Non rationality of some norm one tori

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Abstract. We give a cohomological criterion that ensures the non stable rationality of a norm torus corresponding to a field extension of prime degree $p \geq 5$.

Keywords and Phrases: Norm one torus, rational variety, periodic cohomology.

Introduction, notations

Let l/k be a finite separable field extension, and $T = R^1_{l/k}(\mathbb{G}_\mathrm{m})$ its norm one torus. It is, in general, a fairly difficult problem to determine whether T is a (stably) rational variety. In the paper [LB], Le Bruyn has shown non stable rationality in the case where l/k is of prime degree $p \geq 5$ and generic (meaning that its Galois closure has Galois group S_p). This was later generalized by Cortella and Kunyavskii ([CK]), who proved that a norm torus corresponding to a generic field extension l/k of degree ≥ 4 is not stably rational. In this paper, we give a sharper cohomological criterion for non stable rationality of norm one tori of prime degree $p \geq 5$ (theorem 2.2). In particular, we show the following. Let l/k be a field extension of prime degree $p \geq 5$, such that its Galois closure has Galois group isomorphic to $\mathbb{Z}/p\mathbb{Z} \rtimes H$, where H is a subgroup of $\mathrm{Aut}(\mathbb{Z}/p\mathbb{Z})$ of order ≥ 3. Then, the torus $R^1_{l/k}(\mathbb{G}_\mathrm{m})$ is not a stably rational variety. As a main tool, we use a canonical flasque resolution of the character group of a norm one torus (proposition 1).

In the sequel, we denote by k a commutative field, by k_sep a separable closure of k and by Γ_k the Galois group of k_sep/k. By a Γ_k-lattice, we mean a $\mathbb{Z}$-free abelian group M of finite rank, endowed with a continuous action of Γ_k. The $\mathbb{Z}$-dual of M will be denoted by M'. If T is an algebraic k-torus, we denote by T^* (resp. T_*) the character group (resp. cocharacter group) of T; they are Γ_k-lattices, dual to each other. Let G be a finite group, and M a G-module. For $i \in \mathbb{Z}$, we denote by $\hat{H}^i(G, M)$ the i-th Tate cohomology group of G with values in M. If p is a prime number, M_p (resp. $M/\mathfrak{p}$) stands for the p-primary part of M (resp. for the quotient $M/\mathfrak{p}M$). Finally, if X is a finite set, we denote by $\#X$ its cardinality.

Acknowledgements

Contents

1. A canonical flasque resolution of norm one tori

In this section, we explain how a canonical flasque resolution of a norm one torus can be obtained. Let us first recall a few notions. A permutation lattice is a Γ_k-lattice that admits a $\mathbb{Z}$-base permuted by the action of Γ_k. A stably permutation lattice is a lattice M such that there exists permutation lattices P and Q, together with an isomorphism $M \oplus P \simeq Q$. A flasque lattice is a lattice F that satisfies $H^{-1}(H, M) = 0$ for all open subgroups $H \subset \Gamma_k$. Let M be a Γ_k-lattice. A flasque resolution of M is the data of an exact sequence of Γ_k-lattices

$$0 \longrightarrow M \longrightarrow P \longrightarrow F \longrightarrow 0,$$

such that P is a permutation lattice and F is flasque. Such a resolution always exists. What is more, we have the following remarkable fact, due to Voskresenskii: a torus with character module isomorphic to M is stably rational over k if and only if F is stably permutation. We refer to [CTS] (in particular, section 1 and proposition 6) for proofs and further information on flasque resolutions.

From now on, l/k is an étale algebra, corresponding to a finite Γ_k-set X (i.e. $X = \mathrm{Hom}_k(l, k_{\mathrm{sep}})$). We denote by $T = \mathrm{R}^1_{l/k}(\mathbb{G}_{\mathrm{m}})$ the norm one torus of l/k, i.e. the kernel of the norm map $\mathrm{R}_{l/k}(\mathbb{G}_{\mathrm{m}}) \longrightarrow \mathbb{G}_{\mathrm{m}}$. Its character module T^* canonically fits into an exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}^X \longrightarrow T^* \longrightarrow 0,$$

where the first map sends 1 to the element of $\mathbb{Z}^X$ whose coordinates are all equal to 1. Dually, T_* is the kernel of the augmentation map $\mathbb{Z}^X \longrightarrow Z$.

Consider the permutation lattice P defined by the formula

$$P := \bigoplus_{S,T \subset X; S \cap T = \varnothing} \mathbb{Z},$$

where the sum is taken over all couples (S, T) of nonempty disjoint subsets of X. Consider the exact sequence of Γ_k-lattices:

$$0 \longrightarrow C \longrightarrow P \xrightarrow{\pi} T_* \longrightarrow 0, \tag{1}$$

where the map π is given by

$$\pi(1_{S,T}) = \frac{\#S}{\gcd(\#S, \#T)} \sum_{t \in T} t - \frac{\#T}{\gcd(\#S, \#T)} \sum_{s \in S} s.$$

(here we view T_* as a sublattice of $\mathbb{Z}^X$)

The content of this section is then summed up in the following

Proposition 1.1. *The lattice C is coflasque. Consequently, the dual of the sequence (1) is a flasque resolution of T^*.*

Proof. We have to show that, for every open subgroup $H \subset \Gamma_k$, the cohomology group $H^1(H, C)$ is trivial. Looking at the long cohomology sequence associated to (1), and using the fact that P is coflasque, we see that it suffices to show that the map $P^H \xrightarrow{\pi^H} T_*^H$ is surjective for every such H. Let H be an open subgroup of Γ_k. Let $\mathrm{Orb}(X)$ denote the set of orbits of X under H, and n be the greatest common divisor of the sizes of these orbits. Then the group T_*^H is canonically isomorphic to the kernel of the surjection

$$\mathbb{Z}^{\mathrm{Orb}(X)} \xrightarrow{f} \mathbb{Z},$$

$$S \mapsto \frac{\#S}{n}, S \in \mathrm{Orb}(X).$$

Let us consider the dual map of f, which fits into an exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{f^*} \mathbb{Z}^{\mathrm{Orb}(X)} \longrightarrow \bigwedge^2 \mathbb{Z}^{\mathrm{Orb}(X)},$$

where the map on the right is given by wedging (say, on the left) by f^*. Dualizing, we obtain the exact sequence (a portion of the Koszul complex associated to f)

$$\bigwedge^2 \mathbb{Z}^{\mathrm{Orb}(X)} \longrightarrow \mathbb{Z}^{\mathrm{Orb}(X)} \xrightarrow{f} \mathbb{Z} \longrightarrow 0,$$

where the map on the left is given by

$$S \wedge T \mapsto f(S)T - f(T)S = \frac{\#S}{n}T - \frac{\#T}{n}S.$$

It follows that the elements $\frac{\#S}{n}T - \frac{\#T}{n}S$ generate T_*^H. But they lie in the image of $P^H \longrightarrow T_*^H$; indeed, the basis vector of P corresponding to (S,T) is a fixed point of H, and maps to $\frac{\#S}{\gcd(\#S,\#T)}T - \frac{\#T}{\gcd(\#S,\#T)}S$, a multiple of which equals $\frac{\#S}{n}T - \frac{\#T}{n}S$. □

2. A criterion for non stable rationality of some norm one tori

In this section, we give a necessary condition for a norm one torus to be a stably rational variety. We then apply it to a concrete case. In particular, as a corollary, we recover a result of Le Bruyn. To begin with, we need an easy lemma which ensures the p-periodicity of the cohomology of a group. Recall that a finite group G is said to have p-periodic cohomology, with d as a p-period (d being a non zero integer), if there exists an element $\alpha \in \hat{H}^d(G,\mathbb{Z})_p$ which is invertible in the ring $\hat{H}^*(G,\mathbb{Z})_p$ (which is a quotient of $\hat{H}^*(G,\mathbb{Z})$). We refer to [Br], section VI.9, for elementary properties of p-periodic cohomology.

Lemma 2.1. *Let p be a prime number, and G a finite group, such that p, but not p^2, divides $\#G$. Assume there exists a nonzero integer d such that $\hat{H}^d(G,\mathbb{Z})_p$ is isomorphic to $\mathbb{Z}/p\mathbb{Z}$. Then, G has p-periodic cohomology, with d as a p-period.*

Proof. We mimick the proof of the implication iv) $\Longrightarrow$ i) of [Br], theorem VI 9.1. Choose an isomorphism $\hat{H}^d(G,\mathbb{Z})_p \longrightarrow \mathbb{Z}/p\mathbb{Z} = \hat{H}^0(G,\mathbb{Z})_p$. By *loc. cit.*, theorem VI.7.4, this map is given by the cup product with an element of $\hat{H}^{-d}(G,\mathbb{Z})_p$. Hence, every non zero element of $\hat{H}^d(G,\mathbb{Z})_p$ is invertible in $\hat{H}^*(G,\mathbb{Z})_p$. □

We now have all the material required to prove the main theorem of this section.

Theorem 2.2. *Let l/k be a finite field extension of prime degree p. Let m/k be a Galois closure of l/k, with Galois group G. Set $X = \mathrm{Hom}_k(l,m)$; it is a finite G-set of order p. Assume that the torus $T = \mathrm{R}^1_{l/k}(\mathbb{G}_\mathrm{m})$ is a stably rational variety. Then G has p-periodic cohomology, with 4 as a p-period.*

Proof. Recall that T_* is the kernel of the augmentation $\mathbb{Z}^X \longrightarrow \mathbb{Z}$. Let us consider the canonical resolution

$$0 \longrightarrow C \longrightarrow P \longrightarrow T_* \longrightarrow 0$$

constructed in the previous section. Assume that T is stably rational. Then, there exists finite G-sets A and B and an isomorphism (of G-lattices) $C \oplus \mathbb{Z}^A \simeq \mathbb{Z}^B$. Because a permutation module is self-dual, reducing everything mod p we find that $\hat{H}^i(G,C/p)$ and $\hat{H}^i(G,C'/p)$ should be isomorphic groups for all $i \in \mathbb{Z}$ (recall

that $C' = \mathrm{Hom}_{\mathbb{Z}}(C, \mathbb{Z})$).
First of all, let us show that the groups $\hat{H}^i(G, P/p)$ are trivial for $i \in \mathbb{Z}$. Let (S, T) be a basis vector of P (hence, S and T are disjoint nonempty subsets of X). The stabilizer of (S, T) in G has order prime to p; indeed, assume the contrary. Then, this stabilizer acts transitively on X, since the stabilizer of an element of X is of order prime to p. Hence, S and T, being nonempty, should both equal X itself, which cannot be. This being established, we have, by Shapiro's lemma, that $\hat{H}^i(G, P/p)$ is a direct sum of factors of the type $\hat{H}^i(K, \mathbb{Z}/p\mathbb{Z})$, for K a subgroup of G of order prime to p. Therefore, $\hat{H}^i(G, P/p)$ is trivial for $i \in \mathbb{Z}$. By Shapiro's lemma again, we find that $\hat{H}^i(G, (\mathbb{Z}/p\mathbb{Z})^X) = 0$ for all $i \in \mathbb{Z}$ (this is because the stabilizer of an element of X is of order prime to p). Having those facts at our disposal, let us look at the cohomology sequence associated to the resolution (1), taken mod p. We find that $\hat{H}^i(G, C/p) = \hat{H}^{i-1}(G, T_*/p)$. Considering now the exact sequence

$$0 \longrightarrow T_*/p \longrightarrow (\mathbb{Z}/p\mathbb{Z})^X \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow 0,$$

we find that $\hat{H}^i(G, T_*/p) = \hat{H}^{i-1}(G, \mathbb{Z}/p\mathbb{Z})$, and hence

$$\hat{H}^i(G, C/p) = \hat{H}^{i-2}(G, \mathbb{Z}/p\mathbb{Z}), \forall i \in \mathbb{Z}.$$

Similarly, studying the two exact sequences

$$0 \longrightarrow T^*/p \longrightarrow P/p \longrightarrow C'/p \longrightarrow 0$$

and

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow (\mathbb{Z}/p\mathbb{Z})^X \longrightarrow T^*/p \longrightarrow 0,$$

we find that

$$\hat{H}^i(G, C'/p) = \hat{H}^{i+2}(G, \mathbb{Z}/p\mathbb{Z}), \forall i \in \mathbb{Z}.$$

Therefore, $\hat{H}^i(G, \mathbb{Z}/p\mathbb{Z}) = \hat{H}^{i+4}(G, \mathbb{Z}/p\mathbb{Z}), \forall i \in \mathbb{Z}$.
In particular, $\hat{H}^3(G, \mathbb{Z}/p\mathbb{Z}) = \hat{H}^{-1}(G, \mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$. Now, consider the exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{*p} \mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow 0.$$

Taking its associated long cohomology sequence yields

$$0 \longrightarrow \hat{H}^3(G, \mathbb{Z})/p \longrightarrow \hat{H}^3(G, \mathbb{Z}/p\mathbb{Z}) \longrightarrow \hat{H}^4(G, \mathbb{Z})_p \longrightarrow 0.$$

We claim that the p-torsion of $\hat{H}^3(G, \mathbb{Z})$ is trivial; indeed, by the standard restriction-corestriction argument, it is enough to show that $\hat{H}^3(S_p, \mathbb{Z}) = 0$, where S_p is a p-Sylow of G. But S_p is isomorphic to $\mathbb{Z}/p\mathbb{Z}$, and we have $\hat{H}^3(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}) = \hat{H}^1(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}) = 0$. Therefore, we have a canonical isomorphism $\hat{H}^4(G, \mathbb{Z})_p \simeq \hat{H}^3(G, \mathbb{Z}/p\mathbb{Z}) = \hat{H}^{-1}(G, \mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$. Thus, according to lemma 2.1, G has p-periodic cohomology, with 4 as a period. □

Let us now give an application of this result. Let $p \geq 5$ be a prime number, $X = \mathbb{Z}/p\mathbb{Z}$, H a finite subgroup of $\mathrm{Aut}(\mathbb{Z}/p\mathbb{Z}) = (\mathbb{Z}/p\mathbb{Z})^*$ (which is cyclic of order $p-1$). Assume that $\#H \geq 3$, and set $G = (\mathbb{Z}/p\mathbb{Z}) \rtimes H$, acting on X the obvious way ($\mathbb{Z}/p\mathbb{Z}$ acts by translations and H by group automorphisms). We then have:

Proposition 2.3. *Let m/k be a finite Galois field extension of Galois group G, and l/k the fixed field corresponding to H (i.e. we have $\mathrm{Hom}_k(l, m) = X$ as G-sets). Then, the norm one torus $T = \mathrm{R}^1_{l/k}(\mathbb{G}_\mathrm{m})$ is not a stably rational variety.*

Proof. By theorem 2.2, if T was rational, we would have $\hat{H}^3(G,\mathbb{Z}/p\mathbb{Z}) = \hat{H}^{-1}(G,Z/pZ) = \mathbb{Z}/p\mathbb{Z}$. We will see this is not the case. To this end, apply Hochschild-Serre's spectral sequence

$$H^i(H,H^j(\mathbb{Z}/p\mathbb{Z},\mathbb{Z}/p\mathbb{Z})) \Longrightarrow H^{i+j}(G,\mathbb{Z}/p\mathbb{Z}),$$

which degenerates since $H^i(H,H^j(\mathbb{Z}/p\mathbb{Z},\mathbb{Z}/p\mathbb{Z})) = 0$ for $i \geq 1$ (H is of order prime to p). We then find that

$$H^3(G,\mathbb{Z}/p\mathbb{Z}) = H^3(\mathbb{Z}/p\mathbb{Z},\mathbb{Z}/p\mathbb{Z})^H,$$

and it remains to describe the action of H on $H^3(\mathbb{Z}/p\mathbb{Z},\mathbb{Z}/p\mathbb{Z})$. Let $\alpha \in H^2(\mathbb{Z}/p\mathbb{Z},\mathbb{Z}) = H^1(\mathbb{Z}/p\mathbb{Z},\mathbb{Q}/\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$ be the canonical class. Using the fact that the cup-product

$$\mathbb{Z}/p\mathbb{Z} = H^1(\mathbb{Z}/p\mathbb{Z},\mathbb{Z}/p\mathbb{Z}) \longrightarrow H^3(\mathbb{Z}/p\mathbb{Z},\mathbb{Z}/p\mathbb{Z}),$$

$$x \mapsto x \cup \alpha$$

is an isomorphism, one readily finds that $H^3(\mathbb{Z}/p\mathbb{Z},\mathbb{Z}/p\mathbb{Z})$ is H-isomorphic to $\mathbb{Z}/p\mathbb{Z} \otimes_Z \mathbb{Z}/p\mathbb{Z}$ (the action of H on both factors being the canonical one). In other words, this H^3 is isomorphic to $\mathbb{Z}/p\mathbb{Z}$, on which H acts by the formula $h.x = h(h(x))$. Since H is assumed to have order > 2, this action is non-trivial, and hence its only fixed point is the zero element. This finishes the proof.

□

Remark 2.4. In particular, since G is a subgroup of S_p (the symmetric group on p letters), we obtain another proof of Le Bruyn's theorem which states that a generic norm torus of prime degree $p \geq 5$ is not a rational variety ([LB]).

Corollary 2.5. *Let l/k be a finite separable extension of prime degree $p \geq 5$. Assume that the Galois group G of the Galois closure of l/k has odd order n, and that the torus $T = \mathrm{R}^1_{l/k}(\mathbb{G}_{\mathrm{m}})$ is a stably rational variety. Then l/k is cyclic (i.e. $G = \mathbb{Z}/p\mathbb{Z}$).*

Proof. Let H be the subgroup of G corresponding to l. Let $P = \mathbb{Z}/p\mathbb{Z}$ be a p-Sylow subgroup of G, and N its normalizer. Since G is a subgroup of S_p, N is a subgroup of $\mathbb{Z}/p\mathbb{Z} \rtimes (\mathbb{Z}/p\mathbb{Z})^*$. Because T is stably rational, according to the preceding proposition, we necessarily have $N = \mathbb{Z}/p\mathbb{Z}$ or $N = \mathbb{Z}/p\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$. The second case is to be excluded since G has odd order. Thus, $P = N$ is its own normalizer. But then, the number of elements of G of order p is $(n/p)(p-1) = n - n/p$, and the number of elements of G of order not p is n/p, which is the order of H. Hence, H is normal in G, whence the claim.

□

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