\in\mathrm{PDec}(\mathrm{Mod}'_f(A))$. By proposition 7.1, we have a canonical decomposition $\mathrm{End}(E)\simeq R\otimes T$, where R and T are Azumaya algebras. One readily checks that R does not depend on the choice of (E,ϕ) modulo $\sim$ up to Brauer equivalence. Indeed, if $(E,\phi)=(X\otimes Y,s_X)$ is trivial, then the decomposition given by 7.1 is nothing but $\mathrm{End}(E)\simeq\mathrm{End}(X)\otimes\mathrm{End}(Y)$. We thus get a map $f:\mathrm{Br}(\mathrm{Mod}'_f(A))\longrightarrow\mathrm{Br}(A)$, which is easily seen to be a group homomorphism. In the opposite direction, let R be an Azumaya algebra (over A). There exists a faithfully flat ring extension B/A, together with a free B-module M, and an isomorphism (of Azumaya algebras over B) $R\otimes_A B \simeq \mathrm{End}_B(M) = M\otimes_B M^*$. Denote by $s = s_M$ the switch on the M-factors in $R\otimes_A R\otimes_A B \simeq M\otimes_B M^*\otimes_B M\otimes_B M^*$. We see that s is independent of the choice of M and of the isomorphim $\simeq$; hence it descends (by faithfully flat descent theory) to an involution of $R\otimes_A R$, which we still denote by s. The image of (R, s) in $\mathrm{Br}(\mathrm{Mod}'_f(A))$ does not depend on the choice of R modulo Brauer equivalence. We thus get a map $g : \mathrm{Br}(A) \longrightarrow \mathrm{Br}(\mathrm{Mod}'_f(A))$. It is the inverse of f (verification left to the reader). □

We now concentrate on the case of a field k. Denote by $\mathrm{Mod}'(k)$ the symmetric monoidal category of nonzero k-vector spaces.

Lemma 8.4. *Let $(E, \phi) \in \mathrm{PDec}(\mathrm{Mod}'(k))$. Then (E, ϕ) is trivial if and only if there exists a nonzero $e \in E$ such that $\phi(e, e)$ and (e, e) are collinear.*

Proof. The 'if' part is easy: if (E, ϕ) is isomorphic to $(X \otimes Y, s_X)$, take a nonzero $x \in X$ (resp $y \in Y$) and put $e := x \otimes y$. Assume there exists a nonzero $e \in E$ such that $\phi(e, e)$ and (e, e) are collinear. Up to scaling ϕ by a nonzero scalar (which does not change the isomorphism class of (E, ϕ)), we may as well assume that $\phi(e, e) = (e, e)$. Then $((E, e), \phi)$ belongs to $\mathrm{PDec}(\mathcal{P}(\mathrm{Mod}'(k)))$, hence is trivial in this category by theorem 5. A fortiori, (E, ϕ) is trivial in $\mathrm{PDec}(\mathrm{Mod}'(k))$. □

Lemma 8.5. *Let $(E, \phi), (E', \phi') \in \mathrm{PDec}(\mathrm{Mod}'(k))$. Then (E, ϕ) and (E', ϕ') are stably isomorphic if and only if there exists a nonzero morphism $(E, \phi) \longrightarrow (E', \phi')$ in $\mathrm{PDec}(\mathrm{Mod}'(k))$.*

Proof. Assume given a nonzero $f : (E, \phi) \longrightarrow (E', \phi')$. tensoring by $(E, \phi)^{op} := (E, (12)\phi)$, we get a nonzero $g : T \longrightarrow (E, \phi)^{op} \otimes (E', \phi')$, where $T = (X \otimes Y, s_X)$ is a trivial object. Thus, there exists a nonzero $x \in X$ (resp. $y \in Y$) such that $e := g(x\otimes y) \in E$ is nonzero. But $\phi(e, e)$ and (e, e) are collinear (easy verification), so that by lemma 8.4, we infer that $(E, \phi)^{op} \otimes (E', \phi') = S$ is trivial. Tensoring by (E, ϕ), we get that $T \otimes (E', \phi') = S \otimes (E, \phi)$, qed. The converse implication is easy and left to the reader. □

Remark 8.6. Lemmas 8.4 and 8.5 imply that, in $\mathrm{PDec}(\mathrm{Mod}'(k))$, an object is stably trivial if and only if it is trivial.

Proposition 8.7. *The canonical homomorphism* $\mathrm{Br}(k) = \mathrm{Br}(\mathrm{Mod}'_f(k)) \longrightarrow \mathrm{Br}(\mathrm{Mod}'(k))$ *is an isomorphism.*

Proof. Thanks to the previous remark, only surjectivity remains to be checked. Take $(E, \phi) \in \mathrm{PDec}(\mathrm{Mod}'(k))$. Thanks to lemma 8.5, it is enough to find a nonzero finite-dimensional subspace $F \subset E$ such that $F \otimes F \subset E \otimes E$ is stable by ϕ, since the inclusion $(F, \phi_{|F}) \longrightarrow (E, \phi)$ is then a nonzero homomorphism. Such an F can be built as follows. Choose a nonzero $e \in E$, and put $x := \phi(e \otimes e)$. It is a symmetric vector in $E \otimes E$, since ϕ and (12) commute. As such, there exists a finite dimensional vector space $F \subset E$ such that x belongs to $F \otimes F$ and F is minimal for this property. The space F can be alternatively defined as follows. First, we introduce some notation. Let $1 \le i \le n$ be positive integers. For $f \in E^*$, denote by f_i the linear map

$$E^{\otimes^n} \longrightarrow E^{\otimes^{n-1}},$$

$$e_1 \otimes \ldots \otimes e_n \mapsto f(e_i)e_1 \otimes \ldots \otimes \hat{e_i} \otimes \ldots \otimes e_n.$$

Then F is nothing but the image of the morphism

$$E^* \longrightarrow E,$$

$$f \mapsto f_1(x).$$

I claim that $F \otimes F$ is stable by ϕ. For $f, f \in E^*$, denote by $\Lambda(f, f')$ the composite

$$E^{\otimes^4} \stackrel{\phi_{12}\phi_{34}}{\longrightarrow} E^{\otimes^4} \stackrel{f_1 f_3'}{\longrightarrow} E \otimes E$$

. Then the $\Lambda(f, f')(e \otimes e \otimes e \otimes e)$, for f, f' ranging through E^*, span the space $F \otimes F$. Thus, it suffices to check that $(\phi \circ \Lambda(f, f'))(e \otimes e \otimes e \otimes e)$ always belongs to $F \otimes F$. We compute the composite $\phi \circ \Lambda(f, f')$. It is readily seen to equal

$$E^{\otimes^4} \stackrel{\phi_{24}\phi_{12}\phi_{34}}{\longrightarrow} E^{\otimes^4} \stackrel{f_1 f_3'}{E} \otimes E.$$

But

$$\phi_{24}\phi_{12}\phi_{34} = \phi_{12}\phi_{14}\phi_{34} = \phi_{12}\phi_{13}\phi_{14} = \phi_{13}\phi_{23}\phi_{14},$$

so that

$$(\phi \circ \Lambda(f, f'))(e \otimes e \otimes e \otimes e) = (f_1 \circ f_3' \circ \phi_{13} \circ \phi_{23} \circ \phi_1 4)(e \otimes e \otimes e \otimes e).$$

Since $(\phi_{23}\phi_1 4)(e \otimes e \otimes e \otimes e)$ belongs to $F^{\otimes^4}$, its image by ϕ_{13} belongs to $E \otimes F \otimes E \otimes F$, so that its image by $f_1 \circ f_3' \circ \phi_{13}$ belongs to $F \otimes F$, qed.

□

Bibliographie

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