THE LIE ALGEBRA OF TYPE G_2 IS RATIONAL OVER ITS QUOTIENT BY THE ADJOINT ACTION

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Abstract. Let G be a split simple group of type G_2 over a field k, and let $\mathfrak{g}$ be its Lie algebra. Answering a question of J.-L. Colliot-Thélène, B. Kunyavskiĭ, V. L. Popov, and Z. Reichstein, we show that the function field $k(\mathfrak{g})$ is generated by algebraically independent elements over the field of adjoint invariants $k(\mathfrak{g})^G$.

Résumé. Soit G un groupe algébrique simple et déployé de type G_2 sur un corps k. Soit $\mathfrak{g}$ son algèbre de Lie. On démontre que le corps des fonctions $k(\mathfrak{g})$ est transcendant pur sur le corps $k(\mathfrak{g})^G$ des invariants adjoints. Ceci répond par l'affirmative à une question posée par J.-L. Colliot-Thélène, B. Kunyavskiĭ, V. L. Popov et Z. Reichstein.

I. Introduction. Let G be a split connected reductive group over a field k and let $\mathfrak{g}$ be the Lie algebra of G. We will be interested in the following natural question:

Question 1. Is the function field $k(\mathfrak{g})$ *purely transcendental* over the field of invariants $k(\mathfrak{g})^G$ for the adjoint action of G on $\mathfrak{g}$? That is, can $k(\mathfrak{g})$ be generated over $k(\mathfrak{g})^G$ by algebraically independent elements?

In [5], the authors reduce this question to the case where G is simple, and show that in the case of simple groups, the answer is affirmative for split groups of types A_n and C_n, and negative for all other types except possibly for G_2. The standing assumption in [5] is that $\mathrm{char}(k) = 0$, but here we work in arbitrary characteristic.

The purpose of this note is to settle Question 1 for the remaining case $G = G_2$.

Theorem 2. *Let k be an arbitrary field and G be the simple split k-group of type G_2. Then $k(\mathfrak{g})$ is purely transcendental over $k(\mathfrak{g})^G$.*

Under the same hypothesis, and also assuming $\mathrm{char}(k) = 0$, it follows from Theorem 2 and [5, Theorem 4.10] that the field extension $k(G)/k(G)^G$ is also purely transcendental, where G acts on itself by conjugation.

Apart from settling the last case left open in [5], we were motivated by the (still mysterious) connection between Question 1 and the Gelfand-Kirillov (GK) conjecture [9]. In this context $\mathrm{char}(k) = 0$. A. Premet [11] recently showed that the GK conjecture fails for simple Lie algebras of any type other than A_n, C_n and G_2. His paper relies on the negative results of [5] and their characteristic p analogues ([11], see also [5, Theorem 6.3]). It is not known whether a positive answer to Question 1 for $\mathfrak{g}$ implies the GK conjecture for $\mathfrak{g}$. The GK conjecture has been proved for algebras of type A_n (see [9]), but remains open for types C_n and G_2. While Theorem 2 does not settle the GK conjecture for type G_2, it puts the remaining two open cases—for algebras of type C_n and G_2—on equal footing vis-à-vis Question 1.

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II. Twisting. Temporarily, let W be a linear algebraic group over a field k. (In the sequel, W will be the Weyl group of G; in particular, it will be finite and smooth.) We refer to [7, Section 3], [8, Section 2], or [5, Section 2] for details about the following facts.

Let X be a quasi-projective variety with a (right) W-action defined over k, and let ζ be a (left) W-torsor over k. The diagonal left action of W on $X \times_{\mathrm{Spec}(k)} \zeta$ (by $g.(x,z) = (xg^{-1}, gz)$) makes $X \times_{\mathrm{Spec}(k)} \zeta$ into the total space of a W-torsor $X \times_{\mathrm{Spec}(k)} \zeta \to B$. The base space B of this torsor is usually called the *twist* of X by ζ. We denote it by ${}^{\zeta}X$.

It is easy to see that if ζ is trivial then ${}^{\zeta}X$ is k-isomorphic to X. Hence, ${}^{\zeta}X$ is a k-form of X, i.e., X and ${}^{\zeta}X$ become isomorphic over an algebraic closure of k.

The twisting construction is functorial in X: a W-equivariant morphism $X \to Y$ (or rational map $X \dashrightarrow Y$) induces a k-morphism ${}^{\zeta}X \to {}^{\zeta}Y$ (resp., rational map ${}^{\zeta}X \dashrightarrow {}^{\zeta}Y$).

III. The split group of type G_2. We fix notation and briefly review the basic facts, referring to [13], [1], or [2] for more details. Over any field k, a simple split group G of type G_2 has a faithful seven-dimensional representation V. Following [2, (3.11)], one can fix a basis $f_1, \dots, f_7$, with dual basis $X_1, \dots, X_7$, so that G preserves the nonsingular quadratic norm $N = X_1X_7 + X_2X_6 + X_3X_5 + X_4^2$. (See [1, §6.1] for the case $\mathrm{char}(k) = 2$. In this case V is not irreducible, since the subspace spanned by f_4 is invariant; the quotient $V/(k \cdot f_4)$ is the minimal irreducible representation. However, irreducibility will not be necessary in our context.) The corresponding embedding $G \hookrightarrow \mathrm{GL}_7$ yields a split maximal torus and Borel subgroup $T \subset B \subset G$, by intersecting with diagonal and upper-triangular matrices. Explicitly, the maximal torus is

$$T = \mathrm{diag}(t_1,\, t_2,\, t_1t_2^{-1},\, 1,\, t_1^{-1}t_2,\, t_2^{-1},\, t_1^{-1}); \tag{1}$$

cf. [2, Lemma 3.13].

The Weyl group $W = N(T)/T$ is isomorphic to the dihedral group with 12 elements, and the surjection $N(T) \to W$ splits. The inclusion $G \hookrightarrow \mathrm{GL}_7$ thus gives rise to an inclusion $N(T) = T \rtimes W \hookrightarrow D \rtimes \mathrm{S}_7$, where $D \subset \mathrm{GL}_7$ is the subgroup of diagonal matrices. On the level of the dual basis $X_1, \dots, X_7$, we obtain an isomorphism $W \cong \mathrm{S}_3 \times \mathrm{S}_2$ realized as follows: S_3 permutes the three ordered pairs (X_1, X_7), (X_2, X_6) and (X_3, X_5), and S_2 exchanges the two ordered triples (X_1, X_5, X_6) and (X_7, X_3, X_2). The variable X_4 is fixed by W. For details, see [2, §A.3].

The subgroup $P \subset G$ stabilizing the isotropic line spanned by f_1 is a maximal standard parabolic, and the corresponding homogeneous space $P\backslash G$ is isomorphic to the five-dimensional quadric $\mathcal{Q} \subset \mathbb{P}(V)$ defined by the vanishing of the norm, i.e., by the equation

$$X_1X_7 + X_2X_6 + X_3X_5 + X_4^2 = 0. \tag{2}$$

Note that the quadric $\mathcal{Q}$ is endowed with an action of T. An easy tangent space computation shows that P is smooth regardless of the characteristic of k.

Lemma 3. *The group P is* special, *i.e., $H^1(l, P) = \{1\}$ for every field extension l/k. Moreover, P is rational, as a variety over k.*

Proof. Since the split group of type G_2 is defined over the prime field, we may replace k by the prime field for the purpose of proving this lemma, and in particular, we can assume k is perfect. We begin by briefly recalling a construction of Chevalley [4]. The isotropic line $E_1 \subset V$ stabilized by P is spanned by f_1, and P also preserves an isotropic 3-space E_3 spanned by f_1, f_2, f_3; see, e.g., [2, §2.2]. There is a corresponding map $P \to \mathrm{GL}(E_3/E_1) \cong \mathrm{GL}_2$, which is a split surjection thanks to the block matrix described in [10, p. 13] as the image of "B" in GL_7. The kernel is unipotent, and we have a split exact sequence corresponding to the Levi decomposition:

$$1 \to R_u(P) \to P \to \mathrm{GL}_2 \to 1. \tag{3}$$

Combining the exact sequence in cohomology induced by (3) with the fact that both $R_u(P)$ and GL_2 are special (see [12, pp. 122 and 128]), shows that P is special.

Since P is isomorphic to $R_u(P) \times \mathrm{GL}_2$ as a variety over k, and P is smooth, so is $R_u(P)$. A smooth connected unipotent group over a perfect field is rational [6, IV, §2(3.10)]; therefore $R_u(P)$ is k-rational, and so is P. □

IV. Proof of Theorem 2. Keep the notation of the previous section. By a *W-model* (of $k(\mathcal{Q})^T$), we mean a quasi-projective k-variety Y, endowed with a right action of W, together with a dominant W-equivariant k-rational map $\mathcal{Q} \dashrightarrow Y$ which, on the level of function fields, identifies $k(Y)$ with $k(\mathcal{Q})^T$. Such a map $\mathcal{Q} \dashrightarrow Y$ is called a *(W-equivariant) rational quotient map.* A W-model is unique up to a W-equivariant birational isomorphism; we will construct an explicit one below.

We reduce Theorem 2 to a statement about rationality of a twisted W-model, in two steps. The first holds for general split connected semisimple groups G.

Proposition 4. *Let G be a split connected semisimple group over k, with split maximal k-torus T. Let $K = k(\mathfrak{t})^W$, $L = k(\mathfrak{t})$, and let ζ be the W-torsor corresponding to the field extension L/K. If the twisted variety ${}^\zeta(G_K/T_K)$ is rational over K, then $k(\mathfrak{g})$ is purely transcendental over $k(\mathfrak{g})^G$.*

Proof. Consider the $(G \times W)$-equivariant morphism

$$f\colon G/T \times_{\mathrm{Spec}(k)} \mathfrak{t} \to \mathfrak{g}$$

given by $(\overline{a}, t) \mapsto \mathrm{Ad}(a)t$, where $\mathfrak{t}$ is the Lie algebra of T, $\overline{a} \in G/T$ is the class of $a \in G$, modulo T. Here G acts on $G/T \times \mathfrak{t}$ by translations on the first factor (and trivially on $\mathfrak{t}$), and via the adjoint representation on $\mathfrak{g}$. The Weyl group W naturally acts on $\mathfrak{t}$ and G/T (on the right), diagonally on $G/T \times \mathfrak{t}$, and trivially on $\mathfrak{g}$.

The image of f contains the semisimple locus in $\mathfrak{g}$, so f is dominant and induces an inclusion $f^*\colon k(\mathfrak{g}) \hookrightarrow k(G/T \times \mathfrak{t})$. Clearly $f^*\, k(\mathfrak{g}) \subset k(G/T \times \mathfrak{t})^W$. We will show that in fact

$$f^*\, k(\mathfrak{g}) = k(G/T \times \mathfrak{t})^W . \tag{4}$$

Write $\overline{k}$ for an algebraic closure of k, and note that the preimage of a $\overline{k}$-point of $\mathfrak{g}$ in general position is a single W-orbit in $(G/T \times \mathfrak{t})_{\overline{k}}$. To establish (4), it remains to check that f is smooth at a general point (g, x) of $G/T \times \mathfrak{t}$. (In particular, when $\mathrm{char}(k) = 0$ nothing more is needed.) To carry out this calculation, we may assume without loss of generality that k is algebraically closed and (since f is G-equivariant) $g = 1$. Since $\dim(G/T \times \mathfrak{t}) = \dim(\mathfrak{g})$, it suffices to show that the differential

$$df\colon T_{(1,x)}(G/T \times \mathfrak{t}) \to T_x(\mathfrak{g})$$

is surjective, for any regular semisimple element $x \in \mathfrak{t}$. Equivalently, we want to show that $[x, \mathfrak{g}] + \mathfrak{t} = \mathfrak{g}$. Since x is regular, we have $\dim([x, \mathfrak{g}]) + \dim \mathfrak{t} = \dim \mathfrak{g}$. Thus it remains to show that $[x, \mathfrak{g}] \cap \mathfrak{t} = 0$. To see this, suppose $[x, y] \in \mathfrak{t}$ for some $y \in \mathfrak{g}$. Since x is semisimple, we can write $y = \sum_{i=1}^r y_{\lambda_i}$, where y_λ is an eigenvector for $\mathrm{ad}(x)$ with eigenvalue λ, and $\lambda_1, \dots, \lambda_r$ are distinct. Then $[x, y] = \sum_{i=1}^r \lambda_i y_{\lambda_i} \in \mathfrak{t}$ is an eigenvector for $\mathrm{ad}(x)$ with eigenvalue 0. Remembering that eigenvectors of $\mathrm{ad}(x)$ with distinct eigenvalues are linearly independent, we conclude that $[x, y] = 0$. This completes the proof of (4).

It is easy to see $k(G/T \times \mathfrak{t})^{G \times W} = k(\mathfrak{t})^W$. Summarizing, f^* induces a diagram

$$\begin{array}{ccc} k(G/T \times_{\mathrm{Spec}(k)} \mathfrak{t})^W & \xrightarrow{\sim} & k(\mathfrak{g}) \\ | & & | \\ k(\mathfrak{t})^W & \xrightarrow{\sim} & k(\mathfrak{g})^G, \end{array}$$

where the top row is the G-equivariant isomorphism (4), and the bottom row is obtained from the top by taking G-invariants. Note that

$$k(G/T \times_{\mathrm{Spec}(k)} \mathfrak{t}) \simeq K((G/T)_K \times_{\mathrm{Spec}(K)} \mathrm{Spec}\, L),$$

where $\simeq$ denotes a G-equivariant isomorphism of fields. (Recall that G acts trivially on $\mathfrak{t}$ and hence also on L and K.) Thus the field extension on the left side of our diagram can be rewritten as $K({}^{\zeta}(G_K/T_K))/K$, where ζ is the W-torsor $\mathrm{Spec}(L) \to \mathrm{Spec}(K)$. By assumption, this field extension is purely transcendental; the diagram shows it is isomorphic to $k(\mathfrak{g})/k(\mathfrak{g})^G$. □

For the second reduction, we return to the assumptions of Section III.

Proposition 5. *Let G be a split simple group of type G_2, with maximal torus T and Weyl group W, and let $\mathcal{Q}$ be the quadric defined in Section III. Suppose that for a given W-model Y of $k(\mathcal{Q})^T$, and for some W-torsor ζ over some field K/k, the twisted variety ${}^{\zeta}(Y_K)$ is rational over K. Then the twisted variety ${}^{\zeta}(G_K/T_K)$ is rational over K.*

Proof. For the purpose of this proof, we may view K as a new base field and replace it with k.

We claim that the left action of P on G/T is generically free. Since G has trivial center, the (characteristic-free) argument at the beginning of the proof of [5, Lemma 9.1] shows that in order to establish this claim it suffices to show that the right T-action on $\mathcal{Q} = P\backslash G$ is generically free. The latter action, given by restricting the linear action (1) of T on $\mathbb{P}^6$ to the quadric $\mathcal{Q}$ given by (2), is clearly generically free.

Let Y be a W-model. The W-equivariant rational map $G/T \dashrightarrow Y$ induced by the projection $G \to P\backslash G = \mathcal{Q}$ is a rational quotient map for the left P-action on G/T; cf. [5, p. 458]. Since the P-action is generically free, this map is a P-torsor over the generic point of Y; see [3, Theorem 4.7]. By the functoriality of the twisting operation, after twisting by a W-torsor ζ, we obtain a rational map ${}^{\zeta}(G/T) \dashrightarrow {}^{\zeta}Y$, which is a P-torsor over the generic point of ${}^{\zeta}Y$. This torsor has a rational section, since P is special; see Lemma 3. In particular, ${}^{\zeta}(G/T)$ is k-birationally isomorphic to $P \times {}^{\zeta}Y$. Since P is k-rational (once again, by Lemma 3), ${}^{\zeta}(G/T)$ is rational over ${}^{\zeta}Y$. Since ${}^{\zeta}Y$ is rational over k, we conclude that so is ${}^{\zeta}(G/T)$, as desired. □

It remains to show that the hypothesis of Proposition 5 holds. As before, we may replace the field K with k. The following lemma completes the proof of Theorem 2.

Lemma 6. *Let Y be a W-model. The twisted variety ${}^{\zeta}Y$ is rational over k, for every W-torsor ζ over k.*

Proof. We begin by constructing an explicit W-model. The affine open subset $\mathcal{Q}^{\mathrm{aff}} = \{x_1x_7 + x_2x_6 + x_3x_5 + 1 = 0\} \subset \mathbb{A}^6$ (where $X_4 \neq 0$) is $N(T)$-invariant. Here the affine coordinates on $\mathbb{A}^6$ are $x_i := X_i/X_4$, for $i \neq 4$. The field of rational functions invariant for the T-action on $\mathcal{Q}^{\mathrm{aff}}$ is $k(y_1, y_2, y_3, z_1, z_2)$, where the variables

$$y_1 = x_1x_7, \quad y_2 = x_2x_6, \quad y_3 = x_3x_5, \quad z_1 = x_1x_5x_6, \text{ and } z_2 = x_2x_3x_7$$

are subject to the relations $y_1 + y_2 + y_3 + 1 = 0$ and $y_1y_2y_3 = z_1z_2$. Thus we may choose as a W-model the affine subvariety Λ_1 of $\mathbb{A}^5$ given by these two equations, where $W = \mathrm{S}_2 \times \mathrm{S}_3$ acts on the coordinates as follows: S_2 permutes z_1, z_2, and S_3 permutes y_1, y_2, y_3. (Recall the W-action defined in Section III, and note that the field $k(\mathcal{Q})$ is recovered by adjoining the classes of variables x_1 and x_2.) We claim that Λ_1 is W-equivariantly birationally isomorphic to

$$\Lambda_2 = \{(Y_1 : Y_2 : Y_3 : Z_0 : Z_1 : Z_2) \,:\, Y_1 + Y_2 + Y_3 + Z_0 = 0 \text{ and } Y_1Y_2Y_3 = Z_1Z_2Z_0\} \subset \mathbb{P}^5,$$
$$\Lambda_3 = \{(Y_1 : Y_2 : Y_3 : Z_1 : Z_2) \,:\, Y_1Y_2Y_3 + (Y_1 + Y_2 + Y_3)Z_1Z_2 = 0\} \subset \mathbb{P}^4, \text{ and}$$
$$\Lambda_4 = \{(Y_1 : Y_2 : Y_3 : Z_1 : Z_2) \,:\, Z_1Z_2 + Y_2Y_3 + Y_1Y_3 + Y_1Y_2 = 0\} \subset \mathbb{P}^4\,,$$

where W acts on the projective coordinates $Y_1, Y_2, Y_3, Z_1, Z_2, Z_0$ as follows: S_2 permutes Z_1, Z_2, S_3 permutes Y_1, Y_2, Y_3, and every element of W fixes Z_0. Note that $\Lambda_2 \subset \mathbb{P}^5$ is the projective closure of $\Lambda_1 \subset \mathbb{A}^5$; hence, using $\simeq$ to denote W-equivariant birational equivalence, we have $\Lambda_1 \simeq \Lambda_2$. The isomorphism $\Lambda_2 \simeq \Lambda_3$ is obtained by eliminating Z_0 from the system of equations defining Λ_2. Finally, the isomorphism $\Lambda_3 \simeq \Lambda_4$ comes from the Cremona transformation $\mathbb{P}^4 \dashrightarrow \mathbb{P}^4$ given by $Y_i \to 1/Y_i$ and $Z_j \to 1/Z_j$ for $i = 1, 2, 3$ and $j = 1, 2$.

Let ζ be a W-torsor over k. It remains to be shown that ${}^{\zeta}\Lambda_4$ is k-rational. Since Λ_4 is a W-equivariant quadric hypersurface in $\mathbb{P}^4$, and the W-action on $\mathbb{P}^4$ is induced by a linear representation $W \to \mathrm{GL}_5$, Hilbert's Theorem 90 tells us that ${}^{\zeta}\mathbb{P}^4$ is k-isomorphic to $\mathbb{P}^4$, and ${}^{\zeta}\Lambda_4$ is isomorphic to a quadric hypersurface in $\mathbb{P}^4$ defined over k; see [7, Lemma 10.1]. It is easily checked that Λ_4 is smooth over k, and therefore so is ${}^{\zeta}\Lambda_4$. The zero-cycle of degree 3 given by $(1:0:0:0:0) + (0:1:0:0:0) + (0:0:1:0:0)$ in Λ_4 is W-invariant, so it defines a zero-cycle of degree 3 in ${}^{\zeta}\Lambda_4$. By Springer's theorem, the smooth quadric ${}^{\zeta}\Lambda_4$ has a k-rational point, hence is k-rational. □

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