## Quantum toroidal algebras

### Construction
- Affinizations of Lie algebras:
  \[ \hat{g} \rightarrow C[t, t^{-1}] \otimes g \rightarrow C[z^\pm 1, t^\pm 1] \otimes g \]
  - Simple Lie algebra: Loop algebra
  - Affinizations of quantum groups:
  \[ U_q(g) \rightarrow U_q(C) \rightarrow U_q(U) \]

### Properties
- Some fundamental elements about quantum toroidal algebras:
  - In type A, they are in Schur-Weyl duality with elliptic Cherednik algebras,
  - the quantum affine algebra is a subalgebra of the quantum toroidal algebra,
  - Quantum toroidal algebras have a "coproduct" which involves infinite sums

### Motivations
- **Cherednik algebra**
- **Quantum toroidal algebra**
- **Representations**
- **Geometry**
- **Conformal field theory**
- **Combinatorics**

### State of art
- No example of finite-dimensional representations were known until very recently.

## Aim of my works

**Aim** Construct finite-dimensional representations of quantum toroidal algebras of type A at roots of unity. We have three different constructions:

- Construction via monomial crystals,
- Construction by fusion products,
- Construction via the affinization \( \hat{U}(\hat{g}_{\infty}) \) of type \( A_{\infty} \).

## Extremal representations of Kashiwara

### Facts
- The extremal fundamental representations:
  - are representations \( V_\ell \) of \( U_q(\hat{g}_{\infty}) \) \( (\ell = 1, \ldots, n) \) with crystal bases \( B_\ell \),
  - are isomorphic to the global Weyl modules [Hatayama, Pressley 05],
  - admit an irreducible quotient of finite dimension [Kashiwara 02].

### Idea
- Extend the action of the quantum affine algebra on \( V_\ell \) to an action of the quantum toroidal algebra: the representations of \( \hat{U}(\hat{g}_{\infty}) \) hence obtained should have finite-dimensional quotients.

### First construction
- Crystal bases \( B_\ell \) can be realized by monomial crystals \( M_{\ell} \) [Hernandez, Nakajima 06].
- Monomials occurring in these crystals appear also in the theory of \( q \)-characters of quantum toroidal algebras [Frenkel, Reshetikhin 90].

### Aim
- Construct a representation of \( \hat{U}(\hat{g}_{\infty}) \) satisfying the following properties:
  - its \( q \)-character is the sum of monomials in \( M_{\ell} \),
  - its restriction to the quantum affine subalgebra is \( V_\ell \).

### Theorem
- Such a representation exists if and only if \( \ell \) is one of the nodes 1, \( r+1 \), or \( n \) of the Dynkin diagram, where \( n = 2r+1 \) is odd. It is denoted by \( V_\ell(a) \) with \( a \in \mathbb{C}^* \) and is called extremal loop weight representation.

### Remark
- The extremal loop weight representations \( V_\ell(a) \), also called vector representations, are used in [Feigin, Jimbo, Miwa, Mukhin 13].

## Finite-dimensional representations

### Theorem
- Specializing \( q \) at a particular root of unity in the representations \( V_\ell(a) \), we get irreducible finite-dimensional representations by taking a quotient.

### Remark
- This is the first systematic construction of finite-dimensional representations of quantum toroidal algebras at roots of unity.

## Second construction

### Motivation
- The extremal representations are related to tensor products of highest weight representations and lowest weight representations [Kashiwara 94].

### Theorem
- Process of tensor products of highest weight representations and lowest weight representations of \( \hat{U}(\hat{g}_{\infty}) \).
- We recover the vector representation \( V_\ell(a) \).

### Proof
- Drinfeld coproduct and related methods [Hernandez 07].

### Theorem
- We get extremal loop weight representations as subquotients of \( \otimes V_\ell(a) \).
- We obtain new finite-dimensional representations at roots of unity.

## Third construction

### Conjecture
- Relation between the \( q \)-character of representations \( \hat{U}(\hat{g}_{\infty}) \) and the one of representations \( \hat{U}(\hat{g}_{\infty}) \).

### Theorem
- Construction of \( \ell \)-extremal representations \( V_\ell^{+}\) for \( \hat{U}(\hat{g}_{\infty}) \).
- Proof of the conjecture: we recover the representations \( V_\ell(a) \) of \( \hat{U}(\hat{g}_{\infty}) \).

## Perspectives

- Construction of finite-dimensional representations for quantum toroidal algebras of general type.
- Classification of irreducible finite-dimensional representations of quantum toroidal algebras at roots of unity.
- Description of finite-dimensional representations of elliptic Cherednik algebras at roots of unity by Schur-Weyl duality.

## References
