Adic and perfectoid spaces

IMJ–PRG specialised M2 course, 27 February – 5 April

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Abstract

The main goal of this course is to develop the foundations of the theory of perfectoid spaces (P), more precisely to prove the various tilting correspondences for perfectoid rings, the almost purity theorem, and almost vanishing theorems. We develop simultaneously what is needed from the theory of adic spaces (A).

Contents

Some notation

For the whole course we fix a prime number p. All rings are commutative. Given a ring A of characteristic p, we write $\varphi : A \to A$, $a \mapsto a^p$ for its Frobenius endomorphism (which is a ring homomorphism).

1 (P) Integral perfectoid rings

Let A be a topological ring. We say that A is *integral perfectoid* if and only if there exists a non-zero-divisor $\pi \in A$ such that

(a) the topology on A is the π-adic topology, and A is complete for this topology (i.e., $A \to \varprojlim_n A/\pi^n A$ is an isomorphism of topological rings, where each $A/\pi^n A$ has the discrete topology);

(b) $p \in \pi^p A$;

(c) $\Phi : A/\pi A \to A/\pi^p A$, $a \mapsto a^p$, is an isomorphism

It is convenient for us, even though it is not exactly standard in the literature, to call any such element π a perfectoid pseudo-uniformiser (ppu).

Example 1.1. Here are some easy examples:

(i) The p-adic completions of the rings $\mathbb{Z}_p[p^{1/p^\infty}]$ and $\mathbb{Z}_p[\zeta_{p^\infty}]$ are perfectoid (equipped with the p-adic topology), with perfectoid pseudo-uniformisers $p^{1/p}$ and $\zeta_{p^2} - 1$ respectively.

(ii) $\mathbb{Z}_p$ is not perfectoid.

Construction 1.2. Given any ring A we write

$$\varprojlim_{x \mapsto x^p} A := \{(a_0, a_1, \dots) \in A^{\mathbb{N}} : a_i^p = a_{i-1} \text{ for all } i \geq 1\}$$

for the set of compatible sequences of p-power roots in A. Note that we can multiply two such sequences, so $\varprojlim_{x \mapsto x^p} A$ forms a multiplicative monoid; if A has characteristic p then we can also add two such sequences, so then A is even a ring.

The construction is clearly functorial: a morphism of rings $A \to B$ induces a morphism of monoids $\varprojlim_{x \mapsto x^p} A \to \varprojlim_{x \mapsto x^p} B$ (which is even a morphism of rings if A and B have characteristic p).

The following result is extremely important and will be used repeatedly: if $\pi \in A$ is an element such that (i) $p \in \pi A$ and (ii) A is π-adically complete, then the map $\varprojlim_{x \mapsto x^p} A \to \varprojlim_{x \mapsto x^p} A/\pi A$ is actually a bijection (hence an isomorphism of monoids). We leave the details of the proof to the reader, but give the following recipe for the inverse of the map: given $b = (b_0, b_1, \dots) \in \varprojlim_{x \mapsto x^p} A/\pi A$, let $\widetilde{b_i} \in A$ be an arbitrary lift of $b_i \in A/\pi A$ for each $i \geq 0$. Then set

$$a_i := \lim_{n \to \infty} \widetilde{b_{i+n}}^{p^n}$$

and check that $a := (a_0, a_1, \dots) \in \varprojlim_{x \mapsto x^p} A$ really is a well-defined lift of b.

Lemma 1.3. *Let A be integral perfectoid, and $\pi \in A$ a perfectoid pseudo-uniformiser. Then:*

(i) Every element of $A/\pi p A$ is a p^{th}-power (n.b., $A/\pi p A$ does not necessarily have characteristic p).

(ii) If an element $a \in A[\frac{1}{\pi}]$ satisfies $a^p \in A$, then $a \in A$.

(iii) After multiplying π by a unit it admits a compatible sequence of p-power roots $\pi^{1/p}, \pi^{1/p^2}, \dots \in A$.

Proof. (i): Using the surjectivity of Φ, a simple induction lets us write any $a \in A$ as an infinite sum $a = \sum_{i\geq 0} a_i^p \pi^{pi}$ for some $a_i \in A$; but this is $\equiv (\sum_{i\geq 0} a_i \pi^i)^p \bmod p\pi A$.

(ii): Let $l \geq 0$ be the smallest integer such that $\pi^l a \in A$. Assuming that $l > 0$, we get a contradiction by noting that $\pi^{pl} a^p \in \pi^{pl} A \subseteq \pi^p A$, whence $\pi^l a \in \pi A$ by condition (c), and so $\pi^{l-1} a \in A$.

(iii): Since the Frobenius is surjective on $A/\pi^p A$, there exists an element of $\varprojlim_{x\mapsto x^p} A/\pi^p A$ of the form $(\pi \bmod \pi^p A, ?, ?, \dots)$. Applying the exercise of Construction 1.2, we deduce that the natural map $\varprojlim_{x\mapsto x^p} A \to \varprojlim_{x\mapsto x^p} A/\pi^p A$ is a bijection. Hence there exists $a = (a_0, a_1, \dots) \in \varprojlim_{x\mapsto x^p} A$ such that $a_0 \equiv \pi \bmod \pi^p A$; therefore $a = u\pi$ for some $u \in 1 + \pi^{p-1} A \subseteq A^\times$ (the inclusion $\subseteq$ results from π-adic completeness of A). $\square$

Lemma 1.4. *Let A be integral perfectoid, and $\varpi \in A$ an element satisfying conditions (a) and (b). Then ϖ is a non-zero-divisor satisfying (c), i.e., it is a perfectoid pseudo-uniformiser.*

Proof. We must show that $\Phi : A/\varpi A \to A/\varpi^p A$ is an isomorphism. Let $\pi \in A$ be a perfectoid pseudo-uniformiser.

It follows from Lemma 1.3(i) that every element of A/pA is a p^{th}-power; hence every element of its quotient $A/\varpi^p A$ is a p^{th}-power, i.e., Φ is surjective.

The fact that π and ϖ define the same topology implies that a power of each is divisible by the other, whence ϖ is a non-zero-divisor and $A[\frac{1}{\varpi}] = A[\frac{1}{\pi}]$. If $a \in A$ satisfies $a^p \in \varpi^p A$, then $(a/\varpi) \in A[\frac{1}{\pi}]$ satisfies $(a/\varpi)^p \in A$, and it then follows from Lemma 1.3(ii) that in fact $a \in \varpi A$ as desired. $\square$

Lemma 1.5. *Suppose A is a complete topological ring such that $pA = 0$. Then A is integral perfectoid if and only if it is perfect and the topology is π-adic for some non-zero-divisor $\pi \in A$.*

Proof. Exercise. $\square$

1.1 The tilt of an integral perfectoid ring

Definition 1.6. The *tilt* of an integral perfectoid ring A is $A^\flat := \varprojlim_{x\mapsto x^p} A/pA$, equipped with the inverse limit topology (A/pA is of course given the quotient topology, i.e., the π-adic topology for any choice of pup for A).

Note that $A^\flat$ is a perfect ring of characteristic p; in fact, it is the initial object among all perfect rings of characteristic p mapping to A/pA.

Recalling from Construction 1.2 that the natural map $\varprojlim_{x\mapsto x^p} A \to \varprojlim_{\varphi} A/pA$ is an isomorphism of monoids, we define the *untilting map* $\# : A^\flat \to A$, $b \mapsto b^\#$ to be projection to the 0^{th}-coordinate of $\varprojlim_{x\mapsto x^p} A$; explicitly, the map $\#$ is given by $\varprojlim_{\varphi} A/pA \ni (b_0, b_1, \dots) \mapsto \lim_{i\to\infty} \widetilde{b_i}^{p^i}$, where $\widetilde{b_i} \in A$ are arbitrary lifts of the elements $b_i \in A/pA$.

The untilting map is multiplicative by generally not additive; in fact, given $b, c \in A^\flat$, it transforms under addition as follows:

$$(b+c)^\# = \lim_{i\to\infty} ((b^{1/p^i})^\# + (c^{1/p^i})^\#)^{p^i}.$$

However, note that the composition $A^\flat \overset{\#}{\,} \overset{\bmod p}{\longrightarrow} A/pA$ is a ring homomorphism: indeed, it is the surjective ring homomorphism given by projecting $A^\flat \cong \varprojlim_{\varphi} A/pA$ to the 0^{th}-coordinate. Also, if A is of characteristic p, then the untilting map $\# : A^\flat \to A$ is an isomorphism of rings.

Lemma 1.7. *Let A be an integral perfectoid ring. Then:*

(i) $\# : A^\flat \to A$, *is continuous;*

(ii) the isomorphisms of monoids $\varprojlim_{x\mapsto x^p} A \to A^\flat = \varprojlim_{x\mapsto x^p} A/pA \to \varprojlim_{x\mapsto x^p} A/\pi A$ *are homeomorphisms, where* $\pi \in A$ *is any perfectoid pseudo-uniformiser;*

(iii) $A^\flat$ *is also an integral perfectoid ring.*

Proof. Given $(1, \dots, 1, b_{n+1}, b_{n+2}, \dots) \in \varprojlim_\varphi A/\pi A$, any chosen lifts $\widetilde{b_i}$ satisfy $\widetilde{b_i}^{p^{i-n}} \equiv 1 \bmod \pi A$ for $i > n$, whence $\widetilde{b_i}^{p^i} \equiv 1 \bmod \pi^n A$; taking the limit shows that the untilt is $\equiv 1 \mod \pi^n A$. This proves that the untilting map $\# : \varprojlim_{x\mapsto x^p} A/\pi A \to A$ is continuous (for the inverse limit of discrete topologies on the domain), from which (i) and (ii) easily follow. Filling in the details is left as an exercise.

(iii) We have already noted that $A^\flat$ is a perfect ring of characteristic, and the homeomorphism $A \cong \varprojlim_{x\mapsto x^p} A/\pi A$ shows that A is an inverse limit of discrete rings, whence A is a complete topological ring. According to Lemma 1.5, it remains to prove the following: there exists a non-zero-divisor $\pi^\flat \in A^\flat$ such that the topology on $A^\flat$ is the $\pi^\flat$-adic topology.

Possibly after changing our perfectoid pseudo-uniformiser π, we may assume that it admits compatible p-power roots (by Lemma 1.3(iii)); let $\pi^\flat = (\pi, \pi^{1/p}, \dots,) \in A^\flat$ be the corresponding element of $A^\flat$, which satisfies $(\pi^\flat)^\# = \pi$.

We show first that $\pi^\flat$ is a non-zero-divisor. To do that we note that for each $n \geq 1$ we have an exact sequence

$$0 \longrightarrow \pi^{1-1/p^n} A/\pi A \longrightarrow A/\pi A \overset{\times \pi^{1/p^n}}{\longrightarrow} A/\pi A \overset{\varphi^n}{\longrightarrow} A/\pi A \longrightarrow 0$$

Exactness is easy everywhere except possibly at the second term from the right: but if $a \in A$ satisfies $a^p \in \pi A$ then $a/\pi^{1/p^n} \in A[\frac{1}{\pi}]$ satisfies $(a/\pi^{1/p^n})^{p^n} \in A$, whence Lemma 1.3(ii) implies $a \in \pi^{1/p^n} A$ as desired.

These sequences are moreover compatible in n, with respect to the maps $0, \varphi, \varphi, \mathrm{id}$ respectively. Although it is not always the case that an inverse limit of exact sequences is still exact, in this case the transition map are either surjective (φ and id) or zero, and so taking the inverse limit does yield an exact sequence

$$0 \longrightarrow A^\flat \overset{\times \pi^\flat}{\longrightarrow} A^\flat \overset{\sharp \bmod \pi}{\longrightarrow} A/\pi A \longrightarrow 0.$$

That is, $\pi^\flat$ is a non-zero-divisor of $A^\flat$ and the untilting map induces an isomorphism of rings $A^\flat/\pi^\flat A^\flat \overset{\cong}{\to} A/\pi A$.

Finally we check that the topology on $A^\flat$ is the $\pi^\flat$-adic topology. Since $A^\flat \cong \varprojlim_{x\mapsto x^p} A/\pi A$ is a homeomorphism by part (i), a basis of open neighbourhoods of $0 \in A^\flat$ is given by $\mathrm{Ker}(\mathrm{proj_n})$ for $n \geq 0$, where $\mathrm{proj}_n : \varprojlim_{x\mapsto x^p} A/\pi A \to A/\pi A$, $(b_0, b_1, \dots) \mapsto b_n$ denotes the n^{th}-projection map. Note that proj_0 is the untilting map. Since the composition $A^\flat \overset{\varphi^n \cong}{\longrightarrow} A^\flat \overset{\mathrm{proj}_n}{\longrightarrow} A/\pi A$ is proj_0, the basis of open neighbourhoods is given by

$$\mathrm{Ker}(\mathrm{proj_n}) = \varphi^n(\mathrm{Ker}(\mathrm{proj}_0)) = \varphi^n(\pi^\flat A^\flat) = \pi^{\flat p^n} A^\flat,$$

showing that the topology is indeed $\pi^\flat$-adic. □

We point out explicitly that we showed in the previous proof that the kernel of the surjective ring homomorphism $A^\flat \overset{\# \bmod \pi}{\longrightarrow} A/\pi A$ (i.e., projection to the 0^{th}-coordinate of $A^\flat = \varprojlim_{x\mapsto x^p} A/\pi A$) is $\pi^\flat A^\flat$. This will be used repeatedly.

1.2 Fontaine's map θ

In this section we introduce Fontaine's map $\theta : W(A^\flat) \to A$, which will allow us to recover A from $A^\flat$ in some sense.

Remark 1.8 (Reminder on the ring of Witt vectors)**.** If k is any ring, let $W(k)$ denote its ring of p-typical Witt vectors. Here are some reminders about this object:

(i) There is an identification of sets $W(k) = k^{\mathbb{N}}$. So each element of $W(k)$ may be written uniquely as $(a_0, a_1, \dots)$, with $a_i \in k$.

(ii) Additional and multiplication are given by certain polynomials with integer coefficients (which do not depend on k), for example

$$(a_0, a_1, a_2, \dots) + (b_0, b_1, b_2, \dots) = (a_0 + b_0, a_1 + b_1 - \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} a_0^i b_0^{p-i}, \dots)$$

$$(a_0, a_1, a_2, \dots) \cdot (b_0, b_1, b_2, \dots) = (a_0 b_0, a_0^p b_0 + b_0^p a_1 + p a_1 b_1, \dots)$$

(iii) There is a natural ring homomorphism called the "phantom map"

$$\mathrm{phant} : W(k) \longrightarrow k^{\mathbb{N}}, \qquad (a_0, a_1, \dots) \mapsto (a_0, a_0^p + pa_1, a_0^{p^2} + pa_1^p + p^2 a_2, \dots).$$

In particular, its n^{th}-coordinate $\mathrm{phant}_n : W(k) \to k$, $(a_0, a_1, \dots) \mapsto \sum_{i=0}^n p^i a_i^{p^{n-i}}$ is a ring homomorphism for each $n \geq 0$.

(iv) If $k \supseteq \mathbb{Q}$ then phant is an isomorphism of rings; if k is p-torsion-free then phant is injective.

(v) Given $a \in k$, its *Teichmüller lift* is $[a] := (a, 0, 0, 0, \dots) \in W(k)$; the map $[\cdot] : k \to W(k)$ is multiplicative but not additive.

(vi) Suppose now that $k \supseteq \mathbb{F}_p$. Then $W(k)$ is p-adically complete and

$$\sum_{i=0}^{\infty} [a_i] p^i = (a_0, a_1^p, a_2^{p^2}, \dots),$$

where $a_i \in k$. In particular, if k is perfect then one easily deduces the following: each element of $W(k)$ may be written uniquely as $\sum_{i=0}^{\infty} [a_i] p^i$ for some elements $a_i \in A$, the element $p \in W(k)$ is a non-zero-divisor, and $W(k)/pW(k) \overset{\simeq}{\to} k$, $(a_0, a_1, \dots) \mapsto a_0$.

(vii) <u>Exercise</u>: Continue to suppose that k is a perfect ring of characteristic p; also let $t \in k$ be a non-zero-divisor and $q \in W(k)$ an element such that $q \equiv p \bmod [t]W(k)$.

Check that $[t] \in W(k)$ is a non-zero-divisor (this does not use the hypotheses on k). Using that p is a non-zero-divisor of $W(k)$ and that t is a non-zero-divisor of $k = W(k)/pW(k)$, deduce that p is a non-zero-divisor of $W(k)/[t]W(k)$. Now deduce that q is a non-zero-divisor of $W(k)/[t]W(k)$, hence that $[t]$ is a non-zero-divisor of $W(k)/qW(k)$.

Now suppose further that k is t-adically complete. Prove by induction that $W(k)/p^n W(k)$ is $[t]$-adically complete for all $n \geq 1$, and take the limit to deduce that $W(k)$ is $[t]$-adically complete (even $(p, [t])$-adically complete). Next show that q is a non-zero-divisor of $W(k)$ (this is probably the hardest part of the exercise) and deduce that $W(k)/qW(k)$ is $[t]$-adically complete.

Theorem 1.9 (Fontaine). *Let A be an integral perfectoid ring.*

(i) There is a unique ring homomorphism

$$\theta : W(A^\flat) \longrightarrow A$$

satisfying $\theta([b]) = b^\#$ for all $b \in A^\flat$.

(ii) θ is surjective and its kernel is generated by a non-zero-divisor (usually denoted by $\xi \in W(A^\flat)$).

(iii) A given element $\chi \in \operatorname{Ker}\theta$ is a generator if and only if its Witt vector expansion $\chi = (\chi_0, \chi_1, \dots)$ has the property that $\chi_1 \in A^{\flat\times}$.

Proof. (i): Since any element $b \in W(A^\flat)$ may be written uniquely as a p-adically convergent sum $b = \sum_{i\geq 0}[b_i]p^i$, the content of (i) is the assertion that the well-defined map

$$\theta : W(A^\flat) \longrightarrow A, \qquad b = \sum_{i=0}^{\infty}[b_i]p^i \mapsto \sum_{i=0}^{\infty} b_i^\# p^i$$

is actually a ring homomorphism. Since θ is clearly multiplicative on Teichmüller lifts, an easy argument shows that multiplicativity will follow from additivity. To do this we will use the phantom maps

$$\mathrm{fant}_n : W(A) \xrightarrow{\mathrm{gh}} (A)^{\mathbb{N}} \xrightarrow{\mathrm{proj}_n} A, \qquad (a_0, a_1, \dots) \mapsto a_0^{p^n} + pa_1^{p^{n-1}} + \cdots + p^n a_n,$$

which are ring homomorphisms. Note also that if $a_i \equiv a_i' \bmod pA$ then $p^i a_i^{p^{n-i}} \equiv p^i a_i'^{p^{n-i}} \bmod p^{n+1}A$, so in fact $\mathrm{phant}_n \bmod p^{n+1}$ only depends on the values of the Witt coordinates mod p, i.e., there is a commutative diagram

$$\begin{array}{ccc} W(A) & \xrightarrow{\mathrm{phant}_n} & A \\ \downarrow & & \downarrow \\ W(A/pA) & \xrightarrow[\overline{\mathrm{phant}_n}]{} & A/p^{n+1}A \end{array}$$

in which $\overline{\mathrm{phant}_n}$ must also be a ring homomorphism. But the composition

$$W(A^\flat) \xrightarrow{W(\varphi^{-n})} W(A^\flat) \xrightarrow{W(\#\bmod p)} W(A/pA) \xrightarrow{\overline{\mathrm{phant}_n}} A/p^{n+1}A$$

is exactly $\theta \bmod p^{n+1}$:

$$\sum_{i=0}^{\infty}[b_i]p^i = (b_0, b_1^p, b_2^{p^2}, \dots) \mapsto (b_0^{p^n}, b_1^{p^{1-n}}, b_2^{p^{2-n}}, \dots) \mapsto (b_0^{p^n\#}, b_1^{p^{1-n}\#}, b_2^{p^{2-n}\#}, \dots) \mapsto \sum_{i=0}^{n} p^i b_i^{p^{i-n}\# p^{n-i}} = \sum_{i=0}^{n} p^i b_i^\#$$

Since the first two maps are ring homomorphisms (they are the maps on $W(-)$ induced by the ring homomorphisms $\varphi^{-n} : A^\flat \to A^\flat$ and $\# : A^\flat \to A/pA$, we deduce that $\theta \bmod p^{n+1}$ is a ring homomorphism. But this is true for all n and A is separated for the p-adic topology, so this means θ is a ring homomorphism.

(ii): Since A and $W(A^\flat)$ are p-adically complete, to prove surjectivity of θ it is enough to show that it is surjective mod p. But this follows from the fact that $\# \bmod p : A^\flat \to A/pA$ is surjective.

Now we construct a possible generator ξ of $\operatorname{Ker}\theta$. Let $\pi \in A$ be a perfectoid pseudo-uniformiser admitting p-power roots, and let $\pi^\flat = (\pi, \pi^{1/p}, \dots)$ be the associated perfectoid psedu-uniformiser of $A^\flat$. Since $p \in \pi^p A$ and θ has been shown to be surjective, we may write $p = \pi^p \theta(-z)$ for some $z \in W(A^\flat)$, whence $\xi := p + [\pi^\flat]^p z \in \operatorname{Ker}\theta$. Note that ξ is a non-zero-divisor of $W(A^\flat)$, by applying Remark 1.8(vi) with $k = A^\flat$, $t = \pi^\flat$, $q = \xi$. We next show $\operatorname{Ker}\theta = \xi W(A^\flat)$. Since $W(A^\flat)$ is $[\pi^\flat]$-adically complete and A is $\theta([\pi^\flat]) = \pi$-torsion-free, one easily sees that $\theta : W(A^\flat)/\xi W(A^\flat) \to A$ is an isomorphism if and only if it becomes an isomorphism when we mod out by $[\pi^\flat]$; i.e., we must check that $\theta : W(A^\flat)/(\xi, [\pi^\flat]) \to A/\pi A$ is an isomorphism. But, using $\xi \equiv p \bmod [\pi^\flat]$, this map identifies with $A^\flat/\pi^\flat A^\flat \overset{\#\ \mathrm{mod}\ \pi}{\longrightarrow} A/\pi A$, which we saw was an isomorphism in the proof of Lemma 1.7(iii).

(iii): First note that the Witt vector expansion of our element ξ looks like

$$(\xi_0, \xi_1, \dots) = p + [\pi^\flat]^p x = (0, 1, 0, 0, \dots) + (\pi^{\flat p} x_0, \pi^{\flat p^2} x_1, \dots) = (\pi^{\flat p} x_0, 1 + \pi^{\flat p^2} x_1, \dots),$$

in particular $\xi_1 \in A^{\flat\times}$ (using $\pi^\flat$-adic completeness of $A^\flat$) and $\xi_0 \in \pi^\flat A^\flat$. Now let $\chi = (\chi_0, \chi_1, \dots) \in \operatorname{Ker}\theta$ be another element, and write $\chi = \beta\xi$ for some $\beta = (\beta_0, \beta_1, \dots) \in W(A^\flat)$. Expanding,

$$\chi = \beta\xi = (\beta_0, \beta_1, \dots)(\xi_0, \xi_1, \dots) = (\beta_0\xi_0, \beta_1\xi_0^p + \beta_0^p\xi_1, \dots).$$

Therefore:

$$\begin{aligned}
\operatorname{Ker}\theta = \xi W(A^\flat) &\iff \xi W(A^\flat) = \beta\xi W(A^\flat) \\
&\iff \beta \in W(A^\flat)^\times \text{ (using } \xi \text{ is a n-z-d)} \\
&\iff \beta_0 \in A^{\flat\times} \text{ (using that } W(A^\flat) \text{ is } p\text{-adically complete and } W(A^\flat)/pW(A^\flat) = A^\flat) \\
&\iff \beta_0^p\xi_1 \in A^{\flat\times} \text{ (since we already know } \xi_1 \in A^{\flat\times}) \\
&\iff \beta_1\xi_0^p + \beta_0^p\xi \in A^{\flat\times} \text{ (since } A^\flat \text{ is } \pi^\flat\text{-adically complete and } \xi_0 \in \pi^\flat A^\flat) \\
&\iff \xi \in A^{\flat\times},
\end{aligned}$$

completing the proof of part (iii). □

1.3 Tilting correspondence for integral perfectoid rings

We are now prepared to establish the easiest tilting correspondance, namely that for integral perfectoid rings.

Given an integral perfectoid ring A and an A-algebra B, we always equip B with the canonical topology induced by A, i.e., we give B the π-adic topology where $\pi \in A$ is any perfectoid pseudo-uniformiser (this topology on B does not depend on the chosen π); we say that B is a *perfectoid A-algebra* if and only if B (equipped with the just-defined topology) is an integral perfectoid ring. Note that if B is a perfectoid A-algebra and $\pi \in A$ is any perfectoid pseudo-uniformiser, the (the image in B of) π is also a perfectoid pseudo-uniformiser of the perfectoid ring B; this follows from Lemma 1.4.

Theorem 1.10 (Tilting correspondence for integral perfectoid rings)**.** *Fix an integral perfectoid ring A. Then tilting induces an equivalence of categories*

$$\textit{perfectoid } A\textit{-algebras} \xrightarrow{\simeq} \textit{perfectoid } A^\flat\textit{-algebras}, \qquad B \mapsto B^\flat,$$

with inverse given by sending a perfectoid $A^\flat$-algebra C to $C^\# := W(C) \otimes_{W(A^\flat),\theta} A$.

Proof. Let $\pi \in A$ be a perfectoid pseudo-uniformiser admitting p-power-roots, and $\pi^\flat$ the associated perfectoid pseudo-uniformiser of $A^\flat$; also let $\xi = p + [\pi^\flat]^p z \in W(A^\flat)$ be the generator of the ideal $\mathrm{Ker}(\theta : W(A^\flat) \to A)$ which we constructed in the proof of Theorem 1.9.

Step 1: Letting B be a perfectoid A-algebra, we show $(B^\flat)^\# = B$. We obviously have a commutative diagram (with surjective horizontal arrows by Theorem 1.9(ii))

$$\begin{array}{ccc} W(B^\flat) & \xrightarrow{\theta_B} & B \\ \uparrow & & \uparrow \\ W(A^\flat) & \xrightarrow[\theta=\theta_A]{} & A \end{array}$$

and so the image of ξ in $W(B^\flat)$ lands in $\mathrm{Ker}\,\theta_B$ (this denotes the θ-map for the integral perfectoid ring B). But the first coordinate in the Witt vector expansion of ξ is a unit of $A^\flat$ (by Theorem 1.9(iii)), and so its image in $B^\flat$ is also a unit; therefore Theorem 1.9(iii) (this time for the ring B) implies that $\mathrm{Ker}\,\theta_B = \xi W(A^\flat)$. In other words, the above diagram is a pushout and so the induced map $(B^\flat)^\# = W(B^\flat) \otimes_{W(A^\flat),\theta} A \to B$ is an isomorphism, as required.

Step 2: Letting C be a perfectoid A-algebra, we show that $C^\#$ is a perfectoid $A^\flat$-algebra and that $(C^\#)^\flat = C$. Since θ is surjective with kernel $\xi W(A^\flat)$, we can write $C^\# = W(C)/\xi W(C)$ viewed as an A-algebra via the identification $\theta : W(A^\flat)/\xi W(A^\flat) \overset{\simeq}{\to} A$. Remark 1.8(vi) (with $k = C$, $t = \pi^\flat$, $q = \xi$) therefore shows that $C^\#$ is complete for the π-adic topology and that π is a non-zero-divisor of C. It remains to show that $\Phi : C^\#/\pi C^\flat \to C^\#/\pi^p C^\flat$ is an isomorphism. But again writing $C^\# = W(C)/\xi W(C)$ and recalling that $\xi \equiv p \bmod [\pi^\flat]^p$, this map may be rewritten as $\Phi : C/\pi^\flat C \to C/\pi^{\flat p} C$, which is indeed an isomorphism since $\pi^\flat$ is a perfectoid pseudo-uniformiser of C. This completes the proof that $C^\#$ is a perfectoid A-algebra.

Finally, as we already used in the previous paragraph, we have $C^\#/\pi C^\# = C/\pi^\flat C$. Tilting obtains

$$(C^\#)^\flat = \varprojlim_{x \mapsto x^p} C^\#/\pi C^\# = \varprojlim_{x \mapsto x^p} C/\pi^\flat C = C^\flat = C,$$

where the final equality is the fact that tilting an integral perfectoid ring of characteristic p has no effect. □