

Adic and perfectoid spaces

IMJ–PRG specialised M2 course, 27 February – 5 April

Matthew Morrow
matthew.morrow@imj-prg.fr

Abstract

The main goal of this course is to develop the foundations of the theory of perfectoid spaces, more precisely to prove the various tilting correspondences for perfectoid rings, the almost purity theorem, and almost vanishing theorems. We develop simultaneously what is needed from the theory of adic spaces.

CONTENTS

1	Integral perfectoid rings	3
1.1	The tilt of an integral perfectoid ring	5
1.2	Fontaine’s map θ	7
1.3	Tilting correspondence for integral perfectoid rings	9
1.4	Anneaux perfectoides entiers et basculement : exercices et exemples	10
1.5	Treillis des sous-anneaux perfectoides entiers : exercices	11
2	Huber rings	12
2.1	Bounded sets	13
3	Perfectoid Tate rings	16
3.1	Aside: The language of almost mathematics	18
3.2	Tilting perfectoid Tate rings	19
4	Perfectoid fields	22
4.1	Exercices sur la fondation des corps perfectoides	22
4.1.1	Corps valués	22
4.1.2	Le cas des corps perfectoides	23
4.2	Aside: almost mathematics II	24
4.3	Tilting perfectoid fields	25
5	The adic spectrum of a Huber ring	27
5.1	Completion	30
5.2	Localisation	31

Some notation

For the whole course we fix a prime number p . All rings are commutative. Given a ring A of characteristic p , we write $\varphi : A \rightarrow A$, $a \mapsto a^p$ for its Frobenius endomorphism (which is a ring homomorphism).

1 INTEGRAL PERFECTOID RINGS

This section is devoted to the foundations of the theory of integral perfectoid rings, which is largely of a commutative algebra flavour.

Let A be a topological ring. We say that A is *integral perfectoid*¹ if and only if there exists a non-zero-divisor $\pi \in A$ such that

- (a) the topology on A is the π -adic topology, and A is complete for this topology (i.e., $A \rightarrow \varprojlim_n A/\pi^n A$ is an isomorphism of topological rings, where each $A/\pi^n A$ has the discrete topology);
- (b) $p \in \pi^p A$;
- (c) $\Phi : A/\pi A \rightarrow A/\pi^p A, a \mapsto a^p$, is an isomorphism

It is convenient for us, even though it is not exactly standard in the literature, to call any such element π a perfectoid pseudo-uniformiser (ppu).

Note that \mathbb{Z}_p is “too small” to be perfectoid: it fails to satisfy condition (b).

Example 1.1. Here are some examples which furnish us with lots of integral perfectoid rings.

- (i) Let C be an algebraic extension of \mathbb{Q}_p , and \mathcal{O}_C the integral closure of \mathbb{Z}_p in C ; assume that every element of $\mathcal{O}_C/p\mathcal{O}_C$ has a p^{th} -root and that there exists a non-unit $\pi \in \mathcal{O}_C$ such that $p \in \pi^p \mathcal{O}_C$.

Check that $\Phi : \mathcal{O}_C/\pi\mathcal{O}_C \rightarrow \mathcal{O}_C/\pi^p\mathcal{O}_C, a \mapsto a^p$ is an isomorphism and deduce that the p -adic completion (equivalently the p -adic completion) $\widehat{\mathcal{O}_C}$ is an integral perfectoid ring.

For example, we can apply this to $C = \mathbb{Q}_p(p^{1/p^\infty})$ (in which case $\mathcal{O}_C = \mathbb{Z}_p[p^{1/p^\infty}]$), or $C = \mathbb{Q}_p(\zeta_{p^\infty})$ (in which case $\mathcal{O}_C = \mathbb{Z}_p[\zeta_{p^\infty}]$), or $C = \mathbb{Q}_p^{\text{alg}}$.

- (ii) Let A be an integral perfectoid ring. Its algebra of “perfectoid polynomials” is

$$A\langle T^{1/p^\infty} \rangle := \text{the } \pi\text{-adic completion of } \bigcup_{i \geq 1} A[T^{1/p^i}],$$

where $\pi \in A$ is any perfectoid pseudo-uniformiser. Check that $A\langle T^{1/p^\infty} \rangle$ is an integral perfectoid ring, and more generally for several variables T_1, \dots, T_d .

- (iii) Let A be an integral perfectoid ring and B any étale A -algebra. Show that the π -adic completion \widehat{B} is an integral perfectoid ring. (You will need the following fact: given any ring of characteristic p , the diagram $k' \xrightarrow{\varphi} k'$ is a pushout, i.e., $k' \otimes_{k, \varphi} k \xrightarrow{\sim} k'$.)

$$\begin{array}{ccc} k' & \xrightarrow{\varphi} & k' \\ \uparrow & & \uparrow \\ k & \xrightarrow{\varphi} & k \end{array}$$

- (iv) Let C be as in part (i) and let R be a smooth \mathcal{O}_C -algebra; suppose that there exists an étale morphism $\mathcal{O}_C[T_1, \dots, T_d] \rightarrow R$ (if R is equi-dimensional, this is always true locally on $\text{Spec } R$). Now use (ii) and (iii) to construct an integral perfectoid ring R_∞

¹La meilleure traduction de “integral perfectoid ring” n’est pas évidente, mais on va utiliser *anneau perfectoïde entier*.

Construction 1.2. Given any ring A we write

$$\varprojlim_{x \rightarrow x^p} A := \{(a_0, a_1, \dots) \in A^{\mathbb{N}} : a_i^p = a_{i-1} \text{ for all } i \geq 1\}$$

for the set of compatible sequences of p -power roots in A . Note that we can multiply two such sequences, so $\varprojlim_{x \rightarrow x^p} A$ forms a multiplicative monoid; if A has characteristic p then we can also add two such sequences, so then A is even a ring.

The construction is clearly functorial: a morphism of rings $A \rightarrow B$ induces a morphism of monoids $\varprojlim_{x \rightarrow x^p} A \rightarrow \varprojlim_{x \rightarrow x^p} B$ (which is even a morphism of rings if A and B have characteristic p).

The following result is extremely important and will be used repeatedly: if $\pi \in A$ is an element such that (i) $p \in \pi A$ and (ii) A is π -adically complete, then the resulting map

$$\varprojlim_{x \rightarrow x^p} A \longrightarrow \varprojlim_{x \rightarrow x^p} A/\pi A$$

is actually a bijection (hence an isomorphism of monoids). We leave the details of the proof to the reader, but give the following recipe for the inverse of the map: given $b = (b_0, b_1, \dots) \in \varprojlim_{x \rightarrow x^p} A/\pi A$, let $\tilde{b}_i \in A$ be an arbitrary lift of $b_i \in A/\pi A$ for each $i \geq 0$; then set

$$a_i := \lim_{n \rightarrow \infty} \tilde{b}_{i+n} \widetilde{p^n}$$

and check that $a := (a_0, a_1, \dots) \in \varprojlim_{x \rightarrow x^p} A$ really is a well-defined lift of b .

Lemma 1.3. *Let A be integral perfectoid, and $\pi \in A$ a perfectoid pseudo-uniformiser. Then:*

(i) *Every element of $A/\pi p A$ is a p^{th} -power (n.b., $A/\pi p A$ does not necessarily have characteristic p).*

(ii) *If an element $a \in A[\frac{1}{\pi}]$ satisfies $a^p \in A$, then $a \in A$.*

(iii) *After multiplying π by a unit it has a compatible sequence of p -power roots $\pi^{1/p}, \pi^{1/p^2}, \dots \in A$.*

Proof. (i): Using the surjectivity of Φ , a simple induction lets us write any $a \in A$ as an infinite sum $a = \sum_{i \geq 0} a_i^p \pi^{pi}$ for some $a_i \in A$; but this is $\equiv (\sum_{i \geq 0} a_i \pi^i)^p \pmod{\pi p A}$.

(ii): Let $l \geq 0$ be the smallest integer such that $\pi^l a \in A$. Assuming that $l > 0$, we get a contradiction by noting that $\pi^{pl} a^p \in \pi^{pl} A \subseteq \pi^p A$, whence $\pi^l a \in \pi A$ by condition (c), and so $\pi^{l-1} a \in A$.

(iii): Since the Frobenius is surjective on $A/\pi^p A$, there exists an element of $\varprojlim_{x \rightarrow x^p} A/\pi^p A$ of the form $(\pi \bmod \pi^p A, ?, ?, \dots)$. Applying the exercise of Construction 1.2, we deduce that the natural map $\varprojlim_{x \rightarrow x^p} A \rightarrow \varprojlim_{x \rightarrow x^p} A/\pi^p A$ is a bijection. Hence there exists $a = (a_0, a_1, \dots) \in \varprojlim_{x \rightarrow x^p} A$ such that $a_0 \equiv \pi \bmod \pi^p A$; therefore $a = u\pi$ for some $u \in 1 + \pi^{p-1} A \subseteq A^\times$ (the inclusion \subseteq results from π -adic completeness of A). \square

Lemma 1.4. *Let A be integral perfectoid, and $\varpi \in A$ an element satisfying conditions (a) and (b). Then ϖ is a non-zero-divisor satisfying (c), i.e., it is a perfectoid pseudo-uniformiser.*

Proof. We must show that $\Phi : A/\varpi A \rightarrow A/\varpi^p A$ is an isomorphism. Let $\pi \in A$ be a perfectoid pseudo-uniformiser.

It follows from Lemma 3(i) that every element of A/pA is a p^{th} -power; hence every element of its quotient $A/\varpi^p A$ is a p^{th} -power, i.e., Φ is surjective.

The fact that π and ϖ define the same topology implies that a power of each is divisible by the other, whence ϖ is a non-zero-divisor and $A[\frac{1}{\varpi}] = A[\frac{1}{\pi}]$. If $a \in A$ satisfies $a^p \in \varpi^p A$, then $(a/\varpi) \in A[\frac{1}{\pi}]$ satisfies $(a/\varpi)^p \in A$, and it then follows from Lemma 3(ii) that in fact $a \in \varpi A$ as desired. \square

Lemma 1.5. *Suppose A is a complete topological ring such that $pA = 0$. Then A is integral perfectoid if and only if it is perfect and the topology is π -adic for some non-zero-divisor $\pi \in A$.*

Proof. Exercise. \square

1.1 The tilt of an integral perfectoid ring

Definition 1.6. The *tilt*² of an integral perfectoid ring A is $A^\flat := \varprojlim_{x \rightarrow x^p} A/pA$, equipped with the inverse limit topology (A/pA is of course given the quotient topology, i.e., the π -adic topology for any choice of perfectoid pseudo-uniformiser for A).

Note that A^\flat is a perfect ring of characteristic p ; in fact, it is the initial object among all perfect rings of characteristic p mapping to A/pA .

Recalling from Construction 1.2 that the natural map $\varprojlim_{x \rightarrow x^p} A \rightarrow \varprojlim_{x \rightarrow x^p} A/pA$ is an isomorphism of monoids, we define the *untilting map* $\# : A^\flat \rightarrow A$, $b \mapsto b^\#$ to be projection to the 0th-coordinate of $\varprojlim_{x \rightarrow x^p} A$; explicitly, the map $\#$ is given by $\varprojlim_{x \rightarrow x^p} A/pA \ni (b_0, b_1, \dots) \mapsto \lim_{i \rightarrow \infty} \tilde{b}_i^{p^i}$, where $\tilde{b}_i \in A$ are arbitrary lifts of the elements $b_i \in A/pA$.

The untilting map is multiplicative by generally not additive; in fact, given $b, c \in A^\flat$, it transforms under addition as follows:

$$(b + c)^\# = \lim_{i \rightarrow \infty} ((b^{1/p^i})^\# + (c^{1/p^i})^\#)^{p^i}.$$

However, note that the composition $A^\flat \xrightarrow{\# \bmod p} A/pA$ is a ring homomorphism: indeed, it is the surjective ring homomorphism given by projecting $A^\flat \cong \varprojlim_{x \rightarrow x^p} A/pA$ to the 0th-coordinate. Also, if A is of characteristic p , then the untilting map $\# : A^\flat \rightarrow A$ is an isomorphism of rings.

Lemma 1.7. *Let A be an integral perfectoid ring. Then:*

- (i) *the untilting map $\# : A^\flat \rightarrow A$ is continuous;*
- (ii) *the isomorphisms of monoids $\varprojlim_{x \rightarrow x^p} A \rightarrow A^\flat = \varprojlim_{x \rightarrow x^p} A/pA \rightarrow \varprojlim_{x \rightarrow x^p} A/\pi A$ are homeomorphisms, where $\pi \in A$ is any perfectoid pseudo-uniformiser;*
- (iii) *A^\flat is also an integral perfectoid ring.*

Proof. Given $(1, \dots, 1, b_{n+1}, b_{n+2}, \dots) \in \varprojlim_{x \rightarrow x^p} A/\pi A$, any chosen lifts \tilde{b}_i satisfy $\tilde{b}_i^{p^{i-n}} \equiv 1 \pmod{\pi A}$ for $i > n$, whence $\tilde{b}_i^{p^i} \equiv 1 \pmod{\pi^n A}$; taking the limit shows that the untilt is $\equiv 1 \pmod{\pi^n A}$. This proves that the untilting map $\# : \varprojlim_{x \rightarrow x^p} A/\pi A \rightarrow A$ is continuous (for the inverse limit of discrete topologies on the domain), from which (i) and (ii) easily follow. Filling in the details is left as an exercise.

(iii) We have already noted that A^\flat is a perfect ring of characteristic p , and the homeomorphism $A \cong \varprojlim_{x \rightarrow x^p} A/\pi A$ shows that A is an inverse limit of discrete rings, whence A is a

²Le *basculé* de A en français.

complete topological ring. According to Lemma 1.5, it remains to prove the following: there exists a non-zero-divisor $\pi^b \in A^b$ such that the topology on A^b is the π^b -adic topology.

Possibly after changing our perfectoid pseudo-uniformiser π , we may assume that it admits compatible p -power roots (by Lemma 3(iii)); let $\pi^b = (\pi, \pi^{1/p}, \dots) \in A^b$ be the corresponding element of A^b , which satisfies $(\pi^b)^\# = \pi$.

We show first that π^b is a non-zero-divisor. To do that we note that for each $n \geq 1$ we have an exact sequence

$$0 \longrightarrow \pi^{1-1/p^n} A/\pi A \longrightarrow A/\pi A \xrightarrow{\times \pi^{1/p^n}} A/\pi A \xrightarrow{\varphi^n} A/\pi A \longrightarrow 0$$

Exactness is easy everywhere except possibly at the second term from the right: but if $a \in A$ satisfies $a^p \in \pi A$ then $a/\pi^{1/p^n} \in A[\frac{1}{\pi}]$ satisfies $(a/\pi^{1/p^n})^{p^n} \in A$, whence Lemma 3(ii) implies $a \in \pi^{1/p^n} A$ as desired.

These sequences are moreover compatible in n , with respect to the maps $0, \varphi, \varphi, \text{id}$ respectively. Although it is not always the case that an inverse limit of exact sequences is still exact, in this case the transition maps are either surjective (φ and id) or zero, and so taking the inverse limit does yield an exact sequence

$$0 \longrightarrow A^b \xrightarrow{\times \pi^b} A^b \xrightarrow{\# \bmod \pi} A/\pi A \longrightarrow 0.$$

That is, π^b is a non-zero-divisor of A^b and the untilting map induces an isomorphism of rings $A^b/\pi^b A^b \xrightarrow{\cong} A/\pi A$.

Finally we check that the topology on A^b is the π^b -adic topology. Since $A^b \cong \varprojlim_{x \rightarrow x^p} A/\pi A$ is a homeomorphism by part (i), a basis of open neighbourhoods of $0 \in A^b$ is given by $\text{Ker}(\text{proj}_n)$ for $n \geq 0$, where $\text{proj}_n : \varprojlim_{x \rightarrow x^p} A/\pi A \rightarrow A/\pi A, (b_0, b_1, \dots) \mapsto b_n$ denotes the n^{th} -projection map. Note that proj_0 is the untilting map. Since the composition $A^b \xrightarrow{\varphi^n \cong} A^b \xrightarrow{\text{proj}_n} A/\pi A$ is proj_0 , the basis of open neighbourhoods is given by

$$\text{Ker}(\text{proj}_n) = \varphi^n(\text{Ker}(\text{proj}_0)) = \varphi^n(\pi^b A^b) = \pi^{bp^n} A^b,$$

showing that the topology is indeed π^b -adic. \square

We point out explicitly that we showed in the previous proof that the kernel of the surjective ring homomorphism $A^b \xrightarrow{\# \bmod \pi} A/\pi A$ (i.e., projection to the 0^{th} -coordinate of $A^b = \varprojlim_{x \rightarrow x^p} A/\pi A$) is $\pi^b A^b$, i.e., untilting induces an isomorphism of rings

$$\# : A^b/\pi^b A^b \xrightarrow{\cong} A/\pi A.$$

This will be used frequently.

Example 1.8. Let A be an integral perfectoid ring and $A\langle T^{1/p^\infty} \rangle$ its algebra of perfectoid polynomials from Example 1.1. Note that its tilt $A\langle T^{1/p^\infty} \rangle^b$ contains an element $T^b := (T, T^{1/p}, T^{1/p^2}, \dots)$. Construct an isomorphism of perfectoid A^b -algebras

$$A^b\langle U^{1/p^\infty} \rangle \xrightarrow{\cong} A\langle T^{1/p^\infty} \rangle^b$$

which sends U to T^b , where the left side denotes the algebra of perfectoid polynomials over A^b .

1.2 Fontaine's map θ

In this section we introduce Fontaine's map $\theta : W(A^b) \rightarrow A$ for integral perfectoid rings, which will allow us to recover A from A^b and will therefore play a fundamental role in untilting.

Remark 1.9 (Reminder on the ring of Witt vectors). If k is any ring, let $W(k)$ denote its ring of p -typical Witt vectors. A classical reference is Serre's *Corps locaux*. Here are some reminders about this object:

- (i) There is an identification of sets $W(k) = k^{\mathbb{N}}$. So each element of $W(k)$ may be written uniquely as (a_0, a_1, \dots) , with $a_i \in k$.
- (ii) Addition and multiplication are given by certain polynomials with integer coefficients (which do not depend on k), for example

$$(a_0, a_1, a_2, \dots) + (b_0, b_1, b_2, \dots) = (a_0 + b_0, a_1 + b_1 - \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} a_0^i b_0^{p-i}, \dots)$$

$$(a_0, a_1, a_2, \dots) \cdot (b_0, b_1, b_2, \dots) = (a_0 b_0, a_0^p b_0 + b_0^p a_1 + p a_1 b_1, \dots)$$

- (iii) There is a natural ring homomorphism called the “phantom” or “ghost” map

$$\text{phant} : W(k) \longrightarrow k^{\mathbb{N}}, \quad (a_0, a_1, \dots) \mapsto (a_0, a_0^p + p a_1, a_0^{p^2} + p a_1^p + p^2 a_2, \dots).$$

In particular, its n^{th} -coordinate $\text{phant}_n : W(k) \rightarrow k$, $(a_0, a_1, \dots) \mapsto \sum_{i=0}^n p^i a_i^{p^{n-i}}$ is a ring homomorphism for each $n \geq 0$.

- (iv) If $k \supseteq \mathbb{Q}$ then phant is an isomorphism of rings; if k is p -torsion-free then phant is injective.
- (v) Given $a \in k$, its *Teichmüller lift* is $[a] := (a, 0, 0, 0, \dots) \in W(k)$; the map $[\cdot] : k \rightarrow W(k)$ is multiplicative but not additive.
- (vi) Suppose now that $k \supseteq \mathbb{F}_p$. Then $W(k)$ is p -adically complete and

$$\sum_{i=0}^{\infty} [a_i] p^i = (a_0, a_1^p, a_2^{p^2}, \dots),$$

where $a_i \in k$. In particular, if k is perfect then one easily deduces the following: each element of $W(k)$ may be written uniquely as $\sum_{i=0}^{\infty} [a_i] p^i$ for some elements $a_i \in A$, the element $p \in W(k)$ is a non-zero-divisor, and $W(k)/pW(k) \xrightarrow{\sim} k$, $(a_0, a_1, \dots) \mapsto a_0$.

- (vii) Exercise: Continue to suppose that k is a perfect ring of characteristic p ; also let $t \in k$ be a non-zero-divisor and $q \in W(k)$ an element such that $q \equiv p \pmod{[t]W(k)}$.

Check that $[t] \in W(k)$ is a non-zero-divisor (this does not use the hypotheses on k). Using that p is a non-zero-divisor of $W(k)$ and that t is a non-zero-divisor of $k = W(k)/pW(k)$, deduce that p is a non-zero-divisor of $W(k)/[t]W(k)$. Now deduce that q is a non-zero-divisor of $W(k)/[t]W(k)$, hence that $[t]$ is a non-zero-divisor of $W(k)/qW(k)$.

Now suppose further that k is t -adically complete. Prove by induction that $W(k)/p^n W(k)$ is $[t]$ -adically complete for all $n \geq 1$, and take the limit to deduce that $W(k)$ is $[t]$ -adically complete (even $(p, [t])$ -adically complete). Next show that q is a non-zero-divisor of $W(k)$ (this is probably the hardest part of the exercise) and deduce that $W(k)/qW(k)$ is $[t]$ -adically complete.

Theorem 1.10 (Fontaine). *Let A be an integral perfectoid ring.*

(i) *There is a unique ring homomorphism*

$$\theta : W(A^\flat) \longrightarrow A$$

which satisfies $\theta([b]) = b^\#$ for all $b \in A^\flat$.

(ii) *θ is surjective and its kernel is generated by a non-zero-divisor (usually denoted by $\xi \in W(A^\flat)$).*

(iii) *A given element $\chi \in \text{Ker } \theta$ is a generator of this kernel if and only if its Witt vector expansion $\chi = (\chi_0, \chi_1, \dots)$ has the property that $\chi_1 \in A^{\flat\times}$.*

Proof. (i): Since any element $b \in W(A^\flat)$ may be written uniquely as a p -adically convergent sum $b = \sum_{i \geq 0} [b_i]p^i$, the content of (i) is the assertion that the well-defined map

$$\theta : W(A^\flat) \longrightarrow A, \quad b = \sum_{i=0}^{\infty} [b_i]p^i \mapsto \sum_{i=0}^{\infty} b_i^\# p^i$$

is actually a ring homomorphism (this map makes sense since A is p -adically complete). It is enough to check that this map is a ring homomorphism modulo any power of p (since A is p -adically separated), so fix $n \geq 0$.

We will use the phantom map from Remark 1.9(iii)

$$\text{phant}_n : W(A) \xrightarrow{\text{gh}} (A)^\mathbb{N} \xrightarrow{\text{proj}_n} A, \quad (a_0, a_1, \dots) \mapsto a_0^{p^n} + pa_1^{p^{n-1}} + \dots + p^n a_n,$$

which is a ring homomorphism. Note that if $a_i \equiv a'_i \pmod{pA}$ then $p^i a_i^{p^{n-i}} \equiv p^i a_i'^{p^{n-i}} \pmod{p^{n+1}A}$, so in fact $\text{phant}_n \pmod{p^{n+1}}$ only depends on the values of the Witt coordinates mod p , i.e., there is a commutative diagram

$$\begin{array}{ccc} W(A) & \xrightarrow{\text{phant}_n} & A \\ \downarrow & & \downarrow \\ W(A/pA) & \xrightarrow[\text{phant}_n]{} & A/p^{n+1}A \end{array}$$

in which $\overline{\text{phant}}_n$ must also be a ring homomorphism. But the composition

$$W(A^\flat) \xrightarrow{W(\varphi^{-n})} W(A^\flat) \xrightarrow{W(\# \bmod p)} W(A/pA) \xrightarrow{\overline{\text{phant}}_n} A/p^{n+1}A$$

is exactly $\theta \pmod{p^{n+1}}$:

$$\sum_{i=0}^{\infty} [b_i]p^i = (b_0, b_1^p, b_2^{p^2}, \dots) \mapsto (b_0^{p^n}, b_1^{p^{1-n}}, b_2^{p^{2-n}}, \dots) \mapsto (b_0^{p^n \#}, b_1^{p^{1-n} \#}, b_2^{p^{2-n} \#}, \dots) \mapsto \sum_{i=0}^n p^i b_i^{p^{i-n} \# p^{n-i}} = \sum_{i=0}^n p^i b_i^\#$$

Since the first two maps are ring homomorphisms (they are the maps on $W(-)$ induced by the ring homomorphisms $\varphi^{-n} : A^\flat \rightarrow A^\flat$ and $\# : A^\flat \rightarrow A/pA$), we deduce that $\theta \pmod{p^{n+1}}$ is a ring homomorphism, as required.

(ii): Since A and $W(A^\flat)$ are p -adically complete, to prove surjectivity of θ it is enough to show that it is surjective mod p . But this follows from the fact that $\# \bmod p : A^\flat \rightarrow A/pA$ is surjective.

Now we construct a possible generator ξ of $\text{Ker } \theta$. Let $\pi \in A$ be a perfectoid pseudo-uniformiser admitting p -power roots, and let $\pi^b = (\pi, \pi^{1/p}, \dots)$ be the associated perfectoid pseudo-uniformiser of A^b . Since $p \in \pi^b A$ and θ has been shown to be surjective, we may write $p = \pi^b \theta(-z)$ for some $z \in W(A^b)$, whence $\xi := p + [\pi^b]^p z \in \text{Ker } \theta$. Note that ξ is a non-zero-divisor of $W(A^b)$, by applying Remark 1.9(vi) with $k = A^b$, $t = \pi^b$, $q = \xi$. We next show $\text{Ker } \theta = \xi W(A^b)$. Since $W(A^b)$ is $[\pi^b]$ -adically complete and A is $\theta([\pi^b]) = \pi$ -torsion-free, one easily sees that $\theta : W(A^b)/\xi W(A^b) \rightarrow A$ is an isomorphism if and only if it becomes an isomorphism when we mod out by $[\pi^b]$; i.e., we must check that $\theta : W(A^b)/(\xi, [\pi^b]) \rightarrow A/\pi A$ is an isomorphism. But, using $\xi \equiv p \pmod{[\pi^b]}$, this map identifies with $A^b/\pi^b A^b \xrightarrow{\# \text{ mod } \pi} A/\pi A$, which we saw was an isomorphism in the proof of Lemma 1.7(iii).

(iii): First note that the Witt vector expansion of our element ξ looks like

$$(\xi_0, \xi_1, \dots) = p + [\pi^b]^p x = (0, 1, 0, 0, \dots) + (\pi^{bp} x_0, \pi^{bp^2} x_1, \dots) = (\pi^{bp} x_0, 1 + \pi^{bp^2} x_1, \dots),$$

in particular $\xi_1 \in A^{b \times}$ (using π^b -adic completeness of A^b) and $\xi_0 \in \pi^b A^b$. Now let $\chi = (\chi_0, \chi_1, \dots) \in \text{Ker } \theta$ be another element, and write $\chi = \beta \xi$ for some $\beta = (\beta_0, \beta_1, \dots) \in W(A^b)$. Expanding,

$$\chi = \beta \xi = (\beta_0, \beta_1, \dots)(\xi_0, \xi_1, \dots) = (\beta_0 \xi_0, \beta_1 \xi_0^p + \beta_0^p \xi_1, \dots).$$

Therefore:

$$\begin{aligned} \text{Ker } \theta = \xi W(A^b) &\iff \xi W(A^b) = \beta \xi W(A^b) \\ &\iff \beta \in W(A^b)^\times \text{ (using } \xi \text{ is a n-z-d)} \\ &\iff \beta_0 \in A^{b \times} \text{ (using that } W(A^b) \text{ is } p\text{-adically complete and } W(A^b)/pW(A^b) = A^b) \\ &\iff \beta_0^p \xi_1 \in A^{b \times} \text{ (since we already know } \xi_1 \in A^{b \times}) \\ &\iff \beta_1 \xi_0^p + \beta_0^p \xi_1 \in A^{b \times} \text{ (since } A^b \text{ is } \pi^b\text{-adically complete and } \xi_0 \in \pi^b A^b) \\ &\iff \xi \in A^{b \times}, \end{aligned}$$

completing the proof of part (iii). □

1.3 Tilting correspondence for integral perfectoid rings

We are now prepared to establish the easiest tilting correspondance,³ namely that for integral perfectoid rings.

Given an integral perfectoid ring A and an A -algebra B , we always equip B with the canonical topology induced by A , i.e., we give B the π -adic topology where $\pi \in A$ is any perfectoid pseudo-uniformiser (this topology on B does not depend on the chosen π); we say that B is a *perfectoid A -algebra* if and only if B (equipped with the just-defined topology) is an integral perfectoid ring. Note that if B is a perfectoid A -algebra and $\pi \in A$ is any perfectoid pseudo-uniformiser, then the (image in B of) π is also a perfectoid pseudo-uniformiser of the perfectoid ring B ; this follows from Lemma 1.4.

Theorem 1.11 (Tilting correspondence for integral perfectoid rings). *Fix an integral perfectoid ring A . Then tilting induces an equivalence of categories*

$$\text{perfectoid } A\text{-algebras} \xrightarrow{\cong} \text{perfectoid } A^b\text{-algebras}, \quad B \mapsto B^b,$$

with inverse given by sending a perfectoid A^b -algebra C to $C^\# := W(C) \otimes_{W(A^b), \theta} A$.

³correspondance de basculement

Proof. Let $\pi \in A$ be a perfectoid pseudo-uniformiser admitting p -power-roots, and $\pi^{\flat} = (\pi, \pi^{1/\pi}, \dots) \in A^{\flat}$ the associated perfectoid pseudo-uniformiser of A^{\flat} ; also let $\xi = p + [\pi^{\flat}]^p z \in W(A^{\flat})$ be the generator of the ideal $\text{Ker}(\theta : W(A^{\flat}) \rightarrow A)$ which we constructed in the proof of Theorem 1.10.

Step 1: Letting B be a perfectoid A -algebra, we show $(B^{\flat})^{\#} = B$. We obviously have a commutative diagram (with surjective horizontal arrows by Theorem 1.10(ii))

$$\begin{array}{ccc} W(B^{\flat}) & \xrightarrow{\theta_B} & B \\ \uparrow & & \uparrow \\ W(A^{\flat}) & \xrightarrow{\theta = \theta_A} & A \end{array}$$

and so the image of ξ in $W(B^{\flat})$ lands in $\text{Ker } \theta_B$ (this denotes the θ -map for the integral perfectoid ring B). But the first coordinate in the Witt vector expansion of ξ is a unit of A^{\flat} (by Theorem 1.10(iii)), and so its image in B^{\flat} is also a unit; therefore Theorem 1.10(iii) (this time for the ring B) implies that $\text{Ker } \theta_B = \xi W(A^{\flat})$. In other words, the above diagram is a pushout and so the induced map $(B^{\flat})^{\#} = W(B^{\flat}) \otimes_{W(A^{\flat}), \theta} A \rightarrow B$ is an isomorphism, as required.

Step 2: Letting C be a perfectoid A -algebra, we show that $C^{\#}$ is a perfectoid A^{\flat} -algebra and that $(C^{\#})^{\flat} = C$. Since θ is surjective with kernel $\xi W(A^{\flat})$, we can write $C^{\#} = W(C)/\xi W(C)$ viewed as an A -algebra via the identification $\theta : W(A^{\flat})/\xi W(A^{\flat}) \xrightarrow{\sim} A$. Remark 1.9(vi) (with $k = C$, $t = \pi^{\flat}$, $q = \xi$) therefore shows that $C^{\#}$ is complete for the π -adic topology and that π is a non-zero-divisor of C . It remains to show that $\Phi : C^{\#}/\pi C^{\#} \rightarrow C^{\#}/\pi^p C^{\#}$ is an isomorphism. But again writing $C^{\#} = W(C)/\xi W(C)$ and recalling that $\xi \equiv p \pmod{[\pi^{\flat}]^p}$, this map may be rewritten as $\Phi : C/\pi^{\flat} C \rightarrow C/\pi^{\flat p} C$, which is indeed an isomorphism since π^{\flat} is a perfectoid pseudo-uniformiser of C . This completes the proof that $C^{\#}$ is a perfectoid A -algebra.

Finally, as we already used in the previous paragraph, we have $C^{\#}/\pi C^{\#} = C/\pi^{\flat} C$. Tilting obtains

$$(C^{\#})^{\flat} = \varprojlim_{x \mapsto x^p} C^{\#}/\pi C^{\#} = \varprojlim_{x \mapsto x^p} C/\pi^{\flat} C = C^{\flat} = C,$$

where the final equality is the fact that tilting an integral perfectoid ring of characteristic p has no effect. \square

1.4 Anneaux perfectoïdes entiers et basculement : exercices et exemples

- (i) Soit C une extension algébrique de \mathbb{Q}_p , et \mathcal{O}_C la clôture intégrale de \mathbb{Z}_p dans C . Supposons que tout élément de $\mathcal{O}_C/p\mathcal{O}_C$ admette une racine p -ième et qu'il existe $\pi \in \mathcal{O}_C$ qui n'est pas une unité tel que $p \in \pi^p \mathcal{O}_C$ (par exemple $C = \mathbb{Q}_p(\zeta_{p^\infty})$, $\mathbb{Q}_p(p^{1/p^\infty})$ ou $\overline{\mathbb{Q}_p}$).

Démontrer que $\Phi : \mathcal{O}_C/\pi \mathcal{O}_C \rightarrow \mathcal{O}_C/\pi^p \mathcal{O}_C$, $a \mapsto a^p$ est un isomorphisme et en déduire que le complété p -adique $\widehat{\mathcal{O}_C}$ de \mathcal{O}_C est un anneau perfectoïde entier.

- (ii) Soit A un anneau perfectoïde entier. Son "algèbre de polynômes perfectoïde" est

$$A\langle T^{1/p^\infty} \rangle := \text{le complété } \pi\text{-adique de } \bigcup_{i \geq 1} A[T^{1/p^i}]$$

où π est n'importe quelle pseudo-uniformisante perfectoïde de A . Démontrer que $A\langle T^{1/p^\infty} \rangle$ est un anneau perfectoïde entier (en fait une A -algèbre perfectoïde).

On remarque que $A\langle T^{1/p^\infty} \rangle^b$ (càd le basculé de $A\langle T^{1/p^\infty} \rangle$) contient un élément $T^b := (T, T^{1/p}, T^{1/p^2}, \dots)$. Construire un isomorphisme de A^b -algèbres perfectoides

$$A^b\langle U^{1/p^\infty} \rangle \xrightarrow{\sim} A\langle T^{1/p^\infty} \rangle^b$$

qui envoie U sur T^b , où le membre de gauche est une algèbre de polynômes perfectoïde sur A^b .

Démontrer les résultats analogues pour plusieurs variables T_1, \dots, T_d .

- (iii) Soit A un anneau perfectoïde entier et B une A -algèbre étale. Démontrer que le complété π -adique \widehat{B} est une A -algèbre perfectoïde. (Le fait suivant sera utile : si $k \rightarrow k'$ est un morphisme étale entre des anneaux de caractéristique p , alors le diagramme

$$\begin{array}{ccc} k & \longrightarrow & k' \\ \varphi \uparrow & & \uparrow \varphi \\ k & \longrightarrow & k' \end{array}$$

est un pushout, càd $k' \otimes_{k, \varphi} k \xrightarrow{\sim} k'$.)

- (iv) Soit R une \mathcal{O}_C -algèbre lisse et supposons qu'il existe un morphisme étale $\mathcal{O}_C[T_1, \dots, T_d] \rightarrow R$ (si R est équi-dimensionnelle, cette hypothèse est toujours satisfaite localement sur $\text{Spec } R$). Utiliser (ii) et (iii) pour construire une $\widehat{\mathcal{O}_C}$ -algèbre perfectoïde R_∞ .

1.5 Treillis des sous-anneaux perfectoïdes entiers : exercices

Soit B un anneau perfectoïde entier.

- (i) Soit $\pi \in B$ une pseudo-uniformisante perfectoïde qui admet une suite compatible de racines $\pi^{1/p}, \pi^{1/p^2}, \dots \in B$; soit $B^{\circ\circ}$ l'idéal des éléments topologiquement nilpotents de B . Montrer que $B^{\circ\circ} = \bigcup_{n \geq 0} \pi^{1/p^n} B$.
- (ii) Étant donné un sous-anneau ouvert $A \subseteq B$, montrer que les assertions suivantes sont équivalentes :
- (a) A est un anneau perfectoïde entier ;
 - (b) A est p -clos dans B (càd " $f \in B$ et $f^p \in A \Rightarrow f \in A$ ") ;
 - (c) $A \supseteq B^{\circ\circ}$ et $A/B^{\circ\circ}$ est p -clos dans $B/B^{\circ\circ}$.

En déduire que $A \mapsto A/B^{\circ\circ}$ définit une bijection de {sous-anneaux ouverts de B qui sont des anneaux perfectoïdes entiers} à {sous-anneaux p -clos de $B/B^{\circ\circ}$ }. (De plus, que A est intégralement clos dans B ssi $A/B^{\circ\circ}$ est intégralement clos dans $B/B^{\circ\circ}$.)

- (iii) Soit $A \subseteq B$ un sous-anneau ouvert qui est un anneau perfectoïde entier. Montrer que A^b est un sous-anneau ouvert de B^b .
- (iv) Montrer que l'application $\# : B^b \rightarrow B$ induit un isom. d'anneaux $B^b/B^{\circ\circ} \xrightarrow{\sim} B/B^{\circ\circ}$.
- (v) ("Correspondance de basculement pour les sous-anneaux de B ") Montrer que $A \mapsto A^b$ induit une bijection de {sous-anneaux ouverts de B qui sont des anneaux perfectoïdes entiers} à {idem. pour B^b } (De plus, que A est intégralement clos dans B ssi A^b est intégralement clos dans B^b .)

2 HUBER RINGS

The topological rings from which we will build adic spaces are Huber rings:

Definition 2.1. A *Huber ring* (or *f-adic ring* in the older terminology) is a topological ring R which satisfies the following: there exist an open subring $R_0 \subseteq R$ and a finitely generated ideal I of R_0 such that the topology on R_0 is the I -adic topology. (Warning: I is not usually an ideal of R !) Any such pair $I \subseteq R_0$ is called an *ideal of definition* and *subring of definition* of R .

Example 2.2. The simplest, but least interesting example, is as follows. Let R be a ring and $I \subseteq R$ a finitely generated ideal (even the case $I = 0$ is allowed). Then R is a Huber ring with ideal and subring of definition $I \subseteq R_0 := R$. See Example 2.5 for more interesting examples.

Note that the topology on R is uniquely determined by R_0 and I . However, the converse is more subtle. If R is a ring (without topology), $R_0 \subseteq R$ is a given subring, and I is an ideal of R_0 , then there is obviously a uniquely linear topology on R for which R_0 is open in R and the induced topology on R_0 is the I -adic topology. Indeed, the unique linear topology with these properties has basis $f + I^m$, for $f \in R$ and $m \geq 1$. However, if we equip R with this topology, then it is not necessarily a topological ring (more precisely, multiplication is not necessarily continuous)! We leave it to the reader to construct such an example. Therefore the definition of a Huber ring involves a subtle compatibility between between the topology and the algebra.

However, as we will see in Proposition 2.4, this subtlety does not occur when constructing Tate rings.

Definition 2.3. A *Tate ring* R is a Huber ring with the following additional property: there exists a unit $\pi \in R^\times$ such that $\pi^n \rightarrow 0$ as $n \rightarrow \infty$. Any such element π is called a *pseudo-uniformiser* of R .

Proposition 2.4. (i) Let A be a ring and $\pi \in A$ a non-zero-divisor; equip $A[\frac{1}{\pi}]$ with the topology having basis $f + \pi^m A$ for $f \in A[\frac{1}{\pi}]$, $m \geq 1$. Then $A[\frac{1}{\pi}]$ is a Tate ring with ideal and subring of definition $\pi A \subseteq A$.

(ii) Conversely, let R be a Tate ring; chose a subring of definition R_0 and a pseudo-uniformiser $\pi \in R$. Then there exists $m \geq 1$ such that $\pi^m \in R_0$; moreover, then $\pi^m R_0$ is an ideal of definition, π^m is a non-zero-divisor of R_0 , and $R = R_0[\frac{1}{\pi^m}]$ with topology as in (i).

Proof. (i): Exercise. (ii): Since R_0 is open and π is topologically nilpotent, there exists $m \geq 1$ such that $\pi^m \in R_0$. Since π^m is a unit of the topological ring R and R_0 is open, the ideal $\pi^m R_0$ is also open. Next let $I \subseteq R_0$ be an ideal of definition; since $\pi^m R_0$ is open, we have $I^n \subseteq \pi^m R_0$ for $n \gg 0$. Conversely, since π^m is topologically nilpotent, we also have $\pi^{mn} \in I$ for $n \gg 0$. This shows that $\pi^m R_0$ is also an ideal of definition.

Clearly π^m is a non-zero-divisor of R_0 (since it is a unit in the larger ring R). Moreover, given $f \in R$, continuity of multiplication implies $\pi^{mn} f \rightarrow 0$ as $n \rightarrow \infty$, so $\pi^{mn} f \in R_0$ for $n \gg 0$; this shows $R = R_0[\frac{1}{\pi^m}]$. \square

Example 2.5. Here are some example of Tate rings.

- (i) If A is an integral perfectoid ring and $\pi \in A$ is a perfectoid pseudo-uniformiser, then $A[\frac{1}{\pi}]$ is a Tate ring by Proposition 2.4(i). We will study such *perfectoid Tate rings* in Section 3.
- (ii) \mathbb{Q}_p is a Tate ring, with ideal and subring of definition $p\mathbb{Z}_p \subseteq \mathbb{Z}_p$.

- (iii) More generally, if A is any flat \mathbb{Z}_p -algebra then we may equip it with the p -adic topology and form a Tate ring $A[\frac{1}{p}]$
- (iv) $\mathbb{Q}_p\langle T \rangle := \{\sum_{i=0}^{\infty} a_i T^i : a_i \in \mathbb{Q}_p, a_i \rightarrow 0\} = \widehat{\mathbb{Z}_p[T]}[\frac{1}{p}]$ (where $\widehat{}$ denotes the p -adic completion).

2.1 Bounded sets

Definition 2.6. Let R be a Huber ring. A subset $S \subseteq R$ is called *bounded* if and only if for each open neighbourhood $0 \in U \subseteq R$ there exists an open neighbourhood $0 \in V \subseteq R$ such that $sv \in U$ for all $s \in S$ and $v \in V$.

We will always use the following notation: given subsets $T, S \subseteq A$, we write $ST \subseteq A$ for the subgroup generated by all products st , for $s \in S, t \in T$.

Lemma 2.7. *Let R be a Huber ring and $I \subseteq R_0 \subseteq R$ an ideal and a subring of definition. Then $S \subseteq R$ is bounded if and only if there exists $n \geq 1$ such that $SI^n \subseteq R_0$.*

Proof. \Rightarrow : Suppose $S \subseteq R$ is bounded. Since R_0 is open, there exists an open neighbourhood $0 \in V \subseteq R$ such that $sV \subseteq R_0$ for all $s \in S$. But $0 \in I^n \subseteq V$ for $n \gg 0$, whence $sI^n \subseteq R_0$ for all $s \in S$ and so $SI^n \subseteq R_0$ since R_0 is closed under addition.

\Leftarrow : Suppose there exists $n \geq 1$ such that $SI^n \subseteq R_0$. Then, for any open $0 \in U \subseteq R$, pick $m \geq 1$ such that $0 \in I^m \subseteq U$; then $V := I^{n+m}$ works, since clearly $SI^{n+m} \subseteq R_0 I^m \subseteq I^m \subseteq U$. \square

Corollary 2.8. *Let R be a Huber ring. If $S, T \subseteq R$ are bounded, then so are $S+T, S \cup T$, and ST . Any finite subset of R is bounded. If R_0 is a subring of definition and $M \subseteq R$ is a finitely generated R_0 -module, then M is bounded.*

Proof. This is all easy using Lemma 2.7. For example, pick $m \geq 1$ such that $I^m S \subseteq R_0$ and $I^m T \subseteq R_0$. Then $I^{2m} ST = I^m I^m ST \subseteq I^m R_0 T = I^m T \subseteq R_0$. For the finite set claim it is enough to show that singletons are bounded, i.e., given $f \in R$ there exists $m \geq 1$ such that $fI^m \subseteq R_0$; this follows from the fact that R_0 is an open neighbourhood of 0, that powers of I are an open neighbourhood basis at 0, and that multiplication $R \times R \rightarrow R$ is continuous.

For the finitely generated module note that $M = SR_0$ for some finite set S , so we can apply the previous parts. \square

The following is the first result where we use the fact that the ideal of definition is required to be finitely generated:

Corollary 2.9. *Let R be a Huber ring and $f_1, \dots, f_n \in R$ some elements which generate an open ideal, i.e., $f_1 R + \dots + f_n R$ is open. Let R_0 be a subring of definition. Then $f_1 R_0 + \dots + f_n R_0$ is open.*

Proof. Let $I \subseteq R_0$ be an ideal of definition. By hypothesis $I^m \subseteq f_1 R + \dots + f_n R$ for some $m \geq 1$. But I^m is a finitely generated R_0 -module, so clearly there exists a finitely generated R_0 -module $M \subseteq A$ such that $I^m \subseteq f_1 R + \dots + f_n R$. But M is bounded by the previous corollary, so $I^{m'} M \subseteq R_0$ for some $m' \geq 1$. Then $I^{m+m'} \subseteq f_1 R_0 + \dots + f_n R_0$. \square

The following is fundamental:

Proposition 2.10. *Let R be a Huber ring and $A \subseteq R$ a subring. Then A is a ring of definition if and only if it is bounded and open in R .*

Proof. \Rightarrow : Suppose A is a ring of definition. Then it is open by definition, and bounded by Lemma 2.7 (take $R_0 = A$ and $n = 1$).

\Leftarrow : Suppose A is bounded and open. Let $I \subseteq R_0 \subseteq A$ be an ideal and a subring of definition; let $T \subseteq R_0$ be a finite set such that $I = TR_0$. Since A is open and bounded, there exists (using Lemma 2.7) $m \geq 1$ such that $I^m \subseteq A$ and $I^m A \subseteq R_0$.

Let $J := T^m A$ be the ideal of A generated by the finite set $T^m \subseteq A$. Then $J^2 = T^m T^m A \subseteq T^m R_0 = I^m$ and $J = T^m A \supseteq T^m I^m = I^{2m}$.

Therefore powers of J and powers of I define the same topology on R , so $J \subseteq A$ are indeed an ideal and a subring of definition. \square

Corollary 2.11. *Let R be a Huber ring.*

(i) *If $R_0, R'_0 \subseteq A$ are subrings of definition, then so are $R_0 R'_0$ and $R_0 \cap R'_0$ (note that these really are subrings; also note that $R_0 R'_0 \supseteq R_0, R'_0$); in particular, the subrings of definition form a filtered family.*

(ii) *Suppose that $A \subseteq B$ are subrings of R such that A is bounded and B is open. Then there exists a subring of definition R_0 such that $A \subseteq R_0 \subseteq B$.*

Proof. (i) Easy exercise: check that $R_0 R'_0$ and $R_0 \cap R'_0$ are both open and bounded, then apply the previous proposition.

(ii) Let R_0 be any subring of definition; then $R_0 B$ is open and bounded, so $R_0 B \cap C$ is also open and bounded, hence a subring of definition by the previous proposition. \square

Definition 2.12. Let R be a Huber ring. An element $f \in R$ is called *power bounded* if and only if the set $f^{\mathbb{N}} := \{f^n : n \geq 0\}$ is bounded. Let $R^\circ \subseteq R$ be the subset of power bounded elements.

An element $f \in R$ is called *topologically nilpotent* if and only if $f^n \rightarrow 0$ as $n \rightarrow \infty$. Let $R^{\circ\circ} \subseteq R$ be the subset of topologically nilpotent element.

Lemma 2.13. *Let R be a Huber ring.*

(i) *R° is an open subring, integrally closed in R .*

(ii) *R° is the union of all subrings of definition.*

(iii) *$R^{\circ\circ}$ is an open ideal of R° .*

Proof. (i): Let $f, g \in R^\circ$. Then $f^{\mathbb{N}} g^{\mathbb{N}}$ is bounded by Corollary 2.8. But $f^{\mathbb{N}} g^{\mathbb{N}} \supseteq \{(f+g)^n, (fg)^n : n \geq 0\}$, and therefore $f+g$ and fg are also power bounded.

Next suppose that $x \in R$ is integral over R° , so that there are $a_0, \dots, a_{d-1} \in R^\circ$ such that $x^d + a_{d-1}x^{d-1} + \dots + a_0 = 0$. Then it is easy to see that $x^{\mathbb{N}} \subseteq a_0^{\mathbb{N}} \dots a_{d-1}^{\mathbb{N}} \{1, x, \dots, x^{d-1}\}$. The set on the right is bounded by Corollary 2.8, whence $f^{\mathbb{N}}$ is also bounded, i.e., $f \in R^\circ$.

Given any subring of definition A_0 and $f \in R_0$, we have $f^{\mathbb{N}} \subseteq R_0$ and so $f^{\mathbb{N}}$ is bounded (since R_0 is bounded); therefore $R_0 \subseteq R^\circ$, showing that R° is open.

(ii): We have just noted that any subring of definition R_0 is contained in R° . Conversely, supposing $f \in R^\circ$, we must find a subring of definition containing f . Let R_0 be any subring of definition, and note that $R_0 f^{\mathbb{N}}$ is R_0 -subalgebra of R generated by f ; this is open (since it contains the open subring R_0) and bounded (by Corollary 2.8), hence is a subring of definition (by Proposition 2.10).

(iii): Exercise. \square

It is important to note that R° is not necessarily bounded (e.g., Consider the Tate ring $R = \mathbb{Q}_p[\varepsilon]/\varepsilon^2$, with subring of definition $R_0 = \mathbb{Z}_p[\varepsilon]/\varepsilon^2$ equipped with the p -adic topology. Then any multiple of ε is nilpotent, hence power bounded, and so $R^\circ = \mathbb{Z}_p + \mathbb{Q}_p\varepsilon$; in particular $p^n R^\circ \not\subseteq R_0$ for all $n \geq 1$, whence R° is not bounded.). We say that the Huber ring R is *uniform* if and only if R° is bounded; this will turn out to be the case for perfectoid Tate rings.

Definition 2.14. Let R be a Huber ring. A subring R^+ is called a *subring of integral elements* if and only if it is open, integrally closed subring in R , and $R^+ \subseteq A^\circ$. For example, Lemma 2.13(i) clearly implies that R° is a subring of integral elements (moreover, it is the largest subring of integral elements).

A *Huber pair* (or *affinoid ring* in the older terminology) (R, R^+) is the data of a Huber ring R and a chosen subring of integral elements R^+ .

3 PERFECTOID TATE RINGS

As explained in Proposition 2.4, if we are given an integral perfectoid ring A we can construct a Tate ring $A[\frac{1}{\pi}]$, where π is any perfectoid pseudo-uniformiser of A . Note that $A[\frac{1}{\pi}]$ does not depend on the choice of π : given another π' , the elements π and π' define the same topology on A , hence each divides a power of the other, and so $A[\frac{1}{\pi}] = A[\frac{1}{\pi'}]$; in fact, $A[\frac{1}{\pi}]$ can be written without making any choices as

$$A[\frac{1}{\pi}] = A[\frac{1}{f} : f \in A \text{ and } fA \text{ is open in } A].$$

In any case, the Tate ring $A[\frac{1}{\pi}]$ is called the *generic fibre* of A , and so we have defined a functor

$$\text{integral perfectoid rings} \longrightarrow \text{Tate rings}, \quad A \mapsto A[\frac{1}{\pi}]$$

The image of this functor is precisely the perfectoid Tate rings:

Definition 3.1. A Tate ring R is called *perfectoid* if and only if the following equivalent conditions are satisfied:

- (i) R has a subring of definition which is an integral perfectoid ring;
- (ii) R is in the image of the above functor;
- (iii) the topological ring R° is integral perfectoid;
- (iv) R is uniform and there exists a pseudo-uniformiser $\pi \in R$ such that $p \in \pi^p R^\circ$ and $\Phi : R^\circ/\pi R^\circ \rightarrow R^\circ/\pi^p R^\circ, f \mapsto f^p$ is an isomorphism (Fontaine's Bourbaki definition).

It is convenient to prove the equivalences at the same time as the following proposition:

Proposition 3.2. *Let R be a perfectoid Tate ring and $R_0 \subseteq R$ a subring of definition. Then R_0 is integral perfectoid if and only if it is p -closed in R (i.e., “ $f \in R$ and $f^p \in R_0 \Rightarrow f \in R_0$ ”). In particular, every subring of integral elements $R^\dagger \subseteq R$ is integral perfectoid.*

Proof of equivalences and the proposition. (iv) \Rightarrow (i): If R is uniform then R° is a subring of definition; since $\pi \in R^\circ$, Proposition 2.4 shows that the topology on R° is the π -adic topology, and the other conditions in (iv) show that R° is integral perfectoid.

(i) \Rightarrow (ii): Suppose that $R_0 \subseteq R$ is an integral perfectoid subring of definition. Let $\pi \in R_0$ be a perfectoid pseudo-uniformiser and note that π is necessarily a pseudo-uniformiser of the Tate ring R (The proof is standard theory of Tate rings, similar to Proposition 2.4: Firstly, π is topologically nilpotent since it defines the topology on R_0 ; secondly, fixing a pseudo-uniformiser $\varpi \in R$, the openness of πR_0 implies $\varpi^n \in \pi R_0$ for $n \gg 0$, whence π is a unit in R .) Proposition 2.4 shows that $R = R_0[\frac{1}{\pi}]$, i.e., R is in the image of the above functor.

(iii) \Rightarrow (iv): If R° is integral perfectoid then it is an open subring whose topology is adic for a finitely generated ideal, whence it is a subring of definition; but any subring of definition is bounded, so this shows that R is uniform. Any choice of perfectoid pseudo-uniformiser π for R° is a pseudo-uniformiser for R (by the same argument as in the previous paragraph) with the desired properties in (iv).

To complete the proof of the equivalences, we must show (ii) \Rightarrow (iii), so suppose that $R = A[\frac{1}{\pi}]$, where A is an integral perfectoid ring and $\pi \in R$ is a pseudo-uniformiser. We note first that $R^{\circ\circ} \subseteq A$: indeed, if $f \in R$ is topologically nilpotent then $f^{p^n} \in A$ for $n \gg 0$, and so $f \in A$ by Lemma 3(ii). In particular this shows that $\pi R^\circ \subseteq A$; therefore R° is bounded, i.e., R is uniform.

Now let $R_0 \subseteq R$ be any p -closed subring of definition (e.g., $R_0 = R^\circ$, since we have just shown R° is bounded, hence is a subring of definition); we will prove that R_0 is integral perfectoid. Just as in the previous paragraph, p -closedness implies that $R^{\circ\circ} \subseteq A$; in particular $\pi \in R_0$, whence the topology on R_0 is the π -adic topology by Proposition 2.4 (this proves condition (a) in the definition of integral perfectoid).

We claim that every element of R_0 is a p^{th} -power modulo π (resp. $\pi^{1/2}$ if $p = 2$); let $f \in R_0$. Then $\pi f \in R^{\circ\circ} \subseteq A$ and so there exist $y, z \in A$ such that $\pi x = y^p + \pi^p z$; after multiplying π by a unit we may assume it admits a p^{th} -root in A , and we then deduce that $(y\pi^{-1/p})^p = x - \pi^{p-1}z \in R_0$ (note that $\pi^{p-1}z$ is topologically nilpotent, hence in R_0), whence $y\pi^{-1/p} \in R_0$ by topological nilpotence again. Since $\pi^{p-1}z \in \pi R_0$ (unless $p = 2$, in which case it is $\in \pi^{1/2}R_0$ since $\pi^{1/2}z \in R_0$), we have shown that x is a p^{th} -power modulo πR_0 (resp. $\pi^{1/2}R_0$), which proves the claim.

Next note that $p \in (\pi^{1/p})^p R_0$: indeed, we know that $p \in \pi^p A \subseteq \pi R^{\circ\circ}$ and that $R^{\circ\circ} \subseteq R_0$. Since the p -closedness of R_0 in R easily implies that $\Phi : R_0/\pi^{1/p}R_0 \rightarrow R_0/\pi R_0$ (resp. $R_0/\pi^{1/4}R_0 \rightarrow R_0/\pi^{1/2}R_0$) is injective, we have indeed proved that R_0 is integral perfectoid (with pseudo-uniformiser $\pi^{1/p}$, resp. $\pi^{1/4}$ if $p = 2$).

In conclusion, assuming condition (ii), we have proved that R° is integral perfectoid, and more generally that any p -closed subring of definition is integral perfectoid. This completes the proof that the conditions in Definition 3.1 are equivalent, and establishes the implication \Leftarrow in the proposition; meanwhile, the implication \Rightarrow is a consequence of (ii). For the final sentence in the proposition just note that, since R is uniform, any subring of integral elements is a subring of definition. \square

Corollary 3.3. *Let R be a perfectoid Tate ring. Then any integral perfectoid subring of definition R_0 contains $R^{\circ\circ}$, and the resulting functor $R_0 \mapsto R_0/R^{\circ\circ}$ defines a bijection*

$$\{\text{integral perfectoid subrings of definition of } R\} \xrightarrow{\cong} \{p\text{-closed subrings of } R^\circ/R^{\circ\circ}\}$$

This restricts to a bijection

$$\{\text{subrings of integral elements of } R\} \xrightarrow{\cong} \{\text{integrally closed subrings of } R^\circ/R^{\circ\circ}\}$$

Proof. By basic commutative algebra, $A \mapsto A/R^{\circ\circ}$ defines a bijection

$$\{\text{subrings of } R^\circ \text{ containing } R^{\circ\circ}\} \xrightarrow{\cong} \{\text{subrings of } R^\circ/R^{\circ\circ}\}.$$

Moreover, the reader can easily check that, given a ring A such that $R^{\circ\circ} \subseteq A \subseteq R^\circ$, then A is p -closed (resp. integrally closed) in R° if and only if $A/R^{\circ\circ}$ is p -closed (resp. integrally closed) in $R^\circ/R^{\circ\circ}$.

To complete the proof, it remains only to use Proposition 3.2 and the observation (which already appeared in the previous proof) that any open p -closed subring of R must contain $R^{\circ\circ}$. \square

Remark 3.4. Similarly to Lemma 1.5, one can prove the following: Given a complete Tate ring R of characteristic p , then R is perfectoid if and only if it is perfect. By using Lemma 1.5, the only non-trivial part of the proof is the following: assuming that R is perfect, we will show that R is uniform. This was noticed first by Yves André, whose proof we give here.

Let $R_0 \subseteq R$ be a subring of definition, and $\pi \in R_0$ a pseudo-uniformiser of R . For each $n \geq 1$ set $R_n := \varphi^{-n}(R_0) = \{f \in R : f^{p^n} \in R_0\}$, which is a subring since R has characteristic p . The Frobenius morphism $\varphi : R \rightarrow R$ is a continuous bijection (since R is assumed to be perfect),

hence is a homeomorphism by Banach's open mapping theorem (it is folklore that Banach's open mapping theorem holds in the generality of complete Tate rings; see, e.g., Henkel "An open mapping theorem..."). Therefore $\varphi(R_0)$ is an open subring of R , so there exists $m \geq 1$ such that $\pi^m R_0 \subseteq R_0$; applying φ^{-n} shows that $\pi^{m/p^n} A_n \subseteq A_{n-1}$ for all $n \geq 1$. By a trivial induction it this means that $\pi^{\sum_{i=1}^n m/p^i} A_n \subseteq A_0$, whence $\pi^m A_n \subseteq A_0$ for all $n \geq 0$ (since $m \geq \sum_{i=1}^n m/p^i$). Next, given $f \in R^\circ$, the set $f^{\mathbb{N}}$ is bounded and so there exists $n \geq 1$ such that $\pi^{p^n} f^{\mathbb{N}} \subseteq R_0$; in particular $\pi^{p^n} f^{p^n} \in R_0$, i.e., $\pi f \in R_n$ and so $\pi^{m+1} f \in \pi R_n \subseteq R_0$. This shows that $\pi^{m+1} R^\circ \subseteq R_0$, i.e., R° is bounded.

3.1 Aside: The language of almost mathematics

Before we can discuss tilting perfectoid Tate rings in the next subsection, it is useful to introduce some language of "almost mathematics".

Throughout this subsection we let A be an integral perfectoid ring, and $A^\circ \subseteq A$ the open ideal consisting of topologically nilpotent elements. (Exercise: letting $\pi \in A$ be any perfectoid pseudo-uniformiser admitting compatible p -power roots $\pi^{1/p}, \pi^{1/p^2}, \dots$, check that $A^\circ = \bigcup_{n \geq 0} \pi^{1/p^n} A$. In particular, this shows that the ideal A° is its own square, which is key to the following theory.)

We say that an A -module M is *almost zero* if and only if $A^\circ M = 0$; by the exercise, this is equivalent to saying that $\pi^{1/p^n} M = 0$ for all $n \geq 0$. Similarly, we say that a map of A -modules $M \rightarrow N$ is an *almost injection/surjection/isomorphism* if and only if the kernel/cokernel/both is almost zero.

The following should look surprising at first glance (it is not true if we replace A° by an arbitrary ideal):

Lemma 3.5. *The category of almost zero A -modules is closed under the following operations: sub-modules, quotients, extensions, all limits, all colimits. Given a π -adically complete A -module M , then M is almost zero if and only if $M/\pi M$ is almost zero. Moreover, almost isomorphisms are closed under base change along an arbitrary module.*

Proof. The surprising fact is extensions: suppose that $0 \rightarrow M \rightarrow N \rightarrow P \rightarrow 0$ is a short exact sequence of A -modules such that M and P are killed by A° ; the N is killed by the ideal $(A^\circ)^2$, but we have noted above that $(A^\circ)^2 = A^\circ$. We leave it to the reader as an exercise in almost mathematics to check the other assertions. \square

Lemma 3.6. *Let B be a perfectoid A -algebra. Then $A \rightarrow B$ is an almost isomorphism if and only if $A^b \rightarrow B^b$ is an almost isomorphism.*

Proof. Let $\pi \in A$ be a perfectoid pseudo-uniformiser admitting compatible p -power roots, and $\pi^b = (\pi, \pi^{1/p}, \dots) \in A^b$ the corresponding perfectoid pseudo-uniformiser of A^b . Recall from the discussion before Theorem 1.11 that π is automatically a perfectoid pseudo-uniformiser for B (and similarly π^b is a ppu for B^b).

\Rightarrow : $A \rightarrow B$ being an almost isomorphism means that it is injective (since A has no π -torsion), so we may view B as an extension of A , and that $\pi^{1/p^n} B \subseteq A$ for all $n \geq 0$. From the injectivity it is clear that

$$A^b = \varprojlim_{x \rightarrow x^p} A \longrightarrow \varprojlim_{x \rightarrow x^p} B = B$$

is injective. Moreover, given an element $b = (b_0, b_1, \dots) \in B^b$ and $n \geq 1$, we have

$$\pi^{b1/p^n} b = (\pi^{1/p^n}, \pi^{1/p^{n+1}}, \dots)(b_0, b_1, \dots) = (b_0 \pi^{1/p^n}, b_1 \pi^{1/p^{n+1}}, \dots) \in A^b,$$

showing that $A^b \rightarrow B^b$ is almost surjective.

\Leftarrow : $A^b \rightarrow B^b$ being an almost isomorphism means that $A^b/\pi^b A^b \rightarrow B^b/\pi^b B^b$ is almost an almost isomorphism (by the base change assertion in Lemma 3.5). But we know from the proof of Lemma 1.7 that $A^b/\pi^b A^b = A/\pi A$ and $B^b/\pi^b B^b \rightarrow B/\pi B$; so we have shown that $A \rightarrow B$ is an almost isomorphism modulo π . Since A and B are complete, (a modification of) Lemma 3.5 implies $A \rightarrow B$ is an almost isomorphism. \square

Lemma 3.7. *Let B be a perfectoid A -algebra. Then $A \rightarrow B$ is an almost isomorphism if and only if the induced map of perfectoid Tate rings $A[\frac{1}{\pi}] \rightarrow B[\frac{1}{\pi}]$ (i.e., the generic fibres) is an isomorphism.*

Proof. \Rightarrow : Suppose that $A \rightarrow B$ is an almost isomorphism. Then the kernel and cokernel are in particular killed by π , whence $A[\frac{1}{\pi}] \xrightarrow{\sim} B[\frac{1}{\pi}]$ as desired.

\Leftarrow : Suppose that $A[\frac{1}{\pi}] \xrightarrow{\sim} B[\frac{1}{\pi}]$. Denoting this common perfectoid Tate ring by R , we may therefore view $A \subseteq B$ as integral perfectoid subrings of definition of R . As we stated in Corollary 3.3, this implies $R^{\circ\circ} \subseteq A$; in particular, $\pi^{1/p^n} B \subseteq A$ for all $n \geq 1$, i.e., $A \rightarrow B$ is an almost isomorphism. \square

3.2 Tilting perfectoid Tate rings

The tilt of a perfectoid Tate ring R is defined to be the perfectoid Tate ring of characteristic p given by

$$\begin{aligned} R^b &:= \text{generic fibre of } R_0^b \\ &= R_0^b[\frac{1}{\pi^b}] \end{aligned}$$

where $R_0 \subseteq R$ is any integral perfectoid subring of definition and $\pi \in R_0$ is a perfectoid pseudo-uniformiser with compatible p -power roots. This does not depend on the chosen subring of definition (or perfectoid pseudo-uniformiser): indeed, obviously the two integral perfectoid subrings $R_0 \subseteq R^\circ$ have the same generic fibre, whence Lemmas 3.6 and 3.7 show that the two integral perfectoid rings $R_0^b \subseteq R^{\circ b}$ also have the same generic fibre. In other words, we could canonically define R^b to be the generic fibre of $R^{\circ b}$; from this point of view, we have just shown that any other integral perfectoid subring of definition $R_0 \subseteq R$ tilts to an integral perfectoid subring of definition $R_0^b \subseteq R^b$.

Theorem 3.8 (Tilting correspondence – lattice of subrings). *Let R be a perfectoid Tate ring. Then $R^{\circ b}$ is the subring of power bounded elements of R (i.e., $R^{\circ b} = R^{b^\circ}$). Moreover, tilting $R_0 \mapsto R_0^b$ defines a bijection*

$$\{\text{integral perfectoid subrings of definition of } R\} \xrightarrow{\sim} \{\text{integral perfectoid subrings of definition of } R^b\},$$

which restricts to a bijection

$$\{\text{subrings integral elements of } R\} \xrightarrow{\sim} \{\text{subrings of integral elements of } R^b\}.$$

Proof. We begin by proving that $R^{\circ b} = R^{b^\circ}$. Since $R^{\circ b}$ is a subring of definition for R^b , certainly we have $R^{\circ b} \subseteq R^{b^\circ}$ and so we may view R^{b° as a perfectoid $R^{\circ b}$ -algebra. By Theorem 1.11, untilting gives us a perfectoid R° -algebra B such that $B^b = R^{b^\circ}$. Since $R^\circ \rightarrow B$ induces an isomorphism on generic fibres after tilting (namely $R \xrightarrow{\sim} R$), Lemma 3.7 tells us that it induces an isomorphism on generic fibres before tilting, i.e., we have $R^\circ \subseteq B \subseteq R$ where B is a subring

of definition of R . But B being a subring of R implies $B \subseteq R^\circ$, i.e., $R^\circ = B$; tilting reveals $R^{\circ b} = R^{b^\circ}$, as desired.

Let $\pi \in R^\circ$ be a perfectoid pseudo-uniformiser admitting p -power roots, and $\pi^b \in R^{\circ b} = R^{b^\circ}$ the associated perfectoid pseudo-uniformiser of R^{b° . We know that the untilting map $\#$ induces an isomorphism $R^{b^\circ}/\pi^b R^{b^\circ} \xrightarrow{\sim} R^\circ/\pi R^\circ$. Since $R^{\circ\circ}/\pi R^\circ$ is the ideal of nilpotent elements of $R^\circ/\pi R^\circ$, and similarly on the tilted side, we also get an isomorphism $R^{b^\circ}/R^{b^\circ\circ} \xrightarrow{\sim} R^\circ/R^{\circ\circ}$.

The desired bijections up subrings now follows by applying Corollary 3.3 to both R and R^b . \square

Finally we note that the tilting correspondence for integral perfectoid rings extends to the generic fibres:

Theorem 3.9 (Tilting correspondence – perfectoid Tate rings). *Let R be a perfectoid Tate ring. Then tilting $S \mapsto S^b$ defines an equivalence of categories*

$$\{\text{perfectoid Tate rings over } R\} \simeq \{\text{perfectoid Tate rings over } R^b\}$$

Proof. Exercice. \square

Exercices

Soit B un anneau perfectoïde entier.

- (i) Étant donné un sous-anneau ouvert $A \subseteq B$, montrer que les assertions suivantes sont équivalentes :
 - (a) A est un anneau perfectoïde entier ;
 - (b) A est p -clos dans B (càd “ $f \in B$ et $f^p \in A \Rightarrow f \in A$ ”) ;
 - (c) $A \supseteq B^{\circ\circ}$ et $A/B^{\circ\circ}$ est p -clos dans $B/B^{\circ\circ}$.

En déduire que $A \mapsto A/B^{\circ\circ}$ définit une bijection entre $\{\text{sous-anneaux ouverts de } B \text{ qui sont des anneaux perfectoïdes entiers}\}$ et $\{\text{sous-anneaux } p\text{-clos de } B/B^{\circ\circ}\}$. (De plus, que A est intégralement clos dans B ssi $A/B^{\circ\circ}$ est intégralement clos dans $B/B^{\circ\circ}$.)

- (ii) Soit $A \subseteq B$ un sous-anneau ouvert qui est un anneau perfectoïde entier. Montrer que A^b est un sous-anneau ouvert de B^b .
- (iii) Montrer que l’application $\# : B^b \rightarrow B$ induit un isom. d’anneaux $B^b/B^{b^\circ\circ} \xrightarrow{\sim} B/B^{\circ\circ}$.
- (iv) (“Correspondance de basculement pour les sous-anneaux de B ”) Montrer que $A \mapsto A^b$ induit une bijection entre $\{\text{sous-anneaux ouverts de } B \text{ qui sont des anneaux perfectoïdes entiers}\}$ et $\{\text{idem. pour } B^b\}$ (De plus, que A est intégralement clos dans B ssi A^b est intégralement clos dans B^b .)

Soit R un anneau perfectoïde de Tate ; on va appliquer les résultats ci-dessus à l’anneau perfectoïde entier R° . Rappelons que le basculé de R est par définition la fibre générique R^b de $R^{\circ b}$. On a montré le 8 mars que $R^{\circ b} = R^{b^\circ}$.

- (v) Étant donné un sous-anneau de définition qui est un anneau perfectoïde entier $R_0 \subseteq R$, montrer de (ii) que $R_0^b \subseteq R^b$ l’est aussi et en déduire que R est la fibre générique de R_0^b . (En deux mots, le basculé R^b peut être défini d’être la fibre générique du basculé de n’importe quel sous-anneau de définition R_0 qui est un anneau perfectoïde.)

(vi) D eduire de (iv) que $R_0 \mapsto R_0^b$ induit des bijections

- {sous-anneaux de d efinition de R qui sont des anneaux perfectoides entiers} $\xrightarrow{\sim}$ {idem. pour R^b } ;
- {sous-anneaux ouverts et int egralement clos de R° } $\xrightarrow{\sim}$ {idem. pour R^{b° }.

Utiliser la Correspondance de basculement pour les anneaux perfectoides entiers pour d emontrer la Correspondance de basculement pour les anneaux perfectoides de Tate (D ef : un *anneau perfectoide de Tate sur R* est un anneau perfectoide de Tate S muni d'un morphisme continue $R \rightarrow S$) :

Theorem 3.10. *Le basculement $S \mapsto S^b$ induit une  equivalence de cat egories {anneaux perfectoides de Tate sur R } $\xrightarrow{\sim}$ {anneaux perfectoides de Tate sur R^b }.*

4 PERFECTOID FIELDS

Let K be a topological field. We say that K is a *perfectoid field* if and only if the following conditions are satisfied:

- (a) the topology on K is induced by a valuation $|\cdot| : K \rightarrow \mathbb{R}_{\geq 0}$, and K is complete for this topology;
- (b) there exists a non-unit $\pi \in \mathcal{O}_K$ such that $p \in \pi^p \mathcal{O}_K$;
- (c) every element of $\mathcal{O}_K/p\mathcal{O}_K$ is a p^{th} -power.

Here $\mathcal{O}_K := \{f \in K : |f| \leq 1\}$ denotes the ring of integers of the valuation $|\cdot|$, which depends only on the topological field K and not on the chosen valuation: indeed, $f \notin \mathcal{O}_K$ iff $|f^{-1}| < 1$ iff $f^{-n} \rightarrow 0$ as $n \rightarrow \infty$ (See the exercises of the next section).

4.1 Exercices sur la fondation des corps perfectoides

Ces exercices établissent les fondations de la théorie des corps perfectoides, y compris leur correspondance de basculement de base, sans référence aux anneaux de Tate.

4.1.1 Corps valués

Soit K un corps topologique et supposons que

- (a) la topologie est non discrète et induite par une valuation $|\cdot| : K \rightarrow \mathbb{R}_{\geq 0}$.

On pose

$$\mathcal{O}_K := \{f \in K : |f| \leq 1\} \quad \mathfrak{m}_K := \{f \in K : |f| < 1\}.$$

- (i) Montrer que l'anneau des entiers \mathcal{O}_K est un sous-anneau de valuation de K tel que $\text{Frac}(\mathcal{O}_K) = K$, et que \mathfrak{m}_K est l'unique idéal maximal de \mathcal{O}_K .
- (ii) Montrer que

$$K \setminus \mathcal{O}_K = \{f \in K : f^{-n} \rightarrow 0 \text{ quand } n \rightarrow \infty\}$$

et

$$\mathfrak{m}_K = \{f \in K : f^n \rightarrow 0 \text{ quand } n \rightarrow \infty\}.$$

En déduire que \mathcal{O}_K et \mathfrak{m}_K ne dépendent que du corps topologique K (pas du choix de la valuation $|\cdot|$).

- (iii) Soit $0 \neq \pi \in \mathcal{O}_K$ non unité (pourquoi un tel élément existe-il ?). Montrer que la topologie sur \mathcal{O}_K (induite par la topologie sur K) est la topologie π -adique.

Réciproquement, soit \mathcal{O} un anneau de valuation muni d'une topologie et supposons que

- (A) il existe $0 \neq \pi \in \mathcal{O}$ tel que la topologie sur \mathcal{O} soit la topologie π -adique et $\bigcap_{n \geq 0} \pi^n \mathcal{O} = \{0\}$;
- (B) $\mathcal{O}^{\circ\circ} := \{f \in \mathcal{O} : f^n \rightarrow 0 \text{ quand } n \rightarrow \infty\}$ est l'idéal maximal de \mathcal{O} .

On pose $F := \text{Frac}(\mathcal{O})$.

(iv) Montrer que $F = \mathcal{O}[\frac{1}{\pi}]$. Étant donné un élément $f \in F^\times$, montrer que sa “norme spectrale”

$$|f| := \inf\{p^{-n/m} : n, m \in \mathbb{Z} \text{ t.q. } f^m \in \pi^n \mathcal{O}\} \in \mathbb{R}_{>0}$$

est bien définie et que $|\pi| = p^{-1}$.

(v) Montrer que $|\cdot| : F \rightarrow \mathbb{R}_{\geq 0}$ est une valuation telle que $\mathcal{O} = \{f \in F : |f| \leq 1\}$ et $\mathcal{O}^{\circ\circ} = \{f \in F : |f| < 1\}$.

(vi) Montrer que la topologie induite sur \mathcal{O} par cette valuation $|\cdot|$ est la topologie π -adique.

En déduire que le foncteur

$$\{\text{corps topologiques } K \text{ satisfont (a)}\} \rightarrow \{\text{anneaux de valuations topologiques } \mathcal{O} \text{ satisfont (A) et (B)}\}$$

qui à K associe \mathcal{O}_K est une équivalence de catégories, avec l'inverse $\mathcal{O} \mapsto \text{Frac}(\mathcal{O})$ donné ci-dessus.

4.1.2 Le cas des corps perfectoïdes

Rappelons qu'un *corps perfectoïde* K est un corps topologique tel que la condition (a), K soit complet et

(b) tout élément de $\mathcal{O}_K/p\mathcal{O}_K$ admet une racine p -ième; et

(c) il existe $\pi \in \mathcal{O}_K$ non unité tel que $p \in \pi^p \mathcal{O}_K$.

(vii) Étant donné un corps topologique tel que la topologie soit induite par une valuation $|\cdot| : K \rightarrow \mathbb{R}_{\geq 0}$, montrer que K est un corps perfectoïde si et seulement si \mathcal{O}_K (muni de la topologie induite par K) est un anneau perfectoïde entier.

(viii) Déduire de l'équivalence de catégories ci-dessus que le foncteur $K \mapsto \mathcal{O}_K$ induit une équivalence de catégories

$$\{\text{corps perfectoïdes } K\} \xrightarrow{\sim} \{\text{anneaux perfds entiers } \mathcal{O} \text{ qui sont des anneaux de valuations satisfont (B)}\}$$

(ix) Soit \mathcal{O} un anneau perfectoïde entier.

Montrer que $\# : \mathcal{O}^b/\mathcal{O}^{b\circ\circ} \rightarrow \mathcal{O}/\mathcal{O}^{\circ\circ}$ est un isomorphisme d'anneaux et en déduire que $\mathcal{O}^{\circ\circ}$ est un idéal maximal de \mathcal{O} ssi $\mathcal{O}^{b\circ\circ}$ est un idéal maximal de \mathcal{O} .

On a vu pendant le cours que \mathcal{O} est un anneau de valuation ssi \mathcal{O}^b l'est. En déduire que \mathcal{O} est un anneau de valuation satisfont (B) ssi \mathcal{O}^b l'est.

Le basculé d'un corps perfectoïde K est $K^b := \text{Frac}(\mathcal{O}_K^b) = \mathcal{O}_K^b[\frac{1}{\pi^b}]$. Déduire des résultats ci-dessus que K^b est un corps perfectoïde avec anneau des entiers \mathcal{O}_K^b et que le foncteur

$$\{\text{corps perfectoïdes sur } K\} \rightarrow \{\text{corps perfectoïdes sur } K^b\}, \quad L \mapsto L^b$$

est une équivalence de catégories.

4.2 Aside: almost mathematics II

Let A be an integral perfectoid ring; recall from Section 3.1 that $A^{\circ\circ} = \bigcup_{n \geq 0} \pi^{1/p^n} A$, where as usual $\pi \in A$ denotes a perfectoid pseudo-uniformiser with compatible p -power roots.

We say that an A -module M is *almost free of rank d* if and only if for each element $\varepsilon \in A^{\circ\circ}$ there exists a morphism of A -modules $f_\varepsilon : A^d \rightarrow M$ whose kernel and cokernel are killed by ε . (It suffices to consider the elements $\varepsilon = \pi^{1/p^n}$ for each $n \geq 0$.) Note that the morphism depends on ε ; in general the module M does not even need to be finitely generated.

Similarly, let's say that an $A/\pi A$ -module N is *almost free of rank d* if and only if for each element $\varepsilon \in A^{\circ\circ}$ there exists a morphism of A -modules $(A/\pi A)^d \rightarrow N$ whose kernel and cokernel are killed by ε .

Lemma 4.1. *Let M be an A -module which is π -adically complete and without π -torsion; let $d \geq 0$. Then the A -module M is almost free of rank d if and only if the $A/\pi A$ -module $M/\pi M$ is almost free of rank d .*

Proof. We suppose that $d = 1$ just to simplify the notation. For any $n \geq 1$ the hypothesis implies that there exists an element $e \in M$ such that the kernel and cokernel of $A/\pi A \rightarrow M/\pi M$, $a \mapsto ae$ are killed by π^{1/p^n} . It is then quite straightforward to check directly that the kernel and cokernel of $A \rightarrow M$, $a \mapsto am$ are also killed by π^{1/p^n} ; the details are left as an exercise. \square

The main reason we have introduced this notion is for the sake of the following rather subtle result:

Proposition 4.2. *Let K be a perfectoid field of characteristic p , and L/K a finite field extension. Then the \mathcal{O}_K -module \mathcal{O}_L (= the integral closure of \mathcal{O}_K inside L) is almost free of rank $|L : K|$.*

Proof. We begin with some elementary theory of field extensions. Since L/K is a separable extension (recall that K is perfect), the trace pairing $L \otimes_K L \xrightarrow{\text{mult}} L \xrightarrow{\text{Tr}_{L/K}} K$ is non-degenerate. So let $e_1, \dots, e_d \in L$ be a basis for L as a K -vector space, and let $e_1^*, \dots, e_d^* \in L$ be the dual basis, i.e., $\text{Tr}_{L/K}(e_i e_j^*) = \delta_{ij}$. By elementary linear algebra we have $b = \sum_{i=1}^d e_i \text{Tr}_{L/K}(b_i e_i^*)$ for all $b \in B$, and by the theory of separable extensions the element $e := \sum_{i=1}^d e_i \otimes e_i \in L \otimes_K L$ is an idempotent (though this is not so easy to prove directly).

Let $\pi \in \mathcal{O}_K$ be a perfectoid pseudo-uniformiser, and fix $N \gg 0$ such that $\pi^N e_i \in \mathcal{O}_L$ for $i = 1, \dots, N$ (note that $L = \mathcal{O}_L[\frac{1}{\pi}]$). We now use that the absolute Frobenius $\varphi : x \mapsto x^p$ is an automorphism of K , \mathcal{O}_K , L , \mathcal{O}_L and $L \otimes_K L$; moreover it satisfies $\varphi(e) = e$ since e is an idempotent. Therefore, for any $n \geq 1$, we have

$$\pi^{2N/p^n} e = \varphi^{-n}(\pi^{2N} e) = \varphi^{-n}\left(\sum_{i=1}^d \pi^N e_i \otimes \pi^N e_i^*\right) = \sum_{i=1}^d \varphi^{-n}(\pi^N e_i) \otimes \varphi^{-n}(\pi^N e_i^*),$$

which is in the image of $\mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_L \rightarrow L \otimes_K L$; i.e., we have shown that e is almost in the image of $\mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_L \rightarrow L \otimes_K L$, which is the key idea of the proof. (To use the correct terminology, this shows that $\mathcal{O}_K \rightarrow \mathcal{O}_L$ is “almost étale”.)

Now define morphisms

$$f : \mathcal{O}_K^d \rightarrow \mathcal{O}_L, \quad (a_1, \dots, a_d) \mapsto \sum_{i=1}^d \varphi^{-n}(\pi^N e_i) a_i$$

and

$$g : \mathcal{O}_L \rightarrow \mathcal{O}_K^d, \quad b \mapsto (\mathrm{Tr}_{L/K}(b\varphi^{-n}(\pi^N e_1^*)), \dots, \mathrm{Tr}_{L/K}(b\varphi^{-n}(\pi^N e_d^*))).$$

Using the properties of the basis, it is easy to check that the compositions fg and gf are both given by multiplication by π^{2N/p^n} . Hence the kernel and cokernel of f (and of g) are killed by π^{2N/p^n} ; therefore \mathcal{O}_L is indeed almost free of rank d . \square

4.3 Tilting perfectoid fields

Let K be a perfectoid field; as explained in the previous exercises, the tilt of K is $K^\flat = \mathrm{Frac}(\mathcal{O}_K^\flat)$, where \mathcal{O}_K^\flat is the tilt of the integral perfectoid ring \mathcal{O}_K . Our first goal is to prove:

Lemma 4.3. *Let K be a perfectoid field. Then K is algebraically closed if and only if K^\flat is algebraically closed.*

Proof. We will only prove the implication \Leftarrow , since the implication \Rightarrow is the same argument (and anyway will follow from the tilting correspondance below). So assume that K^\flat is algebraically closed, and let $f(X) \in \mathcal{O}_K[X]$ be an irreducible monic polynomial of degree d ; we must show that $f(X)$ has a root (whence it is linear qed). Fix a valuation $|\cdot| : K \rightarrow \mathbb{R}_{\geq 0}$ defining the topology.

Step 1 is a weak special case: given $x \in \mathcal{O}_K$, we show that there is $y \in \mathcal{O}_K$ such that $|y^d| = |x|$. Proof: we may write $x = \pi^m x'$ where $m \geq 0$ and $x' \in \mathcal{O}_K \setminus \pi\mathcal{O}_K$. Since $\mathcal{O}_{K^\flat}/\pi^\flat = \mathcal{O}_K/\pi$ and K^\flat is algebraically closed, there exists $y \in \mathcal{O}_K$ such that $y^d \equiv x' \pmod{\pi\mathcal{O}_K}$; since $x' \notin \pi\mathcal{O}_K$, the non-archimedean inequality implies $|y^d| = |x'|$. But $\pi^\flat \in K^\flat$ admits a d^{th} root and $\# : K^\flat \rightarrow K$ is multiplicative, so $\pi^{\flat 1/d\#}$ is a d^{th} root of π . In conclusion, the element $\pi^{\flat 1/d\#m} y'$ works.

Step 2: Given $a \in \mathcal{O}_K$ and $n \geq 0$ such that $|f(a)| \leq |\pi|^n$, there exists $\varepsilon \in \mathcal{O}_K$ such that $|\varepsilon| \leq |\pi|^{n/d}$ and $|f(a + \varepsilon)| \leq |\pi|^{n+1}$. Proof: By step 1 there is $y \in \mathcal{O}_K$ such that $|y^d| = |f(a)|$, whence $g(X) := y^{-d}f(a + yX)$ is a monic, irreducible polynomial in $K[X]$ whose constant coefficient $g(0) = y^{-d}f(a)$ lies in \mathcal{O}_K (even in \mathcal{O}_K^\times). A clever application of Hensel's lemma shows that all coefficients of $g(X)$ lie in \mathcal{O}_K . (See, for example, Lemma 3.2.11 of Bhatt's notes.) Since $\mathcal{O}_{K^\flat}/\pi^\flat = \mathcal{O}_K/\pi$ and K^\flat is algebraically closed and K^\flat is algebraically closed, there is therefore $b \in \mathcal{O}_K$ such that $g(b) \equiv 0 \pmod{\pi\mathcal{O}_K}$. We leave it to the reader to check (easily) that $\varepsilon := yb$ has the desired properties.

Step 3: $f(X)$ has a root. Proof: Use step 2 to successively approximate a root and take the limit. \square

Remark 4.4. Let K be a field which is complete under a valuation $|\cdot| : K \rightarrow \mathbb{R}_{\geq 0}$, and $f(X) \in K[X]$ an irreducible monic polynomial such that $f(0) \in \mathcal{O}_K$. We will use Hensel's lemma to prove that $f(X) \in \mathcal{O}_K[X]$.

We argue by contradiction. Write $f(X) = X^d + a_{d-1}X^{d-1} + \dots + a_1X + a_0$, and let $i_0 \in \{0, \dots, d-1\}$ be the largest index such that $|a_{i_0}| = \max_{0 \leq i \leq d-1} |a_i| > 1$. In other words, $|a_i| \geq |a_{i_0}|$ for $i = 0, \dots, i_0$ and $|a_i| < |a_{i_0}|$ for $i = i_0 + 1, \dots, d-1$. Set

$$f_0(X) := a_{i_0}^{-1}f(X), \quad g(X) := X^{i_0} + a_{i_0-1}^{-1}a_{i_0}X^{i_0-1} + \dots + a_{i_0}^{-1}a_0, \quad h(X) := a_{i_0}^{-1}X^{d-i_0} + 1,$$

all of which are in $\mathcal{O}_K[X]$. Letting $\mathfrak{m}_K = \{a \in \mathcal{O}_K : |a| < 1\}$ be the maximal ideal of \mathcal{O}_K , we have $h(X) \equiv 1 \pmod{\mathfrak{m}_K}$, and $g(X) \equiv f_0(X) \equiv X(\dots) \pmod{\mathfrak{m}_K}$. Therefore $f_0(X) \equiv g(X)h(X) \pmod{\mathfrak{m}_K}$, and the polynomials $g(X)$, $h(X)$ are coprime mod \mathfrak{m}_K . So (a strong form of) Hensel's lemma implies that $f_0(X)$ is reducible, whence $f(X)$ is also reducible, giving the desired contradiction.

Theorem 4.5 (Almost purity and the tilting correspondence). *Let K be a perfectoid field. Then (i) any finite field extension L of K (topologised as a finite dimensional K -vector space) is also a perfectoid field, and (ii) $L \mapsto L^\flat$ defines a degree-preserving equivalence of categories*

$$\{\text{finite field extensions of } K\} \xrightarrow{\cong} \{\text{finite field extensions of } K^\flat\}$$

Corollary 4.6. *K a perfectoid field. Then there exists an isomorphism of absolute Galois groups $\text{Gal}_K \cong \text{Gal}_{K^\flat}$.*

Proof. Immediate from Galois theory and the correspondence of the previous theorem. \square

Remark 4.7. Since K is complete with respect to a valuation $|\cdot| : K \rightarrow \mathbb{R}_{\geq 0}$, standard theory of valued fields implies the following facts: the valuation $|\cdot|$ extends uniquely to any finite extension L/K ; the resulting topology on L is the same as its topology viewed as a finite dimensional K -vector space and in particular L is complete with respect to this topology; the ring of integers \mathcal{O}_L with respect to this valuation is the same as the integral closure of \mathcal{O}_K in L .

Therefore the only thing to prove for part (i) of Theorem 4.5 is the following: given a finite extension L/K , then every element of $\mathcal{O}_L/p\mathcal{O}_L$ is a p^{th} -power. It is remarkable that this is not “obvious”! Of course, if K has characteristic p then it is well-known that a finite extension of a perfect field is still perfect, and this proves (i) in this case (also, if K has characteristic p then part (ii) is trivial since tilting does nothing). The idea will be to deduce the result in characteristic zero by tilting into characteristic p , which will simultaneously prove the tilting correspondence.

Proof of Theorem 4.5. From the end of Exercise Section 4.1 we already know that we have an equivalence of categories

$$\{\text{perfectoid fields over } K\} \rightarrow \{\text{perfectoid fields } K^\flat\}, \quad L \mapsto L^\flat,$$

but we know nothing about how it preserves finiteness.

Step 1: Let M be a finite extension of K^\flat (we know M is automatically a perfectoid field by the previous remark); then its untilt M^\sharp (which we know is a perfectoid field over K) is a finite extension of K of the same degree. Proof: Proposition 4.2 tells us that \mathcal{O}_M/π^\flat is an almost free $\mathcal{O}_{K^\flat}/\pi^\flat$ -module of rank $|M : K^\flat|$. But $\mathcal{O}_M/\pi^\flat = \mathcal{O}_{M^\sharp}/\pi$ and $\mathcal{O}_{K^\flat}/\pi^\flat = \mathcal{O}_K/\pi$, so we may now apply Lemma 4.1 to deduce that \mathcal{O}_{M^\sharp} is an almost free \mathcal{O}_K -module of rank $|M : K^\flat|$; inverting π tells us that M^\sharp is a free K -module of rank $|M : K^\flat|$, as desired.

Thanks to step 1 we have fully faithful functors

$$\{\text{finite field extns of } K^\flat\} \xrightarrow{\#} \{\text{finite field extns of } K \text{ which are perfd}\} \subseteq \{\text{finite field extns of } K\}.$$

Step 2: The composition is surjective (which completes the proof of the theorem, since it shows that the two functors are equivalences). Proof: We use Krasner’s Lemma

Let F be a field which is complete wrt a valuation $|\cdot| : F \rightarrow \mathbb{R}_{\geq 0}$, let $\alpha, \beta \in F^{\text{sep}}$, and let $\alpha_1 = \alpha, \alpha_2, \dots, \alpha_d \in F^{\text{sep}}$ be the conjugates of α ; if $|\alpha - \beta| < |\alpha - \alpha_i|$ for $i = 2, \dots, d$, then $\alpha \in F(\beta)$.

and its corollary

Let F be a field which is complete wrt a valuation $|\cdot| : F \rightarrow \mathbb{R}_{\geq 0}$, and $F_0 \subseteq F$ a dense subfield. Then F is separably closed if and only if F_0 is separably closed.

Now let Q be the completion of an algebraic closure of K^\flat . The corollary to Krasner's lemma implies that Q is an algebraically closed (to be precise, it is perfect and separably closed, hence algebraically closed) perfectoid field of characteristic p . By the previous lemma, its untilt $Q^\#$ is an algebraically closed perfectoid field over K . Moreover, for any finite subextension M of Q/K^\flat we have $Q^\# \supseteq M^\# \supseteq K$; let $N := \bigcup_M M^\# \subseteq Q^\#$ be the union, where M runs over all finite subextensions of Q/K^\flat . So N is an algebraic extension of K . We claim that N is dense in $Q^\#$; indeed, at the level of rings of integers we have

$$\mathcal{O}_N/\pi = \varinjlim_M \mathcal{O}_{M^\#}/\pi = \varinjlim_M \mathcal{O}_M/\pi^\flat = \mathcal{O}_{K^{\text{balg}}}/\pi^\flat = \mathcal{O}_Q/\pi^\flat = \mathcal{O}_{Q^\#}/\pi,$$

which proves the density. The corollary to Krasner therefore implies that N is algebraically closed.

In particular, if L is any finite extension of K , it follows that $L \subseteq N$ and hence there exists a finite extension M/K^\flat such that $L \subseteq M^\#$; we may as well replace M/K^\flat by its Galois closure and so we can suppose that M/K^\flat is a finite Galois extension. It remains only to prove that the fully faithful functor

$$\{\text{subextensions of } M/K^\flat\} \xrightarrow{\#} \{\text{subextensions of } M^\#/K\}$$

is surjective. But $|M^\# : K| = |M : K^\flat|$ by step 1 and $\text{Gal}(M^\#/K) = \text{Gal}(M/K^\flat)$ since the functor is fully faithful. Since M/K^\flat is Galois, it follows from Galois theory that $M^\#/K$ is also Galois, and hence that the two categories have the same finite cardinality; so the injection is surjective, as required. \square

5 THE ADIC SPECTRUM OF A HUBER RING

Definition 5.1. A *valuation* (or *absolute value*) x on a ring R is the data of a totally ordered abelian group Γ (written multiplicatively) and a map $x : R \rightarrow \Gamma \cup \{0\}$ ⁴ which satisfies

- $x(0) = 0, x(1) = 1$;
- $x(fg) = x(f)x(g)$;
- $x(f + g) \leq \max(x(f), x(g))$.

The *value group* $\Gamma_x \subseteq \Gamma$ is defined to be the subgroup generated by the monoid $x(R) \setminus \{0\}$.

The *support* of a valuation x is the prime ideal $\mathfrak{p}_x := \{f \in A : x(f) = 0\}$. The associated *residue field* is $k(x) := \text{Frac } R/\mathfrak{p}_x$, on which x obviously induces a valuation

$$x : k(x) \rightarrow \Gamma \cup \{0\}, \quad f/g \mapsto x(f)/x(g)$$

($f, g \in R \setminus \mathfrak{p}_x$) with image $\Gamma_x \cup \{0\}$; let $\mathcal{O}_x := \{f \in k(x) : x(f) \leq 1\} \supseteq R/\mathfrak{p}_x$ be the associated valuation ring.

Lemma 5.2. *Let $x : R \rightarrow \Gamma$ and $y : A \rightarrow \Delta$ be valuations. Then the following are equivalent:*

- (i) *for any $f, g \in R$, we have $x(f) \leq x(g) \iff y(f) \leq y(g)$;*
- (ii) *$\mathfrak{p}_x = \mathfrak{p}_y$ and $\mathcal{O}_x = \mathcal{O}_y$;*
- (iii) *there exists an isomorphism of ordered groups $\iota : \Gamma_x \cong \Delta_y$ such that $y = \iota \circ x$.*

⁴By definition we have $0\gamma = 0$ and $0 < \gamma$ for all $\gamma \in \Gamma$.

When this is true, we say that x and y are *equivalent* valuations.

Proof. (i) \Rightarrow (ii): We immediately deduce $\mathfrak{p}_x = \mathfrak{p}_y$ (since $x(f) = 0$ iff $x(f) \leq x(0)$ and similarly for y). Next, if $f, g \in R \setminus \mathfrak{p}_x$ so that f/g defines an element of $k(x)$, then $f/g \in \mathcal{O}_x$ iff $x(f) \leq x(g)$ iff $y(f) \leq y(g)$ iff $f/g \in \mathcal{O}_y$.

(ii) \Rightarrow (iii): The valuation x induces an isomorphism $x : k(x)^\times / \mathcal{O}_x^\times \xrightarrow{\cong} \Gamma_x$ such that, given $f, g \in k(x)^\times$, we have $x(f) \leq x(g) \iff gf^{-1} \in \mathcal{O}_x$. This is also true for y , whence the isomorphism ι is given by

$$\Gamma_x \cong k(x)^\times / \mathcal{O}_x^\times = k(y)^\times / \mathcal{O}_y^\times \cong \Delta_y.$$

(using the correspondence between valuations on a field and the associated ring of integers).

(iii) \Rightarrow (i) is obvious. \square

Definition 5.3 (The adic spectrum and its topology). Let (R, R^+) be a Huber pair. Its associated *adic spectrum* $\mathrm{Spa}(R, R^+)$ is the set of equivalence classes of valuations $x : R \rightarrow \Gamma$ such that

- $x(f) \leq 1$ for all $f \in R^+$;
- x is continuous with respect to the order topology on Γ ; i.e., for each $\gamma \in \Gamma$, the set $\{f \in R : x(f) < \gamma\}$ is open.

We give $\mathrm{Spa}(R, R^+)$ the coarsest topology for which the subsets

$$\mathrm{Spa}(R, R^+)(\frac{f}{g}) := \{x \in \mathrm{Spa}(R, R^+) : |f(x)| \leq |g(x)| \neq 0\}$$

are open for all $f, g \in R$.

Example 5.4. To add.

Definition 5.5 (Rational subsets). Let (R, R^+) be a Huber pair and $U \subseteq \mathrm{Spa}(R, R^+)$. Then we say that U is a *rational subset* of $\mathrm{Spa}(R, R^+)$ if and only if there exist $f_1, \dots, f_n \in R$ which generate an open ideal and $g \in R$ such that

$$U = \mathrm{Spa}(R, R^+)(\frac{f_1, \dots, f_n}{g}) := \{x \in \mathrm{Spa}(R, R^+) : |f_i(x)| \leq |g(x)| \neq 0 \text{ for all } i = 1, \dots, n\}.$$

(Note that the elements f_1, \dots, f_n, g are not unique.)

Remark 5.6 (Case of Tate–Huber pairs). If (R, R^+) is a Tate–Huber pair, then the only open ideal of R is R itself. So in this case rational subsets are

$$\mathrm{Spa}(R, R^+)(\frac{f_1, \dots, f_n}{g}) := \{x \in \mathrm{Spa}(R, R^+) : |f_i(x)| \leq |g(x)| \text{ for all } i = 1, \dots, n\}$$

for $f_1, \dots, f_n \in R$ generating the unit ideal. Note that we have omitted the condition “ $|g(x)| \neq 0$ ”, because now it is automatic: if $x(g) = 0$ then $x(f_i) = 0$ for all i , but this is impossible since the f_i generate the unit ideal.

We begin with the following basic lemma:

Lemma 5.7. *Let (R, R^+) be a Huber pair. The rational subsets of $\mathrm{Spa}(R, R^+)$ are open and closed under finite intersection.*

Proof. Let $U = \text{Spa}(R, R^+)(\frac{f_1, \dots, f_n}{g})$ be as in the definition. Clearly $U = \bigcap_{i=1}^n \text{Spa}(R, R^+)(\frac{f_i}{g})$, which is open by definition of the topology.

Given another rational subset $U' = \text{Spa}(R, R^+)(\frac{f'_1, \dots, f'_m}{g'})$, write $f_0 = g$ and $f'_0 = g'$ for convenience; then it is not hard to use the axioms of a valuation to check

$$U \cap U' = \text{Spa}(R, R^+)(\frac{f_i f'_j : i=0, \dots, n, j=0, \dots, m}{gg'})$$

Moreover, the ideal generated by $f_i f'_j$, $i = 1, \dots, n$, $j = 1, \dots, m$ is open, since it is the product of two open ideals (note that in a Huber ring the product of two open ideals is again open; to prove this, fix an ideal of definition I and note that a subgroup is open if and only if it contains a power of I). \square

Lemma 5.8. *Let $\varphi : (R, R^+) \rightarrow (S, S^+)$ be a morphism of Huber pairs; then there is an induced map*

$$\text{Spa}(\varphi) : \text{Spa}(R, R^+) \rightarrow \text{Spa}(S, S^+), \quad x \mapsto x \circ \varphi,$$

which is continuous.

Proof. The existence and continuity of $\text{Spa}(\varphi)$ are easy. \square

Definition 5.9. A topological space X is said to be *spectral* if and only if it is quasi-compact, has a basis of quasi-compact opens closed under finite intersections, and is *sober* (:=any irreducible closed subset of X admits a unique generic point).

The following theorem of Huber presents the main fundamental properties of the adic spectrum:

Theorem 5.10. *Let (R, R^+) be a Huber pair.*

- (i) *The topological space $\text{Spa}(R, R^+)$ is spectral.*
- (ii) *The rational subsets form a basis for the topology.*
- (iii) *Any rational subset is quasi-compact.*
- (iv) *$\text{Spa}(R, R^+) = \emptyset$ if and only if the topology on R is trivial (i.e., the only opens are \emptyset and R).*
- (v) $R^+ = \{f \in R : x(f) \leq 1 \forall x \in \text{Spa}(R, R^+)\}$
 $R^{\circ\circ} = \{f \in R : x(f) < 1 \forall x \in \text{Spa}(R, R^+)\}$

Proof. Unfortunately we probably do not have time to prove this result – it is not so difficult but rather long. We recommend looking at Huber’s paper *Continuous valuations* (the proof occupies section 2 and the first half of section 3). \square

Corollary 5.11. *Let (R, R^+) be a complete Huber pair.*

- (i) $R^\times = \{f \in R : x(f) \neq 0 \forall x \in \text{Spa}(R, R^+)\}$.
- (ii) *If $I \subseteq R$ is a proper ideal, then there exists $x \in \text{Spa}(R, R^+)$ such that $x(f) = 0$ for all $f \in I$.*

Proof. (i): The inclusion \subseteq is clear, so suppose that $f \in R$ is not a unit. Then there is a maximal ideal $\mathfrak{m} \subseteq R$ such that $f \in \mathfrak{m}$. We claim that \mathfrak{m} is closed, i.e., $R \setminus \mathfrak{m}$ is open, for which it is enough to prove the following claim.

Claim: $1 + R^{\circ\circ} \subseteq R^\times$. Proof: Let $I \subseteq R_0$ be an ideal and subring of definition. Given $g \in R^{\circ\circ}$ we have $g^m \in I$ for $m \gg 0$. Since R_0 is I^m -adically complete, certainly $1 - g^m \in R_0^\times \subseteq R^\times$. But $(1 - g)(1 + g + \cdots + g^{m-1}) = 1 - g^m$, so $1 - g \in R^\times$.

Set $K := R/\mathfrak{m}$ and let K^+ be the integral closure of R^+ in R . Since \mathfrak{m} is closed, the quotient topology on the Huber ring K is separated, and hence Theorem 5.10(iv) implies that $\text{Spa}(K, K^+) \neq \emptyset$. This gives a valuation x on A satisfying $x(f) = 0$.

(ii): Next suppose that I is a proper ideal; let $I \subseteq \mathfrak{m}$ be a maximal ideal containing it. As above there exists a valuation x killing \mathfrak{m} . \square

Our next real goal is to construct a structure presheaf \mathcal{O}_X on the adic spectrum $X = \text{Spa}(R, R^+)$, mimicking the structure presheaf on the usual spectrum; given a rational subset $U \subseteq X$, the value $\mathcal{O}_X(U)$ will be defined to be a certain completed localisation of R . Therefore we must first discuss completion and localisation of Huber rings.

5.1 Completion

Definition 5.12. We say that a Huber ring R is *complete* if and only if R is complete and Hausdorff in the usual topological sense.

In general, let \widehat{R} be the Hausdorff completion of R . (Recall: if X is a topological space with a countable neighbourhood basis at every point, then we may define its Hausdorff completion to be $\widehat{R} = R^\mathbb{N}/\{\text{Cauchy sequences}\}$.)

In practice, it is more convenient to define \widehat{R} algebraically as $\widehat{R} := \widehat{R_0} \otimes_{R_0} R$, where $\widehat{R_0} := \varprojlim_r R_0/I^r$ is the I -adic completion of a chosen subring of definition R_0 with respect to an ideal of definition I . The next proposition shows that this process really gives the same result:

Proposition 5.13. \widehat{R} is a Huber ring. Moreover, letting $I \subseteq R_0 \subseteq R$ be any ideal and subring of definition:

- (i) $I\widehat{R_0} \subseteq \widehat{R_0}$ are an ideal and subring of definition of \widehat{R} , where $\widehat{R_0} = \varprojlim_n R_0/I^n$ is the I -adic completion of R_0 ;
- (ii) the canonical map $\widehat{R_0} \otimes_{R_0} R \rightarrow \widehat{R}$ is an isomorphism.

Finally, if $R^+ \subseteq R$ is a subring of integral elements, then its Hausdorff completion $\widehat{R^+} \subseteq \widehat{R}$ is a subring of integral elements, whence $(\widehat{R}, \widehat{R^+})$ is a Huber pair (called the completion of the pair (R, R^+)).

Proof. The universal property of tensor product gives us a map $j : \widehat{R_0} \otimes_{R_0} R \rightarrow \widehat{R}$ making the diagram commute:

$$\begin{array}{ccc}
 \widehat{R} & \xleftarrow{i} & R \\
 \uparrow j & & \uparrow f \\
 \widehat{R_0} \otimes_{R_0} R & & \\
 \uparrow h & & \\
 \widehat{R_0} & \xleftarrow{g} & R_0
 \end{array}$$

By elementary theory of topological completions, $i(R)$ is dense in \widehat{R} , and $\widehat{R_0}$ is open in \widehat{R} ; therefore $\widehat{R} = i(R) + \widehat{R_0}$ and $i^{-1}(\widehat{R_0}) = R_0$. Algebraically, this means that the above square is cartesian in the category of R_0 -modules, and therefore there exists a dotted arrow h making the top and left triangles commute (i.e., $hi = f$ and $h|_{\widehat{R_0}} = g$).

Therefore the R_0 -linear map $hj : \widehat{R_0} \otimes_{R_0} R \rightarrow \widehat{R_0} \otimes_{R_0} R$ satisfies $hjf = f$ and $hjpg = g$; so the universal property of tensor product implies $hj = \text{id}$. But j is surjective (since $\widehat{R} = i(R) + \widehat{R_0}$), so therefore j is an isomorphism, proving (ii).

Since we already know that $\widehat{R_0}$ is an open subring of \widehat{R} , the only part left of (i) is that the topology on $\widehat{R_0}$ is the $I\widehat{R_0}$ -adic topology, for which it is enough to check that $\widehat{R_0}/I^n\widehat{R_0} \cong R/I^n$ for all $n \geq 1$. For R Noetherian this is well-known commutative algebra; in general we refer to the Stacks project [Tag 05GG].

The assertions about R^+ are left as an exercise. \square

The completion of a Huber pair obviously satisfies the following universal property:

Corollary 5.14. *Let (R, R^+) be a Huber pair. Then $(R, R^+) \rightarrow (\widehat{R}, \widehat{R^+})$ is an adic morphism with the following universal property: given any complete Huber pair (S, S^+) and morphism $\varphi : (R, R^+) \rightarrow (S, S^+)$, there exists a unique morphism $\widehat{\varphi} : (\widehat{R}, \widehat{R^+}) \rightarrow (S, S^+)$ such that*

$$\begin{array}{ccc} (R, R^+) & \longrightarrow & (\widehat{R}, \widehat{R^+}) \xrightarrow{\widehat{\varphi}} (S, S^+) \\ & \searrow \varphi \curvearrowright & \uparrow \\ & & \end{array}$$

commutes.

5.2 Localisation

The correct way to localise a Huber pair turns out to be slightly subtle:

Definition 5.15. Let R be a Huber ring, let $g \in R$, and let $f_1, \dots, f_n \in R$ be elements which generate an open ideal. Set

$$R\left[\frac{f_1, \dots, f_n}{g}\right] := R\left[\frac{1}{g}\right]$$

with the following topology: fixing an ideal and subring of definition $I \subseteq R_0$, a neighbourhood basis of $R\left[\frac{f_1, \dots, f_n}{g}\right]$ at 0 is given by $I^m R_0\left[\frac{f_1}{g}, \dots, \frac{f_n}{g}\right]$ for $n \geq 1$.

Proposition 5.16. $R\left[\frac{f_1, \dots, f_n}{g}\right]$ is a Huber ring. Moreover:

- (i) It does not depend on the chosen ideal and subring of definition $I \subseteq R_0$.
- (ii) $g \in R\left[\frac{f_1, \dots, f_n}{g}\right]^\times$ and $\frac{f_i}{g} \in R\left[\frac{f_1, \dots, f_n}{g}\right]^\circ$.
- (iii) The canonical map $R \rightarrow R\left[\frac{f_1, \dots, f_n}{g}\right]$ is adic and satisfies the following universal property: if $\varphi : R \rightarrow S$ is a continuous map of Huber rings such that $\varphi(g) \in S^\times$ and $\frac{\varphi(f_i)}{\varphi(g)} \in S^\circ$ for all i , then φ extends uniquely to a continuous map of Huber rings $R\left[\frac{f_1, \dots, f_n}{g}\right] \rightarrow S$.

Proof. To show that $R\left[\frac{f_1, \dots, f_n}{g}\right]$ is a Huber ring, everything is obvious except for multiplication being continuous. We begin by checking the following:

Claim: For any $h \in R\left[\frac{1}{g}\right]$ there exists $m \geq 1$ such that $hI^m R_0\left[\frac{f_1}{g}, \dots, \frac{f_n}{g}\right] \subseteq R_0\left[\frac{f_1}{g}, \dots, \frac{f_n}{g}\right]$.

Proof: Clearly we may reduce to the case $h = \frac{a}{g^s}$ for some $a \in R$ and $s \geq 0$; in fact, since

$aI^m \subseteq R_0$ for $m \gg 1$, we even reduce to the case $h = \frac{1}{g_s}$. Now we use Corollary 2.9, which says that the set $f_1R_0 + \dots + f_nR_0$ is open, hence contains I^m for some $m \geq 1$. Therefore $\frac{1}{g}I^m \subseteq R_0[\frac{f_1}{g}, \dots, \frac{f_n}{g}]$; since the right side is closed under multiplication, we also get $\frac{1}{g^s}I^{ms} \subseteq R_0[\frac{f_1}{g}, \dots, \frac{f_n}{g}]$, which is enough.

Now we can prove that multiplication is continuous. Let $h_1, h_2 \in R[\frac{1}{g}]$, and let $h_1h_2 + I^mR_0[\frac{f_1}{g}, \dots, \frac{f_n}{g}]$ be a typical neighbourhood of h_1h_2 . By the claim there exist $m_1, m_2 \geq 1$ such that $h_1I^{m_1}R_0[\frac{f_1}{g}, \dots, \frac{f_n}{g}] \subseteq R_0[\frac{f_1}{g}, \dots, \frac{f_n}{g}]$ and similarly for h_2 . Then

$$(h_1 + I^{m+m_2}R_0[\frac{f_1}{g}, \dots, \frac{f_n}{g}])(h_2 + I^{m+m_2}R_0[\frac{f_1}{g}, \dots, \frac{f_n}{g}]) \subseteq h_1h_2 + I^mR_0[\frac{f_1}{g}, \dots, \frac{f_n}{g}],$$

which proves continuity of multiplication.

(i): Easy exercise.

(ii): g is obviously a unit in $R[\frac{f_1, \dots, f_n}{g}]$, since the underlying ring is $R[\frac{1}{g}]$. By construction of the topology, each $\frac{f_i}{g}$ belongs to a subring of definition, namely $R_0[\frac{f_1}{g}, \dots, \frac{f_n}{g}]$, hence is power bounded.

(iii): The map $R \rightarrow R[\frac{f_1, \dots, f_n}{g}]$ is clearly continuous and adic by definition. We must check the universal property. Since $\varphi(g) \in S^\times$, there is obviously a unique map of rings $\tilde{\varphi} : R[\frac{f_1, \dots, f_n}{g}] \rightarrow S$ which extends φ . We must show that $\tilde{\varphi}$ is continuous; let $J \subseteq S_0 \subseteq S$ be an ideal and subring of definition, and fix $m \geq 1$. Since $\frac{\varphi(f_i)}{\varphi(g)}$ is power bounded for each i , Corollary 2.8 tells us that the set $S_0[\frac{\varphi(f_1)}{\varphi(g)}, \dots, \frac{\varphi(f_n)}{\varphi(g)}]$ is bounded, so there exists $m' \geq 1$ such that $J^{m'}S_0[\frac{\varphi(f_1)}{\varphi(g)}, \dots, \frac{\varphi(f_n)}{\varphi(g)}] \subseteq J^m$. Next, the continuity of φ implies the existence of $m'' \geq 1$ such that $\varphi(I^{m''}) \subseteq J^{m'}$. Now it is clear that

$$\tilde{\varphi}(I^{m''}R_0[\frac{f_1}{g}, \dots, \frac{f_n}{g}]) \subseteq J^{m'}S_0[\frac{\varphi(f_1)}{\varphi(g)}, \dots, \frac{\varphi(f_n)}{\varphi(g)}] \subseteq J^m,$$

which proves continuity of $\tilde{\varphi}$. □

Definition 5.17. Let (R, R^+) be a Huber pair, let $g \in R$, and let $f_1, \dots, f_n \in R$ be elements which generate an open ideal. We define $R[\frac{f_1, \dots, f_n}{g}]^+$ to be the integral closure of $R^+[\frac{f_1}{g}, \dots, \frac{f_n}{g}]$ in $R[\frac{1}{g}]$. Thus we obtain a Huber pair $(R[\frac{f_1, \dots, f_n}{g}], R[\frac{f_1, \dots, f_n}{g}]^+)$. Let $(R\langle \frac{f_1, \dots, f_n}{g} \rangle, R\langle \frac{f_1, \dots, f_n}{g} \rangle^+)$ be its completion.

Corollary 5.18. *The canonical map of Huber pairs $(R, R^+) \rightarrow (R\langle \frac{f_1, \dots, f_n}{g} \rangle, R\langle \frac{f_1, \dots, f_n}{g} \rangle^+)$ is adic and satisfies the following universal property: given any complete Huber pair (S, S^+) and morphism $\varphi : (R, R^+) \rightarrow (S, S^+)$ such that $\varphi(g) \in S^\times$ and $\frac{\varphi(f_i)}{\varphi(g)} \in S^+$ for all i , then φ extends uniquely to a map of Huber pairs $\tilde{\varphi} : (R\langle \frac{f_1, \dots, f_n}{g} \rangle, R\langle \frac{f_1, \dots, f_n}{g} \rangle^+) \rightarrow (S, S^+)$.*

Proof. Combine Corollary 5.14 and Proposition 5.16. □