On higher trace forms of separable algebras

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Abstract.

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Introduction, notations

In this paper, we denote by k a commutative field, by k_s (resp. $\bar{k}$) a separable (resp. algebraic) closure of k and by Γ_k the Galois group of k_s/k. The objects we study are separable k-algebras, that is to say, finite dimensional k-algebras A such that $A \otimes_k \bar{k}$ has zero Jacobson radical. Such algebras are products of simple k-algebras, i.e. of the form $\mathrm{End}_D(M)$, where D is a skew field, finite dimensional as a k-vector space, and M is a finite dimensional D-vector space. In [Re], it is proven that, if A/k is a central simple algebra, and if $r \geq 3$ is any integer, then the r-form

$$(a_1, ..., a_r) \mapsto \mathrm{tr}(a_1 ... a_r)$$

(tr being the reduced trace) carries a lot of information about the structure of A; in particular, it captures the essential dimension of A. Our aim is to give a cohomological explanation of this phenomenon, and to generalize Reichstein's result to any separable k-algebra.

Acknowledgements

Contents

1. The group of automorphisms of a higher trace form

Let A/k be a separable k-algebra, and tr its reduced trace form. We denote by G the group of automorphisms of A; we view it as a linear algebraic group. The choice of an isomorphism $A \otimes_k k_s \simeq \Pi_{i=1}^s \mathcal{M}_{n_i}^{e_i}(k_s)$ (where the n_i are all distinct integers) gives an isomorphism $G_{k_s} \simeq \Pi_{i=1}^s \mathrm{PGL}_{n_i}^{e_i} \rtimes \mathfrak{S}_{e_i}$. In particular, G is smooth over k. Let $r \geq 3$ be any integer, and consider the r-form

$$F_r : (a_1, ..., a_r) \mapsto \mathrm{tr}(a_1 ... a_r).$$

We want to compute the linear algebraic group H_r consisting of invertible linear transformations of A which leave F_r invariant. Let $\mu_{r,A}$ be the (multiplicative)

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subgroup of $\mathrm{GL}_1(A)$ (the algebraic group of invertible elements of A) consisting of central elements x such that $x^r = 1$ (beware that $\mu_{r,A}$ is not smooth if $\mathrm{char}(k)$ divides r). The action of $\mu_{r,A}$ of A by translations obviously preserves F_r; it follows that $\mu_{r,A} \rtimes G$ is a subgroup of H_r. The following proposition ensures this inclusion is an equality.

Proposition 1.1. *The natural inclusion $\mu_{r,A} \rtimes G \longrightarrow H_r$ is an isomorphism.*

Proof. For simplicity, we work on the level of k-points of the algebraic groups considered, i.e., we show that $(\mu_{r,A} \rtimes G)(k) = H_r(k)$. The proof we give below also works if we replace k by any k-algebra B, and base-change A accordingly. Therefore, we will also have $(\mu_{r,A} \rtimes G)(B) = H_r(B)$, for any B, which is obviously enough to conclude that the closed embedding $\mu_{r,A} \rtimes G \longrightarrow H_r$ is in fact an isomorphism.
Let $f : A \longrightarrow A$ be a k-linear map belonging to $H_r(k)$. We first show that $f(1)$ is central in A. Let x be any element of A. We have to show that $f(1)f(x) = f(x)f(1)$. Pick $a \in A$ such that $f(a)$ is invertible, and let y be any element of A. We compute:

$$\mathrm{tr}(f(1)f(x)f(a)^{r-3}f(y)) = \mathrm{tr}(x * 1 * a^{r-3} * y) = \mathrm{tr}(f(x)f(1)f(a)^{r-3}f(y)).$$

Since this must hold for every y, and since tr is non-degenerate, we conclude that $f(1)f(x)f(a)^{r-3} = f(x)f(1)f(a)^{r-3}$, whence the conclusion.
Then we show that, for all x_1, x_2 in A, $f(x_1x_2)f(1) = f(x_1)f(x_2)$. We proceed as before: it suffices to show that $f(x_1x_2)f(1)f(a)^{r-3} = f(x_1)f(x_2)f(a)^{r-3}$, or that $\mathrm{tr}(f(x_1x_2)f(1)f(a)^{r-3}f(y)) = \mathrm{tr}(f(x_1)f(x_2)f(a)^{r-3}f(y))$, for any y in A. But both sides equal $\mathrm{tr}(x_1x_2a^{r-3}y)$, whence the claim.
Next, we show that $f(1)$ is invertible. This follows from $f(a)f(a) = f(a^2)f(1)$ (where a is, as before, an element of A such that $f(a)$ is invertible).
Let g be the linear automorphism of A defined by $x \mapsto f(1)^{-1}x$. From what we have shown, it follows that $g \circ f$ is a k-algebra automorphism of A, hence preserves F_r. This implies that g also preserves F_r, which ensures that $f(1) \in \mu_{r,A}(k)$. This finishes the proof.

□

We are now able to prove the following:

Theorem 1.2. *Let $r \geq 3$ be an integer, k_0 a subfield of k, and A/k a separable algebra. We have*

$$\mathrm{ed}_{k_0}(A) = \mathrm{ed}_{k_0}(F_r(A)),$$

where $F_r(A)$ is the r-trace form associated to A.

Proof. Let $B = \Pi_{i=1}^{s} \mathcal{M}_{n_i}^{e_i}$ (all n_i distinct) be the split form of A. Let $G = \Pi_{i=1}^{s} \mathrm{PGL}_{n_i}^{e_i} \rtimes \mathfrak{S}_{e_i}$ be the automorphism group of B. Let $a \in H^1(k, G)$ be the class of A, and $f \in H^1_{fppf}(k, \mu_{r,B} \rtimes G)$ that of $F_r(A)$. Assume there exists a subfield l of k, containing k_0, and a r-form $\tilde{F}$, defined over l, such that $\tilde{F}_k$ is isomorphic to $F_r(A)$. Let $\tilde{f}$ be the class of $\tilde{F}$ in $H^1_{fppf}(l, \mu_{r,B} \rtimes G)$. By change of group using the canonical projection $\mu_{r,B} \rtimes G \longrightarrow G$, we obtain a class $\tilde{a}$ in $H^1(l, G)$. Let $\tilde{A}$ be a separable l-algebra which represents this class. It is then straightforward to check that $\tilde{A}_k \simeq A$. We have proved that $\mathrm{ed}_{k_0}(A) \leq \mathrm{ed}_{k_0}(F_r(A))$. The other inequality is obvious.

□

References

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