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Institut de Mathématiques de Jussieu - Paris Rive Gauche, UMR 7586
École doctorale de sciences mathématiques de Paris Centre, ED 386

Dynamical singularities

Matteo Ruggiero

Mémoire d'Habilitation à Diriger des Recherches

Paris, 06/03/2025

A short Curriculum Vitae

Education and diplomas

- 01.2008-12.2010: PhD student in Mathematics at Scuola Normale Superiore (Pisa, Italy).
- 15.03.2011: PhD degree in Mathematics at Scuola Normale Superiore di Pisa, with thesis: "The valuative tree, rigid germs and Kato varieties", advisor M. Abate.

Past academic positions

- 09.2011-08.2013: Post-doc of the FMJH at the École Polytechnique, CMLS lab (Palaiseau, France).

Current academic position

- 09.2013-present: Maître de conférences at the Université Paris Cité, UFR de Mathématiques, research unit IMJ-PRG, group Géométrie et Dynamique.

Dynamical systems

- Discrete: $f\colon X \circlearrowleft$, study the behavior of iterates $f^n = \overbrace{f \circ \cdots \circ f}^{n \text{ times}}$.
- Continuous: $f^t\colon X \circlearrowleft$, $f^{t+t'} = f^t \circ f^{t'}$, t real or complex.

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Additional structure:

- X is a (complex) analytic space : $X = P^{-1}(0)$, $P\colon \mathbb{C}^N \to \mathbb{C}^M$, P analytic.
- $f\colon X \circlearrowleft$ analytic selfmap : $F = (f_1, \ldots, f_M)\colon \mathbb{C}^M \circlearrowleft$ analytic, $f = F|_X\colon X \circlearrowleft$.
- $f^t\colon X \circlearrowleft$ is the flow of an analytic vector field χ: $\chi \underset{\text{loc}}{=} \chi_1 \partial_1 + \cdots + \chi_M \partial_M$ tangent to X.

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Local aspects: $Z \subset X$ an f-invariant subvariety, we study the germ $f\colon (X, Z) \circlearrowleft$ (typically: $Z = \{x_0\}$, $x_0 = f(x_0)$ a fixed point).

Singularities of varieties and maps

Varieties

- $x \in X$ is regular if $(X, x) \cong (\mathbb{C}^d, 0)$.
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Maps

X, Y varieties of dimension d.

- Regular: $$\begin{array}{ccc} (X^{\text{reg}}, x) & \xrightarrow[\simeq]{f} & (Y^{\text{reg}}, y) \\ \psi_x \uparrow \simeq & & \simeq \uparrow \psi_y \\ (\mathbb{C}^d, 0) & \xrightarrow{\text{id}} & (\mathbb{C}^d, 0) \end{array}$$
- Singular : $x \in X^{\text{sing}} \cup f^{-1}(Y^{\text{sing}}) \cup \mathcal{C}(f)$.

  Critical set : $\mathcal{C}(f) = \{x \in X \mid f \text{ is not locally invertible at } x\}$.

Dynamical singularities

Selfmaps

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Vector fields

- Regular point : $\chi(x) \neq 0$: $\chi \cong \partial_1$.
- Singular set: $\text{Sing}(\chi) := \{x \mid \chi(x) = 0\}$.

Conjugacy and normal forms

Let $f\colon (X, x_0) \circlearrowleft$ be a selfmap. Assume x_0 regular point.
Classical strategy: search of normal forms up to conjugacy.

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Simplest candidate normal form: the linear part $\mathrm{d}f_0$ of f (linearization problem).

What happens when $\mathrm{d}f_0$ gives the least informations possible ?

My research interests

Local dynamics:

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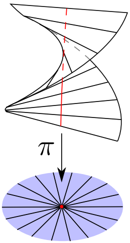
Interplay Geometry-Dynamics:

- compactification of orbit spaces;
- dynamical symmetries: singularities admitting special dynamical systems.

Plan

Resolution of singularities of varieties

We modify the space via blow-ups (modifications):



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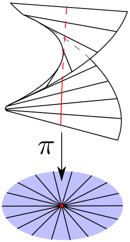

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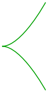

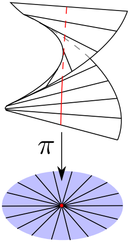

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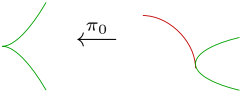

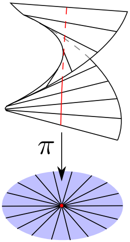

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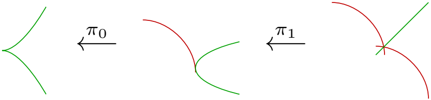

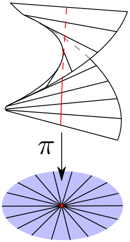

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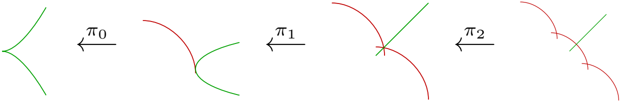

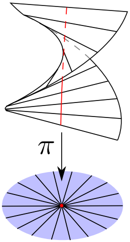

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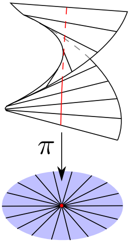

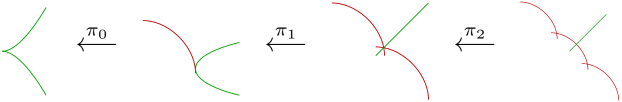

Resolution of singularities of varieties

- 1D (curves) NEWTON-PUISEUX .
- 2D (surfaces) WALKER , ZARISKI , ABHYANKHAR , LIPMAN .
- Local uniformization ZARISKI .
- 3D ZARISKI .
- Algebraic varieties HIRONAKA .
- Analytic varieties AROCA-HIRONAKA-VICENTE .

Other proofs by: VILLAMAYOR ENCINAS-VILLAMAYOR BIERSTONE-MILMAN WŁODARCZYK

Plan

Time-1 flow of vector fields

Let $\chi \in \mathcal{X}^{\geq 2}(\mathbb{C}^d, 0)$ be a vector field, $\mathrm{ord}_0 \chi = h \geq 2$.

Its time-1 flow $f = \exp \chi = \sum_{n=0}^{\infty} \frac{\chi^n}{n!}$ is tangent to the identity:

$$f(z) = z + \overbrace{f^{(h)}(z)}^{\not\equiv 0} + o\big(\|z\|^h\big).$$

Is the converse true?

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Is the converse true? **Almost!**

For any $f \in \mathrm{Diff}_1(\mathbb{C}^d, 0)$, there exists a unique $\chi \in \widehat{\mathcal{X}}^{\geq 2}(\mathbb{C}^d, 0)$ such that $f = \exp \chi$, called the infinitesimal generator.

Remarks

- χ is a formal vector field.
- χ induces (by saturation) a foliation $\mathcal{F}$ by complex curves. The foliation does not retain the informations on the real orbits.

Dynamics 1D

$$\chi(z) = -z^h \partial_z + \text{h.o.t.} \rightsquigarrow f(z) = z(1 - z^{h-1} + \text{h.o.t.}).$$

Goal: describe $\mathcal{B} = \{z \mid z_n := f^n(z) \to 0\}$.

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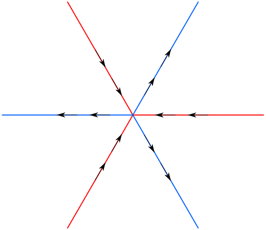

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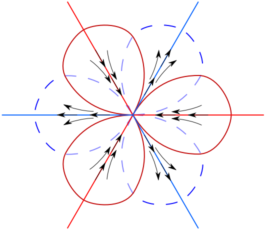

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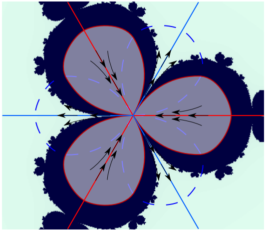

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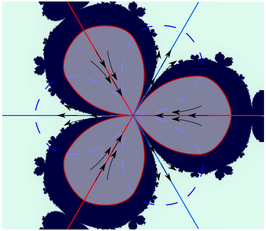

- Topological classification: $f(z) \overset{\text{top}}{\cong} z - z^h$ Camacho , Shcherbakov .
- Analytic classification Écalle , Voronin .

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- Parabolic implosion 1D: Lavaurs, Douady, Shishikura. Applications to:
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  - Construction of wandering domains for polynomial endomorphisms of $\mathbb{P}^2$: Astorg-Buff-Dujardin-Peters-Raissy.

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- Semi-parabolic and parabolic implosion 2D: Bedford-Smillie-Ueda, Bianchi, Astorg-LopezHernanz-Raissy.

$\geq$2 picture: parabolic manifolds

Goal: to describe $\mathcal{B}_\star = \{z \ : \ f^n(z) \xrightarrow{\star} 0\}$, with $\star$ being:

- $\zeta \in \mathbb{S}^{2d-1}$ ($\mathbb{R}$-direction),
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Assume χ non-dicritical, ($\widehat{\chi}_{\pi_0}$ is tangent to E_{π_0}).
A characteristic direction v is called non-degenerate if the eigenvalues of the linear part of $\overline{\chi}_{\pi_0}$ transverse to E_{π_0} is $\neq 0$.

Reduction of singularities of vector fields in 2D

Theorem (Bendixson, Seidenberg)

There exists a sequence $\pi\colon X_\pi \to (\mathbb{C}^2, 0)$ of blow-ups of singular points such that the saturated lift $\widehat{\chi}_\pi$ has only elementary singularities (the linear part is non-nilpotent).

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Any saturated vector field $\chi \in \mathcal{X}(\mathbb{C}^2, 0)$ admits a separatrix.

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Idea: there exist non-degenerate characteristic directions after blow-up.
Abate : any isolated fixed point TId germ admits a parabolic curve.
Parabolic curves are understood in 2D: Lapan, LopezHernanz, Molino, Raissy, Ribon, Rosas, SanzSanchez, Vivas .

What happens in higher dimensions

Positive news:

- HAKIM : Non-degenerate characteristic directions always have parabolic curves along possibly transcendental separatrices.
- LOPEZHERNANZ-RIBON-SANZSANCHEZ-VIVAS : separatrices C always have parabolic manifolds. Ingredients: local uniformization of (χ, C), Ramis-Sibuya normal forms.

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- MCQUILLAN-PANAZZOLO : reduction of singularities in dimension 3; SANCHODESALAS-SANZSANCHEZ : need to allow mild singularities for X_π.
- Separatrices might not exist GÓMEZMONT-LUENGO . ABATE-TOVENA : examples without C-parabolic manifolds (but they have v-parabolic curves).

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Theorem (MONGODI-R.)

There exists examples in $\mathbb{C}^3$ with no non-degenerate characteristic directions for any modifications.

Plan

Goals for superattracting germs

Let $f\colon (\mathbb{C}^2, 0) \circlearrowleft$ be superattracting. Example: $f(x,y) = (y + x^3, x^2 y)$.
Rough dynamics: orbits converge to 0 super-exponentially fast.

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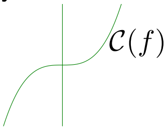

Remark: Cannot get explicit normal forms up to conjugacy, due to the complexity of $\mathcal{C}(f^\infty)$.

We want to control the speed of convergence:

Goals for superattracting germs

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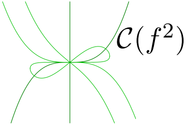

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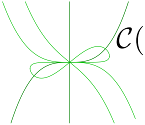

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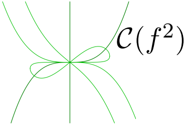

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- Its growth $c_\infty(f) := \lim_n \sqrt[n]{c(f^n)}$: the (first) dynamical degree.

Reduction of singularities of maps: monomialization

$f\colon X \to Y$ a map. We look for modifications π, ϖ that simplify the expression of f (around critical points).

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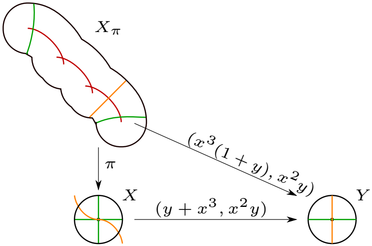

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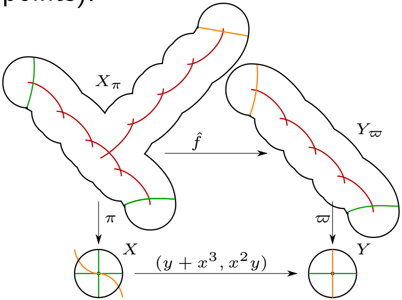

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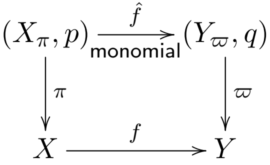

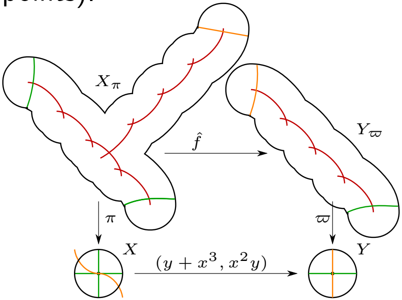

Local monomialization

- Algebraic maps CUTKOSKY .
- Analytic maps CUTKOSKY .
- Quasianalytic maps BELOTTO-BIERSTONE .

Global results: 2D AKBULUT-KING , 3D→2D CUTKOSKY .

Reduction of dynamical singularities, some issues

When consider dynamical systems, we have $X = Y$, and we require $\pi = \varpi$ in order to have a dynamical system f_π acting on X_π.

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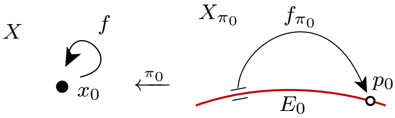

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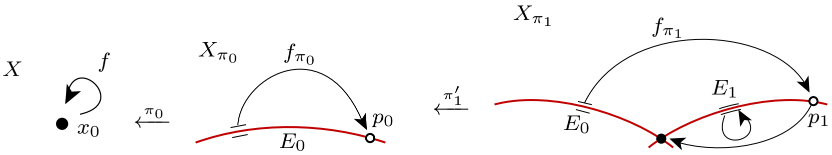

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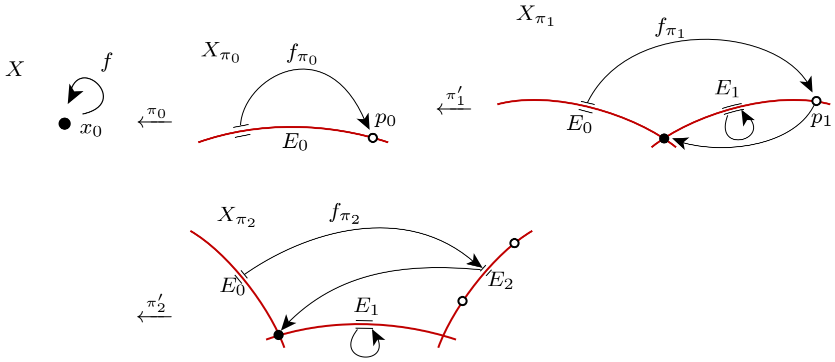

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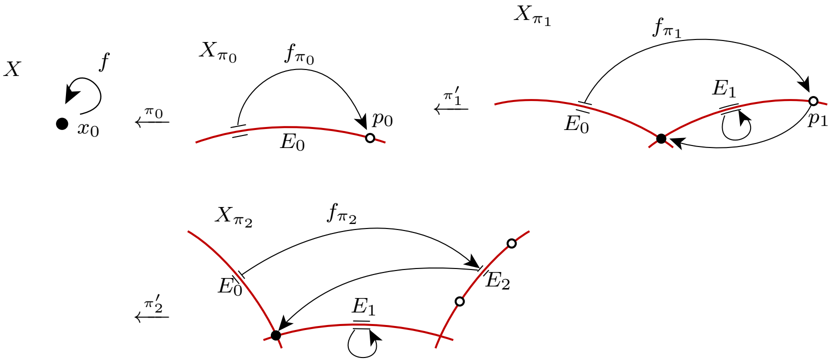
Hope: We can avoid them with large iterates: algebraically stable models.



Motivations

- The sequence $(c_n)_n$ and c_∞ are local analogous of global concepts. They are invariants of conjugacy.
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- Existence and first properties Russakovskii-Shiffman Dinh-Sibony , Truong , Dang ; Khovanskii-Teissier .
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About algebraic stability

- Introduced by Fornaess-Sibony . Allows control to construct invariant objects.
- Existence results: Bedford,Cantat,Dang,Diller, Favre,Jonsson,Lin,Truong,Wulcan .

Valuation spaces

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Related objects and applications

- *Farey blowups* Hubbard-Papadopol ;
- *Picard-Manin space* Manin to study the Cremona group (Cantat , Blanc , Déserti , Lamy , etc.).
- *Hybrid spaces* Berkovich : used to study degenerations Boucksom-Jonsson , Dujardin-Favre ;
- Applications to K-stability Chi Li , Chenyang Xu , Blum .

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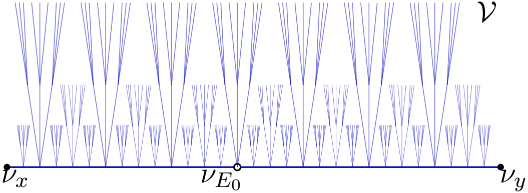

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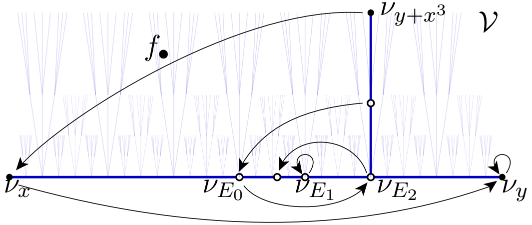

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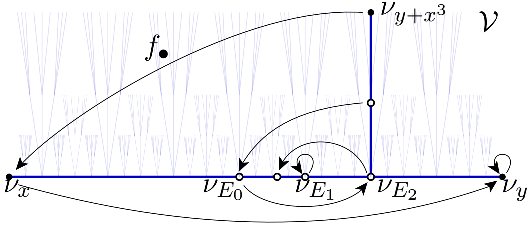

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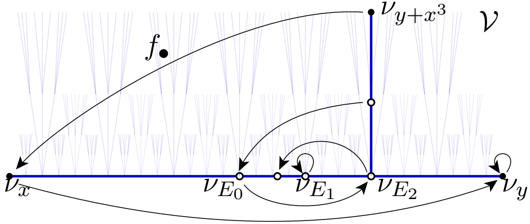


Global dynamics : Favre, Jonsson, Boucksom, Xie, Dang, Abboud.
Local dynamics : Favre, Jonsson, R., Gignac.

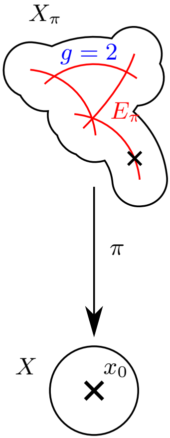
Plan

Modifications and resolutions of singularities

(X, x_0) normal surface singularity,
$(\mathcal{O}_X, \mathfrak{m}_X)$ its associated local ring.

Definition

- A modification is a proper bimeromorphic map $\pi\colon X_\pi \to (X, x_0)$, which is an isomorphism outside of the exceptional divisor $E_\pi := \pi^{-1}(x_0)$.



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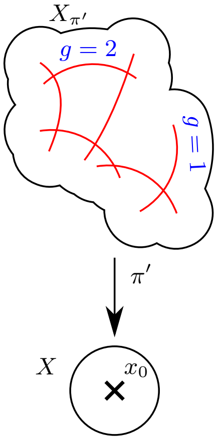

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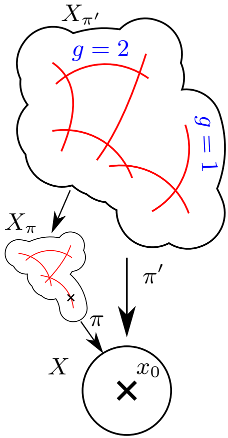
- Another modification π' dominates π if $\pi^{-1} \circ \pi' \colon X_{\pi'} \to X_\pi$ is regular.



Valuations

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A (normalized rank-1 semi-)valuation on (X, x_0) is a map $\nu\colon \widehat{\mathcal{O}}_X \to [0, +\infty]$ such that:

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Other two types: irrational and infinitely singular.

These are points of type II, I, III, IV in the sense of Berkovich.

Dual graphs and divisors

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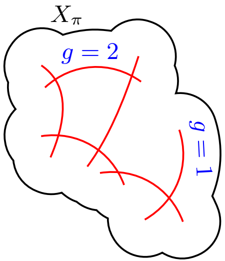

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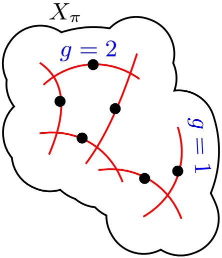

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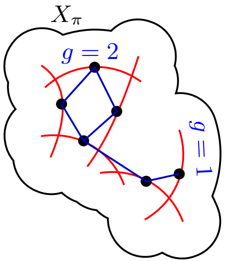

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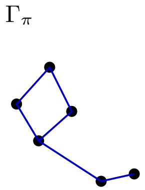
We obtain a graph $\Gamma_\pi \hookrightarrow \mathcal{E}(\pi)_\mathbb{R}$.



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\rightsquigarrow & & \rightsquigarrow & \varprojlim_\pi & \rightsquigarrow \\
Z(\nu_E) & = \big(Z_\pi(\nu_E)\big)_\pi & \in \varprojlim_\pi \Gamma_\pi & \hookrightarrow & b\text{-}\mathcal{E}(X)
\end{array}
$$

- To any $\nu \in \mathcal{V}_X$ is associated $Z(\nu)$ the b-divisor (in the sense of SHOKUROV): if $\pi' = \pi \circ \eta$, then $\eta_* Z_{\pi'}(\nu) = Z_\pi(\nu)$.

Valuations and b-divisors

$$
\begin{array}{ccc}
X_{\pi'} & \supset & E' \\
\downarrow \eta & & \\
X_{\pi} & \supset & E \\
\downarrow \pi & & \\
(X, x_0) & &
\end{array}
\qquad \pi' : X_{\pi'} \to (X, x_0)
$$

$$
\begin{array}{ccccc}
Z_{\pi'}(\nu_{E'}) & = \eta^* Z_{\pi}(\nu_E) & \in \Gamma_{\pi'} & \hookrightarrow & \mathcal{E}(\pi')_{\mathbb{R}} \\
\downarrow \eta_* & & & & \downarrow \eta_* \\
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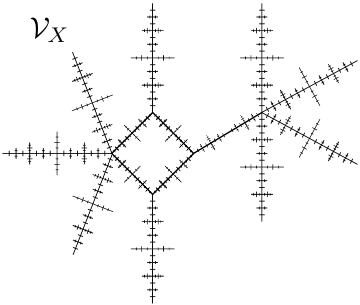
- We have $\mathcal{V}_X \simeq \varprojlim_{\pi} \Gamma_{\pi} \supset \varinjlim_{\pi} \Gamma_{\pi} \simeq \mathcal{V}_X^{\mathrm{qm}}$.

Valuation spaces



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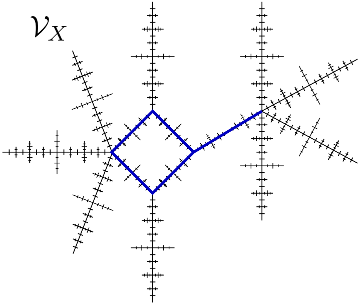

- The dual graph of a good resolution π reads into $\mathcal{V}_X$.

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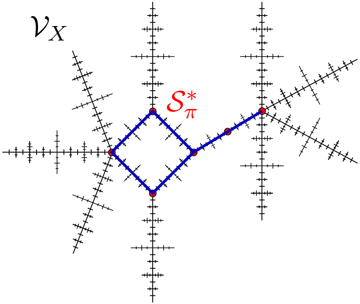

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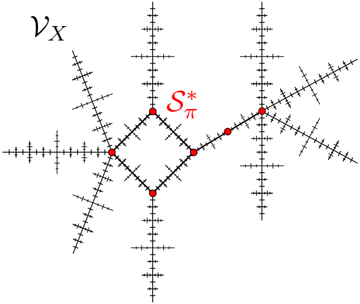

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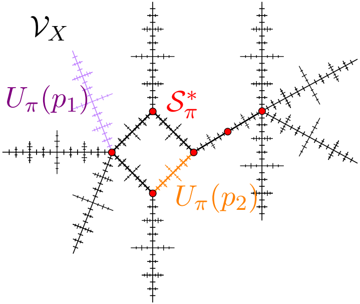

- The dual graph of a good resolution π reads into $\mathcal{V}_X$.
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- Topology: the weakest for which $\nu \mapsto \nu(\phi)$ is continuous $\forall \phi \in \widehat{\mathcal{O}}_X$. Equivalently, generated by connected components (in the sense of graphs) of complements of finite sets.

Intersection theory of valuations

The negative-definite intersection form on $\mathcal{E}(\pi)$ induces a (extended) scalar product on $\mathcal{V}_X$:

$$\langle \nu, \mu \rangle := -Z(\nu) \cdot Z(\mu) := \sup_{\pi} -Z_\pi(\nu) \cdot Z_\pi(\mu) \in (0, +\infty].$$

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Theorem (Gignac-R. , Key technical result)

Let (X, x_0) be a normal surface singularity. Let $\nu, \mu_1, \mu_2 \in \mathcal{V}_X$. Then

$$\langle \nu, \mu_1 \rangle \langle \nu, \mu_2 \rangle \leq \langle \nu, \nu \rangle \langle \mu_1, \mu_2 \rangle \,, \qquad (\star)$$

with equality if and only if ν disconnects μ_1 and μ_2 in $\mathcal{V}_X$.

Independently by García Barroso-González Pérez-Popescu Pampu when $\mathcal{V}_X$ is contractible.

Application to intersection of branches

Definition

Let $\lambda \in \mathcal{V}_X$ be any valuation. For any $\nu_1, \nu_2 \in \mathcal{V}_X$, we set

$$u_\lambda(\nu_1, \nu_2) := \begin{cases} \dfrac{\langle \lambda, \nu_1 \rangle \cdot \langle \lambda, \nu_2 \rangle}{\langle \nu_1, \nu_2 \rangle} & \text{if } \nu_1 \neq \nu_2, \\ 0 & \text{if } \nu_1 = \nu_2. \end{cases}$$

Płoski If $X = (\mathbb{C}^2, 0)$ then u_λ is an ultrametric:
if $\lambda = \mathrm{ord}_0$, $\forall A, B, C$ curve branches, we have up to permutation

$$\frac{A \cdot B}{m(A)m(B)} = \frac{A \cdot C}{m(A)m(C)} \leq \frac{B \cdot C}{m(B)m(C)}.$$

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Theorem (GARCÍABARROSO-GONZÁLEZPÉREZ-POPESCUPAMPU-R.)

The function u_λ is an (extended) ultrametric for a/any λ if and only if $\mathcal{V}_X$ is contractible.

Main tool (★).

Plan

Action induced on $\mathcal{V}_X$

Assume for simplicity that $f\colon (X, x_0) \to (Y, y_0)$ is finite.

$$
\begin{aligned}
f^*\colon \widehat{\mathcal{O}}_Y &\to \widehat{\mathcal{O}}_X \\
\phi &\mapsto \phi \circ f
\end{aligned}
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$\rightsquigarrow$ duality, normalization

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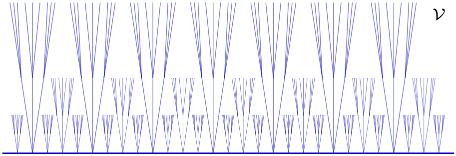

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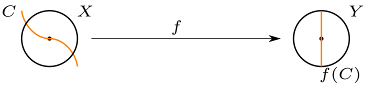

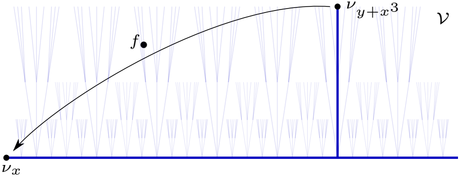

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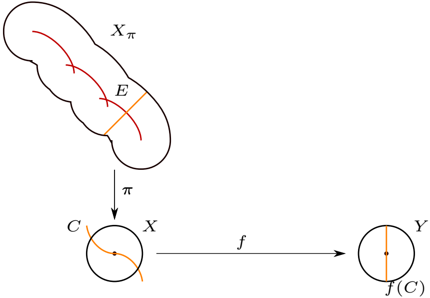

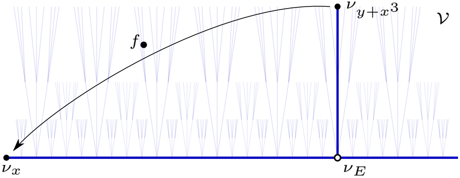

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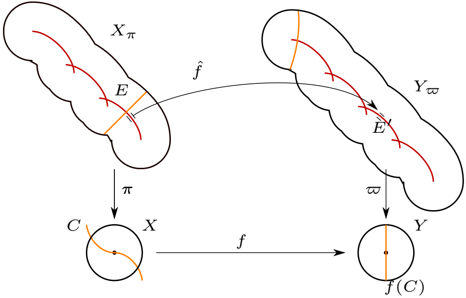

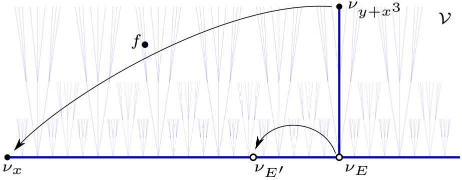

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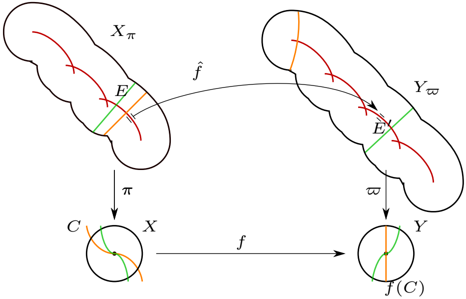

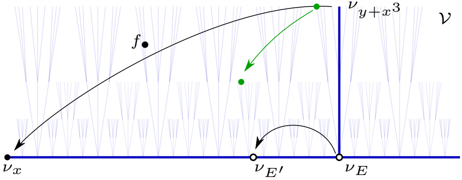

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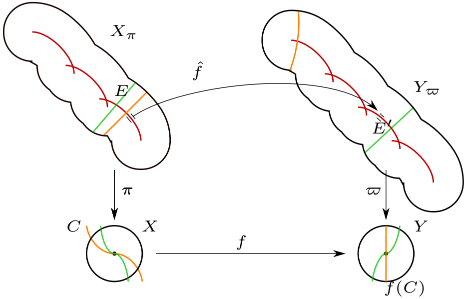

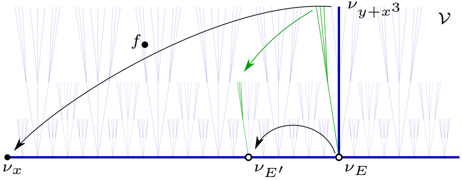

Properness criterium:
$\hat{f}(p) = q \Leftrightarrow f_\bullet U_\pi(p) \subseteq U_\varpi(q)$.

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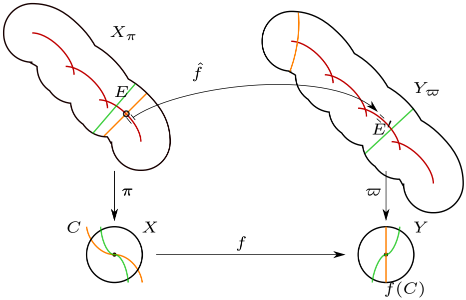

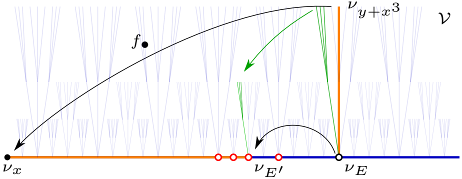

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Eigenvaluations and rigidification

Theorem (Favre-Jonsson)

For any $f\colon (\mathbb{C}^2, 0) \circlearrowleft$ *there exists an eigenvaluation:* $\nu_\star = f_\bullet \nu_\star$ *with an open* $U_\pi(p)$ *in its basin of attraction.*

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Consequences

- There exists $\pi\colon X_\pi \to (\mathbb{C}^2, 0)$, $p \in \pi^{-1}(0)$, such that $f_\pi\colon (X_\pi, p) \circlearrowleft$ is rigid : $\mathcal{C}(f^\infty)$ is SNC and f-invariant.
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Related works

Favre-Jonsson : polynomial endomorphisms of $\mathbb{C}^2$.

Gignac-R. : local version on surface singularities.

Abboud : global version on affine surfaces.

Bell-Diller-Jonsson +Krieger : examples of $\lambda_1(f)$ transcendental.

Dang-Favre : higher dimensional results.

Global attraction properties and algebraically stable models

Theorem (Gignac-R.)

For any superattracting $f\colon (X, x_0) \circlearrowleft$ at a normal surface singularity, there exists an invariant subset $S \subset \mathcal{V}_X$ (either a point, a segment, or a circle) that attracts the orbit $(f^n_\bullet \nu)$ of any quasimonomial valuation $\nu \in \mathcal{V}_X^{qm}$.

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Notable exception

S is a circle, $f_\bullet|_S$ is an irrational rotation $\Rightarrow$ (X, x_0) is a cusp singularity, f is a finite germ.

Angular distance

Definition (Gignac-R.)

The angular distance on $\mathcal{V}_X$ is given by

$$\rho_X(\nu, \mu) := \log \frac{\langle \nu, \nu \rangle \langle \mu, \mu \rangle}{\langle \nu, \mu \rangle^2}.$$

This is an extended distance, that takes finite values on $\mathcal{V}_X^{\mathrm{qm}}$.

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This is an extended distance, that takes finite values on $\mathcal{V}_X^{\mathsf{qm}}$.
It plays the role of the Poincaré distance on valuation spaces.

Theorem (GIGNAC-R.)

For any dominant map $f\colon (X, x_0) \to (Y, y_0)$, *we have*

$$\rho_Y(f_\bullet \nu, f_\bullet \mu) \leq \rho_X(\nu, \mu).$$

Using (★): characterize the case of equality.

Non-finite vs finite

Dichotomy:

- Suppose f is non-finite. Then $f_\bullet$ is a weak contraction:

$$\forall \nu \neq \mu \in \mathcal{V}_X^{\mathsf{qm}}, \quad \rho_X(f_\bullet \nu, f_\bullet \mu) < \rho_X(\nu, \mu).$$

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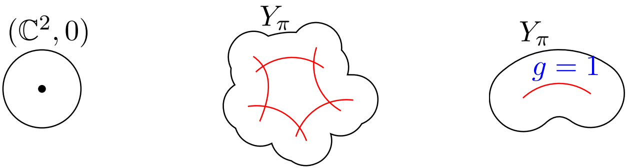
- Suppose f is finite. Wahl : (X, x_0) is log-canonical:
  $(X, x_0) = {}^{(Y, y_0)}\!/_{G}$, G finite group, (Y, y_0) is:

regular   cusp   simple elliptic

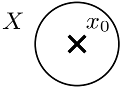


We conclude case by case.

Plan

Singularities admitting special endomorphisms

(X, x_0) singularity.



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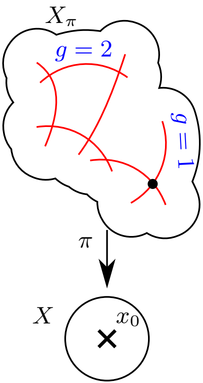

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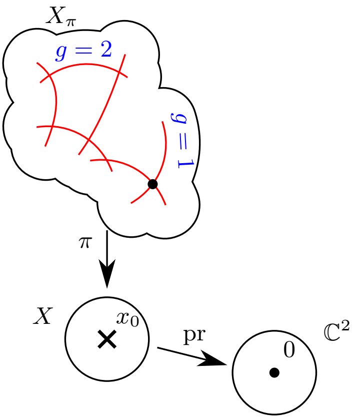

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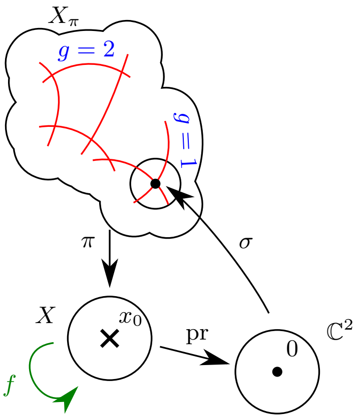

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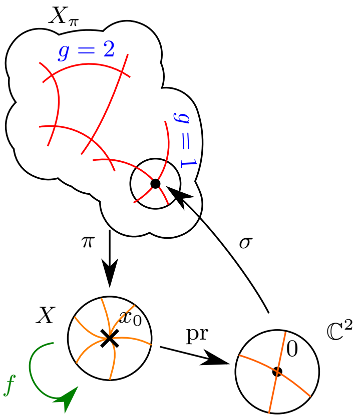
$\sigma(\mathbb{C}^2, 0) \to (X_\pi, p)$ with $p \in \pi^{-1}(x_0)$.

$f = \pi \circ \sigma \circ \mathrm{pr}(X, x_0) \circlearrowleft$.

Singularities admitting special endomorphisms



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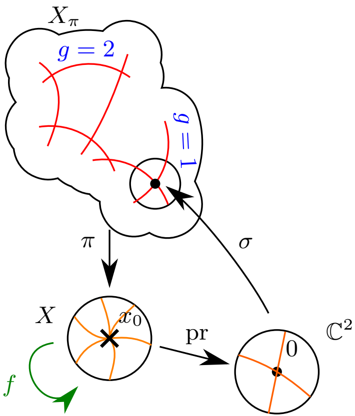
- f contracts curves to 0.

- Finite endomorphisms 2D Wahl, Favre, Gignac-R.

Singularities admitting special endomorphisms



(X, x_0) singularity.

$\pi\colon X_\pi \to (X, x_0)$ resolution.

$\mathrm{pr}\colon (X, x_0) \hookrightarrow (\mathbb{C}^N, 0) \twoheadrightarrow (\mathbb{C}^2, 0)$ generic projection.

$\sigma(\mathbb{C}^2, 0) \to (X_\pi, p)$ with $p \in \pi^{-1}(x_0)$.

$f = \pi \circ \sigma \circ \mathrm{pr}(X, x_0) \circlearrowleft$.

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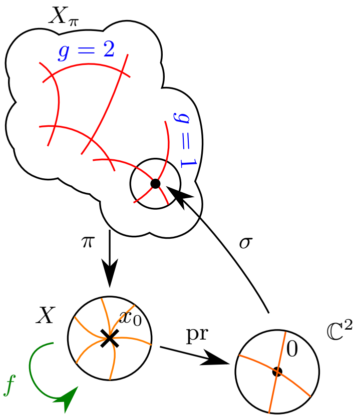

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- f contracts curves to 0.
- f has large topological degree.

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  ≥3D Boucksom-De Fernex-Favre , Broustet-Höring , Zhang .
- Topological degree 1 Fantini-Favre-R. .

Kato germs and sandwiched singularities

Theorem (Fantini-Favre-R.)

A normal surface singularity (X, x_0) admits a non-invertible selfmap of topological degree $1 \iff (X, x_0)$ is sandwiched.

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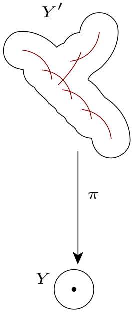
Sandwiched singularities Spivakovsky

- $\varpi : Y' \to Y = (\mathbb{C}^2, 0)$ modification;



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Theorem (FANTINI-FAVRE-R.)

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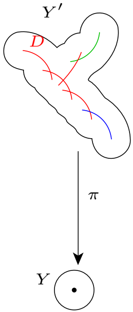

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Sandwiched singularities Spivakovsky

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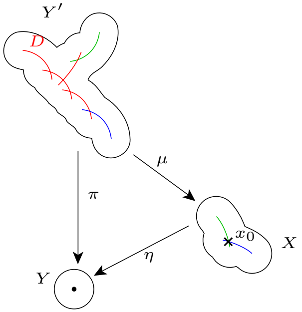

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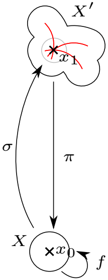
Strategy

- $f = \pi \circ \sigma$, $\pi \colon X' \to (X, x_0)$ modification, σ local isomorphism.



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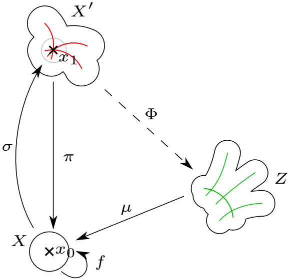

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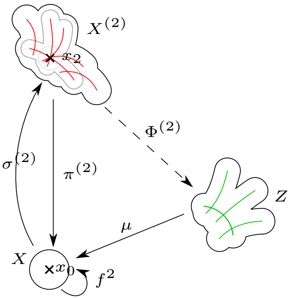
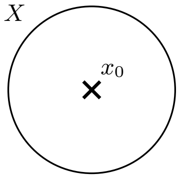
- Yes, if we replace f by f^n, $n \gg 0$.



Singularities admitting contracting automorphisms



(X, x_0) singularity.

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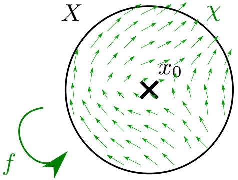
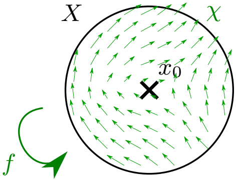


(X, x_0) singularity.
There exists (infinitely many, pairwise non commuting) singular vector fields χ tangent to X (Müller).
Its time-1 flow map defines an automorphism $f\colon (X, x_0) \circlearrowleft$.

Singularities admitting contracting automorphisms



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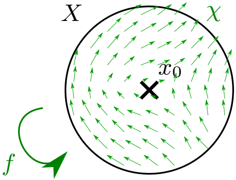
A normal surface singularity (X, x_0) admits a contracting automorphism $\iff (X, x_0)$ is quasihomogeneous.

Singularities admitting contracting automorphisms



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Related results: Orlik-Vagreich , Camacho-Movasati-Scardua , Favre-R. , Morvan .

Publications presented in the HDR (1)

- M.Ruggiero: "Rigidification of holomorphic germs with non-invertible differential". Michigan Mathematical Journal, Volume 61 Issue 1, pp. 161–185, 2012.
- M.Ruggiero: "Contracting rigid germs in higher dimensions". Annales de l'Institut Fourier, Volume 63 Issue 5, pp. 1913–1950, 2013.
- W.Gignac and M.Ruggiero: "Growth of attraction rates for iterates of a superattracting germ in dimension two". Indiana University Mathematics Journal, Volume 63, no.4, pp. 1195–1234, 2014.
- C.Favre and M.Ruggiero: "Normal surface singularities admitting contracting automorphisms". Annales Mathématiques de la faculté des sciences de Toulouse, Volume 23, no. 4, pp. 797–828, 2014.
- M.Ruggiero: "Classification of one dimensional superattracting germs in positive characteristic". Ergodic Theory and Dynamical Systems, Volume 35, Issue 7, pp. 2242–2268, 2015.
- M.Ruggiero and K.Shaw: "Tropical Hopf manifolds and contracting germs". Manuscripta Mathematica, Volume 152, Issue 1-2, pp. 1–60, 2017.

Publications presented in the HDR (2)

- W.Gignac and M.Ruggiero: "Local dynamics of non-invertible maps near normal surface singularities". Memoirs of the AMS 272, no. 1337, xi+100 pages, 2021.
- E.García Barroso, P. González Pérez, P. Popescu-Pampu and M.Ruggiero: "Ultrametric properties for valuation spaces of normal surface singularities". Transactions of the AMS, Volume 372, Issue 12, pp. 8423–8475, 15 December 2019.
- L.Fantini, C.Favre and M.Ruggiero: "Links of sandwiched surface singularities and self-similarity". Manuscripta Mathematica, Volume 162, Issue 1-2, pp. 23–65, 2020.
- N.Istrati, A.Otiman, M.Pontecorvo and M.Ruggiero: "Toric Kato manifolds". Journal de l'École polytechnique, Volume 9, pp. 1347–1395, 2022.
- S.Mongodi and M.Ruggiero: "Birational properties of tangent to the identity germs without non-degenerate singular directions". Journal of the London Mathematical Society, pp. 1–55, 2023.
- R.Dujardin, C.Favre and M.Ruggiero: "On the dynamical Manin-Mumford conjecture for plane polynomial maps". Preprint, pp. 20, 2023.