# ON THE RATIONALITY PROBLEM FOR FORMS OF MODULI SPACES OF STABLE MARKED CURVES OF POSITIVE GENUS 

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#### Abstract

Let $M_{g, n}$ (respectively, $\bar{M}_{g, n}$ ) be the moduli space of smooth (respectively stable) curves of genus $g$ with $n$ marked points. Over the field of complex numbers, it is a classical problem in algebraic geometry to determine whether or not $M_{g, n}$ (or equivalently, $\bar{M}_{g, n}$ ) is a rational variety. Theorems of J. Harris, D. Mumford, D. Eisenbud and G. Farkas assert that $M_{g, n}$ is not even unirational for any $n \geqslant 0$ if $g \geqslant 22$. Moreover, P. Belorousski and A. Logan showed that $M_{g, n}$ is unirational for only finitely many pairs $(g, n)$ with $g \geqslant 1$. Finding the precise range of pairs $(g, n)$, where $M_{g, n}$ is rational, stably rational or unirational, is a problem of ongoing interest.

In this paper we address the rationality problem for twisted forms of $\bar{M}_{g, n}$ defined over an arbitrary field $F$ of characteristic $\neq 2$. We show that all $F$-forms of $\bar{M}_{g, n}$ are stably rational for $g=1$ and $3 \leqslant n \leqslant 4, g=2$ and $2 \leqslant n \leqslant 3, g=3$ and $1 \leqslant n \leqslant 14$, $g=4$ and $1 \leqslant n \leqslant 9, g=5$ and $1 \leqslant n \leqslant 12$.


## 1. Introduction

Let $M_{g, n}$ (respectively $\bar{M}_{g, n}$ ) be the moduli space of smooth (respectively stable) curves of genus $g$ with $n$ marked points. Recall that these moduli spaces are defined over the prime field $\left(\mathbb{Q}\right.$ in characteristic zero and $\mathbb{F}_{p}$ in characteristic $\left.p\right)$. The purpose of this paper is to address the rationality problem for twisted forms of $\bar{M}_{g, n}$. Recall that a form of a scheme $X$ defined over a field $F$ is another scheme $Y$, also defined over $F$, such that $X$ and $Y$ become isomorphic over the separable closure $F^{\text {sep }}$. We will use the terms "form", "twisted form" and " $F$-form" interchangeably. Forms of $\bar{M}_{g, n}$ are of interest because they shed light on the arithmetic geometry of $\bar{M}_{g, n}$, and because they are coarse moduli spaces for natural moduli problems in their own right; see [FR17, Remark 2.4].

This paper is a sequel to [FR17, where two of us considered twisted forms of $\bar{M}_{0, n}$. The main results of [FR17] can be summarized as follows.
Theorem 1.1. Let $F$ be a field of characteristic $\neq 2$ and $n \geqslant 5$ be an integer. Then
(a) all $F$-forms of $\bar{M}_{0, n}$ are unirational.
(b) If $n$ is odd, all $F$-forms of $\bar{M}_{0, n}$ are rational.
(c) If $n$ is even, then there exist fields $E / F$ and $E$-forms of $\bar{M}_{0, n}$ that are not stably rational (or even retract rational) over $E$.

[^0]In the present paper we will study the rationality problem for forms of $\bar{M}_{g, n}$ in the case, where $g \geqslant 1$. Here the rationality problem for the usual (split) moduli space $\bar{M}_{g, n}$ (or equivalently, for $M_{g, n}$ ) over the field of complex numbers is already highly non-trivial. Theorems of J. Harris, D. Mumford, D. Eisenbud [HM82, EH87] and G. Farkas [Fa11] assert that if $g \geqslant 22$, then $M_{g, 0}$ is not unirational (and hence, neither is $M_{g, n}$ for any $n \geqslant 0$ ). Moreover, work of P. Belorousski [Bel98] (for $g=1$ ) and A. Logan [Lo03] (for $g \geqslant 2$ ) tells us that $M_{g, n}$ is unirational for only finitely many pairs $(g, n)$ with $g \geqslant 1$. Finding the precise range of pairs $(g, n)$, where $M_{g, n}$ is rational, stably rational or unirational, is a problem of ongoing interest. In particular, over $\mathbb{C}, M_{g, n}$ is known to be rational for $1 \leqslant n \leqslant r_{g}$ and not unirational for $n \geqslant n_{g}$, where

| $g$ | 1 | 2 | 3 | 4 | 5 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $r_{g}$ | 10 | 12 | 14 | 15 | 12 |
| $n_{g}$ | 11 | - | - | 16 | 15 |

see Lo03 and CF07. Surprisingly, we have not been able to find specific values for $n_{2}$ and $n_{3}$ in the literature, even though Logan showed that they exist; see LLo33, Theorem 2.4]. The main result of the present paper is as follows.

Theorem 1.2. Let $F$ be a field of characteristic $\neq 2$. Then every $F$-form of $\bar{M}_{g, n}$ is stably rational over $F$ if

$$
\begin{aligned}
& g=1 \text { and } 3 \leqslant n \leqslant 4, \\
& g=2 \text { and } 2 \leqslant n \leqslant 3, \\
& g=3 \text { and } 1 \leqslant n \leqslant 14, \\
& g=4 \text { and } 1 \leqslant n \leqslant 9, \\
& g=5 \text { and } 1 \leqslant n \leqslant 12 .
\end{aligned}
$$

Several remarks are in order.
(1) Stable rationality of every form of $\bar{M}_{g, n}$ is a priori much stronger than stable rationality of $\bar{M}_{g, n}$ itself. For example, $\bar{M}_{1,1} \simeq \mathbb{P}^{1}$ is rational, but its forms are conic curves which are not unirational in general.
(2) Theorem 1.2 also holds for $(g, n)=(1,2)$ (respectively, $(2,1))$, provided $\operatorname{char}(F)=0$ (respectively, $\operatorname{char}(F) \neq 2,3$ ); see Remark [2.7.
(3) By [DR15, Theorem 6.1(b)], every $F$-form of $\bar{M}_{1, n}$ is unirational for $3 \leqslant n \leqslant 9$.
(4) The situation we encountered in Theorem 1.1(c), where some forms of $\bar{M}_{0, n}$ are stably rational and others are not, does not arise for any of the pairs $(g, n)$ covered by Theorem 1.2. We do not know if it arises for any pair $(g, n)$ with $g \geqslant 1$ and $2 g+n \geqslant 5$.

A proof of Theorem 1.2 is outlined in Section 3 and completed in Sections 4 and 5 . Our arguments rely on a theorem of B. Fantechi and A. Massarenti [FM14] describing the automorphism group of $\bar{M}_{g, n}$; see Section 2e,

## 2. Preliminaries

All algebraic groups in this paper will be assumed to be affine, and all algebraic varieties to be quasi-projective.

2a. Twisting. Let $G$ be an algebraic group defined over a field $F, X$ be an $F$-variety endowed with a (left) $G$-action, and $P \rightarrow \operatorname{Spec}(F)$ be a (right) $G$-torsor. The twisted variety ${ }^{P} X$ is defined as ${ }^{P} X:=(P \times X) / G$, where $G$ acts on $P \times X$ by $g:(p, x) \rightarrow$ $\left(p \cdot g^{-1}, g \cdot x\right)$. Here $P \times X$ is, in fact, a $G$-torsor over $(P \times X) / G$; in particular, $P \times X \rightarrow$ $(P \times X) / G$ is a geometric quotient. A $G$-equivariant morphism of $F$-varieties $f: X \rightarrow Y$ gives rise to a $G$-equivariant morphism id $\times f: P \times X \rightarrow P \times Y$ which descends to an $F$-morphism ${ }^{P} f:{ }^{P} X \rightarrow{ }^{P} Y$. Similarly a $G$-equivariant rational map $f: X \rightarrow Y$ of $F$ varieties induces a rational map ${ }^{P} f:{ }^{P} X \rightarrow{ }^{P} Y$. Some basic properties of the twisting operation are summarized in Lemma 2.1 below; see also [Flo08, Section 2] or [DR15, Section 3].

Lemma 2.1. Let $G$ be an algebraic group defined over a field $F, f: X \rightarrow Y$ and $f^{\prime}: X^{\prime} \rightarrow$ $Y$ be $G$-equivariant morphisms of $F$-varieties, and $P \rightarrow \operatorname{Spec}(F)$ be a $G$-torsor.
(a) If $f$ is an open (respectively, closed) immersion, then so is ${ }^{P} f$.
(b) If $f$ is a dominant morphism (respectively, an isomorphism or a birational isomorphism), then so is ${ }^{P} f$.
(c) If $f$ is a vector bundle of rank $r$, then so is ${ }^{P} f$. In particular, ${ }^{P} X$ is rational over ${ }^{P} Y$.
(d) ${ }^{P}\left(X \times{ }_{Y} X^{\prime}\right)$ is isomorphic to ${ }^{P} X \times{ }_{P_{Y}}{ }^{P}\left(X^{\prime}\right)$ over ${ }^{P} Y$.
(e) Moreover, if $f$ and $f^{\prime}$ are vector bundles, then ${ }^{P}\left(X \times{ }_{Y} X^{\prime}\right)$ and ${ }^{P} X \times{ }_{P}{ }^{P}{ }^{P}\left(X^{\prime}\right)$ are isomorphic as vector bundles over ${ }^{P} Y$.
(f) If $f$ is a vector bundle of rank $r$, then the twisted Grassmannian bundle ${ }^{P} \operatorname{Gr}(m, X) \simeq$ $\operatorname{Gr}\left(m,{ }^{P} X\right)$ is rational over ${ }^{P} Y$ for any $1 \leqslant m \leqslant r-1$. In particular, ${ }^{P} \mathbb{P}(X) \simeq \mathbb{P}\left({ }^{P} X\right)$ is rational over ${ }^{P} Y$.

Here when we say that $f$ is a vector bundle, we are assuming that $G$ acts on $X$ by vector bundle automorphisms (and similarly for $f^{\prime}$ ). That is, for any $g \in G$ and $y \in Y, g$ restricts to a linear map between the fibers $f^{-1}(y)$ and $f^{-1}(g(y))$.

Proof. For a proof of (a) and (b), see [DR15, Corollary 3.4].
(c) The first assertion is a consequence of Hilbert's Theorem 90. The second assertion follows from the first, since the vector bundle ${ }^{P} f:{ }^{P} X \rightarrow{ }^{P} Y$ becomes trivial after passing to some dense Zariski open subset of ${ }^{P} Y$.
(d) The morphism $\phi: P \times\left(X \times_{Y} X^{\prime}\right) \rightarrow(P \times X) \times_{Y}\left(P \times X^{\prime}\right)$ over $Y$ given by $\left(p, x, x^{\prime}\right) \mapsto\left((p, x),\left(p, x^{\prime}\right)\right)$ descends to a morphism $\bar{\phi}:{ }^{P}\left(X \times_{Y} X^{\prime}\right) \rightarrow{ }^{P} X \times{ }_{P}{ }^{P}\left(X^{\prime}\right)$ over ${ }^{P} Y$. Here the unmarked direct products are assumed to be over $\operatorname{Spec}(F)$. To show that $\bar{\phi}$ is an isomorphism, we may pass to a splitting field $F^{\prime} / F$ for $P$. Over $F^{\prime}$, the $G$-torsor $P \rightarrow \operatorname{Spec}(F)$ becomes split, i.e., $P_{F^{\prime}} \simeq G_{F^{\prime}}$. Thus over $F^{\prime}$, the morphism $P \times X \simeq G \times X \rightarrow X$ given by $(g, x) \mapsto g \cdot x$ is a $G$-torsor. This yields a natural isomorphism between ${ }^{P} X$ and $X$. Identifying ${ }^{P}\left(X^{\prime}\right)$ with $X^{\prime},{ }^{P} Y$ with $Y$, and ${ }^{P}\left(X \times_{Y} X^{\prime}\right)$ with $X \times_{Y} X^{\prime}$ in a similar manner, we see that over $F^{\prime}, \bar{\phi}$ becomes the identity map $X \times_{Y} X^{\prime} \rightarrow X \times_{Y} X^{\prime}$. Hence, $\bar{\phi}$ is an isomorphism over $F$.
(e) To check that $\bar{\phi}$ is an isomorphism of vector bundles over ${ }^{P} Y$, we may, once again, pass to a splitting field $F^{\prime} / F$ for $P$. In the proof of part (d), we identified ${ }^{P}\left(X^{\prime}\right)$ with
$X^{\prime}$ and ${ }^{P}\left(X \times_{Y} X^{\prime}\right)$ with $X \times_{Y} X^{\prime}$ after passing to $F^{\prime}$. Now we observe that these identifications are, in fact, isomorphisms of vector bundles over ${ }^{P} Y$ (which we identified with $Y$ ). Modulo these identifications, $\bar{\phi}$ is the identity map $X \times_{Y} X^{\prime} \rightarrow X \times_{Y} X^{\prime}$, and part (e) follows.
(f) The second assertion follows from the first by setting $m=1$.

Rationality of $\operatorname{Gr}\left(m,{ }^{P} X\right)$ over ${ }^{P} Y$ follows from the fact that any vector bundle, and in particular the vector bundle ${ }^{P} f:{ }^{P} X \rightarrow{ }^{P} Y$, is locally trivial in the Zariski topology.

To show that ${ }^{P} \operatorname{Gr}(m, X)$ is isomorphic to $\operatorname{Gr}\left(m,{ }^{P} X\right)$ over ${ }^{P} Y$, recall that $\operatorname{Gr}(m, X)$ is the quotient of the dense open subset $\left(X^{m}\right)_{0}$ of the $n$-fold fibered product $X^{m}=$ $X \times_{Y} \cdots \times_{Y} X$ consisting of linearly independent $m$-tuples by the group $\mathrm{GL}_{m}$ (over $Y$ ). To construct ${ }^{P} \operatorname{Gr}(m, X)$, we proceed as follows. First take the quotient of the product $P \times\left(X^{m}\right)_{0}$ by the action of $\mathrm{GL}_{m}$. This action is trivial on the first factor, so we obtain $P \times \operatorname{Gr}(m, X)$. Now take the quotient of $P \times \operatorname{Gr}(m, X)$ by $G$ to arrive at ${ }^{P} \operatorname{Gr}(m, X)$.

To construct $\operatorname{Gr}\left(m,{ }^{P} X\right)$, we also start with $P \times\left(X^{m}\right)_{0}$ and take the quotients by the same groups, but in reverse order. First we take the quotient of $P \times\left(X^{m}\right)_{0}$ by $G$ to obtain ${ }^{P}\left(\left(X^{m}\right)_{0}\right) \simeq\left(\left({ }^{P} X\right)^{m}\right)_{0}$ (see parts (d) and (e)); the quotient of $\left(\left(^{P} X\right)^{m}\right)_{0}$ by $\mathrm{GL}_{m}$ is $\operatorname{Gr}\left(m,{ }^{P} X\right)$. Since the actions of $\mathrm{GL}_{m}$ and $G$ on $P \times_{F}\left(X^{m}\right)_{0}$ commute, we conclude that ${ }^{P} \operatorname{Gr}(m, X)$ and $\operatorname{Gr}\left(m,{ }^{P} X\right)$ are isomorphic over ${ }^{P} Y$.

The $F$-forms of a variety $X$ are in a natural bijective correspondence with $H^{1}(F, \operatorname{Aut}(X))$. Here $\operatorname{Aut}(X)$ is a functor which associates to a scheme $S / F$ the abstract group Aut $\left(X_{S}\right)$. In general this functor is not representable by an algebraic group defined over $F$. If it is, one usually says that $\operatorname{Aut}(X)$ is an algebraic group. In this case the bijective correspondence between $H^{1}(F, \operatorname{Aut}(X)$ ) (which may be viewed as a set of $\operatorname{Aut}(X)$-torsors $P \rightarrow \operatorname{Spec}(F)$ ) and the set of $F$-forms of $X$ (up to $F$-isomorphism) can be described explicitly as follows. An $\operatorname{Aut}(X)$-torsor $P \rightarrow \operatorname{Spec}(F)$ corresponds to the twisted variety ${ }^{P} X$, and a twisted form $Y$ of $X$ corresponds to the isomorphism scheme $P=\operatorname{Isom}_{F}(X, Y)$, which is naturally an $\operatorname{Aut}(X)$-torsor over $\operatorname{Spec}(F)$; see [Se97, Section III.1.3], Sp98, Section 11.3].

2b. Étale algebras. An étale algebra $A / F$ is a commutative $F$-algebra of the form $F_{1} \times \cdots \times F_{r}$, where each $F_{i}$ is a finite separable field extension of $F$. $n$-dimensional étale algebras over $F$ are $F$-forms of the split étale algebra $A=F \times \cdots \times F$ ( $n$ times). The automorphism group of this split algebra is the symmetric group $\mathrm{S}_{n}$, permuting the $n$ factors of $F$. Thus $n$-dimensional étale algebras over $F$ are in a natural bijective correspondence with the Galois cohomology set $H^{1}\left(F, \mathrm{~S}_{n}\right)$; see, e.g., Examples 2.1 and 3.2 in Ser03.

2c. Weil restriction. Let $A$ be an étale algebra over $F$ and $X \rightarrow \operatorname{Spec}(A)$ be a variety defined over $A$. The Weil restriction (or Weil transfer) of $X$ to $F$ is, by definition, an $F$-variety $R_{A / F}(X)$ satisfying

$$
\begin{equation*}
\operatorname{Mor}_{F}\left(Y, R_{A / F}(X)\right) \simeq \operatorname{Mor}_{A}\left(Y_{A}, X\right) \tag{2.2}
\end{equation*}
$$

where $Y_{A}:=Y \times_{\operatorname{Spec}(F)} \operatorname{Spec}(A), \operatorname{Mor}_{F}(Y, Z)$ denotes the set of F-morphisms $Y \rightarrow Z$, and $\simeq$ denotes an isomorphism of functors (in $Y$ ). For generalities on this notion we refer the reader to [BLR90, Section 7.6]. For a brief summary, see [Ka00, Section 2]. In particular,
it is shown in BLR90, Theorem 4] that if $X$ is quasi-projective over $A$, then $R_{A / F}(X)$ exists. Note that uniqueness of $R_{A / F}(X)$ follows from (2.2) by Yoneda's lemma.

The following properties of Weil restriction will be helpful in the sequel.
Lemma 2.3. Let $A / F$ be an étale algebra and $X$ be a (quasi-projective) variety defined over $A$.
(a) Let $V$ be a free $A$-module of finite rank, and $X=\mathbb{A}_{A}(V)$ be the associated affine space. Then $R_{A / F}(X)=\mathbb{A}_{F}(V)$, where we view $V$ as an $F$-vector space.
(b) If $X$ and $Y$ are birationally isomorphic over $A$, then $R_{A / F}(X)$ and $R_{A / F}(Y)$ are birationally isomorphic over $F$.
(c) If $X$ is a rational variety over $A$, then $R_{A / F}(X)$ is rational over $F$.

Proof. (a) follows directly from (2.2). For details, see [Ka00, Lemma 1.2].
(b) Since $X$ and $Y$ are birationally isomorphic, there exists a variety $U$ defined over $A$ and open immersions $i: U \hookrightarrow X$ and $j: U \hookrightarrow Y$. After replacing $U$ by an open subvariety, we may assume that $U$ is quasi-projective (we may even assume that $U$ is affine). Since Weil restriction commutes with open immersions, $i$ and $j$ induce open immersions of $R_{A / F}(U)$ into $R_{A / F}(X)$ and $R_{A / F}(Y)$, respectively, and part (b) follows.
(c) By our assumption, $X$ is birationally isomorphic to $Y=\mathbb{A}^{d}$ over $A$, where $d$ is the dimension of $X$. By part (b), $R_{A / F}(X)$ and $R_{A / F}(Y)$ are birationally isomorphic over $F$, and by part (a), $R_{A / F}(Y)$ is an affine space over $F$.

In the special case where $X$ is defined over $F$, the Weil transfer $R_{A / F}\left(X_{A}\right)$ can be explicitly described as follows. The symmetric group $\mathrm{S}_{n}$ acts on the $n$-fold direct product $X^{n}$ by permuting the factors. If $P \rightarrow \operatorname{Spec}(F)$ is a $\mathrm{S}_{n}$-torsor, and $A / F$ is the étale algebra of degree $n$ representing the class of $P$ in $H^{1}\left(F, \mathrm{~S}_{n}\right)$, then $R_{A / F}\left(X_{A}\right)={ }^{P}\left(X^{n}\right)$; see, e.g., [DR15, Proposition 3.2].

2d. Automorphism of marked curves. We shall need the following well-known result in the sequel; see, e.g., in [Ha77, Corollary IV.4.7] for $g=1$ and [Ha77, Exercise V.1.11] for $g \geqslant 2$.

Proposition 2.4. Suppose $2 g+n \geqslant 5$. Then $\operatorname{Aut}\left(C, p_{1}, \ldots, p_{n}\right)=\{1\}$ for a general point ( $C, p_{1}, \ldots, p_{n}$ ) of $\bar{M}_{g, n}$ (or equivalently, of $M_{g, n}$ ).

Note that the inequality $2 g+n \geqslant 5$ is satisfied for every pair of integers $(g, n)$ appearing in Theorem 1.2 .

2e. Automorphisms and forms of $\bar{M}_{g, n}$. The following theorem is the starting point of our investigation.

Theorem 2.5. (A. Massarenti Mas13], B. Fantechi and A. Massarenti [FM14]) Let F be a field of characteristic $\neq 2$. If $g, n \geqslant 1,(g, n) \neq(2,1)$ and $2 g+n \geqslant 5$, then the natural embedding $\mathrm{S}_{n} \rightarrow \operatorname{Aut}_{F}\left(\bar{M}_{g, n}\right)$ is an isomorphism.

Using the bijective correspondence between $F$-forms of $X$ and $\operatorname{Aut}(X)$-torsors $P \rightarrow$ $\operatorname{Spec}(F)$ described at the end of Section 2a, we obtain the following.

Corollary 2.6. For $F, g, n$ as in Theorem 2.5, every $F$-form of $\bar{M}_{g, n}$ is $F$-isomorphic to ${ }^{P} \bar{M}_{g, n}$ for some $\mathrm{S}_{n}$-torsor $P \rightarrow \operatorname{Spec}(F)$.
Remark 2.7. Theorem 1.2 also holds in the following cases.
(a) $g=2$ and $n=1$, and $\operatorname{char}(F)=0$,
(b) $g=1$ and $n=2$ and $\operatorname{char}(F) \neq 2$ or 3 .

In case (a), $\bar{M}_{2,1}$ has no non-trivial automorphisms by [FM14, Theorem 1] and hence, no non-split forms. On the other hand, the split form of $\bar{M}_{2,1}$ is known to be rational; see CF07.

In case (b), the automorphism group $G$ of $\bar{M}_{1,2}$ is non-trivial; however, it is special; see [FM14, Proposition 2.4]. In other words, every $G$-torsor over a field is split. As a consequence, $\bar{M}_{1,2}$ has no non-split forms (see [DR15, Remark 6.4]) and the split form of $\bar{M}_{1,2}$ is rational (see [CF07]).
Remark 2.8. We do not know if $\bar{M}_{g, n}$ can be replaced by $M_{g, n}$ in the statement of Theorem 2.5. If so, then $\bar{M}_{g, n}$ can also be replaced by $M_{g, n}$ in the statements of Theorems 1.2. The proof remains unchanged.

## 3. Proof of Theorem 1.2: the overall strategy

Let $\left(C, p_{1}, \ldots, p_{n}\right)$ be a point of $M_{g, n}$.
Case I. We define a vector space $V$ of dimension $d$ as follows.

- If $g=3,4$ or 5 , then $V=H^{0}\left(C, \omega_{C}\right)^{*}$, where $\omega_{C}$ is the canonical line bundle. Here $d=g$.
- If $g=1$ and $n=3$ or 4 , then $V=H^{0}\left(C, \mathcal{O}_{C}\left(p_{1}+\ldots+p_{n}\right)\right)^{*}$. Here $d=n$.
- If $g=2$ and $n=2$, then $V=H^{0}\left(C, \omega_{C}\left(p_{1}+p_{2}\right)\right)^{*}$. Here $d=3$.

Case II.

- If $g=2$ and $n=3$, then in place of $V$ we define two vector spaces, $V_{1}=H^{0}\left(C, \omega_{C}\right)^{*}$ and $V_{2}=H^{0}\left(C, \mathcal{O}_{C}\left(p_{1}+p_{2}+p_{3}\right)\right)^{*}$. Here $\operatorname{dim}\left(V_{1}\right)=\operatorname{dim}\left(V_{2}\right)=2$.
Remark 3.1. In Case I, for $\left(C, p_{1}, \ldots, p_{n}\right)$ in a suitably defined open subset $\left(M_{g, n}\right)_{0}$ of $M_{g, n}, V$ is the fiber of a vector bundle $E \rightarrow\left(M_{g, n}\right)_{0}$ obtained via push-forward from a vector bundle over the universal curve. In Case II the same is true of both $V_{1}$ and $V_{2}$. After replacing $\left(M_{g, n}\right)_{0}$ by a dense open subset, we may assume without loss of generality that (i) $\left(M_{g, n}\right)_{0}$ is $S_{n}$-invariant and (ii) $\operatorname{Aut}\left(C, p_{1}, \ldots, p_{n}\right)=1$ for any $\left(C, p_{1}, \ldots, p_{n}\right) \in\left(M_{g, n}\right)_{0}$; see Proposition 2.4.

In Case I, set

$$
X=\left\{\left(C, p_{1}, \ldots, p_{n}, B\right) \in\left(M_{g, n}\right)_{0} \mid B \text { is a basis of } V, \text { up to proportionality }\right\}
$$

Here two bases $B=\left(v_{1}, \ldots, v_{d}\right)$ and $B^{\prime}=\left(v_{1}^{\prime}, \ldots, v_{d}^{\prime}\right)$ are called proportional if there exists a $0 \neq c \in k$ such that $v_{i}^{\prime}=c v_{i}$ for every $i=1, \ldots, d$.

In Case II, the vector space $V$ in the definition of $X$ should be replaced by a pair of 2-dimensional vector spaces $V=\left(V_{1}, V_{2}\right)$ and the basis $B$ by a pair $B=\left(B_{1}, B_{2}\right)$, where
$B_{1}$ is a basis of $V_{1}$ and $B_{2}$ is a basis of $V_{2}$. We identify two such pairs, $B=\left(B_{1}, B_{2}\right)$ and $B^{\prime}=\left(B_{1}^{\prime}, B_{2}^{\prime}\right)$, if $B_{1}$ is proportional to $B_{1}^{\prime}$ and $B_{2}$ is proportional to $B_{2}^{\prime}$.

In Case I, choosing a basis in $V$ gives rise to the map $f_{B}: C \rightarrow \mathbb{P}^{d-1}$. Two bases, $B$ and $B^{\prime}$, are proportional if and only if $f_{B}=f_{B^{\prime}}$. In Case II, we obtain two maps, $f_{B_{1}}, f_{B_{2}}: C \rightarrow \mathbb{P}^{1}$, which can be combined into a single morphism $f_{B}=f_{B_{1}} \times f_{B_{2}}: C \rightarrow$ $\mathbb{P}^{1} \times \mathbb{P}^{1}$. Once again, we identify $B=\left(B_{1}, B_{2}\right)$ and $B^{\prime}=\left(B_{1}^{\prime}, B_{2}^{\prime}\right)$ if and only if $f_{B}=f_{B^{\prime}}$.

In each case we will consider a diagram of the form


Here $\alpha$ is the natural projection $\left(C, p_{1}, \ldots, p_{n}, B\right) \rightarrow\left(C, p_{1}, \ldots, p_{n}\right)$ which "forgets" the basis $B$. The second projection $\beta$ "forgets" the curve $C$, and maps $\left(C, p_{1}, \ldots, p_{n}, B\right)$ to a suitable configuration space $Y$ for the remaining data. Specifically, we define $Y$ and $\beta$ as follows.

## Case I.

- If $g=3,4$ or 5 , then $f_{B}: C \rightarrow \mathbb{P}^{g-1}$ is the canonical embedding. We define $Y=\left(\mathbb{P}^{g-1}\right)^{n}$, and $\beta\left(C, p_{1}, \ldots, p_{n}, B\right)=\left(f_{B}\left(p_{1}\right), \ldots, f_{B}\left(p_{n}\right)\right) \in Y$.
- Let $g=1$ and $n=3$ or 4 . The constant function $1 \in H^{0}\left(C, \mathcal{O}_{C}\left(p_{1}+\ldots+p_{n}\right)\right)$ cuts out a hyperplane $L \subset \mathbb{P}^{n-1}$ passing through $f_{B}\left(p_{1}\right), \ldots, f_{B}\left(p_{n}\right)$. We set $Y \subset\left(\left(\mathbb{P}^{n-1}\right)^{*}\right)^{n}$ to be the locally closed subvariety consisting of $n$-tuples $p_{1}, \ldots, p_{n}$ such that $p_{1}, \ldots, p_{n}$ are linearly dependent (i.e., lie in a hyperplane) in $\mathbb{P}^{n-1}$ but any $n-1$ of them are linearly independent and define $\beta\left(C, p_{1}, \ldots, p_{n}\right)=$ $\left(f_{B}\left(p_{1}\right), \ldots, f_{B}\left(p_{n}\right)\right)$.
- If $g=2$ and $n=2$, then for $p_{1}, p_{2}$ in general position on $C$, the image of $f_{B}$ in $\mathbb{P}^{2}$ is a quartic curve $C^{\prime}$ with a node at $p=f_{B}\left(p_{1}\right)=f_{B}\left(p_{2}\right)$, and $f_{B}: C \rightarrow C^{\prime}$ is the normalization map; see [H11, Example 5.15]. Moreover, $C^{\prime}$ has two tangent lines at $p, L_{1}$ and $L_{2}$, which correspond to $p_{1}$ and $p_{2}$ under $f_{B}$. We thus define $Y$ as the open subvariety of $\left(\mathbb{P}^{2}\right)^{*} \times\left(\mathbb{P}^{2}\right)^{*}$ parametrizing pairs of distinct lines and $\beta\left(C, p_{1}, p_{2}, B\right)=\left(L_{1}, L_{2}\right)$.


## Case II.

- Here $g=2$ and $n=3$, and the maps $f_{B_{1}}$ and $f_{B_{2}}: C \rightarrow \mathbb{P}^{1}$ are of degree 2 and 3, respectively; see [H11, Examples 5.11 and 5.13]. For $p_{1}, p_{2}, p_{3}$ in general position, $C^{\prime}:=f_{B}(C)$ is a curve of bidegree (3,2) in $\mathbb{P}^{1} \times \mathbb{P}^{1}, f_{B}=f_{B_{1}} \times f_{B_{2}}$ is an isomorphism between $C$ and $C^{\prime}$, and $f_{B_{2}}\left(p_{1}\right)=f_{B_{2}}\left(p_{2}\right)=f_{B_{2}}\left(p_{3}\right)$ in $\mathbb{P}^{1}$. We set $Y$ to be the open subvariety of $\left(\mathbb{P}^{1}\right)^{3} \times \mathbb{P}^{1}$ consisting of elements of the form $\left(\left(a_{1}, a_{2}, a_{3}\right), b\right)$, where $a_{1}, a_{2}, a_{3} \in \mathbb{P}^{1}$ are distinct and $\beta\left(C, p_{1}, p_{2}, p_{3}, B\right)=$ $\left(\left(f_{B_{1}}\left(p_{1}\right), f_{B_{1}}\left(p_{2}\right), f_{B_{1}}\left(p_{3}\right)\right), f_{B_{2}}\left(p_{1}\right)\right)$.
By Corollary 2.6 it suffices to show that the twisted variety ${ }^{P} M_{g, n}$ is stably rational over $F$ for every $\mathrm{S}_{n}$-torsor $P \rightarrow \operatorname{Spec}(F)$. Here $1 \leqslant g \leqslant 5$, and $(g, n)$ is one of the pairs
appearing in Theorem 1.2. Twisting the diagram (3.2) by $P$ and applying Lemma [2.1, we obtain the following diagram of twisted varieties.


In order to complete the proof of Theorem [1.2, we need to establish the following facts for each pair $(g, n)$ in Theorem 1.2.

Lemma 3.4. The rational map $\beta: X \rightarrow Y$ is dominant.
Lemma 3.5. (a) ${ }^{P} Y$ is rational over $F$,
(b) ${ }^{P} X$ is rational over ${ }^{P} M_{g, n}$,
(c) ${ }^{P} X$ is rational over ${ }^{P} Y$.

## 4. Proof of Lemma 3.4

For $g=3,4,5$ we need to show that there is a canonical curve passing through $n$ points $r_{1}, \ldots, r_{n}$ in general position in $\mathbb{P}^{g-1}$.
$g=3$. Canonical curves of genus 3 are precisely the smooth quartic curves in $\mathbb{P}^{2}$; see Ha77, Example IV.5.2.1]. Since $\operatorname{dim} H^{0}\left(\mathbb{P}^{2}, \mathcal{O}(4)\right)=15$, there is a smooth quartic curve passing through $n$ points in $\mathbb{P}^{2}$ in general position for any $n \leqslant 14$.
$g=4$. We will use the fact that a complete intersection of a smooth quadric surface $Q$ and a smooth cubic surface $S$ in $\mathbb{P}^{3}$ is a canonical curve of genus 4; see Ha77, Example IV.5.2.2]. The dimensions of $H^{0}\left(\mathbb{P}^{3}, \mathcal{O}(2)\right)$ and $H^{0}\left(\mathbb{P}^{3}, \mathcal{O}(3)\right)$ are 10 and 20 , respectively. Hence, as long as $n \leqslant 9$ and $r_{1}, \ldots, r_{n}$ are in general position in $\mathbb{P}^{3}$, there exist a smooth quadric $Q \subset \mathbb{P}^{3}$ and smooth cubic $S \subset \mathbb{P}^{3}$ such that $Q$ and $S$ pass through $r_{1}, \ldots, r_{n}$ and intersect transversely. The intersection $Q \cap S$ is then a canonical curve of genus 4 passing through $r_{1}, \ldots, r_{n}$.
$g=5$. Let $n \leqslant 12$. Since $\operatorname{dim} H^{0}\left(\mathbb{P}^{4}, \mathcal{O}(2)\right)=15$, for $n$ points $r_{1}, \ldots, r_{n}$ in general position in $\mathbb{P}^{4}$, there exist three linearly independent quadrics, $Q_{1}, Q_{2}$ and $Q_{3}$ such that
(i) $Q_{1}, Q_{2}$ and $Q_{3}$ pass through $r_{1}, \ldots, r_{n}$,
(ii) $Q_{1}, Q_{2}$ and $Q_{3}$ intersect transversely, and
(iii) $C=Q_{1} \cap Q_{2} \cap Q_{3}$ is a smooth curve.

By [Ha77, Example IV.5.5.3], $C$ is a canonical curve of genus 5. By (i), $C$ passes through $r_{1}, \ldots, r_{n}$, as desired.
$g=2$ and $n=2$. The projection $\beta$ is equivariant with respect to the natural $\mathrm{GL}_{3^{-}}$ action on $X$ and $Y$. Here $\mathrm{GL}_{3}$ acts on $\left(C, p_{1}, \ldots, p_{n}, B\right) \in X$ by linear changes of the basis $B$, leaving $C$ and $p_{1}, \ldots, p_{n}$ invariant, and on $Y$ via its natural action on $\mathbb{P}^{2}$. Since the $\mathrm{GL}_{3}$-action on $Y$ is transitive, $\beta$ is dominant.
$(g, n)=(2,3),(1,3)$ or $(1,4)$. Let $G=\mathrm{GL}_{2} \times \mathrm{GL}_{2}, \mathrm{GL}_{3}$ or $\mathrm{GL}_{4}$, respectively. In each case, $\beta$ is $G$-equivariant and $G$ acts transitively on $Y$, so the same argument as in the previous case shows that $\beta$ is dominant. This completes the proof of Lemma 3.4.

## 5. Proof of Lemma 3.5

(a) Let $A / F$ be the étale algebra associated to the $\mathrm{S}_{n}$-torsor $P \rightarrow \operatorname{Spec}(F)$. If $g=3,4$ or 5 , then ${ }^{P} Y \simeq R_{A / F}\left(\mathbb{P}^{g-1}\right)$. If $g=2$ and $n=2$, then ${ }^{P} Y \simeq R_{A / F}\left(\left(\mathbb{P}^{2}\right)^{*}\right)$. If $g=2$ and $n=3$, then ${ }^{P} Y \simeq R_{A / F}\left(\mathbb{P}^{1}\right) \times \mathbb{P}^{1}$. Here $\simeq$ stands for "birationally isomorphic over $F$ ". In each case ${ }^{P} Y$ is rational over $F$ by Lemma 2.3(c).

If $g=1$, let $\mathcal{H} \rightarrow\left(\mathbb{P}^{n-1}\right)^{*}$ be the tautological bundle whose fiber over the hyperplane $\{l=0\}$ consists of the points of the affine hyperplane cut out by $l$ in $\mathbb{A}^{n}$. Then $Y$ is $\mathrm{S}_{n^{-}}$ equivariantly birationally isomorphic to the $n$-fold fibered product $\mathbb{P}(\mathcal{H})^{n}$ over $\left(\mathbb{P}^{n-1}\right)^{*}$. Since $\mathrm{S}_{n}$ acts trivially on $\left(\mathbb{P}^{n-1}\right)^{*}$, and $\left(\mathbb{P}^{n-1}\right)^{*}$ is rational over $F$, it suffices to show that ${ }^{P}\left(\mathbb{P}(\mathcal{H})^{n}\right)$ is rational over $\left(\mathbb{P}^{n-1}\right)^{*}$. Choosing projective coordinates $a_{1}, \ldots, a_{n}$ in $\mathbb{P}^{n-1}$, we can identify the function field $K$ of $\left(\mathbb{P}^{n-1}\right)^{*}$ with $F\left(a_{i} / a_{j} \mid i, j=1, \ldots, n\right)$. Over $K, \mathcal{H}$ is isomorphic to the $(n-1)$-dimensional vector subspace on $K^{n}$ given by $a_{1} x_{1}+\cdots+a_{n} x_{n}=0$ and ${ }^{P}\left(\mathbb{P}(\mathcal{H})^{n}\right)$ is isomorphic to $R_{A_{K} / K}\left(\mathbb{P}(\mathcal{H})_{A_{K}}\right)$, where $A_{K}=A \otimes_{F} K$. By Lemma [2.3(c), $R_{A_{K} / K}\left(\mathbb{P}(\mathcal{H})_{A_{K}}\right)$ is rational over $K$. This shows that ${ }^{P}\left(\mathbb{P}(\mathcal{H})^{n}\right)$ is rational over $\left(\mathbb{P}^{n-1}\right)^{*}$, as desired.
(b) Case I. Let $E \rightarrow\left(M_{g, n}\right)_{0}$ be the vector bundle whose fiber over $\left(C, p_{1}, \ldots, p_{n}\right)$ is $V$, as in Remark 3.1. The space of bases in $V$ (up to equivalence) can be identified with a dense open subset of $\mathbb{P}\left(V^{d}\right)$. Hence, $X$ is $\mathrm{S}_{n}$-equivariantly birationally isomorphic to $\mathbb{P}\left(E^{d}\right)$ over $\left(M_{g, n}\right)_{0}$ and consequently ${ }^{P} X$ is birationally isomorphic to ${ }^{P} \mathbb{P}\left(E^{d}\right)$. On the other hand, by Lemma 2.1(f), ${ }^{P} \mathbb{P}\left(E^{d}\right)$ is rational over ${ }^{P}\left(M_{g, n}\right)_{0}$, and by Lemma 2.1(a), ${ }^{P}\left(M_{g, n}\right)_{0}$ is a dense open subset of ${ }^{P}\left(M_{g, n}\right)$. We conclude that ${ }^{P} X \simeq{ }^{P} \mathbb{P}\left(E^{d}\right)$ is rational over ${ }^{P} M_{g, n}$.

Case II. Now suppose $(g, n)=(2,3)$. Let $E_{1}$ and $E_{2}$ be the rank 2 vector bundles over $\left(M_{g, n}\right)_{0}$ whose fibers over $\left(C, p_{1}, \ldots, p_{n}\right)$ are $V_{1}$ and $V_{2}$, respectively, as in Remark 3.1, The space of bases in $V_{i}$ (up to equivalence) can be identified with a dense open subset of $\mathbb{P}\left(V_{i}^{2}\right)$. Hence, $X$ is $S_{3}$-equivariantly birationally isomorphic to a dense open subset of $\mathbb{P}\left(E_{1}^{2}\right) \times_{\left(M_{g, n}\right)_{0}} \mathbb{P}\left(E_{2}^{2}\right)$ over $\left(M_{g, n}\right)_{0}$. By Lemma 2.1(d), ${ }^{P} X$ is birationally isomorphic to ${ }^{P} \mathbb{P}\left(E_{1}^{2}\right) \times^{P}\left(M_{g, n}\right)_{0}{ }^{P} \mathbb{P}\left(E_{2}^{2}\right)$ over ${ }^{P}\left(M_{g, n}\right)_{0}$. By Lemma 2.1(f), each ${ }^{P} \mathbb{P}\left(E_{i}^{2}\right)$ is rational over ${ }^{P}\left(M_{g, n}\right)_{0}$ (or equivalently, over $\left.{ }^{P} M_{g, n}\right)$. Hence, so is ${ }^{P} X \simeq{ }^{P} \mathbb{P}\left(E_{1}^{2}\right) \times{ }_{P\left(M_{g, n}\right)_{0}}{ }^{P} \mathbb{P}\left(E_{2}^{2}\right)$.
(c) Recall that in Case I each $\left(C, p_{1}, \ldots, p_{n}, B\right) \in X$ gives rise to a map $f_{B}: C \rightarrow$ $\mathbb{P}^{d-1}$. Here $d=\operatorname{dim}(V)$. In Case II, $f_{B}$ maps $C$ to $\mathbb{P}^{1} \times \mathbb{P}^{1}$. Now observe that in both cases $f_{B}$ (and thus $B$, up to equivalence) is uniquely determined by the image of $\left(C, p_{1}, \ldots, p_{n}\right)$ under $f_{B}$. Indeed, in each case $f_{B}$ maps $C$ birationally onto its image. If $f_{B}$ and $f_{B^{\prime}}$ have the same image, then composing $f_{B^{\prime}}$ with $f_{B}^{-1}$, we obtain an automorphism of $\left(C, p_{1}, \ldots, p_{n}\right)$. On the other hand, for $\left(C, p_{1}, \ldots, p_{n}\right) \in\left(M_{g, n}\right)_{0}, \operatorname{Aut}\left(C, p_{1}, \ldots, p_{n}\right)=1$; see Remark 3.1.
$g=3$. Using the above observation we can $\mathrm{S}_{n}$-equivariantly identify $X$ with the space of tuples $\left(Q, r_{1}, \ldots, r_{n}\right)$, where $Q \subset \mathbb{P}^{2}$ is a smooth curve of degree 4 and $r_{1}, \ldots, r_{n}$ are $n$ distinct points on $Q$. Let $Y_{0} \subset Y=\left(\mathbb{P}^{2}\right)^{n}$ be the dense open $\mathrm{S}_{n}$-invariant subvariety consisting of $n$-tuples $\left(r_{1}, \ldots, r_{n}\right)$ such that (i) $\left(r_{1}, \ldots, r_{n}\right)$ impose $n$ independent conditions on quartic curves, (ii) there is a smooth quartic curve passing through $r_{1}, \ldots, r_{n}$. Let $W \rightarrow Y_{0} \subset\left(\mathbb{P}^{2}\right)^{n}$ be the vector bundle whose fiber over $\left(r_{1}, \ldots, r_{n}\right)$ is the space of quartic
polynomials in 3 variables vanishing at $r_{1}, \ldots, r_{n}$. Then over $Y_{0}, X$ is $S_{n}$-equivariantly birationally isomorphic to $\mathbb{P}(W)$. By Lemma 2.1, ${ }^{P} X \simeq{ }^{P} \mathbb{P}(W)$ is rational over ${ }^{P} Y$.
$g=4$. Recall that $C \subset \mathbb{P}^{3}$ is a canonically embedded curve of genus 4 if and only if $C$ is a complete intersection of an irreducible quadric surface $Q$ and an irreducible cubic surface $S$ in $\mathbb{P}^{3}$. Moreover, the quadric $Q$ is uniquely determined by $C$, and the cubic polynomial $s$ which cuts out $S$, is uniquely determined up to replacing $s$ by $s^{\prime}=c \cdot s+l q$, where $c \in F^{*}$ is a non-zero constant, $q$ is the quadratic form cutting out $Q$, and $l$ is a linear form. Conversely, any irreducible non-singular curve in $\mathbb{P}^{3}$, which is a complete intersection of an irreducible quadric and an irreducible cubic, is a canonically embedded curve of genus 4; see [Ha77, Example IV.5.2.2].

Let $Y_{0} \subset Y=\left(\mathbb{P}^{3}\right)^{n}$ be the open subset consisting of $n$-tuples of points imposing independent conditions of quadrics and cubics in $\mathbb{P}^{3}$. Let $W$ be the space of $(n+1)$-tuples $\left(q, r_{1}, \ldots, r_{n}\right)$, where $\left(r_{1}, \ldots, r_{n}\right) \in Y_{0}$ and $q \in H^{0}\left(\mathbb{P}^{3}, \mathcal{O}(2)\right)$ vanishes at $r_{1}, \ldots, r_{n}$. The natural projection $W \rightarrow Y_{0}$ given by $\left(q, r_{1}, \ldots, r_{n}\right) \mapsto\left(r_{1}, \ldots, r_{n}\right)$ is a vector bundle of rank $10-n$. The fiber of the projective bundle $\mathbb{P}(W)$ over $\left(r_{1}, \ldots, r_{n}\right) \in Y_{0}$ parametrizes quadric surfaces $Q \subset \mathbb{P}^{3}$ passing through $r_{1}, \ldots, r_{n}$. Now let $W^{\prime}$ be the vector bundle of rank $20-n$ over $\mathbb{P}(W)$, whose fiber over $\left(Q, r_{1}, \ldots, r_{n}\right)$ consists of cubic forms $s \in$ $H^{0}\left(\mathbb{P}^{3}, \mathcal{O}(3)\right)$ vanishing at $r_{1}, \ldots, r_{n}$. Let $W^{\prime \prime} \subset W$ be the subbundle, whose fiber over $\left(Q, r_{1}, \ldots, r_{n}\right)$ consists of cubic forms $l \cdot q$, where $q \in H^{0}\left(\mathbb{P}^{3}, \mathcal{O}(2)\right)$ cuts out $Q$ and $l$ ranges over $H^{0}\left(\mathbb{P}^{3}, \mathcal{O}(1)\right)$. Now set $\bar{W}=W^{\prime} / W^{\prime \prime}$. A general point $\left(S, Q, r_{1}, \ldots, r_{n}\right)$ of $\mathbb{P}(\bar{W})$ gives rise to a canonical curve $Q \cap S \subset \mathbb{P}^{3}$ of genus 4 passing through $r_{1}, \ldots, r_{n}$. Thus $X$ is $\mathrm{S}_{n}$-birationally isomorphic to $\mathbb{P}(\bar{W})$ over $Y_{0}$, and we obtain the following diagram of $\mathrm{S}_{n}$-equivariant maps


Twisting by the $\mathrm{S}_{n}$-torsor $P \rightarrow \operatorname{Spec}(F)$, we obtain a diagram


By Lemma 2.1, ${ }^{P} X$ is rational over ${ }^{P} \mathbb{P}(W)$ and ${ }^{P} \mathbb{P}(W)$ is rational over over ${ }^{P} Y$.
$g=5$. Recall that a general canonical curve $C^{\prime}=f_{B}(C)$ of genus 5 is a complete intersection of three quadric hypersurfaces $Q_{1}, Q_{2}$ and $Q_{3}$ in $\mathbb{P}^{4}$. Let $q_{i} \in H^{0}\left(\mathbb{P}^{4}, \mathcal{O}(2)\right)$ be a defining equation for $Q_{i}$. Then the span of $q_{1}, q_{2}$ and $q_{3}$ is uniquely determined by the
canonical curve $C^{\prime}$, because $H^{0}\left(\mathbb{P}^{4}, \mathcal{I}_{C^{\prime}}(2)\right)$ is 3-dimensional. Conversely, a 3-dimensional subspace of $H^{0}\left(\mathbb{P}^{4}, \mathcal{O}(2)\right)$ in general position cuts out a canonical curve of genus 5 in $\mathbb{P}^{4}$; see [Ha77, Example IV.5.5.3].

Let $Y_{0} \subset Y=\left(\mathbb{P}^{4}\right)^{n}$ be the open subset consisting of $n$-tuples of points imposing independent conditions of quadrics. Let $W$ be the space of $(n+1)$-tuples $\left(q, r_{1}, \ldots, r_{n}\right)$, where $\left(r_{1}, \ldots, r_{n}\right) \in Y_{0}$ and $q \in H^{0}\left(\mathbb{P}^{4}, \mathcal{O}(2)\right)$ vanishes at $r_{1}, \ldots, r_{n}$. The natural projection $W \rightarrow Y_{0} \subset\left(\mathbb{P}^{4}\right)^{n}$ given by $\left(q, r_{1}, \ldots, r_{n}\right) \mapsto\left(r_{1}, \ldots, r_{n}\right)$ is a vector bundle of rank $15-n$. By the above description, $X$ is $\mathrm{S}_{n}$-equivariantly birationally isomorphic to the total space of the Grassmannian bundle $\operatorname{Gr}(3, W)$. Twisting by $P$, we obtain the following diagram


By Lemma 2.1, we conclude that ${ }^{P} X$ is rational over ${ }^{P} Y$.
$g=2$ and $n=2$. Here $Y \subset\left(\mathbb{P}^{2}\right)^{*} \times\left(\mathbb{P}^{2}\right)^{*}$ parametrizes pairs $\left(L_{1}, L_{2}\right)$ of distinct lines in $\mathbb{P}^{2}$. Let $W \rightarrow Y$ be the vector bundle whose fiber over $\left(L_{1}, L_{2}\right)$ consists of quartic forms $q \in H^{0}\left(\mathbb{P}^{2}, \mathcal{O}(4)\right)$ such that both $q_{\mid L_{1}}$ and $q_{\mid L_{2}}$ vanish to second order at $p=L_{1} \cap L_{2}$. Then $X$ is $\mathrm{S}_{2}$-equivariantly birationally isomorphic to $\mathbb{P}(W)$ over $Y$. By Lemma 2.1, we conclude that ${ }^{P} X$ is rational over ${ }^{P} Y$.
$g=2$ and $n=3$. Here $X$ is $S_{3}$-equivariantly birationally isomorphic to $\mathbb{P}(W)$, where $W$ is the vector bundle over a suitable dense open subset of $Y \subset\left(\mathbb{P}^{1}\right)^{3} \times \mathbb{P}^{1}$ whose fiber over $\left(\left(a_{1}, a_{2}, a_{3}\right), b\right)$ consists of bihomogeneous polynomials $\phi \in H^{0}\left(\mathbb{P}^{1} \times \mathbb{P}^{1}, \mathcal{O}(3,2)\right)$ vanishing at $\left(a_{1}, b\right),\left(a_{2}, b\right)$ and $\left(a_{3}, b\right)$. By Lemma 2.1, ${ }^{P} X$ is rational over ${ }^{P} Y$.
$g=1$ and $n=3$. Here $\left(C^{\prime}, r_{1}, r_{2}, r_{3}\right)=f_{B}\left(C, p_{1}, p_{2}, p_{3}\right)$ is a smooth plane cubic curve with three distinct collinear points for every $\left(C, p_{1}, p_{2}, p_{3}\right) \in M_{1,3}$. Conversely, every smooth cubic curve $C^{\prime} \subset \mathbb{P}^{2}$ with three distinct collinear points $r_{1}, r_{2}, r_{3} \in C^{\prime}$ is of the form $f_{B}\left(C, p_{1}, p_{2}, p_{3}\right)$ for some $\left(C, p_{1}, p_{2}, p_{3}, B\right) \in X$, because $\mathcal{O}_{C^{\prime}}\left(r_{1}+r_{2}+r_{3}\right)=\mathcal{O}_{C^{\prime}}(1)$. Thus $X$ is $\mathrm{S}_{3}$-equivariantly birationally isomorphic to $\mathbb{P}(W)$ over $Y$, where $W \rightarrow Y$ is the vector bundle whose fiber over $\left(r_{1}, r_{2}, r_{3}\right)$ consists of cubic forms in $H^{0}\left(\mathbb{P}^{2}, \mathcal{O}(3)\right)$ vanishing at $\left(r_{1}, r_{2}, r_{3}\right)$. Twisting by the $\mathrm{S}_{3}$-torsor $P$, we conclude that ${ }^{P} X \simeq \mathbb{P}\left({ }^{P} W\right)$ is rational over ${ }^{P} Y$ by Lemma 2.1(f).
$g=1$ and $n=4$. For $\left(C, p_{1}, \ldots, p_{4}\right)$ in general position, $\left(C^{\prime}, r_{1}, \ldots, r_{4}\right)=f_{B}\left(C, p_{1}, \ldots, p_{4}\right)$ is a smooth curve of genus 1 in $\mathbb{P}^{3}$ with four coplanar points no three of which are collinear. By Ha77, Exercise IV.3.6(b)], the space $H^{0}\left(\mathbb{P}^{3}, \mathcal{I}_{C^{\prime}}(2)\right)$ of global sections of the ideal sheaf $\mathcal{I}_{C^{\prime}}(2)$ is 2 -dimensional. Moreover, if $q_{1}, q_{2}$ is a basis of this space, then $C^{\prime}$ is a complete intersection of the quadrics $Q_{1}$ and $Q_{2}$ cut out by $q_{1}$ and $q_{2}$. Conversely, a complete intersection of two smooth quadrics in $\mathbb{P}^{3}$ in general position is a smooth curve of genus 1 ; see Ha77, Exercise I.7.2]. We conclude that $X$ is $S_{4}$-equivariantly birationally isomorphic to the total space of the Grassmannian bundle $\operatorname{Gr}(2, W)$ over a suitably defined dense open subvariety $Y_{0} \subset Y$, where $W \rightarrow Y_{0}$ is the vector bundle whose fiber over $\left(r_{1}, \ldots, r_{4}\right) \in Y$ consists of $q \in H^{0}\left(\mathbb{P}^{3}, \mathcal{O}(2)\right)$ vanishing at $\left(r_{1}, \ldots, r_{4}\right)$. Twisting by a $\mathrm{S}_{4}$-torsor $P \rightarrow \operatorname{Spec}(F)$, we see that ${ }^{P} X$ is birational to $\operatorname{Gr}\left(2,{ }^{P} W\right)$ over ${ }^{P} Y$. By Lemma 2.1(f), ${ }^{P} X$ is rational over ${ }^{P} Y$.

This concludes the proof of Lemma 3.5 and thus of Theorem 1.2 ,

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