TRIANGULAR GALOIS REPRESENTATIONS THAT DO NOT LIFT.

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ABSTRACT. Let p be an odd prime. This short note gives an example of a 3-dimensional triangular Galois representation

$$\rho_1 : \mathrm{Gal}(\mathbb{Q}((T))) \longrightarrow \mathbf{B}_3(\mathbb{F}_p),$$

that does not lift to a representation

$$\rho_2 : \mathrm{Gal}(\mathbb{Q}((T))) \longrightarrow \mathbf{B}_3(\mathbb{Z}/p^2).$$

This shows that Theorem 3.3 of the preprint [1] actually fails, as stated. Shortly after I released this note, Merkurjev and Scavia proposed a method to build more disruptive counter-examples; see [2]. The text of this note is, up to minor modifications, the original one (released 24/8/2024).

Contents

1. The counter-example.

Let p be an odd prime. Let F be a field of characteristic zero, whose algebraic closure is denoted by $\overline{F}/F$. Set

$$G^0 := \mathrm{Gal}(\overline{F}/F).$$

Recall the notation $\widehat{\mathbb{Z}}(1) := \varprojlim \mu_n$, for the Tate module of roots of unity. For a finite Galois module M and $n \geq 1$, set $M(1) := M \otimes_{\widehat{\mathbb{Z}}} \widehat{\mathbb{Z}}(1)$, $M(n+1) = M(n)(1)$ and $M(-n) = \mathrm{Hom}(M(n), \mathbb{Q}/\mathbb{Z})$.
Henceforth, we pick F such that the natural map

$$H^1(F, \mathbb{Z}/p^2(2)) \longrightarrow H^1(F, \mathbb{Z}/p(2))$$

is *not* surjective.

Example 1.1. Assume that F is a number field or a local field of characteristic zero, containing the p-th roots of unity but not the p^2-th roots of unity. Fix an isomorphism of trivial G^0-modules $\mathbb{Z}/p(1) \simeq \mathbb{Z}/p$. Then, the assumption above holds. Indeed, the cyclotomic character mod p^2 is then a non-trivial homomorphism

$$\chi \in \mathrm{Hom}(G^0, (1 + p\mathbb{Z}/p^2\mathbb{Z})^\times) = H^1(F, \mathbb{Z}/p) = H^1(F, \mathbb{Z}/p(1)) = F^\times/(F^\times)^p.$$

The connecting homomorphism of the extension

$$0 \longrightarrow \mathbb{Z}/p(2) \longrightarrow \mathbb{Z}/p^2(2) \longrightarrow \mathbb{Z}/p(2) \longrightarrow 0$$

is then just given (up to sign) by the cup-product

$$F^\times/(F^\times)^p = H^1(F, \mathbb{Z}/p(1)) \longrightarrow H^2(F, \mathbb{Z}/p(1)) = \mathrm{Br}(F)[p],$$

$$(x) \mapsto (x) \cup \chi.$$

It is well-known that it is non-trivial, by class field theory. Refining this argument, one can prove that the non-surjectivity assumption also holds when $F = \mathbb{Q}$.

Reformulating in terms of extensions, there is an extension of $(\mathbb{F}_p, G^0)$-bundles

$$(\mathcal{E}^0) : 0 \longrightarrow \mathbb{Z}/p(2) \longrightarrow E^0 \longrightarrow \mathbb{F}_p \longrightarrow 0,$$

that does not lift to an extension of $(\mathbb{Z}/p, G^0)$-bundles of the shape

$$0 \longrightarrow \mathbb{Z}/p^2(2) \longrightarrow * \longrightarrow \mathbb{Z}/p^2 \longrightarrow 0.$$

Consider the extension of $(\mathbb{F}_p, G^0)$-bundles

$$0 \longrightarrow \mathbb{Z}/p(2) \bigoplus \mathbb{Z}/p(1) \longrightarrow V_3 \longrightarrow \mathbb{F}_p \longrightarrow 0,$$

given by the (Baer) sum of $(\mathcal{E}^0)$ and the trivial extension

$$0 \longrightarrow \mathbb{Z}/p(1) \longrightarrow \mathbb{Z}/p(1) \bigoplus \mathbb{F}_p \longrightarrow \mathbb{F}_p \longrightarrow 0.$$

Setting $V_2 := \mathbb{Z}/p(2) \bigoplus \mathbb{Z}/p(1)$, this defines a complete flag

$$\nabla^0 : V_1^0 \subset V_2^0 \subset V_3^0,$$

with graded pieces $L_1 = \mathbb{Z}/p(2)$, $L_2 = \mathbb{Z}/p(1)$ and $L_3 = \mathbb{F}_p$. Both 2-dimensional flags extracted from ∇ are split.
In the sequel, one works over the field of Laurent series $F((T))$. Set

$$G := \mathrm{Gal}(\overline{F((T))}/F((T))) = \widehat{\mathbb{Z}}(1) \rtimes G^0.$$

By Kummer theory, there is an extension of $(\mathbb{F}_p, G)$-bundles (well-defined up to iso)

$$\mathcal{E}^T : 0 \longrightarrow \mathbb{Z}/p(1) \longrightarrow V_2^T \xrightarrow{t} \mathbb{F}_p \longrightarrow 0,$$

with class

$$(T) \in H^1(G, \mathbb{Z}/p(1)) = F((T))^\times / F((T))^{\times p}.$$

Observe that $\mathcal{E}^T$ may be seen as a complete 2-dimensional flag of $(\mathbb{F}_p, G^0)$-bundles,

$$\nabla^T : 0 \subset \mathbb{Z}/p(1) \subset V_2^T.$$

Definition 1.2. *Let A be a commutative ring. If M is an A-module, denote by*

$$\mathrm{Sym}^2(M) = \mathrm{Sym}^2_A(M) := (M \otimes_A M)/ < x \otimes y - y \otimes x >$$

its second symmetric power.

The 3-dimensional $(\mathbb{F}_p, G)$-bundle $\mathrm{Sym}^2(V_2^T)$ naturally fits into the extension

$$0 \longrightarrow V_2^T(1) \longrightarrow \mathrm{Sym}^2(V_2^T) \xrightarrow{\mathrm{Sym}^2(t)} \mathbb{F}_p \longrightarrow 0,$$

providing a natural complete flag

$$\mathbb{Z}/p(2) \subset V_2^T(1) \subset \mathrm{Sym}^2(V_2^T),$$

denoted by

$$\nabla^{T,2} : V_1^{T,2} \subset V_2^{T,2} \subset V_3^{T,2},$$

with graded pieces $L_1 = \mathbb{Z}/p(2)$, $L_2 = \mathbb{Z}/p(1)$ and $L_3 = \mathbb{F}_p$. The extension

$$0 \longrightarrow V_1^{T,2} \longrightarrow V_2^{T,2} \longrightarrow L_2^{T,2} \longrightarrow 0$$

is isomorphic to $\mathcal{E}^T(1)$. Since p is odd, one checks there is a commutative diagram

$$\begin{array}{ccccccccc} & 0 & \longrightarrow & V_2^T(1) & \longrightarrow & \mathrm{Sym}^2(V_2^T) & \longrightarrow & \mathbb{F}_p & \longrightarrow & 0 \\ & & & \downarrow{\scriptstyle t(1)} & & \downarrow & & \downarrow{\scriptstyle 2\mathrm{Id}} & & \\ \mathcal{E}^T : & 0 & \longrightarrow & \mathbb{Z}/p(1) & \longrightarrow & V_2^T & \longrightarrow & \mathbb{F}_p & \longrightarrow & 0, \end{array}$$

where the middle vertical arrow is given by $a \otimes b \mapsto t(a)b + t(b)a$. Therefore, the quotient extension

$$0 \longrightarrow L_2^{T,2} \longrightarrow V_3^{T,2}/V_1^{T,2} \longrightarrow L_3^{T,2} \longrightarrow 0$$

is isomorphic to $2\mathcal{E}^T$. [In short: up to twisting and rescaling, both 2-dimensional complete flags extracted from $\nabla^{T,2}$ are isomorphic to ∇^T.]

Remark 1.3. The construction that was just performed, actually exists mod p^2. Indeed, start with the extension of $(\mathbb{Z}/p^2, G)$-bundles

$$\mathcal{E}_2^T : 0 \longrightarrow \mathbb{Z}/p^2(1) \longrightarrow V_{2,2}^T \longrightarrow \mathbb{Z}/p^2 \longrightarrow 0,$$

whose class is

$$(T) \in H^1(G, \mathbb{Z}/p^2(1)) = F((T))^\times / F((T))^{\times p^2}.$$

In the same way, one builds a lift of $\nabla^{T,2}$, to a complete flag of $(\mathbb{Z}/p^2, G)$-bundles

$$\nabla_2^{T,2} : V_{1,2}^{T,2} \subset V_{2,2}^{T,2} \subset V_{3,2}^{T,2} := \mathrm{Sym}_{\mathbb{Z}/p^2}(V_{2,2}^T),$$

with graded pieces $L_1 = \mathbb{Z}/p^2(2)$, $L_2 = \mathbb{Z}/p^2(1)$ and $L_3 = \mathbb{Z}/p^2$.

DEFINITION 1.4. *Consider the extension (of linear algebraic $\mathbb{Z}$-groups)*

$$1 \longrightarrow \mathbf{U}_3 \longrightarrow \mathbf{B}_3 \longrightarrow \mathbf{T}_3 = \mathbf{G}_m^3 \longrightarrow 1.$$

It has a natural splitting, given by the split diagonal maximal torus $\mathbf{T}_3 \subset \mathbf{B}_3$. For any $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{Z}^3$, one can turn $\mathbf{G}_a$ into (the affine space of) a one-dimensional representation of $\mathbf{T}_3$, by the formula

$$(t_1, t_2, t_3).x = t_1^{\lambda_1} t_2^{\lambda_2} t_3^{\lambda_3} x.$$

As such, denote it by $\mathbf{G}_a(\lambda)$. Let $\mathbf{T}_3$ act on $\mathbf{U}_3$ by conjugation. Consider the $\mathbf{T}_3$-equivariant embedding

$$\iota : \mathbf{G}_a(1, 0, -1) \longrightarrow \mathbf{U}_3,$$

$$x \mapsto \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

identifying $\mathbf{G}_a(1, 0, -1)$ to the center of $\mathbf{U}_3$. Denote by

$$z \in Z^1(G^0, (\mathbb{F}_p^\times)^3) = Z^1(G^0, \mathbf{T}_3(\mathbb{F}_p))$$

the 1-cocycle (a homomorphism) corresponding to the triple $(\mathbb{Z}/p(2), \mathbb{Z}/p(1), \mathbb{Z}/p)$. Taking $\mathbb{F}_p$-points of ι, then twisting by z, one gets an embedding of finite G^0-groups

$$i : \mathbb{Z}/p(2) \hookrightarrow \mathbf{U}_3(\mathbb{F}_p)(2, 1, 0) := \mathbf{U}_3(\mathbb{F}_p)^z,$$

identifying $\mathbb{Z}/p(2)$ to the center of $\mathbf{U}_3(\mathbb{F}_p)(2, 1, 0)$.
The same construction can be performed, with $\mathbb{Z}/p^2$ in place of $\mathbb{F}_p$.

LEMMA 1.5. *The set $H^1(G, \mathbf{U}_3(\mathbb{F}_p)(2, 1, 0))$ parametrises complete 3-dimensional flags of $(\mathbb{F}_p, G)$-modules, with prescribed graded pieces $L_1 = \mathbb{Z}/p(2)$, $L_2 = \mathbb{Z}/p(1)$ and $L_3 = \mathbb{Z}/p$. The same holds mod p^2, replacing $\mathbb{F}_p$ by $\mathbb{Z}/p^2$.*
Via this identification, $[\nabla^0] = i_*([\mathcal{E}^0])$.

PROOF. Standard verification in Galois cohomology. □

DEFINITION 1.6. *Let*

$$\nabla : 0 = V_0 \subset V_1 \subset V_2 \subset V_3$$

be a flag of $(\mathbb{F}_p, G)$-modules as in the preceding Lemma. Denote by

$$\mathrm{End}_{-1}(\nabla) \subset \mathrm{End}(V_3)$$

the $\mathbb{F}_p$-subspace consisting of endomorphisms ϕ that shift degrees of the filtration by -1, i.e. such that $\phi(V_i) \subset V_{i-1}$ for $1 \leq i \leq 3$.
There is a natural extension of $(\mathbb{F}_p, G)$-modules

$$0 \longrightarrow \mathbb{Z}/p(2) \xrightarrow{j = j_\nabla} \mathrm{End}_{-1}(\nabla) \longrightarrow \mathbb{Z}/p(1) \bigoplus \mathbb{Z}/p(1) \longrightarrow 0.$$

It can be obtained by twisting the extension of $\mathbb{F}_p$-representations of $\mathbf{B}_3$,

$$0 \longrightarrow \mathbf{G}_a(1, 0, -1) \xrightarrow{\mathrm{Lie}(\iota)} \mathrm{Lie}(\mathbf{U}_3) \longrightarrow \mathbf{G}_a(1, -1, 0) \bigoplus \mathbf{G}_a(0, 1, -1) \longrightarrow 0,$$

by the $\mathbf{B}_3(\mathbb{F}_p)$-torsor corresponding to ∇.

LEMMA 1.7. *Pick* $[\nabla] \in H^1(G, \mathbf{U}_3(\mathbb{F}_p)(2,1,0))$*. Consider lifting it, to some*

$$[\nabla_2] \in H^1(G, \mathbf{U}_3(\mathbb{Z}/p^2)(2,1,0)).$$

This lifting problem is obstructed by a natural class

$$\mathrm{Obs}_2(\nabla) \in H^2(G, \mathrm{End}_{-1}(\nabla)).$$

Similarly, for $a \in H^1(G, \mathbb{Z}/p(2))$*, denote by* $\mathrm{Obs}_2(a) \in H^1(G, \mathbb{Z}/p(2))$ *the obstruction to lifting* a *to some* $a_2 \in H^1(G, \mathbb{Z}/p^2(2))$*. The following holds.*

(1) There is a canonical iso of $(\mathbb{F}_p, G)$*-bundles* $\mathrm{End}_{-1}(\nabla) \simeq \mathrm{End}_{-1}(\nabla + i_*(a))$*.*
(2) In the group $H^2(G, \mathrm{End}_{-1}(\nabla))$*, via the iso of item (1), one has*

$$\mathrm{Obs}_2(\nabla + i_*(a)) = \mathrm{Obs}_2(\nabla) + j_*(\mathrm{Obs}_2(a)).$$

PROOF. Verification to be added.
□ We are now ready to provide the desired counter-example.
It is the 3-dimensional complete flag of $(\mathbb{F}_p, G)$-bundles

$$\nabla : V_1 \subset V_2 \subset V_3,$$

defined by the formula

$$\nabla := i_*(\mathcal{E}^0) + \nabla^{T,2}.$$

This sum makes sense, because $\mathrm{Im}(i)$ is *central* in $\mathbf{U}_3(\mathbb{F}_p)(2,1,0)$.
The flag ∇ corresponds to a representation

$$\rho_1 : G \longrightarrow \mathbf{B}_3(\mathbb{F}_p).$$

Observe that ∇ is not isomorphic to $\nabla^{T,2}$. However, their 2-dimensional subflags and quotient flags all are all isomorphic to ∇^T, up to twisting and rescaling.

PROPOSITION 1.8. *The flag* ∇ *does not lift to a complete flag of* $(\mathbb{Z}/p^2, G)$*-bundles. Equivalently,* ρ_1 *does not lift to a (continuous) homomorphism*

$$\rho_2 : G \longrightarrow \mathbf{B}_3(\mathbb{Z}/p^2).$$

PROOF. Assume that such a lift exists. Denote it by

$$\nabla_2 : V_{1,2} \subset V_{2,2} \subset V_{3,2},$$

and its graded pieces by $L_{1,2}, L_{2,2}$ and $L_{3,2}$. Write

$$L_{1,2} = \mathbb{Z}/p^2(2) + \epsilon_1,$$

$$L_{2,2} = \mathbb{Z}/p^2(1) + \epsilon_2$$

and

$$L_{3,2} = \mathbb{Z}/p^2 + \epsilon_3,$$

where the ϵ_i's are homomorphisms

$$G \longrightarrow (\mathbb{F}_p, +) = (1 + p\mathbb{Z}/p^2\mathbb{Z})^\times.$$

Upon applying a global twist to the lifted flag, one may assume w.l.o.g. that $\epsilon_2 = 0$.
There is the residue sequence at T in Galois cohomology, reading as

$$0 \longrightarrow H^1(F, \mathbb{F}_p) \longrightarrow H^1(F((T)), \mathbb{F}_p) \xrightarrow{\mathrm{res}} H^0(F, \mathbb{F}_p(-1)) \longrightarrow 0.$$

Since $\mathbb{Z}/p(1) \notin F$, G^0 acts non-trivially on $\mathbb{F}_p(-1)$, whence $H^0(F, \mathbb{F}_p(-1)) = 0$, so that $H^1(F, \mathbb{F}_p) = H^1(F((T)), \mathbb{F}_p)$. Recall the flag $\nabla_2^{T,2}$ from Remark 1.3.
Observe that the extension

$$0 \longrightarrow L_1 \longrightarrow V_2 \longrightarrow L_2 \longrightarrow 0$$

has two lifts to an extension of $(\mathbb{Z}/p^2, G)$-bundles. These are

$$0 \longrightarrow L_{1,2} \longrightarrow V_{2,2} \longrightarrow L_{2,2} \longrightarrow 0$$

and

$$0 \longrightarrow \mathbb{Z}/p^2(2) \longrightarrow V_{2,2}^{T,2} \longrightarrow \mathbb{Z}/p^2(1) \longrightarrow 0.$$

Comparing obstructions in Galois cohomology, one finds that

$$\epsilon_1 \cup (T) = 0 \in H^2(G, \mathbb{Z}/p(1)).$$

Since $\epsilon_1 \in H^1(G^0, \mathbb{F}_p)$, taking residue at T yields $\epsilon_1 = 0$. One proves that $\epsilon_3 = 0$ in a similar way. Thus, graded pieces of ∇_2 are the successive Tate twists $\mathbb{Z}/p^2(2), \mathbb{Z}/p^2(1)$ and $\mathbb{Z}/p^2$. To conclude, we are going to applying Lemma 1.7, of which we adopt notation. As was just proved, one has $\mathrm{Obs}_2(\nabla) = 0$. Also, from the existence of $\nabla_2^{T,2}$ in Remark 1.3, one gets $\mathrm{Obs}_2(\nabla^{T,2}) = 0$. Thus, one has $j_*(\mathrm{Obs}_2(\mathcal{E}^0)) = 0$. To conclude, it remains to show that $\mathrm{Obs}_2(\mathcal{E}^0) = 0$, which will contradict the initial choice of $\mathcal{E}^0$.
Observe that, in the current situation, the extension of $(\mathbb{F}_p, G)$-modules

$$0 \longrightarrow \mathbb{Z}/p(2) \xrightarrow{j} \mathrm{End}_{-1}(\nabla) \longrightarrow \mathbb{Z}/p(1) \bigoplus \mathbb{Z}/p(1) \longrightarrow 0$$

is the sum of two (non-zero scalar multiples of) copies of

$$\mathcal{E}^T(1) : 0 \longrightarrow \mathbb{Z}/p(2) \longrightarrow V_2^T(1) \longrightarrow \mathbb{Z}/p(1) \longrightarrow 0.$$

Thus, its connecting homomorphism reads as

$$H^1(G, \mathbb{Z}/p(1) \bigoplus \mathbb{Z}/p(1)) \longrightarrow H^2(G, \mathbb{Z}/p(2)),$$

$$(u, v) \mapsto (\alpha u + \beta v) \cup (T),$$

for some $\alpha, \beta \in \mathbb{F}_p^\times$. The vanishing of $j_*(\mathrm{Obs}_2(\mathcal{E}^0))$ then implies that

$$\mathrm{Obs}_2(\mathcal{E}^0) = w \cup (T),$$

for some

$$w \in H^1(G, \mathbb{Z}/p(1)) = H^1(G^0, \mathbb{Z}/p(1)) \bigoplus \mathbb{F}_p(T).$$

Since p is odd, $(T) \cup (T) = 0$, so that one may assume w.l.o.g. $w \in H^1(G^0, \mathbb{Z}/p(1))$. Since $\mathrm{Obs}_2(\mathcal{E}^0)$ is unramified (= comes from F), taking residue at T yields $w = 0$, hence $\mathrm{Obs}_2(\mathcal{E}^0) = 0$. This concludes the proof. □

1.1. A concrete description of the flag ∇. To simplify, assume that F is a number field or a local field, containing the p-th roots of unity but not the p^2-th roots of unity- see Example 1.1. Fix an isomorphism of G^0-modules $\mathbb{Z}/p(1) \simeq \mathbb{F}_p$. The extension $(\mathcal{E}^0)$ corresponds (up to iso) to a homomorphism

$$G^0 \longrightarrow \mathbf{B}_2(\mathbb{F}_p),$$

$$g \mapsto \begin{pmatrix} 1 & e^0(g) \\ 0 & 1 \end{pmatrix}.$$

The flag ∇^0 is then simply given by the homomorphism

$$G^0 \longrightarrow \mathbf{B}_3(\mathbb{F}_p),$$

$$g \mapsto \begin{pmatrix} 1 & 0 & e^0(g) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Likewise, the extension $(\mathcal{E}^T)$ corresponds to a group homomorphism of the shape

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},$$

where $G \xrightarrow{t} \mathbb{Z}/p$ is the additive character corresponding, via Kummer theory, to $(T) \in H^1(F((T)), \mathbb{Z}/p(1))$.
A computation then shows that the flag $\nabla^{T,2}$ is given by the homomorphism

$$\begin{pmatrix} 1 & t & t^2 \\ 0 & 1 & 2t \\ 0 & 0 & 1 \end{pmatrix}.$$

The flag ∇ of Proposition 1.8 is then provided by the formula

$$\begin{pmatrix} 1 & t & t^2 + e^0 \\ 0 & 1 & 2t \\ 0 & 0 & 1 \end{pmatrix},$$

where it is understood that e^0 should be precomposed with the natural surjection $G \longrightarrow G^0$. To conclude, we observe that the description above extends to the case where $\mathbb{Z}/p(1) \not\subset F$. In that case, denoting by $\chi : G^0 \longrightarrow \mathbb{F}_p^\times$ the p-th cyclotomic character, ∇ is then given by a homomorphism $G \longrightarrow \mathbf{B}_3(\mathbb{F}_p)$ of the shape

$$\begin{pmatrix} \chi^2 & \chi t & t^2 + e^0 \\ 0 & \chi & 2t \\ 0 & 0 & 1 \end{pmatrix}.$$

Here the maps $t : G \longrightarrow \mathbb{F}_p$ and $e^0 : G^0 \longrightarrow \mathbb{F}_p$ are no longer homomorphisms: they are 1-cocycles.

Remark 1.9. It is likely that the counter-example ρ_1, does not lift to a representation $G \longrightarrow \mathbf{GL}_3(\mathbb{Z}/p^2)$. This would require an extra computation.

Bibliography

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