

THE SMOOTHNESS CONJECTURE

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This short note presents the smoothness conjecture 0.2. It is the current goal of the project “Smooth profinite groups”, started about ten years ago with Charles De Clercq. So far its proof is incomplete, but a large part is already written down. The strategy improved a lot recently. Each step is clear in my mind. Counter-examples are more than welcome: I’ll be happy either way:) Here p is a prime number.

DEFINITION 0.1. *Let G be a profinite group. Let V be a finite-dimensional (continuous) representation of G over $\mathbb{F}_p$.*

For every $i \geq 1$, and every open subgroup $G' \subset G$, there is a cup-product operation

$$H^1(G', \mathbb{F}_p) \otimes_{\mathbb{F}_p} H^i(G', V) \longrightarrow H^{i+1}(G', V)$$

Composing it with the corestriction

$$\text{Cor}_G^{G'} : H^{i+1}(G', V) \longrightarrow H^{i+1}(G, V),$$

one gets a homomorphism

$$N(i, V, G') : H^1(G', \mathbb{F}_p) \otimes H^i(G', V) \longrightarrow H^{i+1}(G, V).$$

CONJECTURE. 0.2. *(Smoothness conjecture for the trivial cyclotomic character.)*

Let G be a profinite group. Assume that, for every $r \geq 1$ and every open subgroup $G' \subset G$, the natural map (given by reduction mod p)

$$\text{Hom}_{\text{cont}}(G', \mathbb{Z}/p^r) \longrightarrow \text{Hom}_{\text{cont}}(G', \mathbb{Z}/p)$$

is surjective. [This assumption is satisfied if $G = \text{Gal}(F_s/F)$ is the absolute Galois group of a field F , containing all p -primary roots of unity.]

Then, for any $i \geq 1$, $H^{i+1}(G, \mathbb{F}_p)$ is additively generated by the subgroups

$$\text{Im}(N(i, \mathbb{F}_p, G')),$$

for $G' \subset G$ running through all open subgroups.

Remark 0.3. The general formulation of the smoothness conjecture, is that the conclusion is true for any G that fits into a $(1, \infty)$ -cyclotomic pair $(G, \mathbb{Z}_p(1))$. In particular, it should be valid for $G = \text{Gal}(F_s/F)$ without any assumption on F .

For the absolute Galois group of a field ($G = \text{Gal}(F_s/F)$), the conjecture implies by induction on i that the cohomology algebra

$$H^*(G, \mathbb{F}_p) := \bigoplus_{i \geq 0} H^i(G, \mathbb{F}_p)$$

is generated in degree one. This is because the corestriction of a pure symbol

$$(x_1) \cup \dots \cup (x_{i+1}) \in H^{i+1}(G', \mathbb{F}_p)$$

is a sum of pure symbols in $H^{i+1}(G, \mathbb{F}_p)$. This fact follows from the existence of the Galois symbol and of the norm in Milnor K-theory, which are both very standard results.

From there, it is known that conjecture 0.2 implies the norm residue isomorphism theorem (Bloch-Kato-Milnor conjecture, proved by Rost and Voevodsky), at a very low cost. It is actually a stronger claim.

One may wonder whether the following strengthened version of the smoothness conjecture holds. For some time I thought it could be true.

STATEMENT 0.4. *Same premises as in the smoothness conjecture.*

Let V be a finite-dimensional representation of G over $\mathbb{F}_p$. Then, for any $i \geq 1$, $H^{i+1}(G, V)$ is additively generated by the subgroups

$$\text{Im}(N(i, V, G')),$$

for $G' \subset G$ running through all open subgroups.

The content of that statement, is that the Galois cohomology of V , as a module over $H^*(G, \mathbb{F}_p)$, is generated by elements of degree one, together with corestrictions maps from all open subgroups.

Assume that $G = \text{Gal}(F_s/F)$, for F a field (say, containing all p -primary roots of unity). We know that classes in

$$H^1(G', \mathbb{F}_p) = H_{et}^1(\text{Spec}(F), \mathbb{F}_p[G/G'])$$

are R -trivial. Thus, statement 0.4 implies the following.

STATEMENT 0.5. *Let V be a finite-dimensional representation of $G = \text{Gal}(F_s/F)$ over $\mathbb{F}_p$. Then, all classes in the Galois cohomology group $H^2(F, V)$ are R -trivial.*

Statement 0.5 is true if $\dim_{\mathbb{F}_p}(V) \leq 2$. However, when $\dim_{\mathbb{F}_p}(V) \geq 3$, classes in $H^1(G, V)$ are in general not R -trivial (e.g. when $F = \mathbb{C}((X, Y))$). This can be proved by adapting computations of section 13 of our text “Smooth profinite groups, I” (dealing with related questions), combined to a classical computation of residues using Fadeev’s exact sequence in Galois cohomology. Under a different perspective, this was known to experts for a long time, as Colliot-Thélène and Gille explained to me.

Very recently, De Clercq showed that, for $F = \mathbb{C}((X_1, \dots, X_{n+1}))$, and for some V , some classes in $H^n(F, V)$ are not R -trivial.

See his preprint here: <https://arxiv.org/abs/2610.22758>

Consequently, statement 0.4 is wrong as well. Knowing this helped me to finalize my strategy to prove the smoothness conjecture. I hope to release it this year.

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