POLYNOMIAL SKEW PRODUCTS WITH SMALL RELATIVE DEGREE

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Abstract. We investigate the local dynamics of a proper superattracting holomorphic germ f in $(\mathbb{C}^2, 0)$ possessing a totally invariant line L such that $f^*L = dL$ with $d \geq 2$, and such that $f|_L$ has a superattracting fixed point at 0 of order $2 \leq c < d$. We prove that any such map is formally conjugated to a skew product of the form $(z^d, P(z, w))$, where $P \in \mathbb{C}[[z]][w]$ is polynomial in w of degree c, hence it induces a natural dynamics on the Berkovich affine line over $\mathbb{C}((z))$. Such non-Archimedean skew products were recently studied by Birkett and Nie-Zhao.

On the non-Archimedean side, we focus on the restriction of the dynamics on the Berkovich open unit ball (which naturally contains all irreducible analytic germs at the origin). We exhibit an invariant compact set $\mathcal{K}$ outside of which all points tend to L, and which supports a natural ergodic invariant measure.

By a careful analysis of local intersection numbers, we prove that the growth of multiplicity of iterated curves is controlled by the recurrence properties of the critical set. In particular, when no critical branch of f belongs to $\mathcal{K}$, any point in $\mathcal{K}$ corresponds to a curve of uniformly bounded multiplicity at 0.

We then return to the complex picture and show the existence of an invariant pluripolar positive closed $(1, 1)$-current T, outside of which all orbits converge to 0 at super-exponential speed c. Under the same assumption on the critical branches as above, we prove that T admits a geometric representation as an average of currents of integration over the curves in $\mathcal{K}$, with respect to the natural invariant measure. In particular, T is uniformly laminar outside the origin.

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Introduction

0.1. **A class of superattracting germs.** In this article we investigate the local dynamics of a class of superattracting points in $\mathbb{C}^2$. It has been known for a long time that, contrary to the one-dimensional case, the local dynamics at superattracting points in higher dimension gives rise to subtle and varied phenomena (see, e.g., [HP94, Fav00, FJ07, Gig14, GR14, MZMS13, Uen19, BEK12]).

Our setting is the following: assume that f is a germ of holomorphic map at $0 \in \mathbb{C}^2$ with $f(0) = 0$, and that there are integers $2 \leq c < d$ satisfying: (i) the line $\{z = 0\}$ is totally invariant and as divisors (or currents) we have $f^*\{z = 0\} = d\{z = 0\}$; and (ii) the restriction of f to $\{z = 0\}$ admits a superattracting point of order c. Then, in local holomorphic coordinates, f can be expressed as

$$f(z, w) = \left(z^d, w^c + zh(z, w)\right), \text{ with } h(0, 0) = 0. \tag{1}$$

This class of mappings arises from our attempts at dealing with the missing case of the dynamical Manin-Mumford conjecture for regular polynomial mappings in [DFR23].

When $p = (z, w)$ is close to 0, $f^n(p)$ converges super-exponentially fast to 0, and there are two distinct regimes: the majority of orbits converge tangentially to $\{z = 0\}$, eventually landing in a region of the form $\{|z| < C|w|^c\}$. For such orbits, the dynamics is dominated by that of $f|_{\{z=0\}}$ and the asymptotic contraction rate

$$c_\infty(p) = \lim_{n\to\infty} \sqrt[n]{\left|\log |f^n(p)|\right|}$$

is equal to c. There is however a thin subset $\mathcal{W}$ of points converging faster to the origin, with asymptotic rate d. The set $\mathcal{W}$ may thus be viewed as a kind of "strong (super-)stable manifold" of the origin. This is a closed pluripolar set with a rich and intriguing structure. More precisely, the following result (which is already partly contained in [FJ07]) is proven in Section 7.

Theorem A. *Let f be a germ of holomorphic mapping in $\mathbb{C}^2$ of the form* (1) *with $2 \leq c < d$. Then for every $p = (z, w)$ close to the origin, the asymptotic contraction rate exists and equals c or d.*

The sequence $\left(\frac{1}{c^n} \log |f^n(z, w)|\right)_{n\geq 0}$ converges to a plurisubharmonic function g with values in $\mathbb{R} \cup \{-\infty\}$, such that e^g is continuous, $g \circ f = cg$, and g is pluriharmonic in $\{g > -\infty\}$.

The closed pluripolar set $\mathcal{W} = \{g = -\infty\} = \mathrm{Supp}(\mathrm{dd}^c g)$ is precisely the locus where $c_\infty = d$. It is not a subvariety unless f is conjugated to a product map.

Our main purpose in this paper is to demonstrate that the structure of the set $\mathcal{W}$ and of the current $\mathrm{Supp}(\mathrm{dd}^c g)$ are encoded by a natural dynamical system whose phase space is of non-Archimedean nature. This space is defined as a suitable completion of the space of germs of irreducible curves in $(\mathbb{C}^2, 0)$, and corresponds to the valuative tree, which was extensively studied in [FJ04, FJ07]. We found it more convenient to view this space as a ball in the Berkovich analytification of the affine line over the field of formal Laurent series $\mathbb{C}((z))$, which makes the connection with other relevant works more transparent (these two points of view are compared in § 1.3.3 below).

0.2. **Polynomial skew products over a non-Archimedean field.** To define our non-Archimedean model we work with mappings of the form

$$f(z,w) = \left(z^d, w^c + \sum_{j=0}^{c-1} h_j(z) w^j\right) \tag{2}$$

where $h_j \in \mathbb{C}[[z]]$ satisfy $h_j(0) = 0$, and as in (1), we assume that $2 \leq c < d$. Any map f as in (2) falls within the class of *polynomial skew products*. It induces a self-map $f_\diamond$ of the field of (formal) Puiseux series $\mathbb{C}\langle\langle z\rangle\rangle$ and of its completion $\mathbb{L}$ for the t-adic norm, defined by the formula

$$f_\diamond \colon \phi(z) \longmapsto \left(\phi\left(z^{1/d}\right)\right)^c + \sum_{j=0}^{c-1} h_j\left(z^{1/d}\right) \left(\phi\left(z^{1/d}\right)\right)^j . \tag{3}$$

Recall that a Puiseux series (resp. an element of $\mathbb{L}$) is a series of the form $\sum_{n=0}^{\infty} a_n z^{\beta_n}$ where $(a_n) \in \mathbb{C}^{\mathbb{N}}$ and (β_n) is an increasing sequence of rational numbers with a fixed denominator (resp. an increasing sequence of rational numbers with $\beta_n \to \infty$).

The map $f_\diamond$ admits a natural continuous extension to the Berkovich analytification of the affine line $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{L}}$ over $\mathbb{L}$, enabling us to exploit analytic and potential theoretic tools to analyze its dynamics. It is crucial to consider at the same time the induced dynamics of $f_\diamond$ on $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}((z))}$. Indeed the latter space is more directly related to geometry, as the classical points lying in the open unit ball correspond to irreducible formal germs in $(\mathbb{C}^2, 0)$ different from $\{z = 0\}$ (while classical points in $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{L}}$ should be thought of as parameterizations of these curves). There is a canonical surjective map $\pi \colon \mathbb{A}^{1,\mathrm{an}}_{\mathbb{L}} \to \mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}((z))}$, and we say that a point (in either space) is of type 1 when it corresponds to a point in $\mathbb{L}$.

To make the discussion clearer, we focus on the dynamics of $f_\diamond$ on $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}((z))}$. We first observe (Theorem 2.6) that the interesting dynamics takes place in the (Berkovich) unit ball $B^{\mathrm{an}}(0,1)$: under iteration, a point $x \in B^{\mathrm{an}}(0,1)$ either converges to the Gauss point ζ_g or it remains in $B^{\mathrm{an}}(0,\rho)$ for some $\rho < 1$. We define

$$\mathcal{K} := \{x \in B^{\mathrm{an}}(0,1) \subset \mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}((z))},\ f^n_\diamond(x) \text{ does not converge to } \zeta_g\} . \tag{4}$$

The ball $B^{\mathrm{an}}(0,1)$ is a $\mathbb{R}$-tree with one (non-compact) end, and the compact set $\mathcal{K}$ consists of the set of points that do not reach a neighborhood of this end under iteration. It thus plays the role of the filled-in Julia set in our context. It will also turn out to be the non-Archimedean counterpart of $\mathcal{W}$. In Section 3 we prove the following:

Theorem B. *Let f be a map of the form* (2) *with $2 \leq c < d$. Then $\mathcal{K}$ is either a Cantor set of type 1 points, or it is reduced to a single point corresponding to a smooth formal curve transverse to $\{z = 0\}$, in which case f is conjugated to a product map.*

There exists a canonical $f_\diamond$-invariant ergodic probability measure μ_{na}, with $\mathrm{Supp}(\mu_{\mathrm{na}}) = \mathcal{K}$ and such that for every $x \in B^{\mathrm{an}}(0,1)$, the sequence $\frac{d^n}{c^n}(f^n_\diamond)^*\delta_x$ converges to μ_{na}.

See below Section 1 for basics on Berkovich theory. Note that in Sections 1 to 6, we work in the slightly more general context of the field $k((z))$, with k an arbitrary algebraically closed field k of characteristic 0.

0.3. **History and related results.** Polynomial and rational dynamics on Berkovich spaces in dimension 1 is now a well-developed subject; a valuable overview is provided in the books of Baker-Rumely [BR10] and Benedetto [Ben19]. Mappings of the form $f_\diamond$ are not directly covered by these references. Their dynamical properties were recently investigated by Birkett [Bir23] and independently by Nie and Zhao [NZ24].

In [NZ24], the authors consider *twisted rational maps*, on $\mathbb{P}^{1,\mathrm{an}}_K$, where K is an algebraically closed field of characteristic 0, that is non-trivially valued. Such maps are of the form $R_\tau = R \circ \tau$, where $R \in K(w)$ is a rational map, and τ is a continuous automorphism of K satisfying the property $|\tau(\cdot)| = |\cdot|^\lambda$ for some $\lambda \in \mathbb{R}_{>0}$. The degree of R is called the *relative degree* of R_τ (this terminology is from [Bir23]). Under the assumption that R_τ has *large relative degree* $\lambda \deg(R) > 1$, they obtain equidistribution results (in the spirit of Favre and Rivera-Letelier [FRL10]) for backward orbits. When moreover R is a polynomial, the authors describe the action of R_τ on the Julia set, as a one-sided shift on $\deg(R)$ symbols.

For mappings of the form (3), if we let τ be the automorphism of $\mathbb{L}$ given by $\tau(z) = z^{1/d}$, we have $f_\diamond = \hat{f} \circ \tau$, where $\hat{f}$ is the polynomial

$$\hat{f}(w) = w^c + \sum_{j=0}^{c-1} h_j(\tau(z)) w^j \ .$$

It follows that in our situation the relative degree equals $\deg\left(\hat{f}\right) = c$ and $\lambda = 1/d$ so that we are in the opposite case $\lambda \deg\left(\hat{f}\right) = c/d < 1$ of *small relative degree.*

In the language of Birkett [Bir23], the value $\lambda_\tau = 1/d$ is the *scale factor*, which places us in the *superattracting* case of that paper. Under this condition, it is shown in [Bir23, Corollary 3.14] that $f_\diamond$ is a contraction on the hyperbolic space, given the Berkovich projective line minus the rigid points, endowed with the hyperbolic distance. This property was investigated in the context of valuation spaces in [GR21].

It is worth mentioning that our set $\mathcal{K}$ lies in the Fatou set of $f_\diamond$ both in the sense of [Bir23, Definition 5.1] and [NZ24, Definition 3.1]. However the set of iterates does not form a normal family near any point of $\mathcal{K}$ in the sense of [FKT12]. This underlines some of the subtleties that may arise in the Fatou-Julia theory of general skew products.

0.4. **Curves of bounded multiplicity in $\mathcal{K}$.** Recall that a type 1 point $x \in \mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}((z))}$ is defined by a series ϕ in $\mathbb{L}$. When ϕ is a Puiseux series, the point x corresponds to an irreducible formal germ $C(x)$, and to comply with the terminology of non-Archimedean geometry, we say that it is *rigid.* We define the *multiplicity* $m(x)$ to be the order of tangency of $C(x)$ with $\{z = 0\}$ (this terminology is consistent with Berkovich theory but it is also a little unfortunate since $m(x)$ is not the multiplicity of $C(x)$ at 0). When x is not defined by a Puiseux series, by definition it is not rigid, and we set $m(x) = \infty$. The geometric interpretation of non-rigid points is less clear (see [FJ05, Proposition 6.9] for an attempt using currents).

In sections 4 and 5 we show that the multiplicity of points in $\mathcal{K}$ is controlled by the recurrence properties of the critical set. More precisely, let $\mathcal{C}$ be the (finite) set consisting of all rigid points associated with the irreducible components of the critical locus of f different from $\{z = 0\}$. We also introduce the restricted critical set $\mathcal{C}^+ \subset \mathcal{C}$ corresponding to critical branches c such that $1+J(c)$ does not divide d, where $J(c)=\mathrm{ord}_c\,\mathrm{Jac}_w(f)$ is the multiplicity of c as a critical branch.

Our second main theorem is the following. Here $\omega(x)$ is the ω-limit set, i.e. the cluster set of the set of iterates of x.

Theorem C. *Let f be a map of the form* (2) *with $2\leq c<d$. Then:*

(i) any point in $\mathcal{K}$ such that $\omega(x)\cap\mathcal{C}^+=\emptyset$ has finite multiplicity;
(ii) if $\mathcal{C}^+$ is non-recurrent in the sense that $\omega(c)\cap\mathcal{C}^+=\emptyset$ for any $c\in\mathcal{C}^+$, then the multiplicity is uniformly bounded on $\mathcal{K}$;
(iii) if $\mathcal{K}$ is not reduced to a point and there is a periodic critical point in $\mathcal{C}^+$, then $\mathcal{K}$ contains non-rigid points and rigid points with arbitrary large multiplicity.

In [Tru14], Trucco studies the dynamics of a polynomial mapping P with coefficients in the field of complex Puiseux series. His Theorem F asserts that the Julia set J_P contains only Puiseux series if and only if no critical point in J_P is recurrent, and in that case the multiplicity is uniformly bounded. Theorem C is thus a generalization of his results to skew-products of small relative degree. Note that it does not say anything in the presence of a non-periodic recurrent critical point in $\mathcal{C}^+$. The very existence of such examples is a non-trivial fact and remains open. We present explicit examples illustrating each case of the theorem in Section 6.

The proof of Theorem C relies on the notion of *generic multiplicity* for type 2 points of $B^{\mathrm{an}}(0,1)$ which we discuss in § 4.2. The key step is to bound from above the generic multiplicity at a point x in terms of the generic multiplicity along its orbit. This delicate point is handled in Proposition 4.7.

0.5. **Back to germs in $\mathbb{C}^2$.** Building on the previous results we come back to the problem of describing the structure of the super-stable set $\mathcal{W}$ for a germ of the form (1). After a formal conjugacy, we can bring f to the form (2) with $2\leq c<d$, so that Theorem A applies. We work under the assumption that no critical branch corresponds to a point in $\mathcal{K}$, or equivalently is contained in $\mathcal{W}$ as a subset of $\mathbb{C}^2$ (see Proposition 8.1). Then by the second conclusion of Theorem C, any point $x\in\mathcal{K}$ is associated with a germ of (a priori) formal curve $C(x)$, and the multiplicity $m(x)$ is uniformly bounded by some integer m.

Recall from Theorem B, that $\mathcal{K}$ supports a canonical ergodic measure μ_{na}. Our main result is the following:

Theorem D. *Assume that f is a holomorphic map of the form* (1) *with $2\leq c<d$, and that no critical branch belongs to $\mathcal{K}$.*

Then there exist an integer $m\in\mathbb{N}^*$ and a constant $r_0>0$, such that for any $x\in\mathcal{K}$ there is a Puiseux series $\phi_x\in\mathbb{C}\langle\langle z\rangle\rangle$ which is convergent in the disk $\mathbb{D}(0,r_0)$, and such that $C(x)$ is parameterized by $t\mapsto(t^m,\phi_x(t^m))$.

Moreover, we have the equality of currents

$$\mathrm{dd}^c g=\int_{\mathcal{K}}\frac{[C(x)]}{m(x)}\,\mathrm{d}\mu_{\mathrm{na}}(x)\ , \tag{5}$$

and $\mathrm{dd}^c g$ is a uniformly laminar current outside the origin.

The proof of this result is given in Section 8, and requires several steps. We first make a suitable base change, that is, we take the pull-back by some branched cover $\beta(z,w) = (z^k, w)$ in order to reduce to the case $m = 1$. Then we construct a suitable geometric model by exhibiting a sequence of point blow-ups over the origin $\kappa\colon X \to (\mathbb{C}^2, 0)$, and a finite set of points $p_i \in \kappa^{-1}(0)$ such that a germ of curve C belongs to $\mathcal{K}$ if and only if the strict transform of $f^n(C)$ intersects $\kappa^{-1}(0)$ at one of the points p_i for every $n \geq 0$.

At the non-Archimedean level, the set of points p_i corresponds to a covering of $\mathcal{K}$ by open balls. More precisely, for any i, the set of all germs of irreducible curves C whose strict transform contains p_i is an open ball B_i. Furthermore, the open cover $\{B_i\}$ can be chosen to form a Markov partition for $f_\diamond$. Introduce the oriented graph Γ whose vertices are balls B_i and an (oriented) edge joins B_i to B_j if and only if $f_\diamond(B_i) \supset B_j$. We prove that the subshift of finite type induced by Γ is canonically conjugated to the dynamics of $f_\diamond$ on $\mathcal{K}$.

Back to the Archimedean picture, let $\widehat{F}\colon X \dashrightarrow X$ be the lift of f. It is a meromorphic map such that all points p_i are indeterminate. In the oriented graph Γ an edge joins B_i to B_j if and only if $\widehat{F}(p_i) \ni p_j$. Our analysis then proceeds as follows: pulling-back curves by $\widehat{F}$ gives rise to a natural *graph transform* $\mathcal{L}_e$ for every edge $e = (p_i p_j)$ of Γ. The superattracting nature of f implies that $\mathcal{L}_e$ sends a curve passing through p_j of a definite size r to a curve through p_i of the same size. Analyzing this graph transform in local coordinates shows that it is a contraction on a suitable function space. Since any point $x \in \mathcal{K}$ is determined by its itinerary $(v_j)_{j\geq 0}$ in the cover $\{B_i\}$ and, at the same time, the sequence $\mathcal{L}_{e_0} \circ \cdots \circ \mathcal{L}_{e_n}(C)$ converges, where $e_j = (v_j v_{j+1})$ and C is any initial data, we conclude that $C(x)$ coincides with (the projection under κ of) its limiting curve. The control of the size r guarantees the convergence of the Puiseux parameterizations over a fixed neighborhood of the origin. Finally, the representation (5) follows from the equidistribution of pull-backs both at the Archimedean and non-Archimedean levels.

To complete the proof, we push this picture down by the branched cover β. A few non-trivial arguments are required to justify the uniform laminarity of the image current and the formula (5). Note that the appearance of the multiplicities in (5) is a feature of our coordinate dependent approach.

0.6. **Related works.** Theorem D implies that $\mathcal{W}$ is a bouquet of analytic curves, and a lamination with Cantor transversals outside the origin. Such a structure first appeared in the work of Yamagishi [Yam01] (see also [Shi10]) who exhibited a simple mechanism for a meromorphic map in $(\mathbb{C}^2, 0)$ to have an invariant Cantor bouquet of analytic curves at an indeterminacy point. Gignac [Gig14] studied in detail the map $(z^4, w^2 - z^4)$ (which is of the form (2)) to build a counter-example to a question of Arnol'd. For this example, he proved that all curves in $\mathcal{K}$ are smooth and intersect $\{z = 0\}$ transversally. In all these works though, there was no explicit connection to non-Archimedean dynamics. Note also that global variants of these constructions for polynomial mappings of $\mathbb{C}^2$ were studied in [DDS05, Vig07].

In a different direction, Kiwi [Kiw06, Theorem 1.6] proved that the locus of non-renormalizable complex cubic polynomials with disconnected Julia set is naturally foliated by analytic curves, corresponding to non-renormalizable cubic polynomials defined over $\mathbb{L}$. It could be interesting to expand on his arguments and see whether the bifurcation current defined in [DF08] can be described similarly as in (5), where the measure $\mu_{\rm na}$ would be replaced by a suitable bifurcation measure in the non-Archimedean cubic parameter space.

Finally, we note that although the laminarity of currents is a pervasive property in holomorphic dynamics, see e.g. [BLS93, BJ00, Duj06, Din05, Duj23], a non-Archimedean interpretation of the set of plaques defining the current is not generally possible. The complex structure needs to degenerate to exhibit a limiting non-Archimedean behaviour, a property which is guaranteed by the fact that we are working locally at a superattracting fixed point. In the global situation which motivated this work, where f is the germ of a regular polynomial mapping at some superattracting point on the line at infinity, the laminar structure of the current $dd^c g$ is reminiscent from that of the Green current obtained in [BJ00], with the dynamics of $f_\diamond$ playing the role of that of $f|_{L_\infty}$.

0.7. **Plan of the paper.** In Section 1, we introduce the necessary tools from Berkovich theory, in particular over the fields $k((z))$ and $\mathbb{L}$, where k is an arbitrary algebraically closed field k of characteristic 0. The non-Archimedean dynamics of the skew-product $f_\diamond$ acting on the Berkovich unit ball is studied in Sections 2 to 6. After some generalities on the topological properties of $f_\diamond$, the structure of the set $\mathcal{K}$ and the measure $\mu_{\rm na}$ is analyzed in Section 3, and the conclusions of Theorem B are contained Theorems 3.2, 3.6, and 3.10. In Sections 4 and 5, we prove Theorem C as a combination of Theorems 4.1 and 5.1. In Section 6, we study the formal conjugacy of maps of the form (1) to maps of the form (2) (Theorem 6.1) and illustrate Theorem C by a few examples. In the last part of the paper (Sections 7 and 8), we turn to complex dynamics. Theorem A follows from Theorem 7.1, and Theorem D corresponds to Theorem 8.2.

1. Dynamics on the Berkovich affine line

In this section we gather the necessary material from Berkovich theory, in particular over the field of Laurent series. The self maps induced by our skew products will be introduced in the next section.

1.1. **The Berkovich affine line.** (See [Ben19, BR10, Jon15] for a more detailed presentation.) Let $(K, |\cdot|)$ be a complete non-Archimedean field. The Berkovich affine line $\mathbb{A}^{1,\mathrm{an}}_K$ over K is by definition the set of all multiplicative seminorms on $K[w]$ whose restriction to K coincides with $|\cdot|$, endowed with the weakest topology making all evaluation maps continuous.

Given a point $x \in \mathbb{A}^{1,\mathrm{an}}_K$ and a polynomial $P \in K[w]$ we write $|P(x)| \in \mathbb{R}_+$ for the value of P at x, that is x is a seminorm $|\cdot|_x$ and $|P(x)| = |P|_x$. Note that when $P = c$ is a constant, then $|c(x)| = |c|$. The space $\mathbb{A}^{1,\mathrm{an}}_K$ is locally compact.

1.1.1. Suppose that K is algebraically closed. We denote by

$$\overline{B}(a,r) = \{b \in K,\ |b-a| \le r\}$$

(resp. $B(a,r) = \{b \in K,\ |b-a| < r\}$) the closed (resp. open) ball of center a and radius r in K. Then Berkovich classified points $x \in \mathbb{A}^{1,\mathrm{an}}_K$ into four types:

Type 1: there exists $a \in K$ such that $|P(x)| = |P(a)|$;
Type 2: there exists $a \in K$, and $r \in |K^*|$ such that $|P(x)| = \sup_{|b-a|\leq r} |P(b)|$;
Type 3: there exists $a \in K$, and $r \in \mathbb{R}_{>0} \setminus |K^*|$ such that $|P(x)| = \sup_{|b-a|\leq r} |P(b)|$;
Type 4: there exists a decreasing sequence of balls (B_n) such that $\bigcap_n B_n = \emptyset$ and $|P(x)| = \inf_n \sup_{b \in B_n} |P(b)|$.

We denote by $\zeta(a, r)$ the point (of type 1, 2 or 3) associated to the ball of center $a \in K$ and radius $r \in \mathbb{R}_{\geq 0}$, and by $\zeta_g = \zeta(0, 1)$ the Gauss point. The norm $|\cdot|$ being non-Archimedean, two (closed) balls $\overline{B}(a, r)$ and $\overline{B}(b, r)$ of same radius $r \geq 0$ are either disjoint or equal. It follows that $\zeta(a, r)$ and $\zeta(b, r)$ are distinct when $r < |b - a|$, and coincide when $r \geq |b - a|$. This phenomenon hints at the $\mathbb{R}$-tree structure of $\mathbb{A}^{1,\mathrm{an}}_K$ (see [FJ04, Chapter 3]), with branching points at points of type 2. More precisely, all points in $\mathbb{A}^{1,\mathrm{an}}_K$ are associated with nested collections of balls in $\mathbb{A}^1(K)$, i.e., filters for the poset of closed balls in $\mathbb{A}^1(K)$, ordered by inclusion, that are closed under the intersection of decreasing sequences (when it gives a ball). The inclusion of nested collections of balls defines a partial order on $\mathbb{A}^{1,\mathrm{an}}_K$, which gives to the latter the structure of a $\mathbb{R}$-tree (see [Jon15, Section 3] for further details). More precisely, we say that $x \leq y \in \mathbb{A}^{1,\mathrm{an}}_K$ iff $|P(x)| \leq |P(y)|$ for all $P \in K[w]$. This order relation endows $\mathbb{A}^{1,\mathrm{an}}_K$ with a tree structure, whose minimal points consist of rigid and type 4 points.

We define the *diameter* of a point $x = \zeta(a, r)$ by $\mathrm{diam}(x) = r$ and observe that it admits a unique extension as a upper semi-continuous and nondecreasing function on $\mathbb{A}^{1,\mathrm{an}}_K$. It follows that $\mathrm{diam}(x) = 0$ if and only if x is of type 1, and if x_n decreases to x, then $\mathrm{diam}(x_n) \to \mathrm{diam}(x)$.

1.1.2. When K is not algebraically closed, let $\widehat{K}$ be the completion of the algebraic closure of K. The absolute Galois group $G = \mathrm{Gal}(K^{\mathrm{alg}}/K)$ (where K^{alg} is an algebraic closure of K) acts on $\widehat{K}$ by isometries so it preserves the radii of balls and acts naturally on $\mathbb{A}^{1,\mathrm{an}}_{\widehat{K}}$. There exists a canonical G-equivariant restriction map $\pi \colon \mathbb{A}^{1,\mathrm{an}}_{\widehat{K}} \to \mathbb{A}^{1,\mathrm{an}}_K$, which identifies $\mathbb{A}^{1,\mathrm{an}}_K$ with the quotient $\mathbb{A}^{1,\mathrm{an}}_{\widehat{K}}/G$.

The action of G preserves the type of points so that we may speak of the type of a point in $\mathbb{A}^{1,\mathrm{an}}_K$. Rigid points in $\mathbb{A}^{1,\mathrm{an}}_K$ are type 1 points defined over a finite extension of K (in general the extension $\widehat{K}/K$ is infinite).

We define the *capacity* of a point $x \in \mathbb{A}^{1,\mathrm{an}}_K$ by

$$\log \mathrm{cap}(x) = \inf \left\{ \frac{\log |P(x)|}{\deg P},\ P \in K[w] \text{ monic non-constant} \right\}. \tag{6}$$

When K is algebraically closed, the infimum above can be taken over polynomials of degree 1 and in this case we deduce that $\mathrm{cap}(x) = \mathrm{diam}(x)$ for every $x \in \mathbb{A}^{1,\mathrm{an}}_K$.

For arbitrary K, we only have $\mathrm{cap}(\zeta(a, r)) \leq r$. In particular the capacity is zero for a rigid point. The computation of the capacity is detailed below in the case of formal Laurent series.

As it is customary, we use the following notation: any $x \in \mathbb{A}^{1,\mathrm{an}}_K$ is a semi-norm $|\cdot|_x$ on $K[w]$, and we denote $|P(x)| = |P|_x$. Abusing slighlty, we write $|x|$ for $|w|_x$.

1.2. **The field of formal Laurent series and its extensions.** Let k be any algebraically closed field of characteristic 0. We endow the field $k((z))$ of formal Laurent series with the z-adic norm normalized by $|z| = e^{-1}$ so that

$$|\phi| = e^{-\mathrm{ord}_z(\phi)}, \text{ where } \mathrm{ord}_z\left(\sum_{n\in\mathbb{Z}} a_n z^n\right) = \min\{n \in \mathbb{Z},\, a_n \neq 0\}.$$

It is a complete metrized field with value group $e^{\mathbb{Z}}$, and ring of integers

$$k((z))^\circ = \{x \in k((z)),\, |x| \leq 1\} = k[[z]].$$

The algebraic closure of $k((z))$ is the field of Puiseux series

$$k\langle\langle z\rangle\rangle = \bigcup_{q\in\mathbb{N}^*} k\left(\left(z^{1/q}\right)\right)$$

This field is not complete, and its completion

$$\mathbb{L} = \left\{ \sum_{n\in\mathbb{N}} a_n z^{\beta_n},\, a_n \in k,\ \text{s.t. } (\beta_n) \in \mathbb{Q}^{\mathbb{N}},\ \forall n\ \beta_n < \beta_{n+1},\ \lim_{n\to\infty} \beta_n = \infty \right\}$$

is algebraically closed.

The absolute Galois group of $k((z))$ is isomorphic to the projective limit $\widehat{\mathbb{U}}$ of the groups of m-th root of unity $\mathbb{U}_m$ under the morphisms $\mathbb{U}_{nm} \to \mathbb{U}_m$ defined by $z \mapsto z^n$. An element $u \in \widehat{\mathbb{U}}$ is a collection of n-th root $u_n \in \mathbb{U}_n$ such that $u_{nm}^n = u_m$ for all n, m. The action of an element $u \in \widehat{\mathbb{U}}$ on z^β is given as follows: write $\beta = p/q$ with p and q coprime, and set $u \cdot z^\beta = u_q^p z^\beta$.

1.3. **The Berkovich affine line over $k((z))$.** Let us describe in more detail the structure of $\mathbb{A}^{1,\mathrm{an}}_{k((z))}$, insisting on its relation with $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{L}}$. This material is mostly taken from [FJ04, Chapter 4] (albeit with a different language).

1.3.1. *Representation by Puiseux series.* Pick any element $\phi \in \mathbb{L}$, and any $r \in [0, +\infty)$. Then we define the point $\zeta(\phi, r) \in \mathbb{A}^{1,\mathrm{an}}_{\mathbb{L}}$ by setting

$$\big|P\big(\zeta(\phi, r)\big)\big| := \prod_i \max\left\{e^{-\mathrm{ord}_z P_i(\phi(z))}, r\right\}, \tag{7}$$

where $P = \prod_i P_i \in \mathbb{L}[w]$ and $\deg(P_i) = 1$ (here, we are implicitly applying the Gauss lemma on the multiplicativity of the norm $|P|_{B(\phi,r)}$). Note that $\zeta(\phi, r)$ is of type 1 if $r = 0$; of type 2 if $r \in e^{\mathbb{Q}}$; and of type 3 if $r \notin e^{\mathbb{Q}}$. Moreover $\zeta(\phi, r) = \zeta(\phi', r')$ iff $r = r'$ and $|\phi - \phi'| \leq r$. Conversely any point in $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{L}}$ which is not of type 4 is of the form $\zeta(\phi, r)$ for some $\phi \in \mathbb{L}$ and some $r \geq 0$. Observe that we have

$$\mathrm{cap}(\zeta(\phi, r)) = \mathrm{diam}(\zeta(\phi, r)) = r \in \mathbb{R}_{\geq 0}.$$

A point x of type 4 can be obtained through series of the form $\phi = \sum_{n\in\mathbb{N}} a_n z^{\beta_n}$ where $a_n \in k^*$, $\beta_n < \beta_{n+1} \in \mathbb{Q}$, and $\beta = \lim_{n\to\infty} \beta_n < \infty$, by the analogue of the formula (7)

$$\big|P\big(\zeta(\phi, r)\big)\big| := e^{-\mathrm{ord}_z P(\phi(z))}.$$

Note that (β_n) has unbounded denominators in this case, so that $P(\phi)$ is never 0. In fact we have $\mathrm{diam}(x) = \beta$.

Pick now any point $x \in \mathbb{A}^{1,\mathrm{an}}_{k((z))}$, and suppose that it is the image of a point of type 1, 2 or 3 under $\pi\colon \mathbb{A}^{1,\mathrm{an}}_{\mathbb{L}} \to \mathbb{A}^{1,\mathrm{an}}_{k((z))}$. Then x is represented by a point $\zeta(\phi, r) \in \mathbb{A}^{1,\mathrm{an}}_{\mathbb{L}}$, for some $\phi \in \mathbb{L}$ and $r \geq 0$. The representative is unique up to the action of the absolute Galois group G. When $\phi \in k\left(\left(z^{1/q}\right)\right)$, the G-orbit of $\zeta(\phi, r)$ is the same as its orbit under the action of $\mathbb{U}_q$. If moreover $\mathrm{ord}_z(\phi) > 0$, i.e., $\phi \in k[[z^{1/q}]]$ and $\phi(0) = 0$, then ϕ is the Puiseux parametrization of a formal curve C in $\mathbb{A}^2$ that intersects the line $\{z = 0\}$ with multiplicity dividing by q and all other representatives give other Puiseux parametrizations of C. In plain words, a rigid point x is attached to a curve C while its preimages under π corresponds to its Puiseux parameterizations. We will say more about the correspondence between germs of curves and points in $\mathbb{A}^{1,\mathrm{an}}_{k((z))}$ in § 8.5.

1.3.2. *Multiplicity and capacity in the closed unit ball* $\mathbb{A}^{1,\mathrm{an}}_{k((z))}$. Pick any point x in the Berkovich closed unit ball $\overline{B}^{\mathrm{an}}(0, 1) = \{|x| \leq 1\} \subset \mathbb{A}^{1,\mathrm{an}}_{k((z))}$. We call the cardinality of $\pi^{-1}(x) \in \mathbb{A}^{1,\mathrm{an}}_{\mathbb{L}}$ the *multiplicity* of x and denote it by $m(x)$. It is finite iff x is either of type 2 or 3, or rigid. Later on, we will define another related quantity, the *generic multiplicity* $b(x)$.

Suppose $x \in \overline{B}^{\mathrm{an}}(0, 1)$ is a rigid point, so that it is represented by a Puiseux series $\phi(z) = \sum_{n\geq 0} a_n z^{n/q} \in k[[z^{1/q}]]$. Then, we have

$$m(x) = \mathrm{lcm}\left\{\frac{q}{\gcd\{n, q\}},\, a_n \neq 0\right\} \tag{8}$$

It can also be interpreted geometrically as follows. Denote by C the formal curve parameterized by $t \mapsto (t^q, \phi(t^q))$. Then $m(x) = (C \cdot \{z = 0\})$ where the latter is the local intersection product between C and the curve $z = 0$ viewed in the formal scheme $\mathrm{Spf}\, k[[z, w]]$ (beware that it is not the multiplicity of C at $(0, 0)$).

Pick any $x \in \mathbb{A}^{1,\mathrm{an}}_{k((z))}$, and choose any $y \in \mathbb{A}^{1,\mathrm{an}}_{\mathbb{L}}$ such that $\pi(y) = x$. Then we set $\mathrm{diam}(x) = \mathrm{diam}(y)$. This is independent on the choice of y since the diameter function is Galois invariant. The capacity of the point x in $\mathbb{A}^{1,\mathrm{an}}_{k((z))}$ is defined as before by

$$\log \mathrm{cap}(x) = \inf\left\{\frac{\log |P(x)|}{\deg P},\, P \in k\left((z)\right)[w] \text{ non-constant}\right\}.$$

Notice that the condition that P is monic is superfluous in this setting, since the absolute value is trivial on k (i.e., $|k^*| = 1$). It is customary to define

$$\alpha(x) = -\log \mathrm{cap}(x) \tag{9}$$

for any point $x \in \mathbb{A}^{1,\mathrm{an}}_{k((z))}$. Note that $\alpha(\zeta_g) = 0$, and α is increasing on each segment $[\zeta_g, x]$ (resp. decreasing for the order $\leq$).

Proposition 1.1. *Let* $x \in \overline{B}^{\mathrm{an}}(0, 1) \subset \mathbb{A}^{1,\mathrm{an}}_{k((z))}$ *be a point of type* 2 *or* 3*. Pick any* $\phi \in k\langle\langle z\rangle\rangle$ *such that* $\zeta(\phi, 0) < x$*. Let* $P_{\min} \in k\left((z)\right)[w]$ *be the minimal polynomial of* ϕ*.*

Then we have

$$\alpha(x) = \frac{-\log|P_{\min}(x)|}{\deg_w P_{\min}}.$$

Proof. We may suppose $\phi \in k[[z^{1/q}]]$ with $q = m(\phi)$. Then the minimal polynomial is equal to $P_{\min}(z,w) = \prod_{u^q=1}(w - u\cdot\phi)$, is monic of degree q, and belongs to $k[[z]][w]$.

The statement then follows from [FJ04, Proposition 3.25]. The latter reference is written in the context of valuations instead of norm. We refer to §1.3.3 below for a quick discussion on how to compare these point of views. □

Proposition 1.2. *Suppose $x \in \overline{B}^{\mathrm{an}}(0,1) \subset \mathbb{A}^{1,\mathrm{an}}_{k((z))}$. Then there exists a decreasing sequence of type 2 points $x_0 = \zeta_g$, $x_{n+1} < x_n$, $\lim_n x_n = x$ such that:*

(i) the multiplicity is constant equal to m_n on the interval $I_n = [x_{n+1}, x_n)$;
(ii) $-\log\frac{\mathrm{diam}(y)}{\mathrm{diam}(y')} = m_n(\alpha(y) - \alpha(y'))$ for all $y, y' \in \overline{I_n}$.

Moreover $m_1 = 1$, and m_n divides m_{n+1} for all n.

This sequence is finite if and only if x has finite multiplicity.

Proof. We only give a sketch of proof, referring to [FJ04, Chapter 5] for more details.

Suppose $x = \pi(\zeta(\phi,r))$ with $\phi = \sum_{n\in\mathbb{N}} a_n z^{\beta_n} \in \mathbb{L}^\circ$ with $a_n \in k^*$, $0 \le \beta_n < \beta_{n+1} \in \mathbb{Q}$ and $r \in [0,1]$.

Write $\beta_n = p_n/q_n$ with p_n, q_n coprime integers. We define by induction a sequence of integers n_j such that $n_0 = 0$, and n_j is the smallest integer such that q_j is not a multiple of $\mathrm{lcm}\{q_l, l \le n_j - 1\}$. Then we set $m_j = \mathrm{lcm}\{q_l, l \le n_j\}$ so that m_j is a multiple of m_{j-1}. One can check that if $\beta_{n_{j-1}} \le -\log r < \beta_{n_j}$, then

$$m\big(\pi(\zeta(\phi,r))\big) = m\Big(\pi\Big(\zeta\Big(\sum_{n<n_j} a_n z^{\beta_n}, r\Big)\Big)\Big) = m_{j-1}$$

which implies (1). Moreover the supremum of $\log|P|/\deg(P)$ on that segment is attained for the minimal polynomial of $\sum_{n<n_j} a_n z^{\beta_n}$. This implies (2). □

Since the multiplicity is increasing on $[\zeta_g, x]$, we obtain:

Corollary 1.3. *The function $-\log\mathrm{diam}|_{[x,\zeta_g]}$ is a convex function of α, or equivalently, α is a concave function of $-\log\mathrm{diam}|_{[x,\zeta_g]}$. In particular*

$$-\log\mathrm{diam} \ge \alpha$$

and if $m = m(x) < \infty$, we have $-\log\mathrm{diam} \le m(x)\alpha$ on $[x,\zeta_g]$.

The second item of Proposition 1.2 implies that the restriction of α on any segment $[y,\zeta_g]$, $y > x$ is continuous, and its concavity implies that it is continuous at x as well. From this we get a canonical parametrization $[0,\alpha(x)] \to [\zeta_g, x]$, $t \mapsto x(t)$ such that $\alpha(x(t)) = t$.

We shall also use the following

Lemma 1.4. *For any $x \leq x' \in \overline{B}^{\mathrm{an}}(0,1) \subset \mathbb{A}^{1,\mathrm{an}}_{k(\!(z)\!)}$, for any monic $P \in k[[z]][w]$ we have*

$$\frac{\alpha(x')}{\alpha(x)} \leq 1 \leq \frac{-\log|P(x)|}{-\log|P(x')|} \leq \frac{\alpha(x)}{\alpha(x')}$$

Proof. The two inequalities on the left follow from the fact that both α and $-\log|P|$ are non-increasing (with respect to the order relation). Let us fix any $\phi \in k\langle\!\langle z\rangle\!\rangle$. For any $t \in [1,+\infty)$, denote by $x(t)$ the unique point in $[\zeta_g, \phi]$ such that $\alpha(x(t)) = t$. Suppose that $x(t_0) = x'$ and $x(t_1) = x$ for some $t_0 < t_1$. It follows from [FJ04, Prop 3.25] and a decomposition into irreducible components that $t \mapsto -\log|P(x(t))|$ is concave. Since P is monic and its coefficients are formal power series in z, it follows that $-\log|P(\zeta_g)| = 0$. It follows that $-\log|P(x(t))| \leq -\frac{t}{t_0}\log|P(x(t_0))|$ which implies the claim. □

1.3.3. *Relation with valuation spaces.* In the literature [FJ07, GR14, GR21], the local analysis of super-attracting analytic dynamical maps in dimension ≥ 2 was mostly formulated in the language of valuations. For the convenience of the reader, we indicate how to translate notions introduced above in the context of the Berkovich affine line to valuation spaces, referring to [FJ04, Chapters 3 & 4] for further details.

Any seminorm $|\cdot|_x \in \mathbb{A}^{1,\mathrm{an}}_K$ defines a semivaluation $\nu_x(\cdot) = -\log|\cdot|_x$ over the ring $K[w]$. When $K = \mathbb{L}$, we get a space of semivaluations over $\mathbb{L}[w]$ which extend the z-adic valuation, defined by

$$\mathrm{ord}_z\left(\sum_{n\in\mathbb{N}} a_n z^{\beta_n}\right) = \min\{\beta_n, a_n \neq 0\}.$$

on $\mathbb{L}^*$ and by $\mathrm{ord}_z(0) = +\infty$. Any such valuation ν_x extends uniquely to a semivaluation, again denoted by ν_x, over $\mathbb{L}[[w]]$ (and all semivaluations over $\mathbb{L}[[w]]$ come from the ones in $\mathbb{L}[w]$). Valuations associated to series $\phi \in \mathbb{L}$ such that $\mathrm{ord}_z(\phi) > 0$ are called *centered*, and the corresponding subset is denoted by $\widehat{\mathcal{V}}^*_z$. They correspond to the Berkovich open unit ball $B^{\mathrm{an}}(0,1) = \{|x| < 1\}$ in $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{L}}$. By adding the non-centered valuation ord_z, corresponding to the Gauss point ζ_g, we get the valuation space $\widehat{\mathcal{V}}_z$.

The absolute Galois group G of $k(\!(z)\!)$ acts on $\widehat{\mathcal{V}}^*_z$, and its quotient $\mathcal{V}^*_z = \widehat{\mathcal{V}}^*_z/G$ can be identified as the set of centered semivaluations over $k(\!(z)\!)[w]$ extending the z-adic valuation on $k(\!(z)\!)[w]$, or equivalently with the set of centered semivaluations over $k[[z,w]]$ that extend the trivial vauation on k, and normalized by the condition $\nu(z) = 1$. These valuations correspond to the Berkovich open unit ball $B^{\mathrm{an}}(0,1)$ in $\mathbb{A}^{1,\mathrm{an}}_{k(\!(z)\!)}$. Again, the whole (normalized) valuation space is given by $\mathcal{V}_z = \mathcal{V}^*_z \sqcup \{\mathrm{ord}_z\}$. This space is also known as the *non-Archimedean link* of $(\mathbb{A}^2_k, 0)$, see [Fan18].

The value $1 - \log\mathrm{cap}(x) = 1 - \log\mathrm{diam}(x)$ of $x = \zeta(\phi, r) \in B(0,1) \subset \mathbb{A}^{1,\mathrm{an}}_{\mathbb{L}}$ is often denoted by $A_z(\nu_x)$, and known as the *relative thinness*, or *relative log-discrepancy* of the valuation $\nu_x \in \mathcal{V}_z$.

For the capacity $\mathrm{cap}(x)$ of $x \in B^{\mathrm{an}}(0,1) \subset \mathbb{A}^{1,\mathrm{an}}_{k(\!(z)\!)}$, we have

$$\alpha_z(\nu_x) := -\log\mathrm{cap}(x) = \sup\left\{\frac{\nu_x(P)}{\deg P}\right\}.$$

Here $\deg P$ is the degree with respect to the w coordinate, which coincides with the intersection multiplicity of $(P = 0)$ with the line $(z = 0)$; this quantity is known as the *relative skewness*, or as the *relative Izumi constant* of $\nu_x \in \mathcal{V}_z$. This quantity is often used to parametrize valuations, in the sense that for any $\phi \in \widehat{\mathbb{L}}$, and for any $t > 0$, there exists a unique semivaluation $\nu_x \in \mathcal{V}_z$ with $x = \pi\big(\zeta(\phi, r)\big)$ and such that $\alpha_z(\nu_x) = t$. This semivaluation is often denoted by $\nu_{\phi,t}$. For convenience in this paper we use the same notation $A(\cdot)$ and $\alpha(\cdot)$ for the corresponding quantities.

When $x \in B^{\mathrm{an}}(0, 1) \subset \mathbb{A}^{1,\mathrm{an}}_{k(\!(z)\!)}$, the sequence $(x_n)_n$ given by Proposition 1.2 corresponds to the approximating sequence in the valuation setting, see [FJ04, Section 3.5].

2. Dynamics of polynomial skew products with small relative degree

2.1. Polynomial skew products and their actions on the Berkovich line. In this section, we fix once for all an algebraically closed field k of characteristic 0 and consider a map of the form

$$f(z, w) = \left(z^d, w^c + \sum_{j=0}^{c-1} h_j(z) w^j\right) \tag{2}$$

where $d, c \geq 2$ and $h_j \in k[[z]]$ satisfy $h_j(0) = 0$. We will often work in the *small relative degree* regime, that is: $c < d$.

Such a map naturally induces a self-map $f_\diamond \colon \mathbb{A}^{1,\mathrm{an}}_{k(\!(z)\!)} \to \mathbb{A}^{1,\mathrm{an}}_{k(\!(z)\!)}$ as follows. Observe first that thanks to the non-Archimedean property, for any point $x \in \mathbb{A}^{1,\mathrm{an}}_{k(\!(z)\!)}$, the assignment $P \mapsto |(P \circ f)(x)|^{1/d}$ defines a multiplicative semi-norm on $k\,(\!(z)\!)\,[w]$ (where P is viewed as a function of (z, w)). In addition if P has degree 0, $|(P \circ f)(x)| = |z|^d$, so we conclude that $P \mapsto |(P \circ f)(x)|^{1/d}$ belongs to $\mathbb{A}^{1,\mathrm{an}}_{k(\!(z)\!)}$. Thus we may define a map $f_\diamond : \mathbb{A}^{1,\mathrm{an}}_{k(\!(z)\!)} \to \mathbb{A}^{1,\mathrm{an}}_{k(\!(z)\!)}$ by setting

$$|P(f_\diamond(x))| := |(P \circ f)(x)|^{1/d}, \text{ for any } P \in k\,(\!(z)\!)\,[w]. \tag{10}$$

It is also convenient to work over $\mathbb{L}$ since the latter is algebraically closed and complete. For this, it is enough to observe that (10) also defines a map $f_\diamond \colon \mathbb{A}^{1,\mathrm{an}}_{\mathbb{L}} \to \mathbb{A}^{1,\mathrm{an}}_{\mathbb{L}}$ such that $f_\diamond \circ \pi = \pi \circ f_\diamond$ where $\pi \colon \mathbb{A}^{1,\mathrm{an}}_{\mathbb{L}} \to \mathbb{A}^{1,\mathrm{an}}_{k(\!(z)\!)}$ denotes the canonical projection.

Lemma 2.1. *For any $\phi \in \mathbb{L}$, we have*

$$f_\diamond(\phi) = \Big(\phi\big(z^{1/d}\big)\Big)^c + \sum_{j=0}^{c-1} h_j\big(z^{1/d}\big)\ \Big(\phi\big(z^{1/d}\big)\Big)^j. \tag{11}$$

Proof. Indeed, since $\mathbb{L}$ is algebraically closed $f_\diamond(\phi)$ is determined by its value on linear polynomials $w - \psi(z)$ with $\psi \in \mathbb{L}$, and for such a polynomial P we have

$$\begin{aligned} |P(f_\diamond(\phi))| = |(P \circ f)(\phi)|^{1/d} &= \left| \phi^c(z) + \sum_{j=0}^{c-1} h_j(z)\phi^j(z) - \psi(z^d) \right|_{\mathbb{L}}^{1/d} \\ &= \left| \phi^c\left(z^{1/d}\right) + \sum_{j=0}^{c-1} h_j(z)\phi^j\left(z^{1/d}\right) - \psi(z) \right|_{\mathbb{L}}, \end{aligned}$$

thereby proving the claim. □

2.2. **Structure of the map $f_\diamond$.**

Theorem 2.2. *Let f be a polynomial map of the form (2). The map $f_\diamond\colon \mathbb{A}^{1,\mathrm{an}}_{k((z))} \to \mathbb{A}^{1,\mathrm{an}}_{k((z))}$ (resp. $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{L}} \to \mathbb{A}^{1,\mathrm{an}}_{\mathbb{L}}$) is open, continuous and proper. It preserves the type of points, the set of rigid points, and the preimage of any point is finite of cardinality at most cd (resp. at most c in $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{L}}$).*

To prove the theorem, we introduce the map $\tau = \tau_d\colon \mathbb{A}^{1,\mathrm{an}}_{\mathbb{L}} \to \mathbb{A}^{1,\mathrm{an}}_{\mathbb{L}}$ defined by

$$\tau\colon x \longmapsto \left(P \mapsto |P(\tau(x))| := \left| P(z^d, w)(x) \right|^{1/d} = \left| P(z^d, w) \right|_x^{1/d} \right). \tag{12}$$

for any $x \in \mathbb{A}^{1,\mathrm{an}}_{\mathbb{L}}$ and any $P \in \mathbb{L}[w]$, (where we interpret P as a series in both variables z and w). In other words, with notation as above, if we put $h(z, w) = (z^d, w)$, then $\tau = h_\diamond$.

Lemma 2.3. *The map τ defines a homeomorphism on $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{L}}$ which preserves the types of points.*

Proof. It is immediate that τ^{-1} is given by

$$\left|P(\tau^{-1}(x))\right| = \left| P\left(z^{1/d}, w\right)(x) \right|^d \text{ for any } P \in \mathbb{L}[w],$$

and the continuity of τ and τ^{-1} follows immediately from the definitions. Lemma 2.1 applied to $h(z, w) = (z^d, w)$ gives $\tau(\phi) = \phi(z^{1/d})$ for any $\phi \in \mathbb{L}$. It follows that type 1 points, and Puiseux series are preserved. Since $|\tau(\phi) - \tau(\psi)| = |\phi - \psi|^{1/d}$, and τ is a homeomorphism, we also have $\tau(B(\phi, r)) = B(\tau(\phi), r^{1/d})$ which implies that τ preserves type 2 and 3 points, hence also type 4 points. □

Remark 2.4.

(1) Since τ restricts to $\mathbb{L}$ as the field automorphism $\phi(z) \mapsto \phi(z^{1/d})$, the notation is consistent with that of [NZ24].

(2) The definition of τ in (12) makes sense on $\mathbb{A}^{1,\mathrm{an}}_{k((z))}$ but it does not define a homeomorphism there. For instance the Puiseux series $\pm z^{1/2}$ parameterize the same curve $w^2 = z$ but their preimages under τ_2 correspond to the two distinct curves $w = \pm z$.

Lemma 2.5. *In $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{L}}$ we have the decomposition $f_\diamond = \hat{f}\circ\tau$ where $\hat{f}$ is the polynomial map*

$$\hat{f}(w) = w^c + \sum_{j=0}^{c-1} h_j\big(z^{1/d}\big)w^j.$$

Proof. Indeed, a direct computation gives

$$\begin{aligned}\left|P\left(\hat{f}(\tau(x))\right)\right| &= \left|P\bigg(z, w^c + \sum_{j=0}^{c-1} h_j\big(z^{1/d}\big)w^j\bigg)(\tau(x))\right| \\ &= \left|P\bigg(z^d, w^c + \sum_{j=0}^{c-1} h_j(z)w^j\bigg)(x)\right|^{1/d} = |P(f_\diamond(x))|\ ,\end{aligned}$$

and we are done. □

Proof of Theorem 2.2. From the basic properties of the action of a polynomial map on the Berkovich affine line (see, e.g., [Ben19]), we infer that $\hat{f}$ is a open, continuous, proper, c-to-one map on $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{L}}$, and that it preserves the type of points. By composition, we conclude that $f_\diamond$ is a open, proper, continuous, c-to-one map on $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{L}}$ that preserves the type of points and rigid points.

The continuity of $f_\diamond$ on $\mathbb{A}^{1,\mathrm{an}}_{k((z))}$ follows directly from formula (10) and the definitions. Since $\pi\colon \mathbb{A}^{1,\mathrm{an}}_{\mathbb{L}} \to \mathbb{A}^{1,\mathrm{an}}_{k((z))}$ is proper (resp. open), the properness (resp. openness) of $f_\diamond$ on $\mathbb{A}^{1,\mathrm{an}}_{k((z))}$ follows from the one of $f_\diamond$ on $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{L}}$.

To see that $f_\diamond$ is (at most) cd-to-one on $\mathbb{A}^{1,\mathrm{an}}_{k((z))}$, it is sufficient to prove that any rigid point has at most cd preimages, since $f_\diamond$ is open and the set of rigid points is dense.

So let $x \in \mathbb{A}^{1,\mathrm{an}}_{k((z))}$ be a rigid point. We only treat the case where $|x| \le 1$. When $|x| > 1$, one needs to work in the coordinates w^{-1}. Geometrically, x is given by a formal germ of irreducible curve we denote by C. It is determined by some Puiseux series $\phi(z) = \sum_{i\ge 0} a_i z^{i/m}$ where $m = \gcd\{i, a_i \neq 0\}$ is the intersection number $C\cdot(z=0)$. A parameterization of C is given by $t \mapsto (t^m, \phi(t^m))$. An equation for C is given by

$$P(z,w) = \prod_{i=1}^{m} (w - \zeta^i\cdot\phi)$$

where ζ is a primitive m-th root of unity and $\zeta\cdot\phi$ is defined as in § 1.2.

Any preimage y_i of x is also a rigid point hence it is determined by a Puiseux series ψ_i of multiplicity m_i, and an equation for the associated branch D_i is denoted by P_i. Since $f_\diamond(y_i) = x$, P_i divides $P\circ f$. We let $r_i = \mathrm{ord}_{P_i}(P\circ f) \in \mathbb{N}^*$. Also, $f(t^{m_i}, \psi(t^{m_i})) = (t^{dm_i}, \phi(t^{dm_i}))$, and $dm_i = e_i m$ for some $e_i \in \mathbb{N}^*$.

Let $\{y_i\}_{i\in I}$ be the set of preimages of x in $\mathbb{A}^{1,\mathrm{an}}_{k((z))}$. We compute the local intersection product

$$(f^*C\cdot f^*\{z=0\})_0 = \sum r_i D_i \cdot d\{z=0\} = \sum d r_i m_i$$

As the left hand side is equal to $cd(C \cdot [z=0])_0 = cdm$, we obtain

$$\sum_I r_i e_i = cd \ , \tag{13}$$

therefore $\# I \leq cd$ and the proof is complete. □

2.3. **Basic dynamical properties of $f_\diamond$.** A closed (resp. open) ball in the Berkovich affine line over $\mathbb{L}$ is defined as $\overline{B}^{\mathrm{an}}(\phi, r) := \{x,\, |(w-\phi(z))(x)| \leq r\}$ (resp. $B^{\mathrm{an}}(\phi, r) := \{x,\, |(w-\phi(z))(x)| < r\}$) for some $\phi \in \mathbb{L}$ and $r \geq 0$. Observe that $\overline{B}^{\mathrm{an}}(\phi, r) \cap \mathbb{L} = \overline{B}(\phi, r)$, and $B^{\mathrm{an}}(\phi, r) \cap \mathbb{L} = B(\phi, r)$,

A ball in $\mathbb{A}^{1,\mathrm{an}}_{k((z))}$ is the image under the canonical projection $\pi \colon \mathbb{A}^{1,\mathrm{an}}_{\mathbb{L}} \to \mathbb{A}^{1,\mathrm{an}}_{k((z))}$ of a ball in $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{L}}$. We abuse notation and identify $\overline{B}^{\mathrm{an}}(\phi, r)$ and $B^{\mathrm{an}}(\phi, r)$ with their images in $\mathbb{A}^{1,\mathrm{an}}_{k((z))}$. Observe that the preimage of a ball under π is a union of balls.

Theorem 2.6. *Let f be a polynomial map of the form (2), with $2 \leq c < d$. The map $f_\diamond \colon \mathbb{A}^{1,\mathrm{an}}_{\mathbb{L}} \to \mathbb{A}^{1,\mathrm{an}}_{\mathbb{L}}$ (resp. $\mathbb{A}^{1,\mathrm{an}}_{k((z))} \to \mathbb{A}^{1,\mathrm{an}}_{k((z))}$) has the following properties.*

(i) The Gauss point ζ_g is totally invariant, and $f_\diamond \colon B^{\mathrm{an}}(0,1) \to B^{\mathrm{an}}(0,1)$ is proper and order preserving.
(ii) The ball $B^{\mathrm{an}}_\infty = \{|x| > 1\}$ is totally invariant, and $f^n(x) \to \zeta_g$ for any $x \in B^{\mathrm{an}}_\infty$.
(iii) The open unit ball is totally invariant.
(iv) For any $a \in k^*$, the map $f_\diamond$ maps $B^{\mathrm{an}}(a,1)$ onto $B^{\mathrm{an}}(a^c,1)$ and $f_\diamond \colon B^{\mathrm{an}}(a,1) \to B^{\mathrm{an}}(a^c,1)$ is proper. In addition, over $\mathbb{L}$, it is a homeomorphism.
  - *When $a \in k^*$ is not a root of unity, then $f^n_\diamond(x) \to \zeta_g$ for any point $x \in B^{\mathrm{an}}(a,1)$.*
  - *When $a \in k^*$ is a root of unity, then $B^{\mathrm{an}}(a,1)$ is eventually mapped to a periodic ball B^{an} which contains a unique periodic cycle lying in $k[[z]]$. Any point in $B^{\mathrm{an}}(a,1)$ is either eventually mapped to this cycle, or converges under iteration to ζ_g.*

Proof. From (11), we infer

$$|f_\diamond(\phi)| = \left| \phi^c\left(z^{1/d}\right) + \sum_{j=0}^{c-1} h_j\left(z^{1/d}\right) \phi^j\left(z^{1/d}\right) \right| = |\phi|^{c/d} \tag{14}$$

for all $\phi \in \mathbb{L}$ such that $|\phi| > 1$ hence for all $|\phi| \geq 1$ by continuity. Conversely, when $|\phi| < 1$ we have $|f_\diamond(\phi)| \leq \max\{|\phi|^{1/d}, |h_0|\} < 1$, and likewise if $|\phi| \leq 1$ then $|f_\diamond(\phi)| \leq 1$. This implies that $B^{\mathrm{an}}(0,1)$ is invariant and $B^{\mathrm{an}}_\infty = \{|x| > 1\}$ is totally invariant. Since by assumption $c/d < 1$, it also follows that $|f^n_\diamond(\phi)| = |\phi|^{(c/d)^n} \to 1$ for all $\phi \in B^{\mathrm{an}}_\infty$, which proves (ii) in $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{L}}$. Since ζ_g is the unique boundary point of B_∞ and $f_\diamond$ is open, we infer that ζ_g is totally invariant too. As $\pi^{-1}(\zeta_g) = \zeta_g$, the Gauss point is also totally invariant and (ii) holds in $\mathbb{A}^{1,\mathrm{an}}_{k((z))}$.

The properness of $f_\diamond \colon B^{\mathrm{an}}(0,1) \to B^{\mathrm{an}}(0,1)$ follows from the fact that it extends continuously to the closure $B^{\mathrm{an}}(0,1) \cup \{\zeta_g\}$ of the open unit ball and that $f_\diamond(\zeta_g) = \zeta_g$. Suppose that $x \leq x' \in B^{\mathrm{an}}(0,1)$ and pick any $P \in k\left((z)\right)[w]$ (or in $\mathbb{L}[w]$). Since z belongs to the base field $k(\!(z)\!)$, we have $|z(x)| = |z(x')|$, so we may multiply P by a suitable power of z and assume that $P \in k[[z]][w]$. In that case

$$|P(f_\diamond(x))| = |(P \circ f)(x)|^{1/d} \leq |(P \circ f)(x')|^{1/d} = |P(f_\diamond(x))|$$

hence $f_\diamond(x) \leq f_\diamond(x')$, and assertion (i) is established (both over $\mathbb{L}$ and $k(\!(z)\!)$).

Suppose now that $\phi \in B(a,1) \subset \mathbb{L}$ for some $a \in k$. Then, writing $a(z) = \sum a_n z^{\beta_n}$, we get $\beta_0 = 0$, $a_0 = a$, and since $h_j(0) = 0$ we get

$$f_\diamond(\phi) = \phi^c\big(z^{1/d}\big) + \sum_{j=0}^{c-1} h_j\big(z^{1/d}\big)\,\phi^j\big(z^{1/d}\big) = a^c + \sum_{n \geq 1} b_j z^{\beta'_j}$$

where $b_j \in k$ and $\beta'_j \in \mathbb{Q}_+^*$. This proves that $f_\diamond(B^{\mathrm{an}}(a,1)) \subset B^{\mathrm{an}}(a^c,1)$ for all $a \in k$, which, together with the total invariance of $\{|x| > 1\}$, implies that $B^{\mathrm{an}}(0,1)$ is totally invariant.

We suppose now that $a \in k^*$. Since ζ_g is the unique boundary point of $B^{\mathrm{an}}(a,1)$, $f_\diamond(\zeta_g) = \zeta_g$, and $f_\diamond$ is continuous we get that $f_\diamond \colon B^{\mathrm{an}}(a,1) \to B^{\mathrm{an}}(a^c,1)$ is proper. We claim that $f_\diamond \colon B^{\mathrm{an}}(a,1) \to B^{\mathrm{an}}(a^c,1)$ is a homeomorphism when working in $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{L}}$. The easiest way to see this is to pre- and post-compose f by translations $T_1(w) = w + a$, and $T_2(w) = w - a^c$ so that we are reduced to analyzing $g_\diamond$ on the open unit ball, where g is defined by

$$g(z,w) = T_2 \circ f \circ T_1(z,w) = f(z, w+a) - (0, a^c) = \left(z^d, cw + \psi(z) + \sum_{j=2}^{c} g_j(z) w^j\right)$$

for some $\psi, g_j \in k[[z]]$ with $\psi(0) = 0$. Observe that the translation by ψ is a homeomorphism so that we can assume that $\psi = 0$. One can then write g as a composition $g = \gamma_1 \circ \gamma_2$ where $\gamma_1(z,w) = (z^d, w)$ and $\gamma_2(z,w) = (z, cw(1 + \sum_{j=1}^{c-1} c^{-1} g_j(z) w^j))$. Observe that γ_2 induces an analytic isomorphism on the Berkovich unit ball (see, e.g., [Ben19, Proposition 3.22] or § 8.1 below), and $(\gamma_1)_\diamond = \tau$ is a homeomorphism on $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{L}}$ preserving the unit ball. We conclude that $g_\diamond = (\gamma_1)_\diamond \circ \gamma_2$ is a homeomorphism of the open unit ball and the first assertion of (iv) follows over $\mathbb{L}$. Over $k(\!(z)\!)$, this still shows that $f_\diamond \colon B^{\mathrm{an}}(a,1) \to B^{\mathrm{an}}(a^c,1)$ is proper and surjective.

If $a \in k^*$ is not a root of unity, the open balls $B^{\mathrm{an}}(f_\diamond^n(a),1)$ are disjoint and the properties of the Berkovich topology imply that for any $x \in B^{\mathrm{an}}(a,1)$, $f_\diamond^n(x) \to \zeta_g$.

Finally, suppose that a is a root of unity, and let n_0 be such that $a^{c^{n_0}}$ is periodic under $w \mapsto w^c$, of period ℓ. Let $B^{\mathrm{an}} = B^{\mathrm{an}}(a^{c^{n_0}},1)$. Then $f^{n_0}(B^{\mathrm{an}}(a,1)) = B^{\mathrm{an}}$, and B^{an} is periodic of period ℓ. Replacing f by a suitable iterate, we may assume that $B^{\mathrm{an}} = B^{\mathrm{an}}(b,1)$ and $\ell = 1$. Observe that all iterates of f bear the same form (2) with c,d replaced by c^n and d^n. We then conjugate by a translation $T(w) := w + b$ so that $g := T^{-1} \circ f \circ T$ has the form

$$g(z,w) = (z^d, cw + O(z, w^2))$$

In this situation, [DFR23, Theorem 2.1] applies, and we infer the existence of a formal power series ϕ vanishing at zero fixed and by g. Translating again by this formal power series, we may assume that $\phi = 0$, which means that $g(z,w) = (z^d, cw(1+O(z,w)))$. A direct computation shows that

$$g_\diamond(az^\alpha + o(z^\alpha)) = acz^{\alpha/d} + o(z^{\alpha/d})$$

for any $a \in k^*$ and $\alpha > 0$, and since $g_\diamond$ is order preserving, this implies $g_\diamond^n(x) \to \zeta_g$ for all $x \in B^{\mathrm{an}} \setminus \{0\}$. Thus, the proof of (iv) is complete, and we are done. □

3. The invariant Cantor set $\mathcal{K}$ and measure μ_{na}

We work under the same assumptions as in the previous section with a map of the form

$$f(z,w) = \left(z^d, w^c + \sum_{j=0}^{c-1} h_j(z) w^j\right), \tag{2}$$

with $d > c \geq 2$, and $h_j(0) = 0$, together with its associated map $f_\diamond$. We focus our attention to the restriction of $f_\diamond$ to the open unit ball $B^{\mathrm{an}}(0,1) \subset \mathbb{A}^{1,\mathrm{an}}_{k((z))}$.

3.1. **The Jacobian formula (I).**

The Jacobian of f is equal to

$$\mathrm{Jac}(f) = dz^{d-1}\,\mathrm{Jac}_w(f)$$

where

$$\mathrm{Jac}_w(f) = cw^{c-1} + \sum_{j=1}^{c-1} jh_j(z)w^{j-1} = c\prod_{j=1}^{c-1}\big(w - c_j(z)\big)\ ,$$

where the Puiseux series $c_j \in k\langle\langle z\rangle\rangle$ are parametrizations of the critical branches of f (other than the branch $(z=0)$). We let $\mathcal{C} = \{c_1, \ldots, c_{c-1}\}$ be the set of critical branches.

The following Jacobian formula is crucial for our investigation.

Lemma 3.1. *For any $x \in B^{\mathrm{an}}(0,1)$ we have*

$$-\log\mathrm{diam}(f_\diamond(x)) = \frac{1}{d}\big(-\log\mathrm{diam}(x) - \log|\mathrm{Jac}_w(f)(x)|\big)\ . \tag{15}$$

Proof. Since the diameter function is Galois equivariant, we can work in $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{L}}$. We use the decomposition $f_\diamond = \hat{f} \circ \tau$ as in the proof of Theorem 2.2. Since $\tau(B(\phi,r)) = B(\tau(\phi), r^{1/d})$, we have $\mathrm{diam}(\tau(x))^d = \mathrm{diam}(x)$. The map $\hat{f} = w^c + \sum_j h_j(z^{1/d})w^j$ is a polynomial function on the Berkovich space, and $\mathbb{L}$ has residual characteristic 0 so that

$$\mathrm{diam}(\hat{f}(x)) = \mathrm{diam}(x) \times |\hat{f}'(x)|$$

by [FRL25, Proposition 3.4], where $\hat{f}'$ is the derivative with respect to w. It follows that

$$\mathrm{diam}(f_\diamond(x)) = \mathrm{diam}(\hat{f}(\tau(x))) = \mathrm{diam}(\tau(x)) \times |\hat{f}'(\tau(x))| = \mathrm{diam}(x)^{1/d} \times |\mathrm{Jac}_w(f)(x)|^{1/d},$$

thereby proving (15). □

3.2. **The invariant Cantor set.** The next theorem describes the topological dynamics of $f_\diamond$ in $B^{\mathrm{an}}(0,1)$ both in $\mathbb{A}^{1,\mathrm{an}}_{k((z))}$ and $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{L}}$. Let us introduce the sets

$$\mathcal{K} := \{x \in B^{\mathrm{an}}(0,1) \subset \mathbb{A}^{1,\mathrm{an}}_{k((z))},\ f^n_\diamond(x) \text{ does not converge to } \zeta_g\}$$

and

$$\mathcal{K}_{\mathbb{L}} := \{x \in B^{\mathrm{an}}(0,1) \subset \mathbb{A}^{1,\mathrm{an}}_{\mathbb{L}},\ f^n_\diamond(x) \text{ does not converge to } \zeta_g\}.$$

Theorem 3.2. *Let f be a polynomial map of the form (2), with $2 \le c < d$. Then $\mathcal{K}_{\mathbb{L}} = \pi^{-1}(\mathcal{K})$, and both sets are compact totally invariant subsets included in $\overline{B}^{\mathrm{an}}(0,\rho_0)$ for some $\rho_0 < 1$. The set $\mathcal{K}_{\mathbb{L}}$ (resp. $\mathcal{K}$) consists of type 1 points (resp. type 1 points such that $\alpha = \infty$).*

We have the following dichotomy: either $\mathcal{K}$ and $\mathcal{K}_{\mathbb{L}}$ are both Cantor sets, or they are both reduced to a single point, that corresponds to a totally invariant smooth (formal) curve transverse to $(z = 0)$. In the latter case, f is of the form $f(z,w) = (z^d, (w + \phi(z))^c - \phi(z^d))$ for some $\phi \in k[[z]]$.

Remark 3.3. Recall that a type 1 point $x \in \mathbb{A}^{1,\mathrm{an}}_{\mathbb{L}}$ is characterized by the condition

$$-\log \mathrm{diam}(x) = \infty.$$

If ϕ is a Puiseux series, then $\alpha(\phi) = \infty$ as well (see, e.g., Corollary 1.3). When $\phi \in \mathbb{L} \setminus k\langle\langle z\rangle\rangle$, letting (x_n) be the sequence of points given by Proposition 1.2, the condition $\alpha = \infty$ is equivalent to

$$\sum_{n\ge 0} \frac{1}{m_n} \log \frac{\mathrm{diam}(x_n)}{\mathrm{diam}(x_{n+1})} = \infty\ .$$

Proof. We first work in $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{L}}$. Set $\rho_0 := \max_{0\le j\le c-1} |h_j|^{1/(c-j)} < 1$. Then, if $\phi \in \mathbb{L}$ is such that $\rho_0 < |\phi| < 1$, we get $|\phi|^c > \max_{0\le j\le c-1}\{|h_j||\phi|^j\}$, so that

$$|f_\diamond(\phi)| = \left|\phi^c\left(z^{1/d}\right) + \sum_{j=0}^{c-1} h_j\left(z^{1/d}\right)\phi^j\left(z^{1/d}\right)\right| = |\phi|^{c/d}. \tag{16}$$

It follows that $U := \{\rho_0 < |x| < 1\} \subset B^{\mathrm{an}}(0,1)$ is a $f_\diamond$-invariant open set and $f^n_\diamond(x) \to \zeta_g$ for any $x \in U$. We thus define

$$\mathcal{K}_{\mathbb{L}} := \bigcap_{n\ge 0} \Big(B^{\mathrm{an}}(0,1) \setminus f^{-n}_\diamond(U)\Big).$$

It is a compact totally invariant set, and $x \in \mathcal{K}_{\mathbb{L}}$ if and only if $f^n_\diamond$ does not converge to ζ_g. Since $\pi^{-1}\{\rho_0 < |x| < 1\} = \{\rho_0 < |x| < 1\}$, we also get that $x \in \pi(\mathcal{K}_{\mathbb{L}})$ iff $|f^n_\diamond(x)| \le \rho_0$ for all n, and $\mathcal{K}_{\mathbb{L}} = \pi^{-1}(\mathcal{K})$.

We now proceed to show that $\mathrm{diam}(x) = 0$ for any $x \in \mathcal{K}_{\mathbb{L}}$. In particular x must be of type 1.

Lemma 3.4. *For any $x \le x' \in B^{\mathrm{an}}(0,1) \subset \mathbb{A}^{1,\mathrm{an}}_{\mathbb{L}}$ of positive diameter, we have*

$$\left|\log \frac{|\,\mathrm{Jac}_w(f)(x')|}{|\,\mathrm{Jac}_w(f)(x)|}\right| \le (c-1) \times \log \frac{\mathrm{diam}(x')}{\mathrm{diam}(x)}\ . \tag{17}$$

Proof. By continuity, we may suppose that $x = \zeta(\phi, r)$ and $x' = \zeta(\phi, r')$ with $\phi \in \mathbb{L}$ and $r \leq r' < 1$. Write $\mathrm{Jac}_w(f) = c\prod_{j=1}^{c-1}(w - c_j(z))$. Then

$$\log \frac{|\mathrm{Jac}_w(f)(x')|}{|\mathrm{Jac}_w(f)(x')|} = \log \frac{\prod_j \max\{|\phi - c_j|, r'\}}{\prod_j \max\{|\phi - c_j|, r\}} \leq (c-1)\log \frac{r'}{r}.$$

Indeed to justify the inequality, we split the products according to the position of $|\phi - c_j|$ with respect to r': if $|\phi - c_j| \leq r'$ then $\frac{\max\{|\phi - c_j|, r'\}}{\max\{|\phi - c_j|, r\}} \leq \frac{r'}{r}$ and otherwise this fraction equals 1. The desired inequality (17) follows. $\square$

Now, let $x \in B^{\mathrm{an}}(0,1)$ be of finite diameter, and observe that $[x, \zeta_g) \cap U$ is non-empty. Pick any $x' \in I \cap U$, so that $\mathrm{diam}(f_\diamond^n(x')) \to 1$ as $n \to \infty$. Then for every $n \geq 1$, Lemma 3.1 (applied to f^n) yields:

$$\begin{aligned}\left|\log \frac{\mathrm{diam}(f_\diamond^n(x'))}{\mathrm{diam}(f_\diamond^n(x))}\right| &\leq \frac{1}{d^n}\left|\log \frac{\mathrm{diam}(x')}{\mathrm{diam}(x)}\right| + \frac{1}{d^n}\left|\log \frac{|\mathrm{Jac}_w(f^n)(x')|}{|\mathrm{Jac}_w(f^n)(x)|}\right| \\ &\leq \frac{1 + (c^n - 1)}{d^n}\left|\log \frac{\mathrm{diam}(x')}{\mathrm{diam}(x)}\right| \longrightarrow 0\end{aligned}$$

which implies $x' \notin \mathcal{K}_{\mathbb{L}}$, as announced.

Note that $\mathrm{diam}(\pi(x)) = \mathrm{diam}(x)$, hence $\mathcal{K} \subset \{\mathrm{diam} = 0\}$ as well. We now work in $\mathbb{A}^{1,\mathrm{an}}_{k((z))}$ and prove $\alpha(x) = \infty$ for every $x \in \mathcal{K}$. We rely on the following lemma.

Lemma 3.5. *For any $x \leq x' \in B^{\mathrm{an}}(0,1) \subset \mathbb{A}^{1,\mathrm{an}}_{k((z))}$, such that $\alpha(x) < \infty$ and for all $n \in \mathbb{N}$, we have*

$$\frac{\alpha(x')}{\alpha(x)} \leq \frac{\alpha(f_\diamond^n(x'))}{\alpha(f_\diamond^n(x))} \leq \frac{\alpha(x)}{\alpha(x')}\ . \tag{18}$$

Proof. By continuity, we may suppose that x and x' are of type 2. Since $f_\diamond^n$ preserves the order, we have $f_\diamond^n(x') \leq f_\diamond^n(x)$ hence there exists $P \in k[[z]][w]$ monic of degree ℓ such that

$$\alpha(f_\diamond^n(x')) = -\frac{1}{\ell}\log|P(f_\diamond^n(x'))| = -\frac{1}{\ell d^n}\log|(P \circ f^n)(x'))|$$

and similarly $\alpha(f_\diamond^n(x)) = -\frac{1}{\ell d^n}\log|(P \circ f^n)(x))|$. We conclude using Lemma 1.4. $\square$

Suppose now that $x \leq \zeta_g$, and $\alpha(x) < \infty$, and recall that $\alpha(\zeta_g) = 0$. Viewing the relation (18) as an equicontinuity property, we infer that the set of points in $[x, \zeta_g]$ lying in the basin of attraction of ζ_g is both open and closed. This entails that $f_\diamond^n(x) \to \zeta_g$ as required.

It remains to prove the dichotomy. Note that, even if for convenience we drop the 'an' superscript, all the balls we consider in the proof are balls in the affine Berkovich line. Let us first work over $\mathbb{A}^{1,\mathrm{an}}_{k((z))}$. To do this, consider the closed ball $\overline{B}_{0,0} := \overline{B}(0, \rho_0)$. It follows from the previous arguments that $\mathcal{K} = \bigcap_n f_\diamond^{-n}(\overline{B}_{0,0})$. Since $f_\diamond$ is finite, type preserving and order-preserving, $f_\diamond^{-n}(\overline{B}_{0,0})$ is a finite union of disjoint closed balls $\overline{\mathcal{B}}_n = \{\overline{B}_{n,i}\}$ such that $f_\diamond^n(\overline{B}_{n,i}) = \overline{B}_{0,0}$ for all n and i.

For any $\phi \in \mathcal{K}$, consider the sequence of balls $\overline{B}_n(\phi) \in \overline{\mathcal{B}}_n$ containing ϕ. It is decreasing hence its intersection $\bigcap_n \overline{B}_n(\phi)$ is included in $\mathcal{K}$ and contains ϕ. It is equal to $\{\phi\}$ because $\mathcal{K}$ contains only type 1 points.

Suppose now that there exists an integer n_0 such that $\#\overline{\mathcal{B}}_{n_0} \geq 2$. We shall prove that $\mathcal{K}$ is a Cantor set. To do so we need to show that $\mathcal{K}$ is a compact, totally disconnected and perfect set. The first property is clear. Pick any point $\phi \in \mathcal{K}$. Then for any n the finite collection of balls $\overline{\mathcal{B}}_n = \{\overline{B}_{n,i}\}$ covers $\mathcal{K}$. The function χ taking value 1 on $\overline{B}_n(\phi) \cap \mathcal{K}$ and 0 on all other pieces $\overline{B}_{n,i} \cap \mathcal{K}$ is continuous since the closed balls are disjoint, hence the connected component of ϕ in $\mathcal{K}$ is included in $\overline{B}_n(\phi)$, so that $\mathcal{K}$ is totally disconnected. To see that $\mathcal{K}$ is perfect, choose any open neighborhood V of some $\phi \in \mathcal{K}$. For n sufficiently large $\overline{B}_n(\phi) \subset V$, and $f_\diamond^n \colon \overline{B}_n(\phi) \to \overline{B}_{0,0}$ is surjective. It follows that $\overline{B}_n(\phi)$ contains a preimage by $f_\diamond^n$ of a point in each set $\overline{B}_{m,i} \cap \mathcal{K}$, and in particular at least 2 points. Thus $\mathcal{K}$ is perfect.

Suppose now on the contrary that for every n, $\overline{\mathcal{B}}_n = \{\overline{B}_{n,0}\}$ consists of a single ball. Then $\mathcal{K}$ is reduced to a singleton $\{x\} = \bigcap_n \overline{B}_{n,0}$. To conclude we need to argue that x is represented by a branch of the critical set, and that this branch is smooth and transversal to $(z = 0)$.

Observe first that we can construct a sequence $\overline{\mathcal{B}}_{n,\mathbb{L}}$ as above over $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{L}}$, and the same analysis proves that either $\mathcal{K}_{\mathbb{L}}$ is a Cantor set or a single point $\{\phi\}$ with $\phi \in \mathbb{L}$.

Suppose that $\mathcal{K}_{\mathbb{L}}$ is a singleton. Since $\mathcal{K}_{\mathbb{L}}$ is Galois invariant, x must be a point of type 1 of (relative) multiplicity equal to 1. We deduce that x is represented by a smooth branch ϕ transversal to $(z = 0)$. We claim that $w - \phi(z)$ is a branch of the critical set. In fact, from the fact that $w - \phi(z)$ is totally invariant we deduce that

$$f^*(w - \phi(z)) = w^c + \sum_{j=0}^{c-1} h_j(z) w^j - \phi(z^d)$$

must be of the form $(w - \phi(z))^e u(z, w)$ for a suitable $e \in \mathbb{N}^*$ and a unit $u = \sum_k u_k(w) z^k$. By computing at $z = 0$ we get $w^e u_0 = w^c$, from which we deduce that $e = c$ and $u_0 \equiv 1$. Assume that there exists $k \geq 1$ so that $u_k \not\equiv 0$. Take k to be minimal, and let h be the order of u_k at 0. Then the expression $(w - \phi(z))^c u(z, w)$ has a non-trivial monomial $z^k w^{h+c}$, while $f^*(w - \phi(z))$ does not, and we get a contradiction. Thus we have showed that $u \equiv 1$ and that $w \circ f(z, w) = (w - \phi(z))^c + \phi(z^d)$. Computing the derivative, we get

$$\mathrm{Jac}_w(f) = c(w - \phi(z))^{c-1},$$

from which we deduce that $w - \phi(z)$ is a critical branch, as desired.

To conclude the proof of Theorem 3.2, it remains to exclude the possibility that $\mathcal{K} = \{x\}$ is a singleton and $\mathcal{K}_{\mathbb{L}}$ is a Cantor set. For convenience, we postpone the proof of this fact until Theorem 4.1 is established. $\square$

3.3. **The non-Archimedean Green function.** In this section, we can work indifferently over $\mathbb{A}^{1,\mathrm{an}}_{k((z))}$ or $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{L}}$.

Recall that a subharmonic function $g\colon U \to [-\infty,\infty)$ on an open set $U \subset \mathbb{A}^{1,\mathrm{an}}_{k(\!(z)\!)}$ (resp. in $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{L}}$) is upper-semicontinuous, not identically $-\infty$, and satisfies the submean value inequalities, see, e.g., [BR10, Definition 8.16]. Suffice it to say that the set of subharmonic functions is stable under taking maxima, sums, multiplication by positive constants and decreasing limits. It contains all functions of the form $q\log|P|$ for any $P \in k(\!(z)\!)[w]$ (resp. in $\mathbb{L}[w]$), and $q \in \mathbb{Q}^*_+$, and the lattice generated by these functions is dense in the space of all subharmonic functions, see, e.g., [Ste19] (or [BR10, Proposition 8.45]). Also, if g is subharmonic in $U \subset \mathbb{A}^{1,\mathrm{an}}_{k(\!(z)\!)}$, then $g \circ \pi$ is subharmonic in $\pi^{-1}(U) \subset \mathbb{A}^{1,\mathrm{an}}_{\mathbb{L}}$.

Theorem 3.6. *Let f be a polynomial map of the form (2), with $2 \le c < d$.*

Then, for every $n \in \mathbb{N}^*$, the function $g_n : x \mapsto \frac{d^n}{c^n}\log|f^n_\diamond(x)|$ is subharmonic on $B^{\mathrm{an}}(0,1) \subset \mathbb{A}^{1,\mathrm{an}}_{k(\!(z)\!)}$. The sequence $(g_n)_{n\ge1}$ converges pointwise to a subharmonic function

$$g_{\mathrm{na}}\colon B^{\mathrm{an}}(0,1) \to \mathbb{R}_- \text{ such that } g_{\mathrm{na}} \circ f_\diamond = \frac{c}{d} g_{\mathrm{na}}.$$

Furthermore the convergence is locally uniform on compact subsets of $B^{\mathrm{an}}(0,1) \setminus \mathcal{K}$, in particular g_{na} is continuous there.

Furthermore $g_{\mathrm{na}}(x) = \log|x|$ on an annulus of the form $\{\rho_0 < |x| < 1\}$, and

$$\mathcal{K} = \{g_{\mathrm{na}} = -\infty\}\ .$$

The same results hold in $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{L}}$ and for the corresponding function $g_{\mathrm{na},\mathbb{L}}$ we have the relation $g_{\mathrm{na},\mathbb{L}} = g_{\mathrm{na}} \circ \pi$.

The first assertion is a consequence of the following basic lemma.

Lemma 3.7. *If g is a subharmonic function on $B^{\mathrm{an}}(0,1)$ then $g \circ f_\diamond$ is subharmonic.*

Proof. For any polynomial $k(\!(z)\!)[w]$, denote by $\log_P$ the function $x \mapsto \log|P(x)|$ on $\mathbb{A}^{1,\mathrm{an}}_{k(\!(z)\!)}$. It is enough to check that $\log_P \circ f_\diamond$ is subharmonic for every P. Writing $f(z,w) = (z^d, f_2(z,w))$, for every $\psi \in \mathbb{L}$ we have

$$(\log_P \circ f_\diamond)(\psi) = \frac{1}{d}\log\Big|P\big(z^d, f_2(z,\psi(z))\big)\Big|, \tag{19}$$

which shows that $\log_P \circ f_\diamond = \frac{1}{d}\log_{P\circ f}$ is subharmonic. □

Proof of Theorem 3.6. From (16), the density of type 1 points, and the continuity of $|\cdot|$, there exists $0 < \rho_0 < 1$ such that denoting $U = \{\rho_0 < |x| < 1\}$, $|f_\diamond(x)| = |x|^{c/d}$ for any $x \in U$. Hence U is $f_\diamond$-invariant and we infer that for any $x \in U$ and any $n \in \mathbb{N}^*$, $g_n(x) = \log|x|$.

Pick any compact set $L \subset B^{\mathrm{an}}(0,1)\setminus\mathcal{K}$. There exists an integer N such that $f^N_\diamond(L) \subset U$. Then for $x \in L$ and $n \ge N$ we have

$$g_n(x) = \left(\frac{d}{c}\right)^{n-N+N}\log\left|f^{n-N}_\diamond\left(f^N_\diamond(x)\right)\right| = \left(\frac{d}{c}\right)^{N}\log\left|f^N_\diamond(x)\right|, \tag{20}$$

thus the sequence is locally stationary, and the local uniform convergence statement follows. In addition, for $x \in \mathcal{K}$ we have that $\log|f^n_\diamond(x)| \le \log\rho_0$ for every n, from which we infer that $g_n(x) \to -\infty$, hence the function $g_{\rm na}$ is well defined, continuous outside $\mathcal{K}$ and $\mathcal{K} = \{g_{\rm na} = -\infty\}$.

It remains to check that $g_{\rm na}$ is subharmonic. Outside $\mathcal{K}$ this follows from the local uniform convergence. Next, fix a large integer $A \gg 1$. Then $\max\{g_n, -A\}$ forms a sequence of bounded subharmonic functions, hence by Hartog's theorem [FRL10, Proposition 2.18], there exists a subsequence $\max\{g_{n_k}, -A\}$ converging pointwise on the complement of type 1 points to a subharmonic function g_A. When A increases, then g_A decreases toward $g_{\rm na}$ which is hence subharmonic too.

Finally, the same proof works in $\mathbb{A}^{1,\rm an}_{\mathbb{L}}$, and the relation $g_{\rm na,\mathbb{L}} = g_{\rm na} \circ \pi$ follows from the fact that $g_n(\pi(x)) = g_{n,\mathbb{L}}(x)$, as follows from the definitions. $\square$

3.4. **The invariant measure.** We now turn to the invariant measure $\mu_{\rm na}$. By default we work in the unit ball in $\mathbb{A}^{1,\rm an}_{k(\!(z)\!)}$, but for reasons that will appear necessary in Section 8, we have to deal at the same time with $\mathbb{A}^{1,\rm an}_{\mathbb{L}}$.

Working in $B^{\rm an}(0,1) \subset \mathbb{A}^{1,\rm an}_{\mathbb{L}}$ first, recall that to any subharmonic function g on $B^{\rm an}(0,1)$ one can associate a positive measure Δg such that the identity

$$\Delta \log_P = \sum_{P(\phi)=0} \delta_\phi$$

holds for any $P \in \mathbb{L}[w]$, where as before we denote $\log_P : x \mapsto \log|P|_x$, and the roots are counted with multiplicity. Conversely, any probability measure μ with compact support in $B^{\rm an}(0,1)$ is of the form Δg, where g is subharmonic and $g(x) = \log|x|$ in some neighborhood of ζ_g: indeed we may take $g_\mu(x) = \int \log\langle x,y\rangle \, \mathrm{d}\mu(y)$ such that $g_\mu(\zeta_g) = 0$. Here $\langle x,y\rangle = |x \wedge y|$ where $x \wedge y$ is the unique supremum of the segment $[x,y]$, see the discussion in [Ben19, §13.3], or [BR10, Prop. 8.66]. Observe that if $\mu_n \to \mu$ weakly, then $g_{\mu_n} \to g_\mu$ pointwise outside the set of Type 1 points.

Now, descending to $\mathbb{A}^{1,\rm an}_{k(\!(z)\!)}$ we can also associate a positive measure Δg to any subharmonic function g on $B^{\rm an}(0,1)$, which satisfies the fundamental identity

$$\pi_* \Delta(g \circ \pi) = \Delta g. \tag{21}$$

For instance, if $P \in k[[z]][w]$, we now have

$$\Delta \log_P = \sum_{P(\phi)=0} \pi_* \delta_\phi, \tag{22}$$

where the roots $\phi \in k\langle\!\langle z \rangle\!\rangle$ of P are counted with multiplicity.

In both situations, for any such positive measure $\mu = \Delta g$, we put $f_\diamond^* \mu = \Delta(g \circ f_\diamond)$, which is a positive measure by Lemma 3.7. The operator $f_\diamond^*$ is continuous for the weak convergence of measures, and Formula (19) entails that if μ is a probability measure having compact support in $B^{\rm an}(0,1)$, then $f_\diamond^* \mu$ is a positive measure with compact support in the open unit ball and has mass c/d.

By duality, we infer the existence of a push-forward operator on the space of continuous function with compact support on $B^{\rm an}(0,1)$ so that

$$\sup|(f_\diamond)_* h| \le \frac{c}{d} \sup|h| \ , \text{ and } \int h \, \mathrm{d}(f_\diamond^* \mu) = \int ((f_\diamond)_* h) \, \mathrm{d}\mu \ , \tag{23}$$

for any continuous h. Namely, $(f_\diamond)_* h(x) = \langle f_\diamond^* \delta_x, h\rangle$.

Remark 3.8. The push-forward operator can be obtained explicitely as follows. Over $\mathbb{L}$ recall that $f_\diamond = \hat{f} \circ \tau$ where $\hat{f}$ is a polynomial map of degree c defined over $\mathbb{L}$, and $\tau(\phi(z)) = \phi(z^{1/d})$. Then we have $(f_\diamond)_* h(x) = \frac{1}{d} \sum_{f_\diamond(y)=x} e(y)h(y)$ where $e(y) \in \{1, \cdots, c\}$ is the local degree of $\hat{f}$ at $\tau(y)$ as defined in [FRL10, Jon15, Ben19]. A similar description can be obtained over $k\,((z))$.

Lemma 3.9. *For any compactly supported measure μ on $\mathbb{A}^{1,\mathrm{an}}_{k((z))}$ (resp. $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{L}}$), we have $(f_\diamond)_*(f_\diamond^*\mu) = \dfrac{c}{d}\mu$.*

Proof. By density and linearity, we only need to check the formula for Dirac masses at type 1 points, and may work over $\mathbb{L}$. Then $\mu = \Delta \log|w - \phi|$ for some $\phi \in \mathbb{L}$, and $f_\diamond^*\mu = \Delta \frac{1}{d} \log|(w-\phi)\circ f|$ whose support equals $f_\diamond^{-1}\{\phi\}$ so that $(f_\diamond)_*(f_\diamond^*\mu) = C\mu$ for some positive constant C. Computing the mass on both sides leads to $C = c/d$ which completes the proof. $\square$

Theorem 3.10. *Keeping the assumptions and notation of Theorem 3.6, define $\mu_{\mathrm{na}} := \Delta g_{\mathrm{na}}$. This is a $f_\diamond$-invariant probability measure satisfying $f_\diamond^*\mu_{\mathrm{na}} = \frac{c}{d}\mu_{\mathrm{na}}$. Its support is equal to $\mathcal{K}$.*

For any point $x \in B^{\mathrm{an}}(0,1)$, we have the weak convergence of measures

$$\frac{d^n}{c^n}(f_\diamond^n)^*\delta_x \to \mu_{\mathrm{na}}\ . \tag{24}$$

Furthermore, the measure μ_{na} is mixing (hence ergodic).

Likewise, in $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{L}}$ we define $\mu_{\mathrm{na},\mathbb{L}} := \Delta g_{\mathrm{na},\mathbb{L}}$, which satisfies $\mu_{\mathrm{na}} = \pi_*\mu_{\mathrm{na},\mathbb{L}}$, and the same properties hold. In addition, the measure $\mu_{\mathrm{na},\mathbb{L}}$ is balanced, that is, for any open ball B in which $f_\diamond$ is injective, we have $\mu(f_\diamond(B)) = c\mu(B)$.

Proof. Since $g_{\mathrm{na}}(x) = \log|x|$ near ζ_g, μ_{na} is a probability measure. Its support contains $\mathcal{K}$ because $g_{\mathrm{na}} = -\infty$ on $\mathcal{K}$. Conversely, pick any point x outside $\mathcal{K}$. Then for large n, $f_\diamond^n(x)$ is close to ζ_g thus g is harmonic in a neighborhood of $f_\diamond^n(x)$. Since $f_\diamond$ preserves harmonic functions, it follows that g_{na} is harmonic near x, and we conclude that $\mathrm{Supp}(\mu_{\mathrm{na}}) = \mathcal{K}$.

By definition μ_{na} satisfies $f_\diamond^*\mu_{\mathrm{na}} = \frac{c}{d}\mu_{\mathrm{na}}$, therefore Lemma 3.9 implies that μ_{na} is invariant.

Let μ be any compactly supported probability measure in $B^{\mathrm{an}}(0,1)$ and write $\mu = \Delta h$ with h subharmonic and $h(x) = \log|x|$ in a $f_\diamond$-invariant neighborhood U of ζ_g. Arguing as in the proof of Theorem 3.6, we deduce that $h_n := \frac{d^n}{c^n} h \circ f_\diamond^n = g_n$ on $f_\diamond^{-n}(U)$, hence h_n converges uniformly on compact sets to g_{na} on the complement of $\mathcal{K}$. But since $h < 0$ and $d > c$ is also clear that $h_n \to -\infty$ on $\mathcal{K}$, hence $h_n \to g_{\mathrm{na}}$ pointwise everywhere. We conclude that $\Delta h_n \to \mu_{\mathrm{na}}$, i.e., $\frac{d^n}{c^n} f_\diamond^{n*}\mu \to \mu_{\mathrm{na}}$. This implies (24).

To prove the mixing property, pick any compactly supported continuous function ψ on $B^{\mathrm{an}}(0,1)$. By the convergence (24) and the duality relation in (23), $\frac{d^n}{c^n}\ (f_\diamond^n)_*\psi(x)$ converges pointwise to $\int \psi\, \mathrm{d}\mu_{\mathrm{na}}$ for any $x \in B^{\mathrm{an}}(0,1)$. Without loss of generality, assume

$\int \psi \,\mathrm{d}\mu_{\mathrm{na}} = 0$. For any other continuous function φ, we get

$$\begin{aligned}\int (\varphi \circ f_\diamond^n)\, \psi \,\mathrm{d}\mu_{\mathrm{na}} &= \left\langle \frac{d^n}{c^n} (f_\diamond)^* (\varphi \mu_{\mathrm{na}}), \psi \right\rangle = \left\langle \varphi\mu_{\mathrm{na}}, \frac{d^n}{c^n} (f_\diamond)_* \psi \right\rangle \\ &= \int \left(\frac{d^n}{c^n} (f_\diamond)_* \psi(x) \right) \varphi(x) \,\mathrm{d}\mu_{\mathrm{na}}(x) \underset{n\to\infty}{\longrightarrow} 0,\end{aligned}$$

where in the last step we use the fact that $\frac{d^n}{c^n}\,(f_\diamond^n)_*\psi$ is uniformly bounded, and dominated convergence. Therefore μ_{na} is mixing, as announced.

The previous arguments can be repeated mutatis mutandis in $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{L}}$, and we get the same properties. Note that the relation $\mu_{\mathrm{na}} = \pi_* \mu_{\mathrm{na},\mathbb{L}}$ follows from $g_{\mathrm{na},\mathbb{L}} = g_{\mathrm{na}} \circ \pi$ and (21). It remains to establish the balancing property of $\mu_{\mathrm{na},\mathbb{L}}$. For this, we fix a type 1 point x, represented by a series $\phi(z)$, and observe that

$$\mu_n := \frac{d^n}{c^n} (f^n_{\mathbb{L}\diamond})^* \delta_x = \frac{d^n}{c^n} \Delta \left(\log_P \circ f^n_{\mathbb{L}\diamond}\right) = \frac{1}{c^n} \sum_{f^n_{\mathbb{L}\diamond}(y)=x} \delta_y, \tag{25}$$

where $P(w) = w - \phi(z)$. If B is as in the statement of the theorem, then, $f_{\mathbb{L}\diamond}$ being open, $f_{\mathbb{L}\diamond}(B)$ is an open ball as well, and $\mu_{\mathrm{na},\mathbb{L}}(\partial B) = \mu_{\mathrm{na},\mathbb{L}}(\partial f_{\mathbb{L}\diamond}(B)) = 0$ because $\mathcal{K}_{\mathbb{L}}$ consists of type 1 points. Hence $\mu_n(B)$ converges to $\mu_{\mathrm{na},\mathbb{L}}(B)$ and likewise for $f_{\mathbb{L}\diamond}(B)$. Finally, by (25), $\mu_n(f_{\mathbb{L}\diamond}(B)) = c\mu_{n+1}(B)$ for all n, and we conclude by letting $n \to \infty$. $\square$

4. Upper bounds for the multiplicity on $\mathcal{K}$

We still denote by $f(z,w) = \left(z^d, w^c + \sum_{j=0}^{c-1} h_j(z) w^j\right)$ a polynomial map as in (2), and recall that the Jacobian determinant is equal to $\mathrm{Jac}(f) = dz^{d-1}\,\mathrm{Jac}_w(f)$ where

$$\mathrm{Jac}_w(f) = cw^{c-1} + \sum_{j=1}^{c-1} j h_j(z) w^{j-1} \in k[[z]][w].$$

We factor $\mathrm{Jac}_w(f)$ in the field of Puiseux series:

$$\mathrm{Jac}_w(f) = c \prod_{i=1}^{c-1} (w - c_i(z))$$

where $c_i(z) \in k\langle\langle z \rangle\rangle$. Since $h_j(0) = 0$ for all j, note that $c_i(0) = 0$ as well.

The critical points $c_i \in \mathcal{C}$ are not necessarily distinct, and we introduce

$$J(c_i) := \mathrm{ord}_{c_i} \mathrm{Jac}_w(f), \tag{26}$$

which is a positive integer.

It will be convenient to use the following notation: if $\alpha = p/q$ is a rational number with p and q coprime, then we denote by $q(\alpha) := q$ its denominator.

To any c_i we associate the integer

$$\nu(c_i) := q\left(\frac{d}{1 + J(c_i)}\right) \in \mathbb{N}_{\geq 1}.$$

Note that if d is even and c is a simple critical point then $\nu(c) = 1$. We denote by $\mathcal{C}^+ \subset \mathcal{C}$ the set of critical points c_i such that $\nu(c_i) \geq 2$

As usual, we denote by $\omega(x)$ the limit set of a point x, that is,

$$\omega(x) = \bigcap_{n\geq 0} \overline{\{f_\diamond^m(x),\, m \geq n\}}\ .$$

Our main result can be stated as follows. Here we work in $\mathbb{A}^{1,\mathrm{an}}_{k((z))}$.

Theorem 4.1. *Let f be a polynomial map of the form (2) with $2 \leq c < d$, and recall that $\mathcal{C}^+ = \{c_i \in \mathcal{C}, \nu(c_i) \geq 2\}$.*

(1) Any point $x \in \mathcal{K}$ such that $\omega(x) \cap \mathcal{C}^+ = \emptyset$ has finite multiplicity, that is, it is a rigid point).
(2) Suppose that $\mathcal{C}^+ \cap \omega(c) = \emptyset$ for every $c \in \mathcal{C}^+$. Then there exists $m_0 \in \mathbb{N}^*$ such that $\mathcal{K} \subset \{m \leq m_0\}$. In particular, any point in $\mathcal{K}$ is rigid.

As an immediate consequence we get the following corollary, which holds in particular when $\mathcal{C}^+ = \emptyset$.

Corollary 4.2. *If $c \notin \mathcal{K}$ for every $c \in \mathcal{C}^+$, then there exists an integer m_0 such that $\mathcal{K} \subset \{m \leq m_0\}$.*

Using Theorem 4.1, we can now complete the proof of a theorem from the previous section.

End of proof of Theorem 3.2. Suppose that $\mathcal{K}_\mathbb{L}$ is a Cantor set, and $\mathcal{K} = \{x\}$ is a single point. Since $\mathcal{K}_\mathbb{L}$ is invariant by the Galois action, we deduce that $\mathcal{K}_\mathbb{L}$ is the Galois orbit of any lift y of x to $\mathbb{A}^{1,\mathrm{an}}_\mathbb{L}$. By assumption $\mathcal{K}_\mathbb{L}$ is infinite, hence $m(x) = +\infty$, in particular x is not a Puiseux series. Yet, the critical branches are Puiseux series, so all critical orbits belong to the basin of attraction of ζ_g, and by Corollary 4.2, the multiplicity is uniformly bounded in $\mathcal{K}$, a contradiction. □

4.1. **The Jacobian formula (II).** It is convenient to set $A(x) := 1 - \log \operatorname{diam}(x)$ for any non rigid point $x \in \mathbb{A}^{1,\mathrm{an}}_\mathbb{L}$, and work systematically over $\mathbb{L}$. With this notation, (15) is equivalent to the formula:

$$A(f_\diamond(x)) = \frac{1}{d}\left(A(x) + g_{\mathrm{Jac}(f)}(x)\right)\ , \tag{27}$$

where $g_{\mathrm{Jac}(f)}(x) := -\log|\mathrm{Jac}(f)(x)|$. Indeed, $\mathrm{Jac}(f) = dz^{d-1}\,\mathrm{Jac}_w(f)$ so $\log|\mathrm{Jac}(f)| = -(d-1) + \log|\mathrm{Jac}_w(f)|$, and the formula follows from (15). It can be checked that the function $g_{\mathrm{Jac}(f)}$ is locally constant outside the convex hull of $\mathcal{C} \cup \{\zeta_g\}$ which we denote by $\mathcal{T}$, and call the critical tree.

Pick any rigid point ϕ in the unit ball, and, for any positive real number $t > 0$, consider the point $\zeta(\phi, e^{-t})$. Then we have

$$g_{\mathrm{Jac}(f)}(\zeta(\phi, e^{-t})) = \sum_i -\max(\log|\phi - c_i|, -t) + (d-1)\ .$$

Observing that $A(\zeta(\phi, e^{-t})) = 1 + t$, we obtain the following result.

Proposition 4.3. *Let $\phi \in B(0,1) \subset \mathbb{L}$ be any rigid point. For any $\tau \in [1, +\infty]$ denote by x_τ the unique point in the segment $[\phi, \zeta_g]$ such that $A(x_\tau) = \tau$ (so that $\phi = x_\infty$ and $\zeta_g = x_1$).*

The function $\tau \mapsto g_{\mathrm{Jac}(f)}(x_\tau)$ is a concave piecewise affine function with integer slope. More precisely, if $x = x_{\tau_0}$, the left derivative

$$J(x) := \left.\frac{d g_{\mathrm{Jac}(f)}(x_\tau)}{d\tau}\right|_{\tau=\tau_0}$$

is an integer, equal to the number of critical curves c_i such that $c_i < x_{\tau_0}$.

We call $J(x)$ the *critical slope* at x. Notice that the notation $J(x)$ is consistent with the one used in (26). If the critical slope is constant on a segment (x_0, x_1) with $x_0 < x_1$, then the function $g_{\mathrm{Jac}(f)}$ is affine on $[x_0, x_1]$ with respect to the parametrization by A. The following consequence of this discussion, together with (27), will be repeatedly used.

Corollary 4.4. *If the critical slope is constant equal to J on an interval (x_0, x_1) with $x_0 < x_1$, then*

$$\frac{A(f_\diamond(x_1)) - A(f_\diamond(x_0))}{A(x_1) - A(x_0)} = \frac{J+1}{d} \ .$$

4.2. **The generic multiplicity.** We now work in $\mathbb{A}^{1,\mathrm{an}}_{k((z))}$. Recall that we defined the multiplicity of a point in §1.3.2. Pick any type 2 point $x \in B^{\mathrm{an}}(0,1)$. We define the *generic multiplicity* of x by the formula

$$b(x) = \mathrm{lcm}\{m(x), q(A(x))\} \ ,$$

where $q(s) = b$ if $s = a/b$ with $a \in \mathbb{Z}, b \in \mathbb{N}^*$ and a and b coprime. Observe that $m(x)$ divides $b(x)$ by definition. A geometric interpretation of this quantity, which also explains the terminology, will be given in § 8.5.

To understand the behaviour of the generic multiplicity, we first extend the multiplicity function defined on $\mathbb{A}^{1,\mathrm{an}}_{k((z))}$ in §1.3.2 to Puiseux series, and then to open and closed balls.

Pick any $\phi \in \mathbb{L}$. If ϕ is not a Puiseux series we set $m(\phi) = \infty$; otherwise $m(\phi)$ is the minimal integer q such that $\phi \in k[[z^{1/q}]]$. In the latter case, if C is the formal curve associated with $\{\phi = 0\}$, then $m(\phi) = (C \cdot \{z = 0\})_0$.

Let B be any (Berkovich) ball in $B^{\mathrm{an}}(0,1)$, which may be closed or open. We define $m(B) = \min\{m(\phi), \phi \in \mathbb{L}, \pi(\phi) \in B\}$. Recall that a closed ball $\overline{B}$ has a unique boundary point x, and when this point is of type 2, then $\overline{B} \setminus \{x\}$ is a union of disjoint open balls B_i such that $\partial B_i = \{x\}$ (corresponding to the tangent directions at x pointing towards B).

Theorem 4.5. *Pick any type 2 point $x \in B^{\mathrm{an}}(0,1) \subset \mathbb{A}^{1,\mathrm{an}}_{k((z))}$, and let $\overline{B}$ be the unique closed ball whose boundary is equal to x. We have $m(\overline{B}) = m(x)$.*

Moreover, the following holds.

- *If $m(x) = b(x)$, then for any open ball B whose boundary equals x, we have $m(B) = m(x)$.*

- *If $m(x) < b(x)$, then there exists a unique open ball B_0 whose boundary equals x such that $m(B_0) = m(x)$. For all other open balls B with $\partial B = \{x\}$, we have $m(B) = b(x)$.*

Corollary 4.6. [FJ04, Proposition 3.44] *Suppose $x \in \overline{B}^{\mathrm{an}}(0,1) \subset \mathbb{A}^{1,\mathrm{an}}_{k((z))}$, and consider the decreasing sequence of type 2 points $x_0 = \zeta_g$, $x_{n+1} < x_n$, $\lim_n x_n = x$ defined in Proposition 1.2.*

Then for each $n \geq 1$, we have the equality $b(x_n) = m(x_{n-1})$.

Proof. Indeed, the set of points $y \leq x_{n-1}$ is a closed ball $\overline{B}$. Consider the open ball B containing x_n and of the same diameter as $\overline{B}$. Pick any Puiseux series ϕ with $\pi(\phi) \in B$ and $m(\phi) = m(B)$. Then $m(B)$ is equal to the multiplicity of all $y \in (x_{n-1}, x_n]$ sufficiently close to x_{n-1}. Since the multiplicity is constant on this interval equal to $m(x_n)$, we conclude that $m(B) = m(x_n) > m(x_{n-1})$. The previous theorem then implies that $b(x_{n-1}) = m(B) = m(x_n)$. □

Sketch of proof of Theorem 4.5. We may suppose that $x = \pi(\zeta(\phi, r))$ where $\phi \in k\langle\langle z\rangle\rangle$ is a Puiseux series, and $r = e^{-t}$ with $t \in \mathbb{Q}_+^*$. Observe that $\overline{B}(\phi, r) = \overline{B}(\psi, r)$ (resp. $B(\phi, r) = B(\psi, r)$) iff $|\phi - \psi| \leq r$ (resp. $|\phi - \psi| < r$) iff $\mathrm{ord}_z(\phi - \psi) \geq t$ (resp. $\mathrm{ord}_z(\phi - \psi) > t$). We may thus suppose that $\phi(z) = \sum_{n=0}^{N} a_n z^{\beta_n}$ with $0 \leq \beta_0 < \cdots < \beta_N < t$. Any open ball having x as boundary points is then of the form $\pi(B(\phi + cz^t, r))$ for some $c \in \mathbb{C}$.

Using (8), we get

$$m(\overline{B}(\phi, r)) = \mathrm{lcm}\{q(\beta_n), 0 \leq n \leq N\}$$

and

$$m(B(\phi + cz^t, r))) = \mathrm{lcm}\{q(t), q(\beta_n), 0 \leq n \leq N\}\ .$$

The first identity together with the description of the Galois action on Puiseux series implies that $m(\overline{B}(\phi, r)) = m(x)$ (recall that $m(x)$ is by definition the cardinality of $\pi^{-1}(x)$). The remaining statements follow from the previous analysis, using the relation $A(\zeta(\phi, e^{-t})) = 1 + t$, which implies $q(t) = q(A(x))$. □

4.3. **Estimates on the multiplicity.** In a dynamical context, the key estimate on multiplicities is given by the following result.

Proposition 4.7. *Pick $x_1 \leq x_0 \in B^{\mathrm{an}}(0,1)$, and suppose that the critical slope is constant equal to J on the interval (x_1, x_0). Set $\nu = q\big(\frac{d}{J+1}\big)$. Then for any type 2 point $x \in [x_1, x_0]$ we have*

$$b(x) \mid \mathrm{lcm}\Big\{b(x_0), \nu b(f_\diamond(x)), \nu b(f_\diamond(x_0))\Big\}$$

The proof relies on the following lemma.

Lemma 4.8. *Let $x_1 \leq x_0 \in B^{\mathrm{an}}(0,1)$ be such that the critical slope is constant equal to J on the interval (x_1, x_0), and the multiplicity is constant on $[f_\diamond(x_1), f_\diamond(x_0))$.*

Then the multiplicity is constant on $[x_1, x_0)$.

Proof. For any point $x' \in [x_1, x_0)$, we have:

$$\frac{\alpha(f_\diamond(x)) - \alpha(f_\diamond(x_0))}{\alpha(x) - \alpha(x_0)} = \frac{\alpha(f_\diamond(x)) - \alpha(f_\diamond(x_0))}{A(f_\diamond(x)) - A(f_\diamond(x_0))} \times \frac{A(f_\diamond(x)) - A(f_\diamond(x_0))}{A(x) - A(x_0)} \times \frac{A(x) - A(x_0)}{\alpha(x) - \alpha(x_0)}$$

By assumption,

$$\frac{\alpha(f_\diamond(x)) - \alpha(f_\diamond(x_0))}{A(f_\diamond(x)) - A(f_\diamond(x_0))}$$

is constant equal to the inverse of the multiplicity on $[f_\diamond(x_1), f_\diamond(x_0))$ (see Proposition 1.2). Since x_0 (hence the whole segment $[x_1, x_0]$) is outside the critical tree by Corollary 4.4 we have

$$\frac{A(f_\diamond(x)) - A(f_\diamond(x_0))}{A(x) - A(x_0)} = \frac{J+1}{d} \ . \tag{28}$$

By Corollary 1.3, A is convex in α hence the function

$$x \mapsto \frac{A(x) - A(x_0)}{\alpha(x) - \alpha(x_0)}$$

is non-decreasing. On the other hand the function

$$x \mapsto \frac{\alpha(f_\diamond(x)) - \alpha(f_\diamond(x_0))}{\alpha(x) - \alpha(x_0)}$$

is non-increasing, because we can write

$$\frac{\alpha(f_\diamond(x)) - \alpha(f_\diamond(x_0))}{\alpha(x) - \alpha(x_0)} = \frac{1}{\deg_w(P)} \frac{-\log|P \circ f(x)| + \log|P \circ f(x_0)|}{\alpha(x) - \alpha(x_0)},$$

where $P \in k[[z]][w]$ is such that $\alpha(x') = -\log|P(x')|/\deg_w(P)$ for every $x' \in [f_\diamond(x_1), f_\diamond(x_0))$ as in Remark 1.1, and $x \mapsto -\log|\Phi \circ f(x)|$ is concave in α by [FJ04, Prop 3.25]. It follows that both functions $\frac{\alpha(f_\diamond(x)) - \alpha(f_\diamond(x))}{\alpha(x) - \alpha(x_0)}$ and $\frac{A(x) - A(x_0)}{\alpha(x) - \alpha(x_0)}$ are constant. Since the multiplicity on $[x_1, x_0)$ is equal to the latter, we conclude that it is constant on this interval. □

Proof of Proposition 4.7. Since $f_\diamond$ is strictly order preserving (see Equation (28)) and continuous, the segment $[x, x_0]$ is mapped bijectively onto the segment $[f_\diamond(x), f_\diamond(x_0)]$. Decompose the latter segment $x'_0 = f_\diamond(x_0) > x'_1 > \cdots > x'_N = f_\diamond(x)$ so that the multiplicity is constant on $(x'_n, x'_{n+1}]$ equal to m'_{n+1} and m'_n strictly divides m'_{n+1}.

Denote by x_n the preimage of x'_n on the segment $[x, x_0]$, and for consistence denote $x = x_N$. By the preceding lemma, the multiplicity is constant on $(x_n, x_{n+1}]$ equal to m_{n+1}. Note that m_n may be equal to m_{n+1}.

Let us prove by induction on $n \leq N$ that

$$b(x_n) \text{ divides } \operatorname{lcm}\{b(x_0), \nu b(f_\diamond(x_0)), \nu b(f_\diamond(x_n))\}, \tag{29}$$

which corresponds to the desired statement for $n = N$. For this, we use the observation:

$$\text{for every } y \leq x,\ q(A(y)) \mid \operatorname{lcm}\{q(A(x)), \nu q(A(f_\diamond(x))), \nu q(A(f_\diamond(y)))\}, \tag{30}$$

which follows from the identity $A(y) = A(x) + \frac{d}{J+1}\big(A(f_\diamond(y)) - A(f_\diamond(x))\big)$ (see (28)).

Suppose first that $n = 1$. Then, we have

$$\begin{aligned} b(x_1) &= \mathrm{lcm}\{m(x_1), q(A(x_1))\} \\ &\mid \mathrm{lcm}\{m(x_1), q(A(x_0)), \nu q(A(f_\diamond(x_1)), \nu q(A(f_\diamond(x_0))\} \\ &\mid \mathrm{lcm}\{b(x_0), \nu b(f_\diamond(x_1)), \nu b(f_\diamond(x_0))\} \end{aligned}$$

where in the second line we use the fact that by definition $q(A(\cdot))|b(\cdot)$ and in the third line we use the fact that either $m(x_1) = m(x_0)$, or $m(x_1) = b(x_0)$, see Corollary 4.6.

Suppose now that (29) is true for some $1 \leq n < N$. The same argument gives

$$\begin{aligned} b(x_{n+1}) &= \mathrm{lcm}\{m(x_{n+1}), q(A(x_{n+1}))\} \\ &\mid \mathrm{lcm}\{m(x_{n+1}), q(A(x_n)), \nu q(A(f_\diamond(x_n))), \nu q(A(f_\diamond(x_{n+1})))\} \\ &\mid \mathrm{lcm}\,\{b(x_n), \nu b(f_\diamond(x_n)), \nu b(f_\diamond(x_{n+1}))\} \end{aligned}$$

since either $m(x_{n+1}) = m(x_n)$, or $m(x_{n+1}) = b(x_n)$. We conclude the proof using the induction hypothesis and the fact that $b(f_\diamond(x_n))|b(f_\diamond(x_{n+1}))$: indeed for $n \geq 1$ we have

$$b(f_\diamond(x_n)) = b(x'_n) = m(x'_{n+1})|b(x'_{n+1}).$$

The proof is complete. □

4.4. **Proof of Theorem 4.1.** Recall from §3.2 that we constructed a nested family $(\overline{\mathcal{B}}_n)_n$ each consisting of finitely many closed (Berkovich) balls $\overline{\mathcal{B}}_n = \{\overline{B}_{n,i}\}$, such that $\mathcal{K} = \bigcap_n \bigcup \overline{\mathcal{B}}_n$, and such that for any $B \in \overline{\mathcal{B}}_n$, $f_\diamond(B) \in \overline{\mathcal{B}}_{n-1}$ and $f_\diamond^{-1}(B)$ is an union of balls in $\overline{\mathcal{B}}_{n+1}$. Here $\bigcup \overline{\mathcal{B}}_n := \bigcup_i \overline{B}_{n,i}$ stands for the union of all the elements in $\overline{\mathcal{B}}_n$.

Let us prove Assertion (1). So, assume that the ω-limit set of the point x does not contain any critical point in $\mathcal{C}^+$. Replacing x by one of its iterates, we may suppose that the closure $O^+(x)$ of the orbit of x is compact and avoids $\mathcal{C}^+$. Denote by $\mathcal{T}^+$ the convex hull of $\mathcal{C}^+$ and ζ_g. Any point in $O^+(x)$ is contained in some ball in $\overline{\mathcal{B}}_n$ that is disjoint from $\mathcal{T}^+$, therefore we can find an integer N and a finite set of balls $\overline{B}_1, \cdots, \overline{B}_m \in \overline{\mathcal{B}}_N$ disjoint from $\mathcal{T}^+$, and whose union contains $O^+(x)$. For the sake of the argument, assume further that for every j, $f_\diamond(\overline{B}_j)$ is disjoint from $\mathcal{T}^+$ (that is, decrease possibly N by 1), and assume that N is so large that each ball in $\overline{\mathcal{B}}_{N-1}$ contains at most one critical point of $\mathcal{C} \setminus \mathcal{C}^+$. Let $y_1, \cdots, y_m$ be the respective boundary points of the $\overline{B}_j$. For any $n \geq N$, let $\hat{x}_n$ be the boundary point of the unique ball in $\overline{\mathcal{B}}_n$ containing x. Then for any $\ell \leq n - N$, the type 2 point $f_\diamond^\ell(\hat{x}_n)$ is the boundary point of a ball in $\overline{\mathcal{B}}_{n-\ell}$, and in particular $f_\diamond^{n-N}(\hat{x}_n) \in \{y_1, \ldots, y_m\}$.

Set $Q = \mathrm{lcm}\{b(y_j), b(f_\diamond(y_j)),\ 1 \leq j \leq m\}$. We inductively apply Proposition 4.7 to the intervals $(f_\diamond^\ell(\hat{x}_n), y_{j(\ell)})$, where $y_{j(\ell)}$ is the unique point $y_i \geq f_\diamond^\ell(\hat{x}_n)$, and observe that $\nu = 1$ on these intervals because they are disjoint from $\mathcal{T}^+$ and contain at most one point of $\mathcal{C} \setminus \mathcal{C}^+$. We thus obtain that

$$b(\hat{x}_n)|\,\mathrm{lcm}\left\{Q, b\big(f_\diamond^\ell(\hat{x}_n)\big)\right\}$$

for any $\ell \leq n - N$. For $\ell = n - N$ we obtain $b(\hat{x}_n)|Q$. In particular, the sequence $b(\hat{x}_n)$ is stationary for large n, which implies that x has multiplicity at most Q.

Let us now establish (2). Since $\mathcal{C}^+ \cap \omega(c) = \emptyset$ for every $c \in \mathcal{C}^+$, one can find a neighborhood U of $\mathcal{C}^+$ and an integer N such that for every $c \in \mathcal{C}^+$, $f_\diamond^n(c) \notin U$ for all

$n \geq N$. We may suppose that U is a finite union of closed balls of $\overline{\mathcal{B}}_M$ for some M. It follows that if $\overline{B} \in \overline{\mathcal{B}}_n$ contains a critical point and $n \geq M$, then $f_\diamond^k(\overline{B}) \in \overline{\mathcal{B}}_{n-k}$ does not contain any critical point for any $N \leq k \leq M - n$. We shall also assume that M is so large that the intersection of $\mathcal{T}$ with any ball $\overline{B} \in \overline{\mathcal{B}}_M$ is either empty or connected.

We introduce some notation. For any point $x \in \mathcal{K}$ and any $n \in \mathbb{N}$, let $b_n(x)$ be the generic multiplicity of the unique boundary point of the ball $\overline{B}_n(x) \in \overline{\mathcal{B}}_n$ containing x (so that with the above notation, $b_n(x) = b(\hat{x}_n)$). Observe that $f_\diamond^k(\overline{B}_n(x)) = \overline{B}_{n-k}(f_\diamond^k(x))$ for all $k \leq n$. For any $n \geq 1$, we define

$$Q_n = \mathrm{lcm}_{x = \partial \overline{B}} \{b(x), b(f_\diamond(x))\} ,$$

where $\overline{B}$ ranges over all balls in $\overline{\mathcal{B}}_n$.

The theorem will follow if we find an upper bound on $b_{n+M}(x)$ independent on $x \in \mathcal{K}$ and n. Pick any $x \in \mathcal{K}$ and any integer $n \geq 0$. Let n_0 (resp. n_1) be the least (resp. greatest) integer $k \leq n$ such that $f_\diamond^k(\overline{B}_{n+M}(x))$ contains a critical point. We know that $n_0 \leq n_1 \leq n_0 + N - 1$.

For any $j \in \{0, \ldots, n\}$, we apply Proposition 4.7 to

$$x_1 = \partial \overline{B}_{n-j+M}(f_\diamond^j(x)) = \partial f_\diamond^j(\overline{B}_{n+M}(x))$$

and to $x_0 = \partial \overline{B}_M(f_\diamond^j(x))$ the boundary point of the unique ball in $\overline{\mathcal{B}}_M$ containing x_1.

When $j < n_0$ or when $j > n_1$, then the ball $f_\diamond^j(\overline{B}_{n+M}(x)) = \overline{B}_{n-j+M}(f_\diamond^j(x))$ does not contain any point of $\mathcal{C}^+$ and at most one critical point of $\mathcal{C} \setminus \mathcal{C}^+$, so

$$b_{n-j+M}(f_\diamond^j(x)) | \, \mathrm{lcm}\{b_{n-j-1+M}(f_\diamond^{j+1}(x)), Q_M\} .$$

When $j \in \{n_0, \ldots, n_1\}$ the critical slope is constant (by our choice of M) on the segment $[x_0, x_1]$ and bounded from above by $c - 1$, so

$$b_{n-j+M}(f_\diamond^j(x)) | \, \mathrm{lcm}\{L b_{n-j-1+M}(f_\diamond^{j+1}(x)), L Q_M\} ,$$

where L is the least common multiple of $q\left(\frac{d}{1+j}\right)$ when j ranges among all integers $0 \leq j \leq (c - 1)$. Since $|n_1 - n_0| \leq N$, we conclude that

$$b_n(x) | L^N Q_M$$

as required.

5. Points of infinite multiplicity

Theorem 5.1. *Suppose that $\mathcal{K}$ is not reduced to a singleton, and that there exists a periodic critical point c_0 with $\nu(c_0) \geq 2$. Then $\mathcal{K}$ contains both rigid and non-rigid points. Moreover, the supremum of the multiplicity of rigid points $x \in \mathcal{K}$ is infinite.*

Remark 5.2. It follows from the equidistribution statement (24) in Theorem 3.10 that the set of non-rigid points is dense in $\mathcal{K}$. Since the multiplicity is lower-semicontinuous, and the preimages of c_0 are rigid and dense in $\mathcal{K}$, we also deduce that for any integer M the set of rigid points in $\mathcal{K} \cap \{m \geq M\}$ is also dense in $\mathcal{K}$.

Observe that $\{m < \infty\}$ is a closed $f_\diamond$-invariant subset, hence $\mu_{\mathrm{na}}(m < \infty)$ is either 0 or 1 by the ergodicity of μ_{na}.

Question 5.3. *Under which dynamical circumstances can we ensure that $\{m < \infty\}$ has zero (resp. full) measure?*

Question 5.4. *Does Theorem 5.1 extend to maps admitting a critical point $c \in \mathcal{C}^+$ such that $\omega(c) \cap \mathcal{C}^+ \neq \emptyset$?*

Proof. As in the previous section, we consider the nested family of finitely many closed (Berkovich) balls $\overline{\mathcal{B}}_n = \{\overline{B}_{n,i}\}$, such that $\mathcal{K} = \bigcap_n \bigcup \overline{\mathcal{B}}_n$. Recall that for any $\overline{B} \in \overline{\mathcal{B}}_n$, we have $f_\diamond(\overline{B}) \in \overline{\mathcal{B}}_{n-1}$, and $f_\diamond^{-1}(\overline{B})$ is an union of balls in $\overline{\mathcal{B}}_{n+1}$. In order to clarify our arguments below, we insist in choosing $\overline{\mathcal{B}}_0 = \{\overline{B}_0\}$ so that $\overline{B}_0$ is the smallest closed ball containing $\mathcal{K}$. It follows that the convex hull $\mathcal{T}(\mathcal{K})$ of $\mathcal{K}$ and ζ_g is a locally finite tree, and $\partial\overline{B}_0$ is a branching point of $\mathcal{T}(\mathcal{K})$ since we assumed $\mathcal{K}$ is not reduced to a singleton.

We may suppose that $c_0 \in \mathcal{C}^+$ is fixed by $f_\diamond$. As before, we let $\hat{c}_n$ be the boundary point of the unique closed ball $\overline{B}_n(c_0) \in \overline{\mathcal{B}}_n$ containing c_0. The sequence $\hat{c}_n$ decreases to c_0 in the sense that $\hat{c}_n \in (c_0, \hat{c}_{n-1})$, and furthermore $f_\diamond(\hat{c}_n) = \hat{c}_{n-1}$.

By assumption, the critical slope J (as defined in §4.1) is constant on the segment $[c_0, \hat{c}_N]$ for some large enough N, and $1 + J$ does not divide d since $\nu(c_0) \geq 2$.

By Corollary 4.4, we have

$$A(\hat{c}_{n+1}) = A(\hat{c}_n) + \frac{d}{J+1}\big(A(\hat{c}_n) - A(\hat{c}_{n-1})\big)$$

for all $n \geq N$. Since $\nu = q\big(\frac{d}{J+1}\big) \geq 2$, we conclude that

$$q(A(\hat{c}_{n+1})) - q(A(\hat{c}_n)) \to \infty, \tag{31}$$

so there is a subsequence (n_j) such that $q(A(\hat{c}_{n_j})) \to \infty$. In particular, $b(\hat{c}_{n_j}) \to \infty$.

Recall that $\hat{c}_0$ is a branched point of $\mathcal{T}(\mathcal{K})$. Since $\mathcal{K}$ is backward invariant, and $f_\diamond$ is open, the relation $f_\diamond(\hat{c}_n) = \hat{c}_{n-1}$ implies by induction that $\hat{c}_n$ is a branched point of $\mathcal{T}(\mathcal{K})$ for every $n \geq 0$.

When j is large enough, $b(\hat{c}_{n_j}) > m(c_0)$, and since the multiplicity of a type 2 point is the minimal multiplicity of rigid points in the corresponding closed ball, we also have $m(c_0) \geq m(\hat{c}_{n_j})$. Since $\hat{c}_{n_j}$ is a branched point of $\mathcal{T}(\mathcal{K})$, there exists an open ball B'_{n_j} whose boundary point is equal to $\hat{c}_{n_j}$ and which does not contain c_0. By Theorem 4.5, there is a unique open ball with $\hat{c}_{n_j}$ as a boundary point, containing a rigid point with multiplicity $< b(\hat{c}_{n_j})$, which must thus be the one containing c_0. It follows that the multiplicity of any rigid point in B'_{n_j} is no less than $b(\hat{c}_{n_j})$. Since $f_\diamond^{n_j+1}$ maps B_{n_j} to an open ball containing the unique closed ball in $\overline{\mathcal{B}}_0$, c_0 admits a preimage $\phi_{n_j} \in B_{n_j}$. This preimage is rigid of multiplicity $\geq b(\hat{c}_{n_j})$, which shows that the multiplicity of rigid points in $\mathcal{K}$ is unbounded.

Let us now argue that $\mathcal{K}$ also contains non-rigid points. Let B_0 be the open ball whose boundary point is $\hat{c}_0$ and contains c_0. We shall construct a decreasing sequence of open balls $\{B'_\ell\}_{\ell\geq 1}$ whose boundary point y_ℓ is a branched point of $\mathcal{T}(\mathcal{K})$, and such that

(i) $f_\diamond^{k_\ell}(B'_\ell) = B_0$ for some $k_\ell \geq 1$;
(ii) $b(B'_\ell) \geq 2b(B'_{\ell-1})$.

Then the sequence of the corresponding closed sets $B'_\ell \cup \{y_\ell\}$ forms a decreasing sequence of compact sets intersecting $\mathcal{K}$ by (i) and its intersection cannot contain any rigid point by the second condition.

Let us explain how to build this sequence of balls. Choose some large n_0 such that $b(\hat{c}_{n_0}) \geq 2b(\hat{c}_0)$. As before, since $\hat{c}_{n_0}$ is a branched point of $\mathcal{T}(\mathcal{K})$ there exists an open ball B whose boundary is equal to $\hat{c}_{n_0}$, which intersects $\mathcal{K}$, and with the property that all rigid points in B have multiplicity at least $b(\hat{c}_{n_0})$. This ball contains a preimage of $\hat{c}_0$ by the equidistribution (24). From this, we infer the existence of an open ball $B'_1 \subset B$ such that $f_\diamond^{k_1}(B'_1) = B_0$ and $b(B'_1) \geq 2b(B_0)$.

Now, suppose that B'_ℓ has been constructed. Since $f_\diamond^{k_\ell}(B'_\ell) = B_0$, there exists a preimage c_ℓ of c_0 under $f_\diamond^{k_\ell}$ in B'_ℓ. For $m \geq 1$, let y_m be the unique preimage of $\hat{c}_m$ by $f_\diamond^{k_\ell}$ lying in $[c_\ell, \partial B'_\ell]$. If $r > 0$ is sufficiently small, all critical points of f^{k_ℓ} in $B^{\mathrm{an}}(c_\ell, r)$ are contained in the orbit of c_ℓ. It then follows from Corollary 4.4 that for m large enough, we have

$$A(\hat{c}_{m+1}) - A(\hat{c}_m) = C\left(A(y_{m+1}) - A(y_m)\right)$$

for some fixed constant $C = C(\ell)$, and since $q(A(\hat{c}_m)) - q(A(c_m)) \to \infty$ by (31), it follows that $q(A(y_{m_j})) \to \infty$ along some subsequence (m_j). We can thus find an index m such that $b(y_m) \geq 2b(B'_\ell)$. As y_m is a branched point of $\mathcal{T}(\mathcal{K})$, by Theorem 4.5, we can choose an open ball B, with $\partial B = y_m$ and such that all rigid points in B have multiplicity at least $b(y_m)$. As before $\hat{c}_0$ admits a preimage z in B under some $f_\diamond^{k_{\ell+1}}$ with $k_{\ell+1} > k_\ell$, which must satisfy $b(z) \geq b(y_m)$, and we choose $B'_{\ell+1}$ to be the corresponding preimage of B_0. □

6. Examples

In this section, we give explicit examples of skew products with uniformly bounded multiplicity on $\mathcal{K}$, as well as examples where there are both rigid points with unbounded multiplicity, and points with infinite multiplicity.

6.1. **Conjugacy to a map of the form** (2)**.**

The class of maps that we have studied so far arises quite naturally, as our next result shows.

Theorem 6.1. *Let $(k, |\cdot|)$ be any algebraically closed complete metrized field of characteristic 0, and pick two integers $d > c \geq 2$. Suppose that $f\colon (\mathbb{A}^2_k, 0) \to (\mathbb{A}^2_k, 0)$ is a finite analytic germ such that:*

(a) the curve $\{z = 0\}$ is (locally) totally invariant and $f^*[z = 0] = d[z = 0]$ as divisors;
(b) the restriction of f to $\{z = 0\}$ has a super-attracting point at 0 of order c.

Then f is formally conjugated to a (formal) map of the form

$$(z, w) \mapsto \left(z^d, w^c + \sum_{j=0}^{c-1} h_j(z) w^j\right) \tag{32}$$

where the h_j are power series vanishing at 0*.*

Remark 6.2. The previous result applies to any polynomial map f of the complex affine plane $\mathbb{A}^2_k$ that extends to a regular endomorphism to $\mathbb{P}^2_k$ of degree $d \geq 2$, and has a super-attracting fixed point on the line at infinity that is not totally invariant.

Remark 6.3. When the formal normal form is analytic, then one can argue that the conjugacy is indeed analytic. Denote by φ the conjugacy between f and its normal form. Up to a polynomial change of coordinates, we may assume that f and its normal form given by (32) coincide up to high order so that φ is tangent to the identity up to any given order, say $\gg c$. Up to a blow-up of the form $\pi(z,w) = (zw^n, w)$ with $n \geq c+1$, the map f lifts to a rigid germ of class 6 in the sense of [Fav00]. Then φ lifts through π and conjugates (zw^n, w) to the lift of the normal form (which is analytic). It implies the lift of φ to be analytic which forces φ to be analytic.

The latter remark raises the following problem.

Question 6.4. *Is it possible to find a holomorphic map*

$$f(z,w) = \Big(z^d, w^c + zh(z,w)\Big)$$

with $d > c \geq 2$ and $h(0) = 0$ which is not analytically conjugated to a map of the form (32)*?*

Remark 6.5. Lemma 6.6 below shows that, under the assumptions of Theorem 6.1, we can at least make sure that the first component of f is z^d after a holomorphic change of coordinates.

The proof of the theorem proceeds in several steps. First observe that by the Böttcher theorem, we can assume that the restriction of f to $\{z = 0\}$ is $w \mapsto w^c$, so that in suitable coordinates any map satisfying the assumptions of the theorem can be written in the form

$$f(z,w) = \Big(z^d(1 + \varepsilon(z,w)), w^c + zh(z,w)\Big), \tag{33}$$

where ε and h are two germs of analytic functions at 0 vanishing at that point.

Lemma 6.6. *There exists a unique analytic germ $\Phi(z,w) = (z(1 + \varphi(z,w)), w)$, with $\varphi(0) = 0$, such that $\Phi \circ f \circ \Phi^{-1}(z,w) = (z^d, \cdots)$.*

Remark 6.7. Note that the assumption $c < d$ is not used in this lemma.

Proof. This is a consequence of [Fav00], see also [Rug13]. We include a proof here for convenience. The conjugacy equation states:

$$z \circ \Phi \circ f = z^d(1 + \varepsilon)(1 + \varphi \circ f) = z^d(1 + \varphi)^d.$$

Since k has characteristic 0, $(1 + z)^{1/\ell}$ is well-defined and analytic in a neighborhood of 0 for any $\ell \geq 1$. A solution of the above equation is then given by

$$1 + \varphi = \prod_{n \geq 1} (1 + \varepsilon \circ f^{n-1})^{d^{-n}}.$$

This expression converges as long as $\sum_{n\geq 1} d^{-n}\left|\varepsilon\circ f^{n-1}(z,w)\right| < +\infty$, a property that holds in a neighborhood of the origin, thanks to the superattracting behavior of f^n. More explicitly, for $\|(z,w)\| \ll 1$, we have $\|f^n(z,w)\| \leq \lambda^n$ for some $\lambda < 1$, while $|\varepsilon(z,w)| \leq M|(z,w)|$ for some $M \gg 0$, so $\|\varepsilon\circ f^n\| = O(\lambda^n)$ and we are done.

The uniqueness follows from the fact that since f is super-attracting, $(1+\varphi\circ f) = (1+\varphi)^d$ implies $\varphi\equiv 0$. □

Proof of Theorem 6.1. By the previous lemma we may assume that

$$f(z,w) = \Big(z^d, w^c + \sum_{n\geq 1} z^n g_n(w)\Big),$$

where $g_n(0)=0$. Write $g_n = g_n^- + g_n^+$ where g_n^- is a polynomial of degree at most $c-1$ and $\mathrm{ord}_w(g_n^+) \geq c$.

We will prove by induction on m that f can be analytically conjugated so that $g_1^+ = \cdots = g_{m-1}^+ = 0$. Indeed, suppose that this condition is satisfied, and for any analytic power series φ vanishing at 0, let us consider the analytic map:

$$\Phi(z,w) = (z, w + z^m\varphi(w))\ .$$

We define $\tilde f$ by $\tilde f = \Phi\circ f\circ\Phi^{-1}$, or $\Phi\circ f = \tilde f\circ\Phi$, and observe that $\tilde f$ is expressed as $\tilde f(z,w) = \Big(z^d, w^c + \sum_{n\geq 1} z^n\tilde g_n(w)\Big)$, where the $\tilde g_n$ are analytic and vanish at 0.

By the inductive hypothesis, and using $d\geq 2$ we get

$$\begin{aligned} w\circ\Phi\circ f(z,w) &= w^c + \sum_{n\geq 1} z^n g_n(w) + z^{md}\varphi\left(w^c + \sum_{n\geq 1} z^n g_n(w)\right) \\ &= w^c + \sum_{n\leq m} z^n g_n^-(w) + z^m g_m^+(w) \mod(z^{m+1}) \end{aligned}$$

On the other hand, we have

$$\begin{aligned} w\circ\tilde f\circ\Phi(z,w) &= (w + z^m\varphi(w))^c + \sum_{n\geq 1} z^n\tilde g_n(w + z^m\varphi(w)) \\ &= w^c + cw^{c-1}z^m\varphi(w) + \sum_{n\leq m} z^n\tilde g_n(w) \mod(z^{m+1}) \end{aligned}$$

Identifying term by term, we infer that $\tilde g_n(w) = g_n^-(w)$ for all $n\leq m-1$, and setting

$$\varphi(w) := \frac{g_m^+(w)}{cw^{c-1}}$$

we get $\tilde g_m = g_m^-$, thereby completing the induction step.

To conclude we observe that for any sequence of analytic maps $\varphi_m(w)$ vanishing at 0 the sequence of analytic diffeomorphisms

$$\hat\Phi_m(z,w) := (z, w + z^m\varphi_m(w))\circ\cdots\circ(z, w + z^2\varphi_2(w))\circ(z, w + z\varphi_1(w))$$

satisfies $\hat\Phi_m(z,w) = \hat\Phi_{m-1}(z,w)\bmod(z^m)$, hence formally converges. □

6.2. **Some explicit examples.** We include here some explicit examples of skew-products $f\colon (\mathbb{C}^2,0)\to(\mathbb{C}^2,0)$ in order to illustrate our results. To simplify notation, we use the following convention: if $\phi\in\widehat{\mathbb{L}}$ is a generalized Puiseux series, we denote the point $\zeta(\phi,e^{-t})$ by $\phi+\star z^t$.

6.2.1. *An example with uniformly bounded multiplicity in $\mathcal{K}$.* Consider the map $f(z,w)=(z^4,w^2-z^4)$. In this case, $\mathrm{Jac}_w(f)=2w$, and we have only one critical branch $c_0=(w=0)$. In this case, $\mathcal{K}$ is contained in $\overline{B}(0;e^{-2}))$, and $f_\diamond(\phi(z))=\phi(z^{1/4})^2-z$ so $f_\diamond(c_0)=f_\diamond(0)=-z\notin\overline{B}(0;2)$, hence the orbit of c_0 converges to ζ_g. In this case the multiplicity is uniformly bounded in $\mathcal{K}$. Moreover, $\overline{\mathcal{B}}_1$ consists of two balls, with boundary points $y_1=z^2+\star z^3$ and $y_2=-z^2+\star z^4$. In particular $\bigcup\mathcal{B}_1$ is disjoint from the critical tree $[\zeta_g,0]$. The generic multiplicities of y_1,y_2 and of their images under $f_\diamond$ are all equal to 1, and we deduce that all elements in $\mathcal{K}$ have multiplicity 1.

This example was studied extensively by Gignac in [Gig14], where the author studies the growth of the intersection of curves in $\mathcal{K}$ under iteration.

6.2.2. *An example with unbounded multiplicity in $\mathcal{K}$.* Consider now the map $f(z,w)=\big(z^5,w^3-3zw^2\big)$. Its jacobian is $\mathrm{Jac}_w(f)=3(w^2-2zw)$, and we get two critical branches $c_0=0$ and $c_1=2z$. Observe that c_0 is fixed, and $f_\diamond(c_1)=-4z^{3/5}$, $f_\diamond^2(c_1)=-64z^{9/25}(1+o(1))$ so that $f_\diamond^n(c_1)\to\zeta_g$. Also c_0 is not totally invariant, hence $\mathcal{K}$ is a Cantor set by Theorem 3.2. It contains non rigid points and rigid points of unbounded multiplicity by Theorem 5.1.

7. Dynamics in the complex basin of attraction

In this section, we suppose that

$$f(z,w)=\Big(z^d,w^c+zh(z,w)\Big) \tag{1}$$

where h is a holomorphic map defined in $D(0,r)^2$ such that $h(0,0)=0$. Under this assumption, the point $(0,0)$ is fixed and superattracting and we denote by Ω its immediate basin of attraction. We write $|p|=|(z,w)|=\max\{|z|,|w|\}$. Recall that the contraction rate $c_\infty(p)$ of a point in Ω is defined by

$$\log c_\infty(p)=\lim_{n\to\infty}\frac{1}{n}\log|\log|f^n(p)||, \tag{34}$$

when the limit exists. The following is a more precise version of Theorem A.

Theorem 7.1. *Let f be of the form (1) with $2\le c<d$. Then the sequence of psh functions $(z,w)\mapsto\frac{1}{c^n}\log|f^n(z,w)|$ converges in L^1_{loc} to a psh function $g\colon\Omega\to\mathbb{R}\cup\{-\infty\}$ such that e^g is continuous, and satisfies*

$$g\circ f=cg\,.$$

The positive closed $(1,1)$ current dd^cg is supported on the locus $\mathcal{W}:=\{g=-\infty\}$ which is a closed pluripolar set. It gives no mass to curves unless f is conjugated to a product map, in which case T is concentrated on a smooth curve intersecting transversally $\{z=0\}$.

Finally, the contraction rate exists at every $p \in \Omega$, and $c_\infty(p) = c$ if $p \notin \mathcal{W}$, and $c_\infty(p) = d$ if $p \in \mathcal{W}$.

Remark 7.2. Note that the existence of the limit g follows from the general result [FJ07, Thm B]. However, for a general superattracting germ, e^g may not be continuous, and the set of possible attraction rates $\{c_\infty(p), p \in \Omega\}$ may not necessarily be finite. A global analog of this phenomenon was exhibited in [DDS05].

Corollary 7.3. *If P is of the form*

$$P(z,w) = P_z(w) = w^k + \sum_{j=0}^{k-1} \alpha_j(z) w^j,$$

with $\alpha_j(0) = 0$ for every $0 \le j \le k-1$, then the sequence $\frac{1}{kc^n} \log P \circ f^n$ converges in $L^1_{\mathrm{loc}}(\Omega)$ to g.

Proof of Theorem 7.1. Writing $f^n(z,w) = (z_n, w_n)$, we have

$$\frac{1}{c^n} \log |f^n(z,w)| = \frac{1}{c^n} \log \max(|z_n|, |w_n|) = \max\left(\frac{d^n}{c^n} \log |z|, \frac{1}{c^n} \log |w_n|\right).$$

Since $\frac{d^n}{c^n} \log |z|$ converges uniformly to $-\infty$, it is enough to focus on the convergence of $g_n \colon (z,w) \mapsto \frac{1}{c^n} \log |w_n|$.

After appropriate scaling of the coordinates, we may assume that $r < 1/2$ and $\|h\| < 1/2$. Let $\Omega_0 := \{(z,w),\ |z| < |w|^c\}$. We claim that $f(\Omega_0) \subset \Omega_0$. Indeed, for $p = (z,w) \in \Omega_0$ we have

$$|w_1| = |w^c + z h(z,w)| \ge |w|^c - \|h\| \cdot |z| \ge \frac{1}{2} |w|^c$$

hence

$$|z_1| = |z|^d < |w|^{cd} \le |w|^{c(d-c)} 2^c |w_1|^c < |w_1|^c.$$

Now, for $n \ge 1$, since $f^n(p) \in \Omega_0$, we have that $|f^n(p)| \sim |w_n|$ and

$$w_{n+1} = w_n^c + z_n h(z_n, w_n) = w_n^c \left(1 + \frac{z_n}{w_n^c} h(z_n, w_n)\right).$$

By definition of Ω_0 we have that $|z_n w_n^{-c} h(z_n, w_n)| \le 1/2$, hence

$$\frac{1}{c^{n+1}} \log |w_{n+1}| - \frac{1}{c^n} \log |w_n| = \frac{1}{c^n} \log \left|1 + \frac{z_n}{w_n^c} h(z_n, w_n)\right| \le \frac{\log 2}{c^n}. \tag{35}$$

Recall that $g_n(z,w) = \frac{1}{c^n} \log |w_n|$. It follows from the previous inequality that the sequence (g_n) converges uniformly in Ω_0 to a pluriharmonic function g such that $g(p) = \log |p| + O(1)$ at the origin. Furthermore, since in Ω_0 $f^n(p) \sim w_n$, $\frac{1}{c^n} \log |f^n|$ converges to g as well, so $g \circ f = cg$.

Let now $\Omega' = \bigcup_{n \ge 0} f^{-n}(\Omega_0) \subset \Omega$ and $\mathcal{W} = \Omega \setminus \Omega'$. Then g extends by iteration to a pluriharmonic function in Ω'. If $p = (z,w) \in \mathcal{W}$ then for every $n \ge 0$ we have $|z_n| \ge |w_n|^c$, and since $z_n = z^{d^n}$ we infer that $c_\infty(p) = d$, and $\lim_{n \to \infty} \frac{1}{c^n} \log |f^n(p)| = -\infty$. Put $g(p) = -\infty$ for $p \in \mathcal{W}$.

Let us show that $g(p) \to -\infty$ when $p \in \Omega$ tends to $\mathcal{W}$. To establish this, for $N \geq 1$, let $\Omega_N = \bigcup_{n=0}^N f^{-n}(\Omega_0)$, which is an increasing sequence of open sets, and fix $p \in \Omega_N \backslash \Omega_{N-1}$. For $0 \leq j \leq N-1$ we have $f^j(p) \notin \Omega_0$ hence

$$\left|f^{N-1}(p)\right| \leq \max(|z_{N-1}|, |w_{N-1}|) \leq |z_{N-1}|^{1/c} \leq |p|^{d^{N-1}/c}.$$

Being f superattracting, it follows that $\left|f^N(p)\right| \leq |p|^{d^{N-1}/c}$ as well. Now, since $f^N(p) \in \Omega_0$, when $n \to \infty$ we have $|f^n(p)| \sim |w_n|$, and

$$\begin{aligned} \frac{1}{c^n} \log|w_n| &= \frac{1}{c^N} \log|w_N| + \sum_{q=N}^{n-1} \left(\frac{1}{c^{q+1}} \log|w_{q+1}| - \frac{1}{c^q} \log|w_q| \right) \\ &\leq \frac{1}{c^N} \log\left|f^N(p)\right| + \sum_{q=N}^{n-1} \frac{\log 2}{c^q} \leq \frac{1}{dc} \frac{d^N}{c^N} \log|p| + \frac{c}{c-1} \log 2, \end{aligned} \tag{36}$$

where in the inequalities we use the estimate on $|f^n(p)|$ and (35). Thus we conclude that

$$g(p) \leq C_1 \frac{d^N}{c^N} \log|p| + C_2$$

for positive constants C_1 and C_2, and the result follows by letting $N \to \infty$.

We claim that (g_n) converges to g pointwise, and even locally uniformly in Ω, in the sense that $|w_n|^{1/c^n}$ converges locally uniformly to e^g. Therefore, the convergence holds in $L^1_{\mathrm{loc}}(\Omega)$, which implies that g is psh, $g \circ f = cg$, and the remaining properties. We already know that the sequence converges locally uniformly outside $\mathcal{W}$ to a continuous pluriharmonic function g and pointwise to $-\infty$ on $\mathcal{W}$. Fix a small $s \neq 0$ and a vertical line $\{z = s\} \subset D(0,r)^2$. In this line, for every n, w_n depends only on w, and $\Omega_N^\complement$ is a subset of the form $w_N^{-1}(\{|w| < s_N\})$, where $s_N = |s|^{d^N/c}$, so it is polynomially convex. Therefore, by the Maximum Principle, since the inequality (36) holds holds at the boundary of $\overline{\Omega_N}$, it holds everywhere in $\Omega_N^\complement$. Thus, the announced local uniform convergence follows.

Suppose that $\mathcal{W}$ is an analytic curve C. Then $T = \mathrm{dd}^c g$ is a current of integration, and since $f^*T = cT$, it follows that C is totally invariant. Let $\{C_i\}$ be the set of branches of C, and denote the associated points in $B(0,1)^{\mathrm{an}}$ by x_{C_i}. Then the finite set $\{x_{C_i}\}$ is totally invariant by $f_\diamond$, which implies $\mathcal{K}$ to be reduced to a singleton by the convergence (24) in Theorem 3.10. This shows that C is irreducible and totally invariant by f. By Theorem 3.2, C is smooth and transversal to $(z = 0)$. Changing coordinates, we may suppose that $C = (w = 0)$ so that $f(z, w) = (z^d, w^c \times \mathrm{unit})$. It then follows from Lemma 6.6 (with coordinates reversed, see Remark 6.7) that f is analytically conjugated to a product map.

Finally, suppose that T carries positive mass on an analytic curve. In this case we claim that $\mathcal{W}$ is a subvariety, in which case the previous paragraph applies. Indeed write the Siu decomposition $T = \sum_{W \in \mathcal{A}} \alpha_W [W] + T_{\mathrm{diffuse}}$, where $\mathcal{A}$ is an at most countable collection of analytic subvarieties and T_{diffuse} gives no mass to curves. Since f^* preserves both the analytic and diffuse part, the current $\sum_{W \in \mathcal{A}} \alpha_W [W]$ is invariant under $c^{-1} f^*$.

Suppose by contradiction that $\mathcal{A}$ is infinite. One can then find $W_0 \in \mathcal{A}$ such that $f^k(W_0)$ is not a critical component for any $k \geq 0$, thus $f^*[f^{k+1}(W_0)] - [f^k(W_0)]$ gives no mass to $[f^k(W_0)]$. The invariance relation then yields

$$\alpha_{f^k(W_0)} = \mathrm{ord}_{f^k(W_0)}\left(c^{-1}f^*\sum_{W\in\mathcal{A}}\alpha_W[W]\right) = c^{-1}\alpha_{f^{k+1}(W_0)}$$

which implies $\alpha_{f^{k+1}(W_0)} \to \infty$, a contradiction. So we are back to the setting of the previous paragraph and the argument is complete. $\square$

Proof of Corollary 7.3. Put

$$\begin{aligned} h_n(z,w) &:= \frac{1}{kc^n}\log P\circ f^n(z,w) = \frac{1}{kc^n}\log P(z_n,w_n) = \frac{1}{kc^n}\log P_{z_n}(w_n) \\ &= \frac{1}{kc^n}\log\left(w_n^k + \sum_{j=0}^{k-1}\alpha_j(z_n)w_n^j\right). \end{aligned}$$

With $g_n = c^{-n}\log|w_n|$ as above, we get

$$h_n(z,w) - g_n(z,w) = \frac{1}{kc^n}\log\left(1 + \sum_{j=0}^{k-1}\alpha_j(z_n)w_n^{j-k}\right).$$

For $(z,w)\notin\mathcal{W}$ we have $z_n = z^{d^n}$ and $|w_n| \gtrsim \beta^{c^n}$ where $\beta < 1$ is locally uniform, so $h_n - g_n$ converges locally uniformly to 0 outside $\mathcal{W}$. Thus, the sequence of psh functions (h_n) is bounded from above and converges to g outside the pluripolar set $\mathcal{W}$, hence it converges to g in $L^1_{\mathrm{loc}}(\Omega)$, and we are done. $\square$

8. Structure of the invariant current

This section is devoted to the proof of Theorem D (see also Theorem 8.2 below) which describes the structure of the invariant current as an average of integration currents over curves in $\mathcal{K}$. We first need to explain how a map of the form (1) acts on $B^{\mathrm{an}}(0,1) \subset \mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}(\!(z)\!)}$.

8.1. **Action of a formal conjugacy on the Berkovich open unit ball.** If K is any complete non-Archimedean field, any power series $\phi = \sum_{n\geq 0} a_n T^n$ such that $|a_0| < 1$ and $|a_i| \leq 1$ for all $i \geq 1$ induces a self map of the (classical) unit ball $B(0,1) \subset K$. If K is algebraically closed, it follows from Hensel's lemma that ϕ is invertible iff $|a_1| = 1$ and $|a_i| < 1$ for all $i \neq 1$ (see e.g. [Ben19, §3.4]). This action extends to the Berkovich unit ball $B^{\mathrm{an}}(0,1)$ (see [BR10] for a discussion of $\overline{B}^{\mathrm{an}}(0,1)$ in these terms, which is readily adapted to this case by using the fact that $B^{\mathrm{an}}(0,1) = \bigcup_{r<1}\overline{B}^{\mathrm{an}}(0,r)$). In particular if $|a_1| = 1$ and $|a_i| < 1$ for $i > 1$, then ϕ induces an analytic isomorphism of $B^{\mathrm{an}}(0,1)$ (whenever K is algebraically closed or not).

Back to $K = k(\!(z)\!)$, recall that a germ of irreducible formal curve C containing 0 and not included in $\{z = 0\}$ defines a a type 1 point $x_C \in B^{\mathrm{an}}(0,1)$ by setting $-\log|P(x_C)| = \dfrac{C\cdot(P=0)}{m(C)}$ for all $P \in \mathbb{C}(\!(z)\!)[w]$, where we recall that $m(C) = C\cdot(z=0)$. The map $C \mapsto x_C$ is a bijection between germs of curves as above and type 1 points in $B^{\mathrm{an}}(0,1)$.

From now on we assume that f is a holomorphic map in $(\mathbb{C}^2, 0)$ of the form (1) with $2 \le c < d$. It follows from Theorem 6.1 that f is conjugated by a *formal* map of the form $\Phi(z,w) = \Big(z, w + \sum_{n\ge 1} z^n \varphi_n(w)\Big)$ with $\varphi_n \in \mathbb{C}[[w]]$ to

$$\tilde{f}(z,w) := \left(z^d, w^c + \sum_{j=0}^{c-1} h_j(z) w^j\right)$$

so that $f = \Phi \circ \tilde{f} \circ \Phi^{-1}$. From the above discussion, the map Φ induces an analytic isomorphism of the Berkovich open unit ball $B^{\mathrm{an}}(0,1) \subset \mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}((z))}$ (resp. $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{L}}$), and for the corresponding actions on $B^{\mathrm{an}}(0,1)$, we have $f_\diamond = \Phi \circ \tilde{f}_\diamond \circ \Phi^{-1}$. Thanks to this conjugacy, all the definitions and results from Sections 2 to 5 apply to $f_\diamond$. In particular we can define $\mathcal{K} := \mathcal{K}(f)$ and $\mathcal{K}_{\mathbb{L}} := \mathcal{K}_{\mathbb{L}}(f)$, and the ergodic measure μ_{na} supported on $\mathcal{K}$ (resp. $\mu_{\mathrm{na},\mathbb{L}}$ on $\mathcal{K}_{\mathbb{L}}$), which are the images of the corresponding objects associated to $\tilde{f}$.

8.2. **The main statement.** Let us first present a basic correspondence between the dynamics of f in $(\mathbb{C}^2, 0)$ and that of $f_\diamond$ in $B^{\mathrm{an}}(0,1)$, which may be viewed as an introduction to Theorem D.

Proposition 8.1. *Let C be any germ of analytic irreducible curve containing* 0 *and not included in* $\{z = 0\}$.

Then $x_C \in \mathcal{K}$ if and only if $C \subset \mathcal{W}$.

Proof. Fix any parametrization $t \mapsto (t^q, \phi(t))$ where $q = m(C)$ and ϕ is analytic. Note that $\phi(z^{1/q})$ is then a Puiseux series determining C.

Suppose first that $x_C \notin \mathcal{K}$. Then for any given $\rho < 1$, there exists an integer n such that $x_{f^n(C)}$ belongs to the annulus $\{\rho < |x| < 1\} \subset B^{\mathrm{an}}(0,1)$. Choose any $c' < 1/c$, and write $\rho = e^{-c'}$. Recall from the proof of Theorem 7.1 that since $h(0,0) = 0$, we may assume $|h| < 1/2$, and for any point $p = (z,w)$ in $\Omega_0 = \{|z| < |w|^c\}$, the inequality

$$|w_1| = |w^c + zh(z,w)| \ge |w|^c - \|h\| \cdot |z| > \frac{1}{2}|w|^c$$

holds, so that in particular $\Omega_0 \subset \{g > -\infty\}$. Replacing C by its image under f^n, we may suppose that $\rho < |x_C| < 1$ which means that $\mathrm{ord}_t(\phi) < qc'$. Now one can find a constant $A > 0$ such that for any small enough t,

$$|\phi(t)|^c \ge A|t|^{c\times \mathrm{ord}_t(\phi)} \ge A(|t|^q)^{c\times \mathrm{ord}_t(\phi)/q} > (|t|^q)^{cc'} > |t|^q \ .$$

Possibly reducing C, this implies that $C \subset \Omega_0 \subset \{g > -\infty\}$, hence C is not included in $\mathcal{W}$.

Conversely suppose that $x_C \in \mathcal{K}$. Define a sequence of negative subharmonic functions on a neighborhood of the unit disk by $g_{n,C} \colon t \mapsto \frac{1}{c^n} \log |w_n(t^q, \phi(t))|$. Suppose by contradiction that $(g_{n,C})$ does not converge to $-\infty$. Then by the Hartogs lemma there exists a subsequence $(g_{n_j,C})$ converging to a subharmonic function $\tilde{g}$. Consider the Lelong number $\nu(g_{n,C}, 0)$ of the subharmonic function $g_{n,C}$ at 0: it is equal to

$$\frac{1}{c^n}\mathrm{ord}_t(w_n(t^q, \phi(t))) = -\frac{1}{c^n} \log |w_n(x_C)| = -\frac{d^n}{c^n} \log |f^n_\diamond(x_C)|$$

which tends to infinity by Theorem 3.6. On the other hand, the upper semicontinuity of the Lelong number implies that

$$\limsup_j -\frac{1}{c^{n_j}} \log |w_{n_j}(x_C)| \leq \nu(\tilde{g}, 0) < \infty.$$

This contradiction shows that $g|_C \equiv -\infty$, hence $C \subset \mathcal{W}$. □

When no critical point of $f_\diamond$ is contained in $\mathcal{K}$, Corollary 4.2 implies that points in $\mathcal{K}$ are associated with formal curves of uniformly bounded multiplicity. By the previous proposition, this condition is equivalent to the statement that no irreducible component of the critical locus of f (besides $\{z = 0\}$) is contained in $\mathcal{W}$. In this situation, we have a description of the current of Theorem 7.1 in terms of non-Archimedean data. The following is a precise version of Theorem D:

Theorem 8.2. *Assume that f is a holomorphic map of the form (1) with $2 \leq c < d$, and that no critical branch is contained in $\mathcal{W}$. Then, $\mathcal{W}$ is a union of analytic curves through the origin. Furthermore, outside the origin, $\mathcal{W}$ is the support of a lamination by Riemann surfaces whose transversals are Cantor sets.*

More precisely, there exists $r_0 > 0$ such that all Puiseux series $\phi \in \mathcal{K}_{\mathbb{L}}$ converge in the disk $\mathbb{D}(0, r_0)$. For any $x \in \mathcal{K}$, if we denote by $C(x)$ the analytic curve in $\mathbb{D}(0, r_0) \times \mathbb{C}$ parameterized by $t \mapsto (t^{m(x)}, \phi(t^{m(x)}))$, where $\phi \in \mathbb{C}\langle\langle z \rangle\rangle$ is such that $x = \pi(\phi)$, we have the integral representation

$$T = \mathrm{dd}^c g = \int_{\mathcal{K}} \frac{[C(x)]}{m(x)} \, \mathrm{d}\mu_{\mathrm{na}}(x) \ .$$

In particular T is a uniformly laminar current outside the origin. See [BLS93] or [Duj23] for this notion and some basic facts that will be needed in § 8.8.

Question 8.3. *Consider any holomorphic map of the form (1) with $2 \leq c < d$. Does there exist $r > 0$ such that for any series $\sum_n a_n z^{\beta_n} \in \mathcal{K}_{\mathbb{L}}$, then $\sum_n |a_n| r^{\beta_n} < \infty$?*

A positive answer to this question could possibly lead to an improvement of the previous result, without any assumption on the critical of f. When $x \in \mathcal{K}_{\mathbb{L}}$, the curve $C(x)$ would be replaced by the positive closed $(1,1)$ current defined in [FJ05, Proposition 6.9].

The remainder of this section will be devoted to the proof of the theorem.

8.3. **The open ball partition and the coding map.** The first step is to construct a sequence of coverings $\mathcal{B}_n$ of $\mathcal{K}$ by open balls. This sequence differs from the one introduced in the proof of Theorem 3.2 which involves closed balls.

We start with any open ball B_0 containing $\mathcal{K}$ whose boundary x_0 is a type 2 point. Its image under $f_\diamond$ is an open ball which also contains $\mathcal{K}$ hence either $f_\diamond(B_0) \subseteq B_0$ or $f_\diamond(B_0) \supsetneq B_0$. Since $(f^n)_\diamond(x_0) \to \zeta_g$, the former case does not happen, and we infer that $f_\diamond^{-1}(B_0)$ is a finite union of open balls whose closures are contained in B_0.

Set $\mathcal{B}_0 = \{B_0\}$, and let $\mathcal{B}_n$ be the collection of connected components of $f_\diamond^{-n}(B_0)$. Each $\mathcal{B}_n$ is a finite collection of disjoint open balls, such that $\mathcal{K} \subset \bigcup \mathcal{B}_n$, and $\bigcup \mathcal{B}_n \subset \bigcup \mathcal{B}_{n-1}$. More precisely, for any $x \in \mathcal{K}$, denote by $B_n(x)$ the unique open ball belonging to $\mathcal{B}_n$ and containing x. Then for any $x \in \mathcal{K}$, $\bigcap_{n \in \mathbb{N}} B_n(x) = \{x\}$. This follows for instance from the fact that $(d/c)^n(f^n)^*_\diamond\delta_{x_0}$ converges weakly to a probability measure whose support is $\mathcal{K}$: indeed, if $\bigcap_{n\in\mathbb{N}} B_n(x)$ were strictly bigger than $\{x\}$, it would contain an open set intersecting $\mathcal{K}$, but disjoint from $(d/c)^n(f^n)^*_\diamond\delta_{x_0}$ for every n.

By our assumption together with compactness, we can find $N \in \mathbb{N}$ such that $\bigcup \mathcal{B}_{N-1}$ does not contain any critical branch. This integer N will be fixed from now on, and we will use the two covering of balls $\mathcal{B}_{N-1}$ and $\mathcal{B}_N$ to encode the dynamics. From now on we denote $\mathcal{B} = \mathcal{B}_N$ and $\mathcal{B}' = \mathcal{B}_{N-1}$.

Let us introduce an oriented graph Γ whose vertex set $V(\Gamma)$ is the collection of open balls $\mathcal{B}$ by declaring that we draw an oriented edge from B_1 to B_2 when $B_2 \subset f_\diamond(B_1)$. We then consider the set Σ of all infinite sequences $(B_i)_{i\geq 0} = B_0B_1B_2\cdots$, with $B_i \in \mathcal{B}$ such that (B_i, B_{i+1}) is an edge of Γ for all i. Then (Σ, σ) is a *subshift of finite type*, where $\sigma\colon (B_i)_{i\geq 0} \mapsto (B_{i+1})_{i\geq 0}$ is the left shift.

Denote by $A = (A_{vv'})_{v,v'\in V(\Gamma)}$ the transition matrix of Γ (that is, $A_{vv'} = 1$ if vv' is an edge, and 0 otherwise). The construction of the sequence of ball coverings $(\mathcal{B}_n)_n$ guarantees that A is primitive, that is A^k has positive entries for some k. Let M be the unique positive (right) eigenvector of A, normalized so that $\sum_{v\in V(\Gamma)} M_v = 1$. The Parry measure μ_Σ on (Σ, σ) is its unique measure of maximal entropy. When the topological degree is constant and equal to some c, the matrix A has exactly c entries equal to 1 in each column, and the left Perron Frobenius eigenvector is $(1, \ldots, 1)$, associated to the dominating eigenvalue c. Then the structure of μ_Σ is easy to describe: for any cylinder $[B_m \cdots B_n] \subset \Sigma$ we have

$$\mu_\Sigma([B_m \cdots B_n]) = c^{-(n-m)} A_{B_mB_{m+1}} \cdots A_{B_{n-1}B_n} \mu_\Sigma([B_n]) \tag{37}$$

and furthermore $\mu_\Sigma([B]) = M_B$ is the entry of M corresponding to B. In particular μ is balanced, that is, $\sigma^*\mu_\Sigma = c\mu_\Sigma$.

There is a canonical coding map $\mathsf{c}\colon \mathcal{K} \to \Sigma$ sending a point x to the sequence $\big(B(f^i_\diamond(x))\big)_{i\geq 0} \in \Sigma$, which semi-conjugates $f_\diamond$ to the left shift σ.

Recall that we have a continuous surjective map $\pi\colon B(0,1)^{\mathrm{an}}_{\mathbb{L}} \to B(0,1)^{\mathrm{an}}$. We also denote by $\mathcal{B}_{n,\mathbb{L}}$ the collection of open balls $\pi^{-1}(B)$ with $B \in \mathcal{B}_n$. For the same integer N as above, we get an oriented graph $\Gamma_{\mathbb{L}}$, a set of itineraries $\Sigma_{\mathbb{L}}$, and a coding map $\mathsf{c}_{\mathbb{L}}\colon \mathcal{K}_{\mathbb{L}} \to \Sigma_{\mathbb{L}}$ which semi-conjugates $f_\diamond$ to the left shift.

The absolute Galois group G of $\mathbb{C}[[z]]$ also acts on $\Gamma_{\mathbb{L}}$, hence on $\Sigma_{\mathbb{L}}$, and its action on $\Sigma_{\mathbb{L}}$ is compatible with the one on $\mathcal{K}_{\mathbb{L}}$. To summarize the discussion, we have a commutative diagram which is G-equivariant and respects the dynamics:

$$\begin{tikzcd} \mathcal{K}_{\mathbb{L}} \arrow[r, "\pi"] \arrow[d, "\mathsf{c}_{\mathbb{L}}"] & \mathcal{K} \arrow[d, "\mathsf{c}"'] \\ \Sigma_{\mathbb{L}} \arrow[r] & \Sigma \end{tikzcd} \tag{38}$$

Under our assumption that $\mathcal{B}$ does not contain any critical branch, the map $\mathsf{c}_{\mathbb{L}}$ becomes a homeomorphism. More precisely, we have

Proposition 8.4.

(1) The coding map $\mathsf{c}\colon \mathcal{K} \to \Sigma$ is a surjective continuous map, semi-conjugating $f_\diamond\colon \mathcal{K} \to \mathcal{K}$ to the left shift $\sigma\colon \Sigma \to \Sigma$.

(2) *The coding map* $\mathsf{c}_{\mathbb{L}}\colon \mathcal{K}_{\mathbb{L}} \to \Sigma_{\mathbb{L}}$ *is a homeomorphism conjugating* $f_\diamond\colon \mathcal{K}_{\mathbb{L}} \to \mathcal{K}_{\mathbb{L}}$ *to the left shift* $\sigma\colon \Sigma_{\mathbb{L}} \to \Sigma_{\mathbb{L}}$.
(3) *The space* $\mathcal{K}_{\mathbb{L}}/G$ *(resp.* $\Sigma_{\mathbb{L}}/G$*) is homeomorphic to* $\mathcal{K}$ *(resp. to* Σ*).*

Recall from § 4.2 that if the boundary point of an open ball $B \subset B(0,1)^{\mathrm{an}}$ has generic multiplicity 1, then $m(B) = 1$ as well, hence $\pi^{-1}(B)$ is a single open ball in $B(0,1)^{\mathrm{an}}_{\mathbb{L}}$.

Corollary 8.5. *Suppose that the boundary point of any open ball from the open covers* $\mathcal{B}$ *and* $\mathcal{B}'$ *has generic multiplicity* 1*. Then* $\pi\colon \mathcal{K}_{\mathbb{L}} \to \mathcal{K}$ *and* $\mathsf{c}\colon \mathcal{K} \to \Sigma$ *are homeomorphisms conjugating the dynamics. Furthermore,* $\mathsf{c}_*\mu_{\mathrm{na}} = \mu_\Sigma$.

Proof. Our assumption implies that the oriented graphs Γ and $\Gamma_{\mathbb{L}}$ are isomorphic under π, so that the left shifts on Σ and $\Sigma_{\mathbb{L}}$ are conjugated. Therefore the bottom arrow $\Sigma_{\mathbb{L}} \to \Sigma$ in the commutative diagram (38) is a homeomorphism and so are $\pi\colon \mathcal{K}_{\mathbb{L}} \to \mathcal{K}$ and $\mathsf{c}\colon \mathcal{K} \to \Sigma$.

The last assertion follows from the fact that if $\mathcal{K}$ is homeomorphic to $\mathcal{K}_{\mathbb{L}}$, then $f_\diamond : \mathcal{K} \to \mathcal{K}$ is c-to-1 and by Theorem 3.10 the measure μ_{na} is balanced, thus $\mathsf{c}_*\mu_{\mathrm{na}}$ must coincide with μ_Σ. □

Proof of Proposition 8.4. The continuity of c follows directly from the description of $\mathcal{K}$ as $\mathcal{K} = \bigcap_{n\geq 0} \bigcup \mathcal{B}_n$ and the continuity of $f_\diamond$. Its surjectivity follows from the fact that given any allowed sequence $(B_i) \in \Sigma$, we can define a nested sequence of balls $\bigcap_{i\geq 0} f_{-i}(B_i)$ whose intersection is non-empty, connected and contained in $\mathcal{K}$, hence reduced to a point, whose image under c is the given itinerary.

The second assertion follows from the claim that in $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{L}}$, for any open ball B which does not contain any critical branch, then $f_\diamond\colon B \to f_\diamond(B)$ is injective. Indeed, this implies that $f_\diamond^{-1}(B) \cap B'$ is an open ball, for any pair of balls B and B' belonging to $\mathcal{B}_n$ and any $n \geq N$, which in turn forces $\mathsf{c}_{\mathbb{L}}$ to be injective, hence a homeomorphism.

To prove our claim, with notation as in Lemma 2.5, since τ is a homeomorphism, we can work with $\tilde{f}$ instead of f, and we are reduced to the statement that a polynomial $P(w) = w^c + \sum_{j=0}^{c-1} h_j(z)w^j$ with $h_j \in z\mathbb{C}[[z]]$ is injective on any open ball B containing no critical point. This is established e.g. in [Tru14, Prop. 2.6].

Finally, the third assertion follows from Theorem 3.2, the commutative diagram (38) and the discussion preceding it. □

8.4. **Base change and reduction to the multiplicity** 1 **case.** The base change of order $k \in \mathbb{N}^*$ is the map $\beta_k(z,w) = (z^k, w)$. Thanks to the special form of f, we have the following commutative diagram

$$\begin{tikzcd} (\mathbb{C}^2,0) \arrow[r, "{(z^d, w^c + z^k h(z^k,w))}"] \arrow[d, "\beta_k"'] & (\mathbb{C}^2,0) \arrow[d, "\beta_k"] \\ (\mathbb{C}^2,0) \arrow[r, "{(z^d, w^c + z h(z,w))}"] & (\mathbb{C}^2,0) \end{tikzcd} \tag{39}$$

Observe that the top map $f_k\colon (z,w) \mapsto (z^d, w^c + z^k h(z^k,w))$ is also of the form (1). By Lemma 2.3, $\beta_{k,\diamond}\colon \phi \mapsto \phi(z^{1/k})$ is a homeomorphism on $B(0,1)^{\mathrm{an}}_{\mathbb{L}}$, such that $\beta_{k,\diamond} \circ f_{k,\diamond} =$

$f_\diamond \circ \beta_{k,\diamond}$. It is also well defined over $\mathbb{C}((z))$ but there it is not necessarily a homeomorphism (see Remark 2.4). Note also that $\mathcal{C}(f_k) = (\beta_{k,\diamond})^{-1}(\mathcal{C}(f))$.

Lemma 8.6. *Given any type 2 point $x \in B(0,1)^{\mathrm{an}} \subset \mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}((z))}$, there exists an integer $k \geq 1$ such that for any multiple ℓ of k, every point in $\beta^{-1}_{\ell,\diamond}(x)$ has generic multiplicity* 1.

Proof. We can suppose that $x = \pi(\zeta(\phi, r))$ with $\phi \in \mathbb{C}\langle\langle z\rangle\rangle$, and $r \in e^{\mathbb{Q}_+}$ (see § 1.3). Then any point $y \in \beta^{-1}_{\ell,\diamond}(x)$ is equal to $\pi(\zeta(u \cdot \phi(z^\ell), r^\ell))$ for some u in the absolute Galois group of $\mathbb{C}[[z]]$. To conclude the proof, by definition of the generic multiplicity it is enough to choose k such that $\phi(z^k)$ is a power series and $r^k \in e^{\mathbb{Z}}$. □

We now return to the proof of the theorem. Recall that we fixed an integer N, and two finite covers $\mathcal{B}$ and $\mathcal{B}'$ of $\mathcal{K}(f)$ by open balls. For any given k, then we let $\mathcal{B}^{(k)}$ (resp. $\mathcal{B}'^{(k)}$ be the set of all connected components of $\beta^{-1}_{k,\diamond}(B)$ with $B \in \mathcal{B}$ (resp. $B \in \mathcal{B}'$). This forms a finite open cover of $\mathcal{K}(f_k)$, and since $\mathcal{C}(f) = \beta_k(\mathcal{C}(f_k))$, we get $\mathcal{B}'^{(k)} \cap \mathcal{C}(f_k) = \emptyset$.

By the previous lemma, we can find a sufficiently divisible integer k such that for all points $x = \partial B$ with $B \in \mathcal{B} \cup \mathcal{B}'$, and for all $y \in \beta^{-1}_{k,\diamond}(x)$, then we have $b(y) = 1$. In other words, all boundary points of the open balls in $\mathcal{B}^{(k)} \cup \mathcal{B}'^{(k)}$ have generic multiplicity 1.

8.5. **Construction of the geometric model.** From this point and up to § 8.7, we do the base change of the previous paragraph, and for notational ease, we replace f by f_k. In particular the boundary points of all balls in $\mathcal{B} \cup \mathcal{B}'$ have generic multiplicity 1 and the conclusions of Corollary 8.5 hold for f.

A *model* $\kappa\colon X \to (\mathbb{C}^2, 0)$ is a smooth complex surface X endowed with a proper bimeromorphic map κ to a neighborhood of 0 in $\mathbb{C}^2$ which is an isomorphism away from 0. Any such map can be decomposed into a finite sequence of point blow-ups.

A point $p \in \kappa^{-1}(0)$ is said to be *free* if it is a smooth point of the curve $\pi^{-1}(z = 0)$. Observe that $0 \in (\mathbb{C}^2, 0)$ is free. When X is obtained by successively blowing-up free points only, we say that X is a *free model.*

Any irreducible component E of $\kappa^{-1}(0)$ determines a type 2 point x_E such that for any $P \in \mathbb{C}((z))[w]$,

$$|P(x_E)| = \exp(-b_E^{-1}\mathrm{ord}_E(P)) \text{ where } b_E = \mathrm{ord}_E(z \circ \kappa) \tag{40}$$

(note that here we abuse notation and write $\mathrm{ord}_E(P)$ for $\mathrm{ord}_E(P \circ \kappa)$). One proves by induction on the number of blow-ups that $b_E = b(x_E)$, where b is the generic multiplicity from § 4.2 (see [FJ04, §6]). If C is any germ of curve transverse to E at a free point (a *curvette*, in the terminology of [FJ04]), then $\kappa(C)$ has multiplicity b_E at the origin, hence the terminology (see [FJ04, Rmk 3.41]).

Conversely, pick a type 2 point $x \in B(0,1)^{\mathrm{an}}$. Then there exists a unique sequence of blow-ups $\mathrm{bl}_{p_i}\colon X_i \to X_{i-1}$, $i = 0, \cdots, n$ such that p_i belongs to the exceptional divisor E_i of $\mathrm{bl}_{p_{i-1}}$ for all i, $X_0 = (\mathbb{C}^2, 0)$ and $x = x_{E_n}$. When $b(x) = 1$, X_n is a free model, i.e, all p_i are free. Furthermore, $\mathrm{diam}(x_{E_n}) = e^{-n}$.

More generally, a model $\kappa\colon X \to (\mathbb{C}^2, 0)$ is free iff $b(x_E) = 1$ for every component E of $\kappa^{-1}(0)$.

Pick any free point $p \in \kappa^{-1}(0)$ lying on some irreducible component E. Choose any smooth curve C' passing through p and intersecting E transversally. Denote by $\mathbb{C}\{z\}$ the ring of convergent power series, and let $P_C \in \mathbb{C}\{z\}[w]$ be any equation for $C = \kappa(C')$. Then the set

$$U(p) := \{x \in B(0,1)^{\mathrm{an}}, |P_C(x)| < \exp(-\mathrm{ord}_E(P_C))/b_E) = |P_C(x_E)|\}$$

is the open ball whose boundary point is equal to x_E and containing x_C. This ball can be characterized geometrically as follows: a germ of irreducible curve D belongs to $U(p)$ if and only if its strict transform in X contains p.

Pick any type 2 point x_0 of generic multiplicity 1. Then $m(x_0) = 1$, and the set of type 2 points $x \in B(0,1)^{\mathrm{an}}$ such that $x \geq x_0$ and $b(x) = 1$ is finite. Indeed on $[x_0, \zeta_g]$ the function A is strictly decreasing, and the multiplicity is equal to 1 so that $b(x) = q(A(x))$; since A is bounded from above, the claim follows. Geometrically, these type 2 points correspond to the generic points of divisors located under x_0 in the corresponding sequence of blow-ups.

The next proposition is a variation on [DF16, Proposition 4.6].

Proposition 8.7. *Let $\{B_i\}$ be any finite collection of disjoint open balls in $B(0,1)^{\mathrm{an}}$ such that $b(\partial B_i) = 1$ for all i. Write $x_i = \partial B_i$.*

Then there exists a unique (up to isomorphism) free model $\kappa\colon X \to (\mathbb{C}^2, 0)$ such that the following holds:

- *for each i, there exists a free point p_i such that $B_i = B(p_i)$;*
- *the set of type 2 points $\mathcal{D}(X) := \{x_E\}$ where E ranges over all irreducible components of $\kappa^{-1}(0)$ coincides with the set of all type 2 points x such that $b(x) = 1$ and $x \geq x_i$ for some i.*

Proof. Let us fix an arbitrary model $\kappa\colon X \to (\mathbb{C}^2, 0)$, and first review the contents of [FJ04, Corollary 6.34]. The open set $B(0,1)^{\mathrm{an}} \setminus \mathcal{D}(X)$ is a union of finitely many annuli (i.e., a connected open set having two boundary points), and infinitely many disjoint open balls. More precisely, if U is a connected component of $B(0,1)^{\mathrm{an}} \setminus \mathcal{D}(X)$ whose boundary point is x_E, then there exists a (unique) free point $p \in E$ such that $U = U(p)$. When U is a connected component of $B(0,1)^{\mathrm{an}} \setminus \mathcal{D}(X)$ having two boundary points $x_E \leq x_{E'}$, then E intersects E' at a point p, and a curve C belongs to U iff its strict transform in X contains p.

We now construct X by induction on the number of balls. When the collection of balls is empty, then we set $\kappa = \mathrm{id}$. Assume we have built the model X for a collection $\{B_i\}$ and pick any open ball B whose boundary point x is a type 2 point of generic multiplicity 1, and such that $B \cap B_i = \emptyset$ for all i.

Let U be the connected component of $B(0,1)^{\mathrm{an}} \setminus \mathcal{D}(X)$ containing B. Since the generic multiplicity of x equals 1, then U cannot be an annulus. Indeed if $\partial U = \{x_E, x_{E'}\}$, then $b(x) \geq b(x_E) + b(x_{E'})$ for every type 2 point $x \in U$, as every type 2 point $x \in U$ is of the form x_F, where F is a divisor above the non-free point $E \cap E'$ so $b(x) \geq 2$.

Therefore, U is an open ball $U = U(p)$ where p is a free point lying on some component E of the exceptional divisor of $\kappa : X \to (\mathbb{C}^2, 0)$. Let $X_1 \to X$ be the blow-up at p with exceptional divisor E_1. Then X_1 is a free model, and $x_{E_1} \leq x_E$. We then repeat the same argument. We obtain a sequence of blow-ups at free points $p_\ell \in E_{\ell-1}$ and type 2 points $x_{E_\ell} \le x_{E_{\ell-1}}$ such that x belongs to the closed ball $B(\ell) = \{y \le x_{E_\ell}\}$. The process stops because $\mathrm{diam}(x) \le \mathrm{diam}(B(\ell)) = e^{-1}\mathrm{diam}(B(\ell-1))$, and we conclude that $x = x_{E_\ell}$ for some ℓ.

The model X_ℓ then satisfies all our requirements. The details are left to the reader, see [FJ04, Chapter 6]. $\square$

We say that a model $\kappa\colon X \to (\mathbb{C}^2, 0)$ dominates $\kappa'\colon X' \to (\mathbb{C}^2, 0)$ when $(\kappa')^{-1} \circ \kappa$ is regular, i.e., X is obtained by blowing-up points above X'. Suppose that X dominates X'. Then the set of type 2 points $\mathcal{D}(X')$ is included in $\mathcal{D}(X)$. Conversely, when $\mathcal{D}(X') \subset \mathcal{D}(X)$ then X dominates X', as follows from [FJ07, Proposition 3.2] applied to the identity map.

We denote by $\kappa\colon X \to (\mathbb{C}^2, 0)$ (resp. $\kappa'\colon X' \to (\mathbb{C}^2, 0)$) the model obtained by applying the previous proposition to $\mathcal{B}$ (resp. to $\mathcal{B}'$). Denote by $\{p_j\}$ (resp. $\{p'_i\}$) the collection of free points in X (resp. in X') such that $\{U(p_j)\} = \mathcal{B}$ (resp. $\{U(p'_i)\} = \mathcal{B}'$). By construction, $\mathcal{D}(X)$ contains all type 2 points x of generic multiplicity 1 such that $x \ge \partial B$ for some $B \in \mathcal{B}$, hence in particular the boundary points of all open balls in $\mathcal{B}'$. Thus from the above discussion we see that X dominates X'.

We thus have two commutative diagrams:

$$\begin{tikzcd} X \arrow[r, "\theta"] \arrow[d, "\kappa"'] & X' \arrow[dl, "\kappa'"] \\ (\mathbb{C}^2,0) & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, dashed, "F"] \arrow[d, "\kappa"] & X' \arrow[d, "\kappa'"] \\ (\mathbb{C}^2,0) \arrow[r, "f"] & (\mathbb{C}^2,0) \end{tikzcd}$$

where F is a meromorphic map.

Proposition 8.8. *With notation as above:*

- *The map θ is a finite sequence of free blow-ups. Moreover, we have $\theta(p_j) = p'_i$ where $U(p'_i)$ is the unique ball in $\mathcal{B}'$ containing $U(p_j)$.*
- *The map F is holomorphic at all points p_j. Moreover, we have $F(p_j) = p'_\ell$, where p'_ℓ is such that $f_\diamond(U(p_j)) = U(p'_\ell)$.*

Proof. The first assertion follows from the definitions. The second one is a consequence of [FJ07, Proposition 3.2] applied to f (where divisorial valuations correspond to type 2 points and $U(p)$ has the same meaning). $\square$

8.6. **The graph transform.** We work with the free models X and X' constructed in the previous subsection. For each p'_i we fix coordinates (z'_i, w'_i) centered at p'_i such that $(z'_i = 0)$ is the exceptional divisor.

Pick any p_j, and let $p'_i = \theta(p_j)$. Since θ is a finite sequence of free blow-ups, then there exists a unique choice of coordinates (z_j, w_j) centered at p_j and such that $(z_j = 0)$ is the exceptional divisor at p_j, and

$$(z'_i, w'_i) = \theta(z_j, w_j) = (z_j, z_j^{n_j} w_j + P_j(z_j)) \tag{41}$$

where $n_j \in \mathbb{N}^*$ and P_j is a polynomial of degree at most n_j vanishing at 0. Indeed, each free blow-up can be expressed in coordinates as $(z, w) \mapsto (z, zw + w_0)$, by choosing the exceptional divisor to be $\{z = 0\}$, from which (41) readily follows by composition.

On the other hand, if $p'_\ell = F(p_j)$, then there exists other coordinates $(\hat{z}_j, \hat{w}_j)$ centered at p_j such that $(\hat{z}_j = 0)$ is the exceptional divisor and

$$(z'_\ell, w'_\ell) = F(\hat{z}_j, \hat{w}_j) = (\hat{z}_j^{m_j}, \hat{w}_j) \tag{42}$$

with $m_j \in \mathbb{N}^*$. Indeed, since $\mathcal{C}(f) \cap \mathcal{B} = \emptyset$, close to p_j, the critical set of F is reduced to $(z_j = 0)$ and $F^*(z'_i = 0)$ is a multiple of $(z_j = 0)$. Using these assumptions we get that in the coordinates (z_j, w_j), F is of the form $(z_j^{m_j}(c_1 + g(z_j, w_j)), c_2 w_j + h(z_j, w_j))$, with $c_1 c_2 \neq 0$, $g(0,0) = h(0,0) = 0$ and $\frac{\partial h}{\partial w}(0,0) = 0$ and we choose $\hat{z}_j = z_j(c_1 + g(z_j, w_j))^{1/m_j}$ and $\hat{w}_j = c_2 w_j + h(z_j, w_j)$ to arrive at the form (42).

The graph Γ can be re-interpreted as follows: the vertex set is in bijection with $\mathcal{B}$ hence with the set of free points p_j in X, and an oriented vertex joins p_j to p_ℓ iff $F(p_j) = \theta(p_\ell)$.

Fix some sufficiently small $r > 0$, and consider the space $\mathcal{H}_r$ of holomorphic functions $z \mapsto h(z)$ defined in the open disk of radius r, vanishing at 0, and whose sup norm is bounded by r. In other words, $\mathcal{H}_r$ is the closed ball of radius r in the Hardy space $H^\infty(\mathbb{D}(0,r))$.

For each vertex p_j and every $h \in \mathcal{H}_r$, we define the smooth curve $C_j(h)$ parameterized by $(z_j(t), w_j(t)) = (t, h(t))$ in the bidisk of radius r centered at p_j, in the coordinates (z_j, w_j), which thus intersects $(z_j = 0)$ transversally at p_j.

For any edge $e = (p_j \to p_\ell)$, we can define a graph transform map $\mathcal{L}_e$ on $\mathcal{H}_r$ by the property that $C_j(\mathcal{L}_e(h))$ is the pull-back by F of the image by θ of $C_\ell(h)$. For this, let $p'_i := \theta(p_j)$. Then $\theta(C_j(h))$ is parametrized by $(z'_i(t), w'_i(t)) = (t, t^{n_j} h(t) + P_j(t))$, and we obtain an equation for $C_\ell(\mathcal{L}_e(h))$ of the form

$$\hat{w}_j = (\hat{z}_j^{m_j})^{n_j} h(\hat{z}_j^{m_j}) + P_j(\hat{z}_j^{m_j})$$

To obtain the explicit form of $\mathcal{L}_e(h)$ in terms of h, it is then necessary to invert (z_j, w_j) in terms of $(\hat{z}_j, \hat{w}_j)$. The formulas are not particularly convenient, but we still can prove the following.

Proposition 8.9. *For any $r > 0$ small enough, $\mathcal{L}_e$ maps $\mathcal{H}_r$ into itself, and it is $1/2$-Lipshitz for the sup norm. Moreover, for any $h_1, h_2 \in \mathcal{H}_r$ we have*

$$\mathrm{ord}_t(\mathcal{L}_e(h_1) - \mathcal{L}_e(h_2)) \geq 1 + \mathrm{ord}_t(h_1 - h_2) \tag{43}$$

Proof. The map $\mathcal{L}_e$ is the composition of two maps $\mathcal{L}_e = T_2 \circ T_1$ where

$$T_1(h) = t^{m_j n_j} h(t^{m_j}) + P_j(t^{m_j}),$$

and T_2 is defined by the following condition: the curve parameterized by $(t, T_2(\gamma)(t))$ in the coordinates (z_j, w_j) is identical to the one defined by $(\tau, \gamma(\tau))$ in the coordinates $(\hat{z}_j, \hat{w}_j)$. Write $\hat{z}_j = z_j A_j(z_j, w_j)$ with $A_j(0) \neq 0$, and $w_j = B_j(\hat{z}_j, \hat{w}_j) = b_j \hat{w}_j + O(\hat{z}_j, \hat{w}_j^2)$ with $b_j \neq 0$. Then we have $\tau = t A_j(t, h(t))$ and $T_2(h)(t) = B_j(\tau, h(\tau))$, so that

$$T_2(h)(t) = B_j(t A_j(t, h(t)), h(t A_j(t, h(t)))) \ .$$

The map T_1 is easy to analyze, and it is strictly contracting both for the complex and the non-Archimedean norm because $m_j n_j \geq 1$. More precisely, on the disk of radius $r > 0$, we have that $|T_1(h_1) - T_1(h_2)|_\infty \leq r|h_1 - h_2|_\infty$, and

$$\mathrm{ord}_t(T_1(h_1) - T_1(h_2)) \geq m_j n_j + m_j \mathrm{ord}_t(h_1 - h_2).$$

On the other hand, T_2 is C-Lipschitz with C depending on the sup norm of the partial derivative of A_j and B_j (details are left to the reader) so that the first statement follows when $Cr < 1/2$. Observe also that $\mathrm{ord}_t(A_j(t, h_1) - A_j(t, h_2)) \geq \mathrm{ord}_t(h_1 - h_2)$ which implies $\mathrm{ord}_t(T_2(h_1) - T_2(h_2)) \geq \mathrm{ord}_t(h_1 - h_2)$. Combining this with $\mathrm{ord}_t(T_1(h_1) - T_1(h_2)) \geq 1 + \mathrm{ord}_t(h_1 - h_2)$, we get the estimate (43), and the proof is complete. □

8.7. **Conclusion in the multiplicity 1 case.** As in the previous step, we consider the two models X and X' and the directed graph Γ whose vertices are in bijection with the set of balls $\mathcal{B}$. In order to clarify notation, we shall write B_v for the ball in $\mathcal{B}$ associated with a vertex v. Similarly, we write p_v for the point in X corresponding to B_v. Recall that $f_\diamond(B_v)$ is a ball in $\mathcal{B}'$, and an edge e joins v to w iff $B_w \subset f_\diamond(B_v)$.

Write $\mu_v := \mu_{\mathrm{na}}(B_v)$. Then $\sum_v \mu_v = 1$ since μ_{na} is a probability measure, and, as explained in § 8.3, μ_v is the v-entry of the Perron-Frobenius eigenvector of the transition matrix, associated to the eigenvalue c. From this we get that

$$\frac{1}{c} \sum_{v' \colon v \to v' \in E(\Gamma)} \mu_{v'} = \mu_v.$$

Pick any $\underline{v} = (v_0 v_1 \cdots) \in \Sigma$, and set $e_i = (v_i v_{i+1})$ for every i. We then define the map $\mathcal{L}^{(n)}_{\underline{v}} \colon \mathcal{H}_r \to \mathcal{H}_r$ by

$$\mathcal{L}^{(n)}_{\underline{v}}(h) = \mathcal{L}_{e_0} \circ \mathcal{L}_{e_1} \circ \cdots \circ \mathcal{L}_{e_{n-1}}(h)\ .$$

Given any vertex $v \in \Gamma$, and any analytic map $h \in \mathcal{H}_r$, recall that $C_v(h)$ is the analytic curve defined as the graph of h in the coordinates (z_v, w_v) at p_v, as introduced in the previous subsection.

Lemma 8.10. *For any $\underline{v} = (v_n)_{n\geq 0} \in \Sigma$, the sequence $(\mathcal{L}^{(n)}_{\underline{v}}(h))_{n\geq 0}$ converges in $\mathcal{H}_r$ to a limit function $h_{\underline{v}}$ independent of h, hence the corresponding graphs $C_{v_0}(\mathcal{L}^{(n)}_{\underline{v}}(h))$ converge in the sense of currents. Likewise, the sequence of germs $C_{v_0}(\mathcal{L}^{(n)}_{\underline{v}}(h))$ converges in $B^{\mathrm{an}}(0,1)$ to $C_{v_0}(h_{\underline{v}})$*

Proof. This is a simple consequence of the contraction statement in Proposition 8.9. Indeed, since for any $m \geq n$

$$\left\| \mathcal{L}^{(n)}_{\underline{v}}(h) - \mathcal{L}^{(m)}_{\underline{v}}(h) \right\| \leq 2^{-n} \left\| h - \mathcal{L}_{e_n} \circ \cdots \circ \mathcal{L}_{e_{m-1}}(h) \right\| \leq 2^{1-n} r,$$

we see that $(\mathcal{L}^{(n)}_{\underline{v}}(h))_{n\geq 0}$ is a Cauchy sequence in $\mathcal{H}_r$. The independence of the limit with respect to h follows from the contraction as well.

For the second statement, we argue similarly, this time by using the contraction property of the graph transform operator in the t-adic norm. □

Consider now the positive closed $(1,1)$-current

$$T_0 = \sum_v \mu_v [\kappa(C_v(0))]$$

and for $n \geq 0$, set $T_n = \frac{1}{c^n}(f^n)^*(T_0)$. By Corollary 7.3, the sequence of currents (T_n) converges to $\mathrm{dd}^c g$. For any vertex v, the structure of f as explained in § 8.5 and 8.6 shows that

$$(f^n)^*[\kappa\left(C_v(0)\right)] = \sum_{\underline{v}\in\Sigma,\, \underline{v}_n = v} \left[\kappa\left(C_{v_0}\left(\mathcal{L}_{\underline{v}}^{(n)}(0)\right)\right)\right] ,$$

so that

$$T_n = \frac{1}{c^n}\sum_v \mu_v \sum_{\underline{v}\in\Sigma,\, \underline{v}_n = v} \left[\kappa\left(C_{v_0}\left(\mathcal{L}_{\underline{v}}^{(n)}(0)\right)\right)\right] .$$

If we denote by $\Sigma(n, \underline{v})$ the cylinder of sequences in Σ coinciding with $\underline{v}$ on their first n entries, by the formula (37), this can be rewritten as

$$\begin{aligned} T_n &= \frac{1}{c^n}\sum_v \mu_v \sum_{\underline{v}\in\Sigma,\, \underline{v}_n = v} \mu\left(\Sigma(n,\underline{v})\right) \left[\kappa\left(C_{v_0}\left(\mathcal{L}_{\underline{v}}^{(n)}(0)\right)\right)\right] \\ &= \int_\Sigma \left[\kappa\left(C_{v_0}\left(\mathcal{L}_{\underline{v}}^{(n)}(0)\right)\right)\right] \mathrm{d}\mu_\Sigma\left(\underline{v}\right). \end{aligned}$$

By Lemma 8.10 above, $[C_{v_0}(\mathcal{L}_{\underline{v}}^{(n)}(0))]$ converges to $\left[C_{v_0}(h_{\underline{v}})\right]$, so by the dominated convergence theorem we conclude that

$$\mathrm{dd}^c g = \lim_{n\to\infty} T_n = \int_\Sigma \left[\kappa\left(C_{v_0}\left(h_{\underline{v}}\right)\right)\right] \mathrm{d}\mu_\Sigma\left(\underline{v}\right). \tag{44}$$

On the other hand, again by Lemma 8.10, $C_{v_0}(\mathcal{L}_{\underline{v}}^{(n)}(0))$ converges to $C_{v_0}\left(h_{\underline{v}}\right)$ in $B^{\mathrm{an}}(0,1)$. Denote by $\widehat{F} : X \dashrightarrow X$ the lift of f to X. By construction, for every $n \geq 0$, the curve $\widehat{F}^n\left(C_{v_0}(h_{\underline{v}})\right)$ intersects the exceptional divisor $\kappa^{-1}(0)$ in X at the point p_{v_n}, therefore $\kappa(C_{v_0}(h_{\underline{v}}))$ belongs to $\mathcal{K}$ and its itinerary is given by $\underline{v}$. Therefore, $\kappa(C_{v_0}(h_{\underline{v}})) = C\left(\mathsf{c}^{-1}\left(\underline{v}\right)\right)$, where $C(\cdot)$ is as in Theorem 8.2 and $\mathsf{c}\colon \mathcal{K} \to \Sigma$ is the conjugacy from § 8.3. In particular, $C(x)$ is a closed analytic curve in a fixed neighborhood of the origin, and a graph over the z coordinate. Thus, from (44), we get

$$\mathrm{dd}^c g = \int_\Sigma \left[C\left(\mathsf{c}^{-1}\left(\underline{v}\right)\right)\right] \mathrm{d}\mu_\Sigma\left(\underline{v}\right) = \int_{\mathcal{K}} [C(x)]\, \mathrm{d}\mu_{\mathrm{na}}(x). \tag{45}$$

To complete the proof, it remains to verify the assertions on the laminar structure of $\mathcal{W}$ and T. For this, we observe that if $\underline{v}$ and $\underline{v'}$ are distinct itineraries, the corresponding curves $C_{v_0}\left(h_{\underline{v}}\right)$ and $C_{v'_0}\left(h_{\underline{v'}}\right)$, constructed above, get separated after finitely many iterations, so they are disjoint in X. Pushing forward under κ, we see that their images intersect only at the origin so the curves $C(x)$ for $x \in \mathcal{K}$ form a lamination outside the origin. From the integral representation (45), the support of this lamination is $\mathrm{Supp}(T) = \mathcal{W}$. Since $\mathcal{W}$ is a closed pluripolar set, this lamination is transversally polar, hence transversally totally disconnected, and it has no isolated leaf for otherwise T would give positive mass to a curve.

8.8. **Uniform laminarity of the current** $T(f)$**.** To complete the proof of the theorem, we push forward the previous picture by the base change β_k from § 8.4, and need to verify the assertions of the theorem. We resume the notation from § 8.4.

Since $g(z,w) = \lim_{n\to\infty} \frac{1}{c^n}\log|w_n|$ we see that $g(f)\circ\beta_k = g(f_k)$, so $\beta_k^*\mathrm{dd}^c g(f) = \mathrm{dd}^c g(f_k)$, hence $(\beta_k)_* T(f_k) = (\beta_k)_*\beta_k^* T(f) = kT(f)$ because β_k has topological degree k.

From this we see that the current $T(f)$ is the image of a uniformly laminar current outside the origin under β_k. While this still implies that $\mathrm{dd}^c g$ is laminar (because $\mathcal{W}$ is pluripolar), to get uniform laminarity we need to argue that the images of the curves $C(x)$ do not intersect each other outside the origin also in this case. Fix an open ball U such that $\{z=0\}\cap\overline{U} = \emptyset$. Since β_k is a covering outside $\{z=0\}$, the preimage of U is a union of k disjoint open sets, so that $T(f)$ is actually the sum of k uniformly laminar currents S_i in U, and $\mathcal{W}(f)$ is the union of the corresponding laminations.

Assume that two of these currents (say S_1 and S_2) have non-trivially intersecting plaques, then by [BLS93, Lemma 6.4] some of these intersections are transverse, and by reducing U further we may assume that in U, the plaques are graphs of definite size over two different directions, and every plaque of S_1 intersects every plaque of S_2.

We will argue that the fact (from Corollary 7.3) that $T(f)$ is the limit of pull-backs of smooth curves makes such a local picture impossible. For this, we first work with f_k. By construction of the model, there is a neighborhood of the points p_j in X that is disjoint from the (proper transform of) the critical set. By pushing under κ, there is a f_k^{-1}-invariant "neighborhood" (not open at the origin) $\mathcal{N}(f_k)$ of $\mathcal{W}(f_k)$ disjoint from $\mathcal{C}(f_k)$, and, since $\mathcal{C}(f_k) = (\beta_{k,\diamond})^{-1}(\mathcal{C}(f))$ we get a corresponding neighborhood $\mathcal{N}(f) = \beta_k(\mathcal{N}(f_k))$ for f.

From our choice of r in § 8.6, a curve of the form $C_j = C_j(h)$ passing by some p_j is disjoint from the critical set. Let $\tilde{\Gamma} = \kappa(C_j(h))$ and $\Gamma = \beta_k(\tilde{\Gamma})$, so that Γ is contained in $\mathcal{N}(f)$. If necessary, we work in a smaller neighborhood of 0 so that Γ is smooth outside 0, and we conclude that all curves $f^{-n}(\Gamma)$ are smooth outside the origin. Furthermore $f^{-n}(\Gamma)\cap U$ is made up of plaques of uniformly bounded geometry (since they descend from X via β_k), so any transverse intersection between S_1 and S_2 at the limit must originate from a self-intersection of $f^{-n}(\Gamma)$. This contradiction completes the proof that $\mathcal{W}(f)$ is a lamination, which must have Cantor transverse structure as before, and that $T(f)$ is a uniformly laminar current. □

8.9. **Integral representation of the current** $T(f)$**.** It remains to justify the integral representation of $T(f)$. To simplify notation we write $\beta = \beta_k$. For a probability measure μ on $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}((z))}$, we put $\beta^*\mu = k\beta_\diamond^*\mu$, which is a probability measure (see § 3.4).

Lemma 8.11. *Let μ be a positive measure with finite support on $B^{\mathrm{an}}(0,1)$ on a set of points corresponding to germs of irreducible analytic curves. Then we have*

$$\beta_*\left(\int_{u\in B^{\mathrm{an}}(0,1)} \frac{[C(u)]}{m(u)}\, d\beta^*\mu(u)\right) = k\int_{v\in B^{\mathrm{an}}(0,1)} \frac{[C(v)]}{m(v)}\, d\mu(v) \tag{46}$$

Admitting this result for the moment, we complete the proof of the theorem. Since $\beta_\diamond$ semi-conjugates $f_{k,\diamond}$ to $f_\diamond$, from the equidistribution Theorem 3.10, and the definition of β^* we deduce that $\beta^*\mu_{\mathrm{na}}(f) = \mu_{\mathrm{na}}(f_k)$. In addition, μ_{na} is a limit of atomic measures as in the Lemma, and by the continuity of the operators β^* and β_* for the respective weak topologies on $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}((z))}$ and the space of (1,1) currents, we infer that (46) also holds for the measure $\mu = \mu_{\mathrm{na}}$. Recall that by construction we have that $m(C(v)) = 1$ for all curves $C(v)$ in the support of $\mathcal{K}(f_k)$, and using the relation $\beta_*T(f_k) = kT(f)$ we obtain

$$\begin{aligned} T(f) = \frac{1}{k}\,\beta_* T(f_k) &= \frac{1}{k}\,\beta_* \left(\int_{u\in B^{\mathrm{an}}(0,1)} [C(u)] d\beta^*(\mu_{\mathrm{na}}(f))(u) \right) \\ &= \int_{v\in B^{\mathrm{an}}(0,1)} \frac{[C(v)]}{m(v)} d\mu_{\mathrm{na}}(f)(v) \end{aligned}$$

as was to be shown. □

Proof of Lemma (8.11). We can assume that μ is a Dirac mass at the point x_C where C is an irreducible analytic germ. In terms of currents, we may write $\beta^*[C] = \sum_i k_i[C_i]$ where C_i are the irreducible components of $\beta^{-1}(C)$, and $k_i \in \mathbb{N}^*$. Observe that if $e_i \in \mathbb{N}^*$ denotes the local topological degree of the restriction $\beta\colon C_i \to C$, then we have

$$\beta_*\beta^*[C] = \sum_i k_i \beta_*[C_i] = \sum_i k_i e_i [C] = k[C]$$

since the topological degree of β is equal to k.

Pick any equation of C of the form $C = \{P(w) = 0\}$, where P is a monic polynomial of degree $m(C)$ in w with coefficients in the ring of holomorphic functions in z. Then, with notation as in § 3.4, by (22), we get $\Delta \log_P = m(C)\delta_{x_C}$. We can write $P\circ\beta = \prod_i P_i^{k_i}$ where P_i are equations of C_i, so that

$$\begin{aligned} \beta^*(m(C)\delta_{x_C}) &= k\beta^*_\diamond(m(C)\delta_{x_C}) = k\Delta\left(\log_P \circ \beta_\diamond\right) \\ &= \Delta\left(x \mapsto \log|P(\beta(x))|\right) = \sum_i k_i\, \Delta \log_{P_i} = \sum_i k_i m(C_i)\delta_{x_{C_i}}, \end{aligned}$$

therefore

$$\int_{u\in B^{\mathrm{an}}(0,1)} \frac{[C(u)]}{m(u)} d\beta^*\delta_{x_C}(u) = \frac{1}{m(C)}\left(\sum_i k_i[C_i]\right),$$

and finally

$$\beta_*\left(\int_{u\in B^{\mathrm{an}}(0,1)} \frac{[C(u)]}{m(u)} d\beta^*\delta_{x_C}(u)\right) = \beta_*\left(\frac{1}{m(C)}\sum_i k_i[C_i]\right) = \frac{k}{m(C)}[C]$$

which concludes the proof. □

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