Adic and perfectoid spaces

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Abstract

The main goal of this course is to develop the foundations of the theory of perfectoid spaces (P), more precisely to prove the various tilting correspondences for perfectoid rings, the almost purity theorem, and almost vanishing theorems. We develop simultaneously what is needed from the theory of adic spaces (A).

Contents

Some notation

For the whole course we fix a prime number p. All rings are commutative. Given a ring A of characteristic p, we write $\varphi : A \to A$, $a \mapsto a^p$ for its Frobenius endomorphism (which is a ring homomorphism).

1 (P) Integral perfectoid rings

Let A be a topological ring. We say that A is *integral perfectoid*[1] if and only if there exists a non-zero-divisor $\pi \in A$ such that

(a) the topology on A is the π-adic topology, and A is complete for this topology (i.e., $A \to \varprojlim_n A/\pi^n A$ is an isomorphism of topological rings, where each $A/\pi^n A$ has the discrete topology);

(b) $p \in \pi^p A$;

(c) $\Phi : A/\pi A \to A/\pi^p A$, $a \mapsto a^p$, is an isomorphism

It is convenient for us, even though it is not exactly standard in the literature, to call any such element π a perfectoid pseudo-uniformiser (ppu).

Example 1.1. Here are some easy examples:

(i) The p-adic completions of the rings $\mathbb{Z}_p[p^{1/p^\infty}]$ and $\mathbb{Z}_p[\zeta_{p^\infty}]$ are perfectoid (equipped with the p-adic topology), with perfectoid pseudo-uniformisers $p^{1/p}$ and $\zeta_{p^2} - 1$ respectively.

(ii) $\mathbb{Z}_p$ is not perfectoid.

Construction 1.2. Given any ring A we write

$$\varprojlim_{x \mapsto x^p} A := \{(a_0, a_1, \dots) \in A^{\mathbb{N}} : a_i^p = a_{i-1} \text{ for all } i \geq 1\}$$

for the set of compatible sequences of p-power roots in A. Note that we can multiply two such sequences, so $\varprojlim_{x \mapsto x^p} A$ forms a multiplicative monoid; if A has characteristic p then we can also add two such sequences, so then A is even a ring.

The construction is clearly functorial: a morphism of rings $A \to B$ induces a morphism of moniods $\varprojlim_{x \mapsto x^p} A \to \varprojlim_{x \mapsto x^p} B$ (which is even a morphism of rings if A and B have characteristic p).

The following result is extremely important and will be used repeatedly: if $\pi \in A$ is an element such that (i) $p \in \pi A$ and (ii) A is π-adically complete, then the map $\varprojlim_{x \mapsto x^p} A \to \varprojlim_{x \mapsto x^p} A/\pi A$ is actually a bijection (hence an isomorphism of monoids). We leave the details of the proof to the reader, but give the following recipe for the inverse of the map: given $b = (b_0, b_1, \dots) \in \varprojlim_{x \mapsto x^p} A/\pi A$, let $\widetilde{b_i} \in A$ be an arbitrary lift of $b_i \in A/\pi A$ for each $i \geq 0$. Then set

$$a_i := \lim_{n \to \infty} \widetilde{b_{i+n}}^{p^n}$$

and check that $a := (a_0, a_1, \dots) \in \varprojlim_{x \mapsto x^p} A$ really is a well-defined lift of b.

Lemma 1.3. *Let A be integral perfectoid, and $\pi \in A$ a perfectoid pseudo-uniformiser. Then:*

(i) Every element of $A/\pi p A$ is a p^{th}-power (n.b., $A/\pi p A$ does not necessarily have characteristic p).

(ii) If an element $a \in A[\frac{1}{\pi}]$ satisfies $a^p \in A$, then $a \in A$.

(iii) After multiplying π by a unit it admits a compatible sequence of p-power roots $\pi^{1/p}, \pi^{1/p^2}, \dots \in A$.

[1] La meilleure traduction de "integral perfectoid ring" n'est pas évidente, mais on va utiliser *anneau perfectoïde entier.*

Proof. (i): Using the surjectivity of Φ, a simple induction lets us write any $a \in A$ as an infinite sum $a = \sum_{i\geq 0} a_i^p \pi^{pi}$ for some $a_i \in A$; but this is $\equiv (\sum_{i\geq 0} a_i\pi^i)^p \bmod p\pi A$.

(ii): Let $l \geq 0$ be the smallest integer such that $\pi^l a \in A$. Assuming that $l > 0$, we get a contradiction by noting that $\pi^{pl}a^p \in \pi^{pl}A \subseteq \pi^p A$, whence $\pi^l a \in \pi A$ by condition (c), and so $\pi^{l-1}a \in A$.

(iii): Since the Frobenius is surjective on $A/\pi^p A$, there exists an element of $\varprojlim_{x\mapsto x^p} A/\pi^p A$ of the form $(\pi \bmod \pi^p A, ?, ?, \dots)$. Applying the exercise of Construction 1.2, we deduce that the natural map $\varprojlim_{x\mapsto x^p} A \to \varprojlim_{x\mapsto x^p} A/\pi^p A$ is a bijection. Hence there exists $a = (a_0, a_1, \dots) \in \varprojlim_{x\mapsto x^p} A$ such that $a_0 \equiv \pi \bmod \pi^p A$; therefore $a = u\pi$ for some $u \in 1 + \pi^{p-1}A \subseteq A^\times$ (the inclusion $\subseteq$ results from π-adic completeness of A). □

Lemma 1.4. *Let A be integral perfectoid, and $\varpi \in A$ an element satisfying conditions (a) and (b). Then ϖ is a non-zero-divisor satisfying (c), i.e., it is a perfectoid pseudo-uniformiser.*

Proof. We must show that $\Phi : A/\varpi A \to A/\varpi^p A$ is an isomorphism. Let $\pi \in A$ be a perfectoid pseudo-uniformiser.

It follows from Lemma 3(i) that every element of A/pA is a p^{th}-power; hence every element of its quotient $A/\varpi^p A$ is a p^{th}-power, i.e., Φ is surjective.

The fact that π and ϖ define the same topology implies that a power of each is divisible by the other, whence ϖ is a non-zero-divisor and $A[\frac{1}{\varpi}] = A[\frac{1}{\pi}]$. If $a \in A$ satisfies $a^p \in \varpi^p A$, then $(a/\varpi) \in A[\frac{1}{\pi}]$ satisfies $(a/\varpi)^p \in A$, and it then follows from Lemma 3(ii) that in fact $a \in \varpi A$ as desired. □

Lemma 1.5. *Suppose A is a complete topological ring such that $pA = 0$. Then A is integral perfectoid if and only if it is perfect and the topology is π-adic for some non-zero-divisor $\pi \in A$.*

Proof. Exercise. □

1.1 The tilt of an integral perfectoid ring

Definition 1.6. The *tilt*[2] of an integral perfectoid ring A is $A^\flat := \varprojlim_{x\mapsto x^p} A/pA$, equipped with the inverse limit topology (A/pA is of course given the quotient topology, i.e., the π-adic topology for any choice of pup for A).

Note that $A^\flat$ is a perfect ring of characteristic p; in fact, it is the initial object among all perfect rings of characteristic p mapping to A/pA.

Recalling from Construction 1.2 that the natural map $\varprojlim_{x\mapsto x^p} A \to \varprojlim_{\varphi} A/pA$ is an isomorphism of monoids, we define the *untilting map* $\# : A^\flat \to A$, $b \mapsto b^\#$ to be projection to the 0^{th}-coordinate of $\varprojlim_{x\mapsto x^p} A$; explicitly, the map $\#$ is given by $\varprojlim_{\varphi} A/pA \ni (b_0, b_1, \dots) \mapsto \lim_{i\to\infty} \widetilde{b_i}^{p^i}$, where $\widetilde{b_i} \in A$ are arbitrary lifts of the elements $b_i \in A/pA$.

The untilting map is multiplicative by generally not additive; in fact, given $b, c \in A^\flat$, it transforms under addition as follows:

$$(b+c)^\# = \lim_{i\to\infty} ((b^{1/p^i})^\# + (c^{1/p^i})^\#)^{p^i}.$$

However, note that the composition $A^\flat \overset{\#}{\longrightarrow} \overset{\bmod p}{\longrightarrow} A/pA$ is a ring homomorphism: indeed, it is the surjective ring homomorphism given by projecting $A^\flat \cong \varprojlim_{\varphi} A/pA$ to the 0^{th}-coordinate. Also, if A is of characteristic p, then the untilting map $\# : A^\flat \to A$ is an isomorphism of rings.

[2]Le *basculé* de A en français.

Lemma 1.7. *Let A be an integral perfectoid ring. Then:*

(i) $\# : A^\flat \to A$, *is continuous;*

(ii) the isomorphisms of monoids $\varprojlim_{x\mapsto x^p} A \to A^\flat = \varprojlim_{x\mapsto x^p} A/pA \to \varprojlim_{x\mapsto x^p} A/\pi A$ *are homeomorphisms, where $\pi \in A$ is any perfectoid pseudo-uniformiser;*

(iii) $A^\flat$ *is also an integral perfectoid ring.*

Proof. Given $(1,\dots,1,b_{n+1},b_{n+2},\dots) \in \varprojlim_\varphi A/\pi A$, any chosen lifts $\widetilde{b_i}$ satisfy $\widetilde{b_i}^{p^{i-n}} \equiv 1 \mod \pi A$ for $i > n$, whence $\widetilde{b_i}^{p^i} \equiv 1 \mod \pi^n A$; taking the limit shows that the untilt is $\equiv 1 \mod \pi^n A$. This proves that the untilting map $\# : \varprojlim_{x\mapsto x^p} A/\pi A \to A$ is continuous (for the inverse limit of discrete topologies on the domain), from which (i) and (ii) easily follow. Filling in the details is left as an exercise.

(iii) We have already noted that $A^\flat$ is a perfect ring of characteristic, and the homeomorphism $A \cong \varprojlim_{x\mapsto x^p} A/\pi A$ shows that A is an inverse limit of discrete rings, whence A is a complete topological ring. According to Lemma 1.5, it remains to prove the following: there exists a non-zero-divisor $\pi^\flat \in A^\flat$ such that the topology on $A^\flat$ is the $\pi^\flat$-adic topology.

Possibly after changing our perfectoid pseudo-uniformiser π, we may assume that it admits compatible p-power roots (by Lemma 3(iii)); let $\pi^\flat = (\pi, \pi^{1/p}, \dots,) \in A^\flat$ be the corresponding element of $A^\flat$, which satisfies $(\pi^\flat)^\# = \pi$.

We show first that $\pi^\flat$ is a non-zero-divisor. To do that we note that for each $n \geq 1$ we have an exact sequence

$$0 \longrightarrow \pi^{1-1/p^n} A/\pi A \longrightarrow A/\pi A \overset{\times \pi^{1/p^n}}{\longrightarrow} A/\pi A \overset{\varphi^n}{\longrightarrow} A/\pi A \longrightarrow 0$$

Exactness is easy everywhere except possibly at the second term from the right: but if $a \in A$ satisfies $a^p \in \pi A$ then $a/\pi^{1/p^n} \in A[\frac{1}{\pi}]$ satisfies $(a/\pi^{1/p^n})^{p^n} \in A$, whence Lemma 3(ii) implies $a \in \pi^{1/p^n} A$ as desired.

These sequences are moreover compatible in n, with respect to the maps $0, \varphi, \varphi, \mathrm{id}$ respectively. Although it is not always the case that an inverse limit of exact sequences is still exact, in this case the transition map are either surjective (φ and id) or zero, and so taking the inverse limit does yield an exact sequence

$$0 \longrightarrow A^\flat \overset{\times \pi^\flat}{\longrightarrow} A^\flat \overset{\sharp \bmod \pi}{\longrightarrow} A/\pi A \longrightarrow 0.$$

That is, $\pi^\flat$ is a non-zero-divisor of $A^\flat$ and the untilting map induces an isomorphism of rings $A^\flat/\pi^\flat A^\flat \overset{\cong}{\to} A/\pi A$.

Finally we check that the topology on $A^\flat$ is the $\pi^\flat$-adic topology. Since $A^\flat \cong \varprojlim_{x\mapsto x^p} A/\pi A$ is a homeomorphism by part (i), a basis of open neighbourhoods of $0 \in A^\flat$ is given by $\mathrm{Ker}(\mathrm{proj_n})$ for $n \geq 0$, where $\mathrm{proj}_n : \varprojlim_{x\mapsto x^p} A/\pi A \to A/\pi A$, $(b_0, b_1, \dots) \mapsto b_n$ denotes the n^{th}-projection map. Note that proj_0 is the untilting map. Since the composition $A^\flat \overset{\varphi^n \cong}{\longrightarrow} A^\flat \overset{\mathrm{proj}_n}{\longrightarrow} A/\pi A$ is proj_0, the basis of open neighbourhoods is given by

$$\mathrm{Ker}(\mathrm{proj_n}) = \varphi^n(\mathrm{Ker}(\mathrm{proj}_0)) = \varphi^n(\pi^\flat A^\flat) = \pi^{\flat p^n} A^\flat,$$

showing that the topology is indeed $\pi^\flat$-adic. □

We point out explicitly that we showed in the previous proof that the kernel of the surjective ring homomorphism $A^\flat \overset{\# \bmod \pi}{\longrightarrow} A/\pi A$ (i.e., projection to the 0^{th}-coordinate of $A^\flat = \varprojlim_{x\mapsto x^p} A/\pi A$) is $\pi^\flat A^\flat$. This will be used repeatedly.

1.2 Fontaine's map θ

In this section we introduce Fontaine's map $\theta : W(A^\flat) \to A$, which will allow us to recover A from $A^\flat$ in some sense.

Remark 1.8 (Reminder on the ring of Witt vectors)**.** If k is any ring, let $W(k)$ denote its ring of p-typical Witt vectors. Here are some reminders about this object:

(i) There is an identification of sets $W(k) = k^{\mathbb{N}}$. So each element of $W(k)$ may be written uniquely as $(a_0, a_1, \dots)$, with $a_i \in k$.

(ii) Additional and multiplication are given by certain polynomials with integer coefficients (which do not depend on k), for example

$$(a_0, a_1, a_2, \dots) + (b_0, b_1, b_2, \dots) = (a_0 + b_0, a_1 + b_1 - \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} a_0^i b_0^{p-i}, \dots)$$

$$(a_0, a_1, a_2, \dots) \cdot (b_0, b_1, b_2, \dots) = (a_0 b_0, a_0^p b_0 + b_0^p a_1 + p a_1 b_1, \dots)$$

(iii) There is a natural ring homomorphism called the "phantom map"

$$\mathrm{phant} : W(k) \longrightarrow k^{\mathbb{N}}, \qquad (a_0, a_1, \dots) \mapsto (a_0, a_0^p + p a_1, a_0^{p^2} + p a_1^p + p^2 a_2, \dots).$$

In particular, its n^{th}-coordinate $\mathrm{phant}_n : W(k) \to k$, $(a_0, a_1, \dots) \mapsto \sum_{i=0}^n p^i a_i^{p^{n-i}}$ is a ring homomorphism for each $n \geq 0$.

(iv) If $k \supseteq \mathbb{Q}$ then phant is an isomorphism of rings; if k is p-torsion-free then phant is injective.

(v) Given $a \in k$, its *Teichmüller lift* is $[a] := (a, 0, 0, 0, \dots) \in W(k)$; the map $[\cdot] : k \to W(k)$ is multiplicative but not additive.

(vi) Suppose now that $k \supseteq \mathbb{F}_p$. Then $W(k)$ is p-adically complete and

$$\sum_{i=0}^{\infty} [a_i] p^i = (a_0, a_1^p, a_2^{p^2}, \dots),$$

where $a_i \in k$. In particular, if k is perfect then one easily deduces the following: each element of $W(k)$ may be written uniquely as $\sum_{i=0}^{\infty} [a_i] p^i$ for some elements $a_i \in A$, the element $p \in W(k)$ is a non-zero-divisor, and $W(k)/pW(k) \xrightarrow{\simeq} k$, $(a_0, a_1, \dots) \mapsto a_0$.

(vii) Exercise: Continue to suppose that k is a perfect ring of characteristic p; also let $t \in k$ be a non-zero-divisor and $q \in W(k)$ an element such that $q \equiv p \bmod [t]W(k)$.

Check that $[t] \in W(k)$ is a non-zero-divisor (this does not use the hypotheses on k). Using that p is a non-zero-divisor of $W(k)$ and that t is a non-zero-divisor of $k = W(k)/pW(k)$, deduce that p is a non-zero-divisor of $W(k)/[t]W(k)$. Now deduce that q is a non-zero-divisor of $W(k)/[t]W(k)$, hence that $[t]$ is a non-zero-divisor of $W(k)/qW(k)$.

Now suppose further that k is t-adically complete. Prove by induction that $W(k)/p^n W(k)$ is $[t]$-adically complete for all $n \geq 1$, and take the limit to deduce that $W(k)$ is $[t]$-adically complete (even $(p, [t])$-adically complete). Next show that q is a non-zero-divisor of $W(k)$ (this is probably the hardest part of the exercise) and deduce that $W(k)/qW(k)$ is $[t]$-adically complete.

Theorem 1.9 (Fontaine)**.** *Let A be an integral perfectoid ring.*

(i) There is a unique ring homomorphism

$$\theta : W(A^\flat) \longrightarrow A$$

satisfying $\theta([b]) = b^\#$ for all $b \in A^\flat$.

(ii) θ is surjective and its kernel is generated by a non-zero-divisor (usually denoted by $\xi \in W(A^\flat)$).

(iii) A given element $\chi \in \operatorname{Ker}\theta$ is a generator if and only if its Witt vector expansion $\chi = (\chi_0, \chi_1, \dots)$ has the property that $\chi_1 \in A^{\flat\times}$.

Proof. (i): Since any element $b \in W(A^\flat)$ may be written uniquely as a p-adically convergent sum $b = \sum_{i\geq 0}[b_i]p^i$, the content of (i) is the assertion that the well-defined map

$$\theta : W(A^\flat) \longrightarrow A, \qquad b = \sum_{i=0}^{\infty}[b_i]p^i \mapsto \sum_{i=0}^{\infty} b_i^\# p^i$$

is actually a ring homomorphism. Since θ is clearly multiplicative on Teichmüller lifts, an easy argument shows that multiplicativity will follow from additivity. To do this we will use the phantom maps

$$\mathrm{phant}_n : W(A) \xrightarrow{\mathrm{gh}} (A)^{\mathbb{N}} \xrightarrow{\mathrm{proj}_n} A, \qquad (a_0, a_1, \dots) \mapsto a_0^{p^n} + pa_1^{p^{n-1}} + \dots + p^n a_n,$$

which are ring homomorphisms. Note also that if $a_i \equiv a_i' \bmod pA$ then $p^i a_i^{p^{n-i}} \equiv p^i a_i'^{p^{n-i}} \bmod p^{n+1}A$, so in fact $\mathrm{phant}_n \bmod p^{n+1}$ only depends on the values of the Witt coordinates mod p, i.e., there is a commutative diagram

$$\begin{array}{ccc} W(A) & \xrightarrow{\mathrm{phant}_n} & A \\ \downarrow & & \downarrow \\ W(A/pA) & \xrightarrow[\overline{\mathrm{phant}}_n]{} & A/p^{n+1}A \end{array}$$

in which $\overline{\mathrm{phant}}_n$ must also be a ring homomorphism. But the composition

$$W(A^\flat) \xrightarrow{W(\varphi^{-n})} W(A^\flat) \xrightarrow{W(\#\bmod p)} W(A/pA) \xrightarrow{\overline{\mathrm{phant}}_n} A/p^{n+1}A$$

is exactly $\theta \bmod p^{n+1}$:

$$\sum_{i=0}^{\infty}[b_i]p^i = (b_0, b_1^p, b_2^{p^2}, \dots) \mapsto (b_0^{p^n}, b_1^{p^{1-n}}, b_2^{p^{2-n}}, \dots) \mapsto (b_0^{p^n\#}, b_1^{p^{1-n}\#}, b_2^{p^{2-n}\#}, \dots) \mapsto \sum_{i=0}^{n} p^i b_i^{p^{i-n}\#p^{n-i}} = \sum_{i=0}^{n} p^i b_i^\#$$

Since the first two maps are ring homomorphisms (they are the maps on $W(-)$ induced by the ring homomorphisms $\varphi^{-n} : A^\flat \to A^\flat$ and $\# : A^\flat \to A/pA$), we deduce that $\theta \bmod p^{n+1}$ is a ring homomorphism. But this is true for all n and A is separated for the p-adic topology, so this means θ is a ring homomorphism.

(ii): Since A and $W(A^\flat)$ are p-adically complete, to prove surjectivity of θ it is enough to show that it is surjective mod p. But this follows from the fact that $\# \bmod p : A^\flat \to A/pA$ is surjective.

Now we construct a possible generator ξ of $\operatorname{Ker}\theta$. Let $\pi \in A$ be a perfectoid pseudo-uniformiser admitting p-power roots, and let $\pi^\flat = (\pi, \pi^{1/p}, \dots)$ be the associated perfectoid pseudo-uniformiser of $A^\flat$. Since $p \in \pi^p A$ and θ has been shown to be surjective, we may write $p = \pi^p\theta(-z)$ for some $z \in W(A^\flat)$, whence $\xi := p + [\pi^\flat]^p z \in \operatorname{Ker}\theta$. Note that ξ is a non-zero-divisor of $W(A^\flat)$, by applying Remark 1.8(vi) with $k = A^\flat$, $t = \pi^\flat$, $q = \xi$. We next show $\operatorname{Ker}\theta = \xi W(A^\flat)$. Since $W(A^\flat)$ is $[\pi^\flat]$-adically complete and A is $\theta([\pi^\flat]) = \pi$-torsion-free, one easily sees that $\theta : W(A^\flat)/\xi W(A^\flat) \to A$ is an isomorphism if and only if it becomes an isomorphism when we mod out by $[\pi^\flat]$; i.e., we must check that $\theta : W(A^\flat)/(\xi, [\pi^\flat]) \to A/\pi A$ is an isomorphism. But, using $\xi \equiv p \bmod [\pi^\flat]$, this map identifies with $A^\flat/\pi^\flat A^\flat \overset{\#\bmod \pi}{\longrightarrow} A/\pi A$, which we saw was an isomorphism in the proof of Lemma 1.7(iii).

(iii): First note that the Witt vector expansion of our element ξ looks like

$$(\xi_0, \xi_1, \dots) = p + [\pi^\flat]^p x = (0, 1, 0, 0, \dots) + (\pi^{\flat p} x_0, \pi^{\flat p^2} x_1, \dots) = (\pi^{\flat p} x_0, 1 + \pi^{\flat p^2} x_1, \dots),$$

in particular $\xi_1 \in A^{\flat\times}$ (using $\pi^\flat$-adic completeness of $A^\flat$) and $\xi_0 \in \pi^\flat A^\flat$. Now let $\chi = (\chi_0, \chi_1, \dots) \in \operatorname{Ker}\theta$ be another element, and write $\chi = \beta\xi$ for some $\beta = (\beta_0, \beta_1, \dots) \in W(A^\flat)$. Expanding,

$$\chi = \beta\xi = (\beta_0, \beta_1, \dots)(\xi_0, \xi_1, \dots) = (\beta_0\xi_0, \beta_1\xi_0^p + \beta_0^p\xi_1, \dots).$$

Therefore:

$$\begin{aligned}
\operatorname{Ker}\theta = \xi W(A^\flat) &\iff \xi W(A^\flat) = \beta\xi W(A^\flat) \\
&\iff \beta \in W(A^\flat)^\times \text{ (using } \xi \text{ is a n-z-d)} \\
&\iff \beta_0 \in A^{\flat\times} \text{ (using that } W(A^\flat) \text{ is } p\text{-adically complete and } W(A^\flat)/pW(A^\flat) = A^\flat) \\
&\iff \beta_0^p\xi_1 \in A^{\flat\times} \text{ (since we already know } \xi_1 \in A^{\flat\times}) \\
&\iff \beta_1\xi_0^p + \beta_0^p\xi \in A^{\flat\times} \text{ (since } A^\flat \text{ is } \pi^\flat\text{-adically complete and } \xi_0 \in \pi^\flat A^\flat) \\
&\iff \xi \in A^{\flat\times},
\end{aligned}$$

completing the proof of part (iii). □

1.3 Tilting correspondence for integral perfectoid rings

We are now prepared to establish the easiest tilting correspondance,[3] namely that for integral perfectoid rings.

Given an integral perfectoid ring A and an A-algebra B, we always equip B with the canonical topology induced by A, i.e., we give B the π-adic topology where $\pi \in A$ is any perfectoid pseudo-uniformiser (this topology on B does not depend on the chosen π); we say that B is a *perfectoid A-algebra* if and only if B (equipped with the just-defined topology) is an integral perfectoid ring. Note that if B is a perfectoid A-algebra and $\pi \in A$ is any perfectoid pseudo-uniformiser, the (image in B of) π is also a perfectoid pseudo-uniformiser of the perfectoid ring B; this follows from Lemma 1.4.

Theorem 1.10 (Tilting correspondence for integral perfectoid rings)**.** *Fix an integral perfectoid ring A. Then tilting induces an equivalence of categories*

$$\textit{perfectoid } A\textit{-algebras} \xrightarrow{\simeq} \textit{perfectoid } A^\flat\textit{-algebras}, \qquad B \mapsto B^\flat,$$

with inverse given by sending a perfectoid $A^\flat$-algebra C to $C^\# := W(C) \otimes_{W(A^\flat),\theta} A$.

[3] correspondance de basculement

Proof. Let $\pi \in A$ be a perfectoid pseudo-uniformiser admitting p-power-roots, and $\pi^\flat$ the associated perfectoid pseudo-uniformiser of $A^\flat$; also let $\xi = p + [\pi^\flat]^p z \in W(A^\flat)$ be the generator of the ideal $\operatorname{Ker}(\theta : W(A^\flat) \to A)$ which we constructed in the proof of Theorem 1.9.

Step 1: Letting B be a perfectoid A-algebra, we show $(B^\flat)^\# = B$. We obviously have a commutative diagram (with surjective horizontal arrows by Theorem 1.9(ii))

$$\begin{array}{ccc} W(B^\flat) & \xrightarrow{\theta_B} & B \\ \uparrow & & \uparrow \\ W(A^\flat) & \xrightarrow[\theta=\theta_A]{} & A \end{array}$$

and so the image of ξ in $W(B^\flat)$ lands in $\operatorname{Ker}\theta_B$ (this denotes the θ-map for the integral perfectoid ring B). But the first coordinate in the Witt vector expansion of ξ is a unit of $A^\flat$ (by Theorem 1.9(iii)), and so its image in $B^\flat$ is also a unit; therefore Theorem 1.9(iii) (this time for the ring B) implies that $\operatorname{Ker}\theta_B = \xi W(A^\flat)$. In other words, the above diagram is a pushout and so the induced map $(B^\flat)^\# = W(B^\flat) \otimes_{W(A^\flat),\theta} A \to B$ is an isomorphism, as required.

Step 2: Letting C be a perfectoid A-algebra, we show that $C^\#$ is a perfectoid $A^\flat$-algebra and that $(C^\#)^\flat = C$. Since θ is surjective with kernel $\xi W(A^\flat)$, we can write $C^\# = W(C)/\xi W(C)$ viewed as an A-algebra via the identification $\theta : W(A^\flat)/\xi W(A^\flat) \xrightarrow{\simeq} A$. Remark 1.8(vi) (with $k = C$, $t = \pi^\flat$, $q = \xi$) therefore shows that $C^\#$ is complete for the π-adic topology and that π is a non-zero-divisor of C. It remains to show that $\Phi : C^\#/\pi C^\flat \to C^\#/\pi^p C^\flat$ is an isomorphism. But again writing $C^\# = W(C)/\xi W(C)$ and recalling that $\xi \equiv p \bmod [\pi^\flat]^p$, this map may be rewritten as $\Phi : C/\pi^\flat C \to C/\pi^{\flat p} C$, which is indeed an isomorphism since $\pi^\flat$ is a perfectoid pseudo-uniformiser of C. This completes the proof that $C^\#$ is a perfectoid A-algebra.

Finally, as we already used in the previous paragraph, we have $C^\#/\pi C^\# = C/\pi^\flat C$. Tilting obtains

$$(C^\#)^\flat = \varprojlim_{x\mapsto x^p} C^\#/\pi C^\# = \varprojlim_{x\mapsto x^p} C/\pi^\flat C = C^\flat = C,$$

where the final equality is the fact that tilting an integral perfectoid ring of characteristic p has no effect. □

1.4 Anneaux perfectoïdes entiers et basculement : exercices et exemples

(i) Soit C une extension algébrique de $\mathbb{Q}_p$, et $\mathcal{O}_C$ la clôture intégrale de $\mathbb{Z}_p$ dans C. Supposons que tout élément de $\mathcal{O}_C/p\mathcal{O}_C$ admette une racine p-ième et qu'il existe $\pi \in \mathcal{O}_C$ qui n'est pas une unité tel que $p \in \pi^p \mathcal{O}_C$ (par exemple $C = \mathbb{Q}_p(\zeta_{p^\infty})$, $\mathbb{Q}_p(p^{1/p^\infty})$ ou $\overline{\mathbb{Q}}_p$).

Démontrer que $\Phi : \mathcal{O}_C/\pi\mathcal{O}_C \to \mathcal{O}_C/\pi^p\mathcal{O}_C$, $a \mapsto a^p$ est un isomorphisme et en déduire que le complété p-adique $\widehat{\mathcal{O}_C}$ de $\mathcal{O}_C$ est un anneau perfectoïde entier.

(ii) Soit A un anneau perfectoïde entier. Son "algèbre de polynômes perfectoïde" est

$$A\langle T^{1/p^\infty}\rangle := \text{le complété } \pi\text{-adique de } \bigcup_{i\geq 1} A[T^{1/p^i}]$$

où π est n'importe quelle pseudo-uniformisante perfectoïde de A. Démontrer que $A\langle T^{1/p^\infty}\rangle$ est un anneau perfectoïde entier (en fait une A-algèbre perfectoïde).

On remarque que $A\langle T^{1/p^\infty}\rangle^\flat$ (càd le basculé de $A\langle T^{1/p^\infty}\rangle$) contient un élément $T^\flat := (T, T^{1/p}, T^{1/p^2}, \dots)$. Construire un isomorphisme de $A^\flat$-algèbres perfectoïdes

$$A^\flat\langle U^{1/p^\infty}\rangle \xrightarrow{\simeq} A\langle T^{1/p^\infty}\rangle^\flat$$

qui envoie U sur $T^\flat$, où le membre de gauche est une algèbre de polynômes perfectoïde sur $A^\flat$.

Démontrer les résultats analogues pour plusieurs variables $T_1, \dots, T_d$.

(iii) Soit A un anneau perfectoïde entier et B une A-algèbre étale. Démontrer que le complété π-adique $\widehat{B}$ est une A-algèbre perfectoïde. (Le fait suivant sera utile : si $k \to k'$ est un morphisme étale entre des anneaux de caractéristique p, alors le diagramme

$$\begin{array}{ccc} k & \longrightarrow & k' \\ {\scriptstyle\varphi}\uparrow & & \uparrow{\scriptstyle\varphi} \\ k & \longrightarrow & k' \end{array}$$

est un pushout, càd $k' \otimes_{k,\varphi} k \xrightarrow{\simeq} k'$.)

(iv) Soit R une $\mathcal{O}_C$-algèbre lisse et supposons qu'il existe un morphisme étale $\mathcal{O}_C[T_1, \dots, T_d] \to R$ (si R est équi-dimensionnelle, cette hypothèse est toujours satisfaite localement sur Spec R). Utiliser (ii) et (iii) pour construire une $\widehat{\mathcal{O}_C}$-algèbre perfectoïde R_∞.

2 (A) Huber rings

The topological rings from which we will build adic spaces are Huber rings:

Definition 2.1. A *Huber ring* (or *f-adic ring* in the older terminology) is a topological ring R which satisfies the following: there exist an open subring $R_0 \subseteq A$ and a finitely generated ideal I of R_0 such that the topology on R_0 is the I-adic topology. (Warning: I is not usually an ideal of R!) Any such pair $I \subseteq R_0$ is called an *ideal of definition* and *subring of definition* of R.

Example 2.2. The simplest, but least interesting example, is as follows. Let R be a ring and $I \subseteq R$ a finitely generated ideal (even the case $I = 0$ is allowed). Then R is a Huber ring with ideal and subring of definition $I \subseteq R_0 := R$. See Example 2.5 for more interesting examples.

Note that the topology on R is uniquely determined by R_0 and I. However, the converse is more subtle. If R is a ring (without topology), $R_0 \subseteq R$ is a given subring, and I is an ideal of R_0, then there is obviously a uniquely linear topology on R for which R_0 is open in R and the induced topology on R_0 is the I-adic topology. Indeed, the unique linear topology with these properties has basis $f + I^m$, for $f \in R$ and $m \geq 1$. However, if we equip R with this topology, then it is not necessarily a topological ring (more precisely, multiplication is not necessarily continuous)! We leave it to the reader to construct such an example. Therefore the definition of a Huber ring involves a subtle compatibility between between the topology and the algebra.

However, as we will see in Proposition 2.4, this subtlety does not occur when constructing Tate rings.

Definition 2.3. A *Tate ring* R is a Huber ring with the following additional property: there exists a unit $\pi \in R^\times$ such that $\pi^n \to 0$ as $n \to \infty$. Any such element π is called a *pseudo-uniformiser* of R.

Proposition 2.4. *(i) Let A be a ring and $\pi \in A$ a non-zero-divisor; equip $A[\frac{1}{\pi}]$ with the topology having basis $f + \pi^m A$ for $f \in A[\frac{1}{\pi}]$, $m \geq 1$. Then $A[\frac{1}{\pi}]$ is a Tate ring with ideal and subring of definition $\pi A \subseteq A$.*

(ii) Conversely, let R be a Tate ring; chose a subring of definition R_0 and a pseudo-uniformiser $\pi \in R$. Then there exists $m \geq 1$ such that $\pi^m \in R_0$; moreover, then $\pi^m R_0$ is an ideal of definition, π^m is a non-zero-divisor of R_0, and $R = R_0[\frac{1}{\pi^m}]$ with topology as in (i).

Proof. (i): Exercise. (ii): Since R_0 is open and π is topologically nilpotent, there exists $m \geq 1$ such that $\pi^m \in R_0$. Since π^m is a unit of the topological ring R and R_0 is open, the ideal $\pi^m R_0$ is also open. Next let $I \subseteq R_0$ be an ideal of definition; since $\pi^m R_0$ is open, we have $I^n \subseteq \pi^m R_0$ for $n \gg 0$. Conversely, since π^m is topologically nilpotent, we also have $\pi^{mn} \in I$ for $n \gg 0$. This shows that $\pi^m R_0$ is also an ideal of definition.

Clearly π^m is a non-zero-divisor of R_0 (since it is a unit in the larger ring R). Moreover, given $f \in R$, continuity of multiplication implies $\pi^{mn} f \to 0$ as $n \to \infty$, so $\pi^{mn} f \in R_0$ for $n \gg 0$; this shows $R = R_0[\frac{1}{\pi^m}]$. □

Example 2.5. Here are some example of Tate rings.

(i) If A is an integral perfectoid ring and $\pi \in A$ is a perfectoid pseudo-uniformiser, then $A[\frac{1}{\pi}]$ is a Tate ring by Proposition 2.4(i). We will study such *perfectoid Tate rings* in Section 3.

(ii) $\mathbb{Q}_p$ is a Tate ring, with ideal and subring of definition $p\mathbb{Z}_p \subseteq \mathbb{Z}_p$.

(iii) More generally, if A is any flat $\mathbb{Z}_p$-algebra then we may equip it with the p-adic topology and form a Tate ring $A[\frac{1}{p}]$

(iv) $\mathbb{Q}_p\langle T\rangle := \{\sum_{i=0}^{\infty} a_i T^i : a_i \in \mathbb{Q}_p, a_i \to 0\} = \widehat{\mathbb{Z}_p[T]}[\frac{1}{p}]$ (where $\widehat{}$ denotes the p-adic completion).

2.1 Bounded sets

Definition 2.6. Let R be a Huber ring. A subset $S \subseteq R$ is called *bounded* if and only if for each open neighbourhood $0 \in U \subseteq R$ there exists an open neighbourhood $0 \in V \subseteq R$ such that $sv \in U$ for all $s \in S$ and $v \in V$.

We will always use the following notation: given subsets $T, S \subseteq A$, we write $ST \subseteq A$ for the subgroup generated by all products st, for $s \in S$, $t \in T$.

Lemma 2.7. *Let R be a Huber ring and $I \subseteq R_0 \subseteq R$ an ideal and a subring of definition. Then $S \subseteq R$ is bounded if and only if there exists $n \geq 1$ such that $SI^n \subseteq R_0$.*

Proof. $\Rightarrow$: Suppose $S \subseteq R$ is bounded. Since R_0 is open, there exists an open nhood $0 \in V \subseteq R$ such that $sV \subseteq R_0$ for all $s \in S$. But $0 \in I^n \subseteq V$ for $n \gg 0$, whence $sI^n \subseteq R_0$ for all $s \in S$ and so $SI^n \subseteq A_0$ since R_0 is closed under addition.

$\Leftarrow$: Suppose there exists $n \geq 1$ such that $SI^n \subseteq R_0$. Then, for any open $0 \in U \subseteq R$, pick $m \geq 1$ such that $0 \in I^m \subseteq U$; then $V := I^{n+m}$ works, since clearly $SI^{n+m} \subseteq R_0 I^m \subseteq I^m \subseteq U$. □

Corollary 2.8. *Let R be a Huber ring. If $S, T \subseteq R$ are bounded, then so are $S + T$, $S \cup T$, and ST. Any finite subset of R is bounded. If R_0 is a subring of definition and $M \subseteq R$ is a finitely generated R_0-module, then M is bounded.*

Proof. This is all easy using Lemma 2.7. For example, pick $m \geq 1$ such that $I^m S \subseteq R_0$ and $I^m T \subseteq R_0$. Then $I^{2m} ST = I^m I^m ST \subseteq I^m R_0 T = I^m T \subseteq R_0$. For the finite set claim it is enough to show that singletons are bounded, i.e., given $f \in R$ there exists $m \geq 1$ such that $fI^m \subseteq R_0$; this follows from the fact that R_0 is an open neighbourhood of 0, that powers of I are an open nhood basis at 0, and that multiplication $R \times R \to R$ is continuous.

For the finitely generated module note that $M = SR_0$ for some finite set S, so we can apply the previous parts. □

The following is the first result where we use the fact that the ideal of definition is required to be finitely generated:

Corollary 2.9. *Let R be a Huber ring and $f_1, \dots, f_n \in R$ some elements which generate an open ideal, i.e., $f_1R + \cdots + f_nR$ is open. Let R_0 be a subring of definition. Then $f_1R_0 + \cdots + f_nR_0$ is open.*

Proof. Let $I \subseteq R_0$ be an ideal of definition. By hypothesis $I^m \subseteq f_1R + \cdots + f_nR$ for some $m \geq 1$. But I^m is a finitely generated R_0-module, so clearly there exists a finitely generated R_0-module $M \subseteq A$ such that $I^m \subseteq f_1R + \cdots + f_nR$. But M is bounded by the previous corollary, so $I^{m'} M \subseteq R_0$ for some $m' \geq 1$. Then $I^{m+m'} \subseteq f_1R_0 + \cdots + f_nR_0$. □

The following is fundamental:

Proposition 2.10. *Let R be a Huber ring and $A \subseteq R$ a subring. Then A is a ring of definition if and only if it is bounded and open in R.*

Proof. $\Rightarrow$: Suppose A is a ring of definition. Then it is open by definition, and bounded by Lemma 2.7 (take $R_0 = A$ and $n = 1$).

$\Leftarrow$: Suppose A is bounded and open. Let $I \subseteq R_0 \subseteq A$ be an ideal and a subring of definition; let $T \subseteq R_0$ be a finite set such that $I = TR_0$. Since A is open and bounded, there exists (using Lemma 2.7) $m \geq 1$ such that $I^m \subseteq A$ and $I^m A \subseteq R_0$.

Let $J := T^m A$ be the ideal of A generated by the finite set $T^m \subseteq A$. Then $J^2 = T^m T^m A \subseteq T^m R_0 = I^m$ and $J = T^m A \supseteq T^m I^m = I^{2m}$.

Therefore powers of J and powers of I define the same topology on R, so $J \subseteq A$ are indeed an ideal and a subring of definition. □

Corollary 2.11. *Let R be a Huber ring.*

(i) If $R_0, R_0' \subseteq A$ are subrings of definition, then so are R_0R_0' and $R_0 \cap R_0'$ (note that these really are subrings; also note that $R_0R_0' \supseteq R_0, R_0'$); in particular, the subrings of definition form a filtered family.

(ii) Suppose that $A \subseteq B$ are subrings of R such that A is bounded and B is open. Then there exists a subring of definition R_0 such that $A \subseteq A_0 \subseteq B$.

Proof. (i) Easy exercise: check that R_0R_0' and $R_0 \cap R_0'$ are both open and bounded, then apply the previous proposition.

(ii) Let R_0 be any subring of definition; then R_0B is open and bounded, so $R_0B \cap C$ is also open and bounded, hence a subring of definition by the previous proposition. □

Definition 2.12. Let R be a Huber ring. An element $f \in R$ is called *power bounded* if and only if the set $f^{\mathbb{N}} := \{f^n : n \geq 0\}$ is bounded. Let $R^\circ \subseteq R$ be the subset of power bounded elements.

An element $f \in R$ is called *topologically nilpotent* if and only if $f^n \to 0$ as $n \to \infty$. Let $R^{\circ\circ} \subseteq R$ be the subset of topologically nilpotent element.

Lemma 2.13. *Let R be a Huber ring.*

(i) R° is an open subring, integrally closed in R.

(ii) R° is the union of all subrings of definition.

(iii) $R^{\circ\circ}$ is an open ideal of R°.

Proof. (i): Let $f, g \in R^\circ$. Then $f^{\mathbb{N}} g^{\mathbb{N}}$ is bounded by Corollary 2.8. But $f^{\mathbb{N}} g^{\mathbb{N}} \supseteq \{(f+g)^n, (fg)^n : n \geq 0\}$, and therefore $f+g$ and fg are also power bounded.

Next suppose that $x \in R$ is integral over R°, so that there are $a_0, \ldots, a_{d-1} \in R^\circ$ such that $x^d + a_{d-1}x^{d-1} + \cdots + a_0 = 0$. Then it is easy to see that $x^{\mathbb{N}} \subseteq a_0^{\mathbb{N}} \cdots a_{d-1}^{\mathbb{N}} \{1, x, \ldots, x^{d-1}\}$. The set on the right is bounded by Corollary 2.8, whence $f^{\mathbb{N}}$ is also bounded, i.e., $f \in R^\circ$.

Given any subring of definition A_0 and $f \in R_0$, we have $f^{\mathbb{N}} \subseteq R_0$ and so $f^{\mathbb{N}}$ is bounded (since R_0 is bounded); therefore $R_0 \subseteq R^\circ$, showing that R° is open.

(ii): We have just noted that any subring of definition R_0 is contained in R°. Conversely, supposing $f \in R^\circ$, we must find a subring of definition containing f. Let R_0 be any subring of definition, and note that $R_0 f^{\mathbb{N}}$ is R_0-subalgebra of R generated by f; this is open (since it containes the open subring R_0) and bounded (by Corollary 2.8), hence is a subring of definition (by Proposition 2.10).

(iii): Exercise. □

It is important to note that R° is not necessarily bounded (e.g., Consider the Tate ring $R = \mathbb{Q}_p[\varepsilon]/\varepsilon^2$, with subring of definition $R_0 = \mathbb{Z}_p[\varepsilon]/\varepsilon^2$ equipped with the p-adic topology. Then any multiple of ε is nilpotent, hence power bounded, and so $R^\circ = \mathbb{Z}_p + \mathbb{Q}_p\varepsilon$; in particular $p^n R^\circ \not\subseteq R_0$ for all $n \geq 1$, whence R° is not bounded.). We say that the Huber ring R is *uniform* if and only if R° is bounded; this will turn out to be the case for perfectoid Tate rings.

Definition 2.14. Let R be a Huber ring. A subring R^+ is called a *subring of integral elements* if and only if it is open, integrally closed subring in R, and $R^+ \subseteq A^\circ$. For example, Lemma 2.13(i) clearly implies that R° is a subring of integral elements (moreover, it is the largest subring of integral elements).

A *Huber pair* (or *affinoid ring* in the older terminology) (R, R^+) is the data of a Huber ring R and a chosen subring of integral elements R^+.

3 (P) Perfectoid Tate rings

As explained in Proposition 2.4, if we are given an integral perfectoid ring A we can construct a Tate ring $A[\frac{1}{\pi}]$, where π is any perfectoid pseudo-uniformiser of A. Note that $A[\frac{1}{\pi}]$ does not depend on the choice of π: given another π', the elements π and π' define the same topology on A, hence each divides a power of the other, and so $A[\frac{1}{\pi}] = A[\frac{1}{\pi}']$; in fact, $A[\frac{1}{\pi}]$ can be written without making any choices as

$$A[\tfrac{1}{\pi}] = A[\tfrac{1}{f} : f \in A \text{ and } fA \text{ is open in } A].$$

In any case, the Tate ring $A[\frac{1}{\pi}]$ is called the *generic fibre* of A, and so we have defined a functor

$$\text{integral perfectoid rings} \longrightarrow \text{Tate rings}, \qquad A \mapsto A[\tfrac{1}{\pi}]$$

The image of this functor is precisely the perfectoid Tate rings:

Definition 3.1. A Tate ring R is called *perfectoid* if and only if the following equivalent conditions are satisfied:

(i) R has a subring of definition which is an integral perfectoid ring;

(ii) R is in the image of the above functor;

(iii) the topological ring R° is integral perfectoid;

(iv) R is uniform and there exists a pseudo-uniformiser $\pi \in R$ such that $p \in \pi^p R^\circ$ and $\Phi : R^\circ/\pi R^\circ \to R^\circ/\pi^p R^\circ$, $f \mapsto f^p$ is an isomorphism (Fontaine's Bourbaki definition).

It is convenient to prove the equivalences at the same time as the following proposition:

Proposition 3.2. *Let R be a perfectoid Tate ring and $R_0 \subseteq R$ a subring of definition. Then R_0 is integral perfectoid if and only if it is p-closed in R (i.e., "$f \in R$ and $f^p \in R_0 \Rightarrow f \in R_0$"). In particular, every subring of integral elements $R^+ \subseteq R$ is integral perfectoid.*

Proof of equivalences and the proposition. (iv)⇒(i): If R is uniform then R° is a subring of definition; since $\pi \in R^\circ$, Proposition 2.4 shows that the topology on R° is the π-adic topology, and the other conditions in (iv) show that R° is integral perfectoid.

(i)⇒(ii): Suppose that $R_0 \subseteq R$ is an integral perfectoid subring of definition. Let $\pi \in R_0$ be a perfectoid pseudo-uniformiser and note that π is necessarily a pseudo-uniformiser of the Tate ring R (The proof is standard theory of Tate rings, similar to Proposition 2.4: Firstly, π is topologically nilpotent since it defines the topology on R_0; secondly, fixing a pseudo-uniformiser $\varpi \in R$, the openness of πR_0 implies $\varpi^n \in \pi R_0$ for $n \gg 0$, whence π is a unit in R.). Proposition 2.4 shows that $R = R_0[\frac{1}{\pi}]$, i.e., R is in the image of the above functor.

(iii)⇒(iv): If R° is integral perfectoid then it is an open subring whose topology is adic for a finitely generated ideal, whence it is a subring of definition; but any subring of definition is bounded, so this shows that R is uniform. Any choice of perfectoid pseudo-uniformiser π for R° is a pseudo-uniformiser for R (by the same argument as in the previous paragraph) with the desired properties in (iv).

To complete the proof of the equivalences, we must show (ii)⇒(iii), so suppose that $R = A[\frac{1}{\pi}]$, where A is an integral perfectoid ring and $\pi \in R$ is a pseudo-uniformiser. We note first that $R^{\circ\circ} \subseteq A$: indeed, if $f \in R$ is topologically nilpotent then $f^{p^n} \in A$ for $n \gg 0$, and so $f \in A$ by Lemma 3(ii). In particular this shows that $\pi R^\circ \subseteq A$; therefore R° is bounded, i.e., R is uniform.

Now let $R_0 \subseteq R$ be any p-closed subring of definition (e.g., $R_0 = R^\circ$, since we have just shown R° is bounded, hence is a subring of definition); we will prove that R_0 is integral perfectoid. Just as in the previous paragraph, p-closedness implies that $R^{\circ\circ} \subseteq A$; in particular $\pi \in R_0$, whence the topology on R_0 is the π-adic topology by Proposition 2.4 (this proves condition (a) in the definition of integral perfectoid).

We claim that every element of R_0 is a p^{th}-power modulo π (resp. $\pi^{1/2}$ if $p = 2$); let $f \in R_0$. Then $\pi f \in R^{\circ\circ} \subseteq A$ and so there exist $y, z \in A$ such that $\pi x = y^p + \pi^p z$; after multiplying π by a unit we may assume it admits a p^{th}-root in A, and we then deduce that $(y\pi^{-1/p})^p = x - \pi^{p-1}z \in R_0$ (note that $\pi^{p-1}z$ is topologically nilpotent, hence in R_0), whence $y\pi^{-1/p} \in R_0$ by topological nilpotence again. Since $\pi^{p-1}z \in \pi R_0$ (unless $p = 2$, in which case it is $\in \pi^{1/2}R_0$ since $\pi^{1/2}z \in R_0$), we have shown that x is a p^{th}-power modulo πR_0 (resp. $\pi^{1/2}R_0$), which proves the claim.

Next note that $p \in (\pi^{1/p})^p R_0$: indeed, we know that $p \in \pi^p A \subseteq \pi R^{\circ\circ}$ and that $R^{\circ\circ} \subseteq R_0$. Since the p-closedness of R_0 in R easily implies that $\Phi : R_0/\pi^{1/p}R_0 \to R_0/\pi R_0$ (resp. $R_0/\pi^{1/4}R_0 \to R_0/\pi^{1/2}R_0$) is injective, we have indeed proved that R_0 is integral perfectoid (with pseudo-uniformiser $\pi^{1/p}$, resp. $\pi^{1/4}$ if $p = 2$).

In conclusion, assuming condition (ii), we have proved that R° is integral perfectoid, and more generally that any p-closed subring of definition is integral perfectoid. This completes the proof that the conditions in Definition 3.1 are equivalent, and establishes the implication $\Leftarrow$ in the proposition; meanwhile, the implication $\Rightarrow$ is a consequence of (ii). For the final sentence in the proposition just note that, since R is uniform, any subring of integral elements is a subring of definition. □

Corollary 3.3. *Let R be a perfectoid Tate ring. Then any integral perfectoid subring of definition R_0 contains $R^{\circ\circ}$, and the resulting functor $R_0 \mapsto R_0/R^{\circ\circ}$ defines a bijection*

$$\{\textit{integral perfectoid subrings of definition of } R\} \xrightarrow{\simeq} \{p\textit{-closed subrings of } R^\circ/R^{\circ\circ}\}$$

This restricts to a bijection

$$\{\textit{subrings of integral elements of } R\} \xrightarrow{\simeq} \{\textit{integrally closed subrings of } R^\circ/R^{\circ\circ}\}$$

Proof. By basic commutative algebra, $A \mapsto A/R^{\circ\circ}$ defines a bijection

$$\{\text{subrings of } R^\circ \text{ containing } R^{\circ\circ}\} \xrightarrow{\simeq} \{\text{subrings of } R^\circ/R^{\circ\circ}\}.$$

Moreover, the reader can easily check that, given a ring A such that $R^{\circ\circ} \subseteq A \subseteq R^\circ$, then A is p-closed (resp. integrally closed) in R° if and only if $A/R^{\circ\circ}$ is p-closed (resp. integrally closed) in $R^\circ/R^{\circ\circ}$.

To complete the proof, it remains only to use Proposition 3.2 and the observation (which already appeared in the previous proof) that any open p-closed subring of R must contain $R^{\circ\circ}$. □

Remark 3.4. Similarly to Lemma 1.5, one can prove the following: Given a complete Tate ring R of characteristic p, then R is perfectoid if and only if it is perfect. By using Lemma 1.5, the only non-trivial part of the proof is the following: assuming that R is perfect, we will show that R is uniform. This was noticed first by Yves André, whose proof we give here.

Let $R_0 \subseteq R$ be a subring of definition, and $\pi \in R_0$ a pseudo-uniformiser of R. For each $n \geq 1$ set $R_n := \varphi^{-n}(R_0) = \{f \in R : f^{p^n} \in R_0\}$, which is a subring since R has characteristic p. The Frobenius morphism $\varphi : R \to R$ is a continuous bijection (since R is assumed to be perfect), hence is a homeomorphism by Banach's open mapping theorem (it is folklore that Banach's open mapping theorem holds in the generality of complete Tate rings; see, e.g., Henkel "An open mapping theorem..."). Therefore $\varphi(R_0)$ is an open subring of R, so there exists $m \geq 1$ such that $\pi^m R_0 \subseteq R_0$; applying φ^{-n} shows that $\pi^{m/p^n} A_n \subseteq A_{n-1}$ for all $n \geq 1$. By a trivial induction it this means that $\pi^{\sum_{i=1}^n m/p^i} A_n \subseteq A_0$, whence $\pi^m A_n \subseteq A_0$ for all $n \geq 0$ (since $m \geq \sum_{i=1}^n m/p^i$). Next, given $f \in R^\circ$, the set $f^{\mathbb{N}}$ is bounded and so there exists $n \geq 1$ such that $\pi^{p^n} f^{\mathbb{N}} \subseteq R_0$; in particular $\pi^{p^n} f^{p^n} \in R_0$, i.e., $\pi f \in R_n$ and so $\pi^{m+1} f \in \pi R_n \subseteq R_0$. This shows that $\pi^{m+1} R^\circ \subseteq R_0$, i.e., R° is bounded.

3.1 Aside: The language of almost mathematics

Before we can discuss tilting perfectoid Tate rings in the next subsection, it is useful to introduce some language of "almost mathematics".

Throughout this subsection we let A be an integral perfectoid ring, and $A^{\circ\circ} \subseteq A$ the open ideal consisting of topologically nilpotent elements. (Exercise: letting $\pi \in A$ be any perfectoid pseudo-uniformiser admitting compatible p-power roots $\pi^{1/p}, \pi^{1/p^2}, \dots$, check that $A^{\circ\circ} = \bigcup_{n \geq 0} \pi^{1/p^n} A$. In particular, this shows that the ideal $A^{\circ\circ}$ is its own square, which is key to the following theory.)

We say that an A-module M is *almost zero* if and only if $A^{\circ\circ} M = 0$; by the exercise, this is equivalent to saying that $\pi^{1/p^n} M = 0$ for all $n \geq 0$. Similarly, we say that a map of A-modules $M \to N$ is an *almost injection/surjection/isomorphism* if and only if the kernel/cokernel/both is almost zero.

The following should look surprising at first glance (it is not true if we replace $A^{\circ\circ}$ by an arbitrary ideal):

Lemma 3.5. *The category of almost zero A-modules is closed under the following operations: sub-modules, quotients, extensions, all limits, all colimits. Given a π-adically complete A-module M, then M is almost zero if and only if $M/\pi M$ is almost zero. Moreover, almost isomorphisms are closed under base change along an arbitrary module.*

Proof. The surprising fact is extensions: suppose that $0 \to M \to N \to P \to 0$ is a short exact sequence of A-modules such that M and P are killed by $A^{\circ\circ}$; the N is killed by the ideal $(A^{\circ\circ})^2$, but we have noted above that $(A^{\circ\circ})^2 = A^{\circ\circ}$. We leave it to the reader as an exercise in almost mathematics to check the other assertions. ☐

Lemma 3.6. *Let B be a perfectoid A-algebra. Then $A \to B$ is an almost isomorphism if and only if $A^\flat \to B^\flat$ is an almost isomorphism.*

Proof. Let $\pi \in A$ be a perfectoid pseudo-uniformiser admitting compatible p-power roots, and $\pi^\flat = (\pi, \pi^{1/p}, \dots) \in A^\flat$ the corresponding perfectoid pseudo-uniformiser of $A^\flat$. Recall from the discussion before Theorem 1.10 that π is automatically a perfectoid pseduo-uniformiser for B (and similarly $\pi^\flat$ is a ppu for $B^\flat$).

$\Rightarrow$: $A \to B$ being an almost isomorphism means that it is injective (since A has no π-torsion), so we may view B as an extension of A, and that $\pi^{1/p^n} B \subseteq A$ for all $n \geq 0$. From the injectivity it is clear that

$$A^\flat = \varprojlim_{x \mapsto x^p} A \longrightarrow \varprojlim_{x \mapsto x^p} B = B$$

is injective. Moreover, given an element $b = (b_0, b_1, \dots) \in B^\flat$ and $n \geq 1$, we have

$$\pi^{\flat 1/p^n} b = (\pi^{1/p^n}, \pi^{1/p^{n+1}}, \dots)(b_0, b_1, \dots) = (b_0 \pi^{1/p^n}, b_1 \pi^{1/p^{n+1}}, \dots) \in A^\flat,$$

showing that $A\flat \to B^\flat$ is almost surjective.

$\Leftarrow$: $A^\flat \to B^\flat$ being an almost isomorphism means that $A^\flat/\pi^\flat A^\flat \to B^\flat/\pi^\flat B^\flat$ is almost an almost isomorphism (by the base change assertion in Lemma 3.5). But we know from the proof of Lemma 1.7 that $A^\flat/\pi^\flat A^\flat = A/\pi A$ and $B^\flat/\pi^\flat B^\flat \to B/\pi B$; so we have shown that $A \to B$ is an almost isomorphism modulo π. Since A and B are complete, (a modification of) Lemma 3.5 implies $A \to B$ is an almost isomorphism. □

Lemma 3.7. *Let B be a perfectoid A-algebra. Then $A \to B$ is an almost isomorphism if and only if the induced map of perfectoid Tate rings $A[\frac{1}{\pi}] \to B[\frac{1}{\pi}]$ (i.e., the generic fibres) is an isomorphism.*

Proof. $\Rightarrow$: Suppose that $A \to B$ is an almost isomorphism. Then the kernel and cokernel are in particular killed by π, whence $A[\frac{1}{\pi}] \xrightarrow{\simeq} B[\frac{1}{\pi}]$ as desired.

$\Leftarrow$: Suppose that $A[\frac{1}{\pi}] \xrightarrow{\simeq} B[\frac{1}{\pi}]$. Denoting this common perfectoid Tate ring by R, we may therefore view $A \subseteq B$ as integral perfectoid subrings of definition of R. As we stated in Corollary 3.3, this implies $R^{\circ\circ} \subseteq A$; in particular, $\pi^{1/p^n} B \subseteq A$ for all $n \geq 1$, i.e., $A \to B$ is an almost isomorphism. □

3.2 Tilting perfectoid Tate rings

The tilt of a perfectoid Tate ring R is defined to be the perfectoid Tate ring of characteristic p given by

$$\begin{aligned} R^\flat :=& \text{generic fibre of } R_0^\flat \\ =& R_0^\flat[\tfrac{1}{\pi^\flat}] \end{aligned}$$

where $R_0 \subseteq R$ is any integral perfectoid subring of definition and $\pi \in R_0$ is a perfectoid pseudo-uniformiser with compatible p-power roots. This does not depend on the chosen subring of definition (or perfectoid pseudo-uniformiser): indeed, obviously the two integral perfectoid subrings $R_0 \subseteq R^\circ$ have the same generic fibre, whence Lemmas 3.6 and 3.7 show that the two integral perfectoid rings $R_0^\flat \subseteq R^{\circ\flat}$ also have the same generic fibre. In other words, we could canonically define $R^\flat$ to be the generic fibre of $R^{\circ\flat}$; from this point of view, we have just shown that any other integral perfectoid subring of definition $R_0 \subseteq R$ tilts to an integral perfectoid subring of definition $R_0^\flat \subseteq R^\flat$.

Theorem 3.8 (Tilting correspondence – lattice of subrings)**.** *Let R be a perfectoid Tate ring. Then $R^{\circ\flat}$ is the subring of power bounded elements of R (i.e., $R^{\circ\flat} = R^{\flat\circ}$). Moreover, tilting $R_0 \mapsto R_0^\flat$ defines a bijection*

$$\{\textit{integral perfectoid subrings of definition of } R\} \xrightarrow{\simeq} \{\textit{integral perfectoid subrings of definition of } R^\flat\},$$

which restricts to a bijection

$$\{\textit{subrings integral elements of } R\} \xrightarrow{\simeq} \{\textit{subrings of integral elements of } R^\flat\}.$$

Proof. We begin by proving that $R^{\circ\flat} = R^{\flat\circ}$. Since $R^{\circ\flat}$ is a subring of definition for $R^\flat$, certainly we have $R^{\circ\flat} \subseteq R^{\flat\circ}$ and so we may view $R^{\flat\circ}$ as a perfectoid $R^{\circ\flat}$-algebra. By Theorem 1.10, untilting gives us a perfectoid R°-algebra B such that $B^\flat = R^{\flat\circ}$. Since $R^\circ \to B$ induces an isomorphism on generic fibres after tilting (namely $R \xrightarrow{=} R$), Lemma 3.7 tells us that it induces an isomorphism on generic fibres before tilting, i.e., we have $R^\circ \subseteq B \subseteq R$ where B is a subring of definition of R. But B being a subring of R implies $B \subseteq R^\circ$, i.e., $R^\circ = B$; tilting reveals $R^{\circ\flat} = R^{\flat\circ}$, as desired.

Let $\pi \in R^\circ$ be a perfectoid pseudo-uniformiser admitting p-power roots, and $\pi^\flat \in R^{\circ\flat} = R^{\flat\circ}$ the associated perfectoid pseudo-uniformiser of $R^{\flat\circ}$. We know that the untilting map # induces an isomorphism $R^{\flat\circ}/\pi^\flat R^{\flat\circ} \xrightarrow{\simeq} R^\circ/\pi R^\circ$. Since $R^{\circ\circ}/\pi R^\circ$ is the ideal of nilpotent elements of $R^\circ/\pi R^\circ$, and similarly on the tilted side, we also get an isomorphism $R^{\flat\circ}/R^{\flat\circ\circ} \xrightarrow{\simeq} R^\circ/R^{\circ\circ}$.

The desired bijections up subrings now follows by applying Corollary 3.3 to both R and $R^\flat$. ☐

Finally we note that the tilting correspondence for integral perfectoid rings extends to the generic fibres:

Theorem 3.9 (Titling correspondence – perfectoid Tate rings)**.** *Let R be a perfectoid Tate ring. Then tilting $S \mapsto S^\flat$ defines an equivalence of categories*

$$\{\textit{perfectoid Tate rings over } R\} \xrightarrow{\simeq} \{\textit{perfectoid Tate rings over } R^\flat\}$$

Proof. To add. ☐