

# ANALOGUES OF GERSTEN'S CONJECTURE FOR SINGULAR SCHEMES

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ABSTRACT. We formulate analogues, for Noetherian local  $\mathbb{Q}$ -algebras which are not necessarily regular, of the injectivity part of Gersten's conjecture in algebraic  $K$ -theory, and prove them in various cases. Our results suggest that the algebraic  $K$ -theory of such a ring should be detected by combining the algebraic  $K$ -theory of both its regular locus and the infinitesimal thickenings of its singular locus.

## 1. INTRODUCTION

If  $A$  is a regular local ring, then Gersten's conjecture, which is a theorem if  $A$  contains a field, predicts that the map  $K_n(A) \rightarrow K_n(\text{Frac}A)$  is injective for all  $n \geq 0$ , where  $\text{Frac}A$  denotes the field of fractions of  $A$ . The aim of this note is to explore certain analogues of this injectivity conjecture in the case when  $A$  is singular, by taking into account nilpotent thickenings of its singular locus. In the case that  $A$  is one-dimensional, this problem was first considered by the first author when  $n = 2$  [4], and later by the second author in general [8]. In this note, we extend some of our earlier results to higher dimensions. We hope that our results indicate the existence of, and stimulate work towards, a more general, undiscovered framework for a form of Gersten's conjecture in the presence of singularities.

An illustrative example of such an analogous injection is our following result for cone singularities.

**Theorem 1.1.** *Let  $k$  be a field of characteristic zero and  $Y \hookrightarrow \mathbb{P}_k^N$  a smooth projective variety; let  $C \hookrightarrow \mathbb{A}_k^{N+1}$  be the cone over  $Y$  and  $(A, \mathfrak{m})$  the local ring at the unique singular point of  $C$ . Then, for any  $n \geq 0$ , the map*

$$K_n(A) \longrightarrow K_n(A/\mathfrak{m}^r) \oplus K_n(\text{Spec}(A) \setminus \{\mathfrak{m}\})$$

*is injective for  $r \gg 0$ .*

Unfortunately, as we shall see in Example 2.4, the conclusion of the theorem does not hold for general isolated singularities, even in dimensional one. To obtain a conjecture which is plausible in general, one should replace  $\text{Spec}(A) \setminus \{\mathfrak{m}\}$  by a resolution of singularities  $X \rightarrow \text{Spec}(A)$ , and then consider  $K$ -groups relative to the exceptional locus and fibre.

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2010 *Mathematics Subject Classification.* Primary 19E08; Secondary 14B05.

*Key words and phrases.* Gersten's conjecture, Algebraic  $K$ -theory, Singular schemes.

**Conjecture 1.2.** *Let  $A$  be a Noetherian local  $\mathbb{Q}$ -algebra,  $I \subseteq A$  an ideal,  $n \in \mathbb{Z}$ , and*

$$\begin{array}{ccc} Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A/I) & \longrightarrow & \mathrm{Spec}(A) \end{array}$$

*an abstract blow-up square in which  $X$  is regular. Then the canonical map*

$$K_n(A, I) \longrightarrow K_n(A/I^r, I/I^r) \oplus K_n(X, Y_{\mathrm{red}})$$

*is injective for  $r \gg 0$ .*

The conjecture is true by elementary  $K$ -theory if  $n = 0$  or  $1$ ; see Example 2.2. We prove the following cases of this conjecture in this paper.

**Theorem 1.3.** *The conjecture is true if  $A$  is quasi-excellent and*

- (i)  $n = 2$ ; or
- (ii)  $Y_{\mathrm{red}}$  is regular and  $n \geq 2$ .

In both cases of the theorem, we prove the stronger assertion that the canonical sequence of pro abelian groups

$$0 \longrightarrow K_n(A, I) \longrightarrow \{K_n(A/I^r, I/I^r)\}_r \oplus K_n(X, Y_{\mathrm{red}}) \longrightarrow \{K_n(rY, Y_{\mathrm{red}})\}_r \longrightarrow 0$$

is short exact. Indeed, pro cdh descent [6, Theorem 3.7] for  $K$ -theory implies the existence of a long exact Mayer–Vietoris sequence of pro abelian groups

$$\cdots \xrightarrow{\partial} K_n(A, I) \longrightarrow \{K_n(A/I^r, I/I^r)\}_r \oplus K_n(X, Y_{\mathrm{red}}) \longrightarrow \{K_n(rY, Y_{\mathrm{red}})\}_r \longrightarrow \cdots,$$

and therefore the injectivity assertion of the conjecture is equivalent to the vanishing of the boundary maps  $\partial$ . In the cases of the theorem we can show firstly, using pro Hochschild–Kostant–Rosenberg theorems, that  $\{K_{n+1}(rY, Y_{\mathrm{red}})\}$  is supported in Adams degrees  $> n$  and secondly, by classical results of Soulé [11] and Nesterenko–Suslin [10], that  $K_n(A, I)$  is supported in Adams degrees  $\leq n$  up to bounded torsion; see Section 3 for the details. This forces  $\partial$  to be zero.

**1.1. Notations and hypotheses.** We work primarily in the generality of quasi-excellent, Noetherian  $\mathbb{Q}$ -algebras, since restricting attention to finite type algebras over a characteristic zero field would only slightly simplify some of the proofs. All rings appearing are commutative and Noetherian.

All (relative) Hochschild, (relative) cyclic and André–Quillen homology groups will be assumed to be over  $\mathbb{Q}$  unless we specify the base ring explicitly.

The Adams eigenspaces of a  $K$ -group  $K_n(A)$  are denoted by  $K_n^{(i)}(A) := \{x \in K_n(A) : \psi^k(x) = k^i x \text{ for all } k \geq 0\}$ , and similarly for relative groups, for schemes, and for Hochschild and cyclic homology.

Pro abelian groups (always indexed over  $\mathbb{N}$ ) are denoted by  $\{G_r\}_r$ . We will repeatedly use, without explicit mention, that if  $G$  is an abelian group and  $G \rightarrow \{G_r\}_r$  is a map of pro abelian groups (i.e., there are compatible maps  $G \rightarrow G_r$  for all  $r \geq 1$ ), then  $G \rightarrow \{G_r\}_r$  is injective if and only if  $G \rightarrow G_r$  is injective for  $r \gg 0$ .

**Acknowledgements.** The second author would like to thank the Tata Institute of Fundamental Research for its hospitality during a visit in November 2012.

## 2. INJECTIVITY ASSERTIONS FOR $K$ -GROUPS OF DESINGULARIZATIONS

In this section, we state various injectivity assertions for the algebraic  $K$ -theory of singular local schemes in terms of their desingularizations. We explain the relations between these assertions and give examples showing that some of them do not hold in general.

Let  $A$  be a Noetherian local ring and let  $I \subset A$  be an ideal such that  $V(I)$  contains the singular locus of  $\text{Spec}(A)$ . We shall say that a commutative diagram

$$(2.1) \quad \begin{array}{ccc} Y & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \text{Spec}(A/I) & \longrightarrow & \text{Spec}(A) \end{array}$$

of schemes is a *resolution square*, or *resolution of  $A$* , if and only if it is an abstract blow-up square (i.e., it is Cartesian,  $f$  is proper, and the map  $f : X \setminus Y \rightarrow \text{Spec}(A) \setminus V(I)$  is an isomorphism) and  $X$  is regular.

Fixing  $n \in \mathbb{Z}$ , we may then consider the following statements, each of which asserts in some sense that the  $n^{\text{th}}$   $K$ -group  $K_n$  is captured by a combination of generic regular and nilpotent singular data:

- (1) $_n$  the map  $K_n(A) \rightarrow K_n(A/I^r) \oplus K_n(X)$  is injective for  $r \gg 0$ ;
- (2) $_n$  the map  $K_n(A, I) \rightarrow K_n(A/I^r, I/I^r) \oplus K_n(X, Y_{\text{red}})$  is injective for  $r \gg 0$ ;
- (3) $_n$  the map  $K_n(A) \rightarrow K_n(A/I^r) \oplus K_n(\text{Spec } A \setminus V(I))$  is injective for  $r \gg 0$ ;
- (4) $_n$  the map  $K_n(X) \rightarrow K_n(rY) \oplus K_n(X \setminus Y)$  is injective for  $r \gg 0$ , where  $rY := X \times_{\text{Spec}(A)} \text{Spec}(A/I^r)$ .

Although assertion (3) $_n$  does not require the existence of the resolution square, it appears to be difficult to study in any other way.

Associated to the resolution square there is a commutative diagram of pro spectra

$$(2.2) \quad \begin{array}{ccc} K(A) & \longrightarrow & K(X) \\ \downarrow & & \downarrow \\ \{K(A/I^r)\}_r & \longrightarrow & \{K(rY)\}_r. \end{array}$$

We will say that the square (2.1) satisfies the *pro Mayer–Vietoris property* in  $K$ -theory if and only if the square (2.2) of pro spectra is homotopy Cartesian (concretely, this means that the associated pro abelian relative  $K$ -groups are isomorphic). This is known to be true if  $A$  is a quasi-excellent  $\mathbb{Q}$ -algebra [6, Thm. 3.7], or if  $A$  is essentially of finite type over an infinite perfect field having strong resolution of singularities [6, Thm. 3.7], or if  $X \rightarrow \text{Spec } A$  is a finite morphism (a consequence of [7, Corol. 0.4]), and conjecturally it is true in general for any abstract blow-up square of Noetherian schemes. Assuming that the square (2.1) satisfies the pro Mayer–Vietoris property in  $K$ -theory, there are resulting long exact Mayer–Vietoris sequences of pro abelian

groups:

$$(2.3) \quad \cdots \longrightarrow K_n(A) \longrightarrow \{K_n(A/I^r)\}_r \oplus K_n(X) \longrightarrow \{K_n(rY)\}_r \longrightarrow \cdots$$

and

$$(2.4) \quad \cdots \longrightarrow K_n(A, I) \longrightarrow \{K_n(A/I^r, I/I^r)\}_r \oplus K_n(X, Y_{\text{red}}) \longrightarrow \{K_n(rY, Y_{\text{red}})\}_r \longrightarrow \cdots.$$

**Lemma 2.1.** *For a resolution of  $A$  as in (2.1), and  $n \in \mathbb{Z}$ , the following implications hold:*

- (i) *If the map  $K_n(A) \rightarrow K_n(A/I)$  is surjective, then  $(1)_n \implies (2)_n$ .*
- (ii)  *$(1)_n \mathcal{E} (4)_n \implies (3)_n \implies (1)_n$ .*

*If we assume moreover that the resolution square satisfies the pro Mayer–Vietoris property, then the following implications also hold:*

- (iii) *If the map  $K_{n+1}(X) \rightarrow K_{n+1}(Y)$  is surjective, then  $(2)_n \implies (1)_n$ .*
- (iv)  *$(3)_n \implies (4)_n$ .*

*Proof.* The claims (i) and (ii) are completely elementary, noting in (i) that the assumption implies  $K_n(A, I) \subseteq K_n(A)$  and in (ii) that  $X \setminus Y = \text{Spec } A \setminus V(I)$ . Now assume that the resolution square has the pro Mayer–Vietoris property.

(iii): By (2.4) and  $(2)_n$ , the map

$$\{K_{n+1}(A/I^r, I/I^r)\}_r \oplus K_{n+1}(X, Y_{\text{red}}) \rightarrow \{K_{n+1}(rY, Y_{\text{red}})\}_r$$

is surjective. But the assumed surjectivity of  $K_{n+1}(X) \rightarrow K_{n+1}(Y_{\text{red}})$  implies that the maps  $K_{n+1}(X) \rightarrow K_{n+1}(rY)$  and  $K_{n+1}(rY, Y_{\text{red}}) \rightarrow K_{n+1}(rY)$  are jointly surjective for any  $r \geq 1$ , and so it follows that  $\{K_{n+1}(A/I^r)\}_r \oplus K_{n+1}(X) \rightarrow \{K_{n+1}(rY)\}_r$  is also surjective. Now (2.3) completes the proof of  $(1)_n$ .

(iv): It follows from (2.4) that the map

$$\{\text{Ker}(K_n(A) \rightarrow K_n(A/I^r))\}_r \longrightarrow \{\text{Ker}(K_n(X) \rightarrow K_n(rY))\}_r$$

is surjective, after which the implication is an elementary consequence of the identification  $X \setminus Y = \text{Spec } A \setminus V(I)$ .  $\square$

*Example 2.2* ( $n = 0, 1$ ). We claim that  $(1)_1$ ,  $(2)_1$ , and  $(3)_1$  are true. Indeed, the kernel  $\Lambda$  of the restriction map

$$A = \Gamma(\text{Spec } A, \mathcal{O}_{\text{Spec } A}) \longrightarrow \Gamma(\text{Spec } A \setminus V(I), \mathcal{O}_{\text{Spec } A})$$

is supported on  $V(I)$ , hence annihilated by a power of  $I$ . It then follows from the Artin–Rees lemma that  $\Lambda \cap I^r = 0$  for  $r \gg 0$ , i.e., that the map  $A \rightarrow A/I^r \oplus \Gamma(\text{Spec } A \setminus V(I), \mathcal{O}_{\text{Spec } A})$  is injective. Taking units and using the split determinant map  $K_1 \rightarrow \mathbb{G}_m$ , which is an isomorphism for any local ring, proves that assertion  $(3)_1$  is true. Hence assertions  $(1)_1$  and  $(2)_2$  are true by Lemma 2.1, noting that  $K_1(A) = A^\times \rightarrow K_1(A/I) = A/I^\times$  is surjective.

We claim also that  $(1)_0$ ,  $(2)_0$ , and  $(3)_0$  are true; these are easy consequences of  $K_0(A) = K_0(A/I^r) = \mathbb{Z}$ .

Assuming that the resolution square satisfies the pro Mayer–Vietoris property for  $K$ -theory, Lemma 2.1(iv) implies that assertions  $(4)_1$  and  $(4)_0$  are also true.

*Example 2.3* ( $n < 0$ ). Assume  $n < 0$ . Then  $(4)_n$  becomes vacuously true since  $K_n(X) = 0$ . Also,  $(1)_n$  and  $(3)_n$  become the identical statement that  $K_n(A) \rightarrow K_n(A/I)$  is injective, since negative  $K$ -groups of rings are nil-invariant. Similarly,  $(2)_n$  becomes the assertion that  $K_n(A, I) \rightarrow K_n(X, Y_{\text{red}}) \cong K_{n+1}(Y_{\text{red}})$  is injective.

Based on the known results when  $A$  is one-dimensional, which we will review in the following subsection, it remains plausible to conjecture that  $(2)_n$  and  $(4)_n$  might be true in general. Our goal in this note is to prove two cases of  $(2)_n$  and to prove  $(1)_n - (4)_n$  for cone singularities.

**2.1. The one-dimensional case.** We now collect together the known results in the case that  $A$  is one-dimensional and its desingularization is obtained by normalisation.

Let  $(A, \mathfrak{m})$  be a one-dimensional, Noetherian local ring such that  $\text{Spec } A \setminus \{\mathfrak{m}\}$  is regular, i.e.,  $A_{\mathfrak{p}}$  is a field for each minimal prime ideal  $\mathfrak{p} \subset A$ ; assume also that the normalisation map  $A_{\text{red}} \rightarrow B := \widetilde{A_{\text{red}}}$  is finite. Then  $B$  is a regular, one-dimensional, semi-local ring, and

$$\begin{array}{ccc} \text{Spec } B/\mathfrak{m}B & \longrightarrow & \text{Spec } B \\ \downarrow & & \downarrow \\ \text{Spec } A/\mathfrak{m} & \longrightarrow & \text{Spec } A \end{array}$$

is an resolution square which satisfies the pro Mayer–Vietoris property. Assuming that  $B$  contains a field, we make the following assertions about this resolution square:

- $(1)_2$  is not necessarily true;
- $(2)_n$  is true for all  $n \in \mathbb{Z}$ ;
- $(3)_2$  is not necessarily true;
- $(4)_n$  is true for all  $n \in \mathbb{Z}$ ;

Firstly,  $(4)_n$  is true since the validity of Gersten's conjecture in the geometric case implies that  $K_n(B) \rightarrow K_n(\text{Spec } B \setminus V(\mathfrak{m}B))$  is injective.

Secondly, it now follows from Lemma 2.1(ii) that  $(1)_2$  and  $(3)_2$  are equivalent; in the next example we will provide a particular choice of  $A$  for which  $(3)_2$  fails.

We now show that  $(2)_n$  is true for all  $n \in \mathbb{Z}$ ; in fact, this was proved in [8, Thm. 2.7], inspired by the case  $n = 2$  [4, Thm. 3.6], under the extraneous assumption that  $A$  was reduced. Indeed, it was shown [8, Corol. 2.6] that the canonical map  $K_n(B, \mathfrak{M}) \rightarrow \{K_n(B/\mathfrak{M}^r, \mathfrak{M}/\mathfrak{M}^r)\}_r$  is surjective for all  $n \geq 0$ , where  $\mathfrak{M} = \sqrt{\mathfrak{m}B}$  is the Jacobson radical of  $B$ ; it is also surjective if  $n < 0$ , since the domain vanishes by nil-invariance of negative  $K$ -theory for rings. Hence (2.3) breaks into short exact sequences, proving  $(2)_n$ .

*Example 2.4.* Let  $k$  be a field of char  $\neq 2$  and let  $A = k[X, Y]_{(X, Y)}/(Y^2 - X^2(X + 1))$  be the local ring at the singular point of the nodal curve  $Y^2 = X^2(X + 1)$ . Then  $A$  is a domain, with finite normalisation map since it is essentially of finite type over a field. However, assertion  $(3)_2$  fails: the map  $K_2(A) \rightarrow K_2(A/\mathfrak{m}^r) \oplus K_2(\text{Frac } A)$  is not injective for any  $r \geq 1$ . The proof may be found in [9, Prop. 2.12] and, as Lemma 2.1(iii) suggests, relies on the fact that  $K_3(\tilde{A}) \rightarrow K_3(\tilde{A}/\mathfrak{m}\tilde{A}) = K_3(k) \oplus K_3(k)$  can be shown not to be surjective.

## 3. VANISHING OF SOME RELATIVE HOCHSCHILD AND CYCLIC HOMOLOGY

This section, where we establish some vanishing results for relative Hochschild and cyclic homology groups, is at the heart of the proofs of our main results. The following preliminary lemma may be ignored by readers who are only interested in the case of finite type algebras (“smooth” means “localisation of smooth, finite type”):

**Lemma 3.1.** *Let  $f : A \rightarrow B$  be a surjection of regular, local  $\mathbb{Q}$ -algebras. Then it is possible to write  $A = \varinjlim_i A_i$  and  $B = \varinjlim_i B_i$  as filtered inductive limits of smooth, local  $\mathbb{Q}$ -algebras, in such a way that  $f = \varinjlim_i f_i$  where  $f_i : A_i \rightarrow B_i$  are compatible surjections.*

*Proof.* Choose a regular system of parameters  $t_1, \dots, t_d$  of  $A$  such that  $I := \text{Ker } f$  is generated by  $t_1, \dots, t_c$  for some  $0 \leq c \leq d = \dim A$ . According to Néron–Popescu desingularization,  $A$  may be written as a filtered inductive limit  $\varinjlim_{i \in I} A_i$  of smooth, local  $\mathbb{Q}$ -algebras such that the homomorphisms  $\phi_i : A_i \rightarrow A$  are local. Let  $\mathfrak{m}_i = \phi_i^{-1}(\mathfrak{m}_A)$  denote the maximal ideal of  $A_i$ . Possibly after discarding part of the bottom of this inductive system, we may assume that  $I$  has a minimal element  $i_0$  and that the elements  $t_1, \dots, t_c$  may be lifted to  $\tilde{t}_1, \dots, \tilde{t}_c \in \mathfrak{m}_{i_0}$ .

We claim that, for any  $i \in I$ , the images of  $\tilde{t}_1, \dots, \tilde{t}_c$  in the local ring  $A_i$  form part of a regular system of parameters. Indeed, otherwise their images mod  $\mathfrak{m}_i^2$  would be linearly dependent over  $A_i/\mathfrak{m}_i$ , and hence  $t_1, \dots, t_c$  mod  $\mathfrak{m}_A^2$  would be linearly dependent over  $\phi_i(A_i)/\phi_i(\mathfrak{m}_i) \subseteq A/\mathfrak{m}_A$ ; but this would contradict the fact that  $t_1, \dots, t_c$  are part of a regular system of parameters in  $A$ .  $\square$

**Proposition 3.2.** *Let  $Y \hookrightarrow X$  be a closed embedding of regular, Noetherian  $\mathbb{Q}$ -schemes of finite Krull dimension. Then the pro abelian group  $\{HC_n^i(rY, Y)\}_r$  vanishes for  $0 \leq i < n$ .*

*Proof.* By the Zariski descent of cyclic homology [1, Thm 2.9], we may assume that  $X = \text{Spec } R$  is affine, with  $Y$  defined by an ideal  $I \subseteq R$ ; note that  $R$  and  $R/I$  are regular.

The usual map of mixed complexes  $(C_\bullet^\mathbb{Q}(-), b, B) \rightarrow (\Omega_\bullet, 0, d)$  is an isomorphism on the associated Hochschild homologies both for  $R$  and  $R/I$ , by the usual HKR theorem [5, Thm. 3.4.4], and for the pro ring  $R/I^\infty$  by the pro HKR theorem [7, Thm. 3.23]. There are therefore induced isomorphisms of the associated cyclic homologies and of the relative homologies; in particular,

$$\{HC_n^{(i)}(R/I^r, I/I^r)\}_r \xrightarrow{\cong} \{H^{2n-i}(\text{Ker}(\Omega_{R/I^r}^\bullet \rightarrow \Omega_{R/I}^\bullet))\}_r$$

for  $0 \leq i < n$ .

Hence, to complete the proof, we may show that the canonical projection  $\Omega_{R/I^r}^\bullet \rightarrow \Omega_{R/I}^\bullet$  is a quasi-isomorphism for each  $r \geq 1$ . If  $I$  is generated by a regular sequence and  $R$  is essentially of finite type over a subfield  $K \subseteq R$ , whence  $S := R/I$  is formally smooth over  $K$ , then it is well-known (e.g., [2, Lem. II.1.2]) that the  $I$ -adic completion of  $R$  is isomorphic to  $S[[T_1, \dots, T_c]]$ . In particular,  $R/I^r \cong S[[T_1, \dots, T_c]]/(T_1, \dots, T_c)^r$  admits the structure of a positively graded  $K$ -algebra with degree zero component  $S$ , and so it follows from the Poincaré Lemma [13, Corol. 9.9.3] that  $\Omega_{R/I^r}^\bullet \xrightarrow{\cong} \Omega_S^\bullet$  (even

though the base field for these Kähler differentials is  $\mathbb{Q}$ , not  $K$ ). In general, we can easily reduce to this case, by using Zariski descent to assume that  $R$  is local and by then applying Lemma 3.1.  $\square$

The following two results are a modification of the previous proposition when  $Y$  is replaced by a normal crossing divisor. A *strict normal crossing divisor* on a regular affine scheme  $\text{Spec}(R)$  is a closed subscheme defined by a non-zero-divisor of the form  $t_1 \cdots t_c$ , where  $t_1, \dots, t_c \in R$  have the property that for each prime ideal  $\mathfrak{p} \in V(t_1 \cdots t_c)$ , those of the  $t_i$  which belong to  $\mathfrak{p}R_{\mathfrak{p}}$  form part of a regular system of parameters of  $R_{\mathfrak{p}}$ . A *normal crossing divisor* on a regular scheme is a divisor which étale locally is a strict normal crossing divisor.

**Lemma 3.3.** *Let  $R$  be a regular, local  $\mathbb{Q}$ -algebra and let  $f \in R$  be a non-zero-divisor for which the de Rham differential  $d : fR/f^2R \rightarrow \Omega_R^1 \otimes_R R/fR$  is injective. Then:*

- (i)  $HH_n^{(i)}(R/fR) = 0$  if  $i \leq n/2$  (unless  $i = n = 0$ ).
- (ii)  $\{HH_n^{(i)}(R/f^rR, fR/f^rR)\}_r = 0$  if  $i \leq (n+1)/2$  (unless  $i = n = 1$  or  $i = n = 0$ ).
- (iii)  $\{HC_n^{(1)}(R/f^rR, fR/f^rR)\}_r = 0$  if  $n \geq 2$ .

*Proof.* (i): Since  $R$  is geometrically regular over  $\mathbb{Q}$  (since it is regular) and  $f$  is a non-zero-divisor, the cotangent complexes  $\mathbb{L}_R$  and  $\mathbb{L}_{R/fR|R}$  of  $\mathbb{Q} \rightarrow R$  and  $R \rightarrow R/fR$  are respectively equal to  $\Omega_R^1$  and  $(fR/f^2R)[1]$ , whence it follows from the Jacobi–Zariski sequence that the cotangent complex  $\mathbb{L}_{R/fR}$  of  $\mathbb{Q} \rightarrow R/fR$  is quasi-isomorphic to the following chain complex of flat  $R/fR$ -modules:

$$0 \leftarrow \Omega_R^1 \otimes_R R/fR \leftarrow fR/f^2R \leftarrow 0.$$

More generally, for any  $i \geq 1$ , the exterior powers  $\mathbb{L}_R^i$  and  $\mathbb{L}_{R/fR|R}^i$  are respectively equal to  $\Omega_R^i$  and  $(f^iR/f^{i+1}R)[i]$ , and it follows from the Kassel–Sletsf oe spectral sequence [3, Thm. 3.2] that  $\mathbb{L}_{R/fR}^i$  is quasi-isomorphic to the following chain complex of flat  $R/fR$ -modules:

$$0 \leftarrow \Omega_R^i \otimes_R R/fR \leftarrow \Omega_R^{i-1} \otimes_R fR/f^2R \leftarrow \cdots \leftarrow \Omega_R^1 \otimes_R f^{i-1}R/f^iR \leftarrow f^iR/f^{i+1}R \leftarrow 0$$

(where  $\Omega_R^{i-j} \otimes_R f^jR/f^{j+1}R$  sits in degree  $j$ ). Hence, part of the Andr e–Quillen homology of  $\mathbb{Q} \rightarrow R/fR$  is given by

$$D_n^i(R/fR) = \begin{cases} \text{Ker}(d : f^iR/f^{i+1}R \rightarrow \Omega_R^1 \otimes_R f^{i-1}R/f^iR) & \text{if } i = n, \\ 0 & \text{if } i > n. \end{cases}$$

It follows easily from the hypothesis on  $f$  that this kernel is zero. Finally, recall that  $D_n^i(R/fR) = HH_{n+i}^{(i)}(R/fR)$  to complete the proof.

(ii): There are short exact sequences of pro  $R$ -modules

$$HH_{n+1}^{(i)}(R/fR) \longrightarrow \{HH_n^{(i)}(R/f^rR, fR/f^rR)\}_r \longrightarrow \{HH_n^{(i)}(R/f^rR)\}_r,$$

where the right term vanishes for  $i < n$  by the pro HKR theorem [7, Thm. 3.23]. Moreover, the left term vanishes if  $i \leq (n+1)/2$  by (i), and hence the central term also vanishes.

(iii): To save space, we will temporarily use the notation  $HH_n^{(i)}$  for the pro abelian group  $\{HH_n^{(i)}(R/f^r R, fR/f^r R)\}_r$  and similarly for cyclic homology. For any  $n \geq 3$ , we see from (ii) and the SBI sequence  $HH_n^{(1)} \rightarrow HC_n^{(1)} \rightarrow HC_{n-2}^{(0)} = 0$  that  $HC_n^{(1)} = 0$ . It remains to treat the difficult case when  $n = 2$ .

In the SBI sequence  $HH_2^{(1)} \rightarrow HC_2^{(1)} \rightarrow HC_0^{(0)} \rightarrow HH_1^{(1)}$ , it follows from (i) that the left term vanishes and that the right term embeds into  $\{HH_1^{(1)}(R/f^r R)\}_r = \{\Omega_{R/f^r R}^1\}_r$ , where we have again applied the pro HKR theorem. Since  $HC_0^{(0)} = \{fR/f^r R\}_r$  with the final arrow  $HC_0^{(0)} \rightarrow HH_1^{(1)}$  corresponding to the de Rham differential, there is an induced isomorphism

$$HC_2^{(1)} \xrightarrow{\cong} \{\text{Ker}(d : fR/f^r R \rightarrow \Omega_R^1 \otimes_R R/f^{r-1} R)\}_r$$

(here we have implicitly used the Leibnitz rule to identify  $\{\Omega_R^1 \otimes_R R/f^{r-1} R\}_r$  and  $\{\Omega_{R/f^r R}^1\}_r$ ). It follows easily from the hypothesis on  $f$  that this kernel is zero, and so we deduce that  $HC_2^{(1)} = 0$ .  $\square$

*Example 3.4.* Suppose that  $R$  is a regular  $\mathbb{Q}$ -algebra and that  $t \in R$  is a non-zero-divisor for which  $R/tR$  is also regular; then we claim that the hypothesis of the previous lemma is satisfied, i.e., that  $d : tR/t^2 R \rightarrow \Omega_R^1 \otimes_R R/tR$  is injective.

By Lemma 3.1 and the same reductions as in Proposition 3.2, it is enough to prove that  $d : TS[[T]]/T^2 S[[T]] \rightarrow \Omega_{S[[T]]}^1 \otimes_{S[[T]]} S$  is injective for any regular  $\mathbb{Q}$ -algebra  $S$ . But this is clear, since the canonical differential  $d/dt$  induces a map of  $S[[T]]$ -modules  $\Omega_{S[[T]]}^1 \rightarrow S[[T]]$  such that the resulting composition  $TS[[T]]/T^2 S[[T]] \rightarrow \Omega_{S[[T]]}^1 \otimes_{S[[T]]} S \rightarrow S[[T]] \otimes_{S[[T]]} S = S$  is given by  $sT \mapsto s$ .

**Proposition 3.5.** *Let  $X$  be a regular, Noetherian  $\mathbb{Q}$ -scheme of finite Krull dimension, and let  $Y \hookrightarrow X$  be a normal crossings divisor. Then  $\{HC_n^{(1)}(rY, Y)\}_r = 0$  for all  $n \geq 2$ .*

*Proof.* Since relative cyclic homology (and each of its Adams summands) satisfies étale descent (see [1, Theorem 2.9]), the usual induction on the size of an affine cover (note that  $X$  is quasi-separated) allows us to assume that  $X = \text{Spec}(R)$  is affine and that  $Y$  is a strict normal crossings divisor, defined by  $f = t_1 \cdots t_c \in R$ .

Then part (iii) of the previous lemma will complete the proof, as soon as we show that the de Rham differential  $d : fR/f^2 R \rightarrow \Omega_R^1 \otimes_R R/fR$  is injective; to check this we may assume that  $R$  is local. Considering the commutative diagram

$$\begin{array}{ccc} fR/f^2 R & \xrightarrow{d} & \Omega_R^1 \otimes_R R/fR \\ \downarrow & & \downarrow \\ \bigoplus_{i=1}^c t_i R/t_i^2 R & \xrightarrow{d} & \bigoplus_{i=1}^c \Omega_R^1 \otimes_R R/t_i R, \end{array}$$

it is enough to show that the left vertical and bottom horizontal arrows are injective. The left vertical arrow is injective since  $R$  is a unique factorisation domain and the elements  $t_1, \dots, t_c$  are all distinct irreducibles. The bottom horizontal arrow is injective by the previous example.  $\square$



## 4. THE MAIN RESULTS

We now collect together the vanishing results of Section 3 to prove our main theorems.

**4.1. Proof of Theorem 1.3.** Throughout this section we consider a quasi-excellent, Noetherian, local  $\mathbb{Q}$ -algebra  $A$ , an ideal  $I \subseteq A$  and a resolution square as in (2.1).

We begin with the following result about the relative  $K$ -groups (which in fact holds for any ideal of any local Noetherian ring):

**Lemma 4.1.** *For any  $i > n \geq 1$ , the group  $K_n^{(i)}(A, I)$  is torsion of bounded exponent.*

*Proof.* Let  $\mathcal{T}$  be the category of torsion groups of bounded exponent. By standard properties of lambda and Adams operators, the sequence of relative  $K$ -groups associated to  $A \rightarrow A/I$  yields to an exact sequence

$$K_{n+1}^{(i)}(A) \longrightarrow K_{n+1}^{(i)}(A/I) \longrightarrow K_n^{(i)}(A, I) \longrightarrow K_n^{(i)}(A)$$

in the category of abelian groups modulo  $\mathcal{T}$ .

Moreover, since  $A$  is local the group  $K_n^{(i)}(A)$  is torsion of bounded exponent if  $i > n$  by Soulé [11, §2.8]; the same applies to  $K_{n+1}^{(i)}(A/I)$  if  $i > n+1$ . This completes the proof in the case  $i > n+1$ , and shows that  $K_n^{(n+1)}(A, I) = \text{Coker}(K_{n+1}^{(n+1)}(A) \rightarrow K_{n+1}^{(n+1)}(A/I))$  modulo  $\mathcal{T}$ . But this cokernel is zero since  $K_{n+1}^{(n+1)}(A) = K_{n+1}^M(A)$  and  $K_{n+1}^{(n+1)}(A/I) = K_{n+1}^M(A/I)$  modulo  $\mathcal{T}$  by Nesterenko–Suslin [10].  $\square$

**Lemma 4.2.** *Fix  $n \geq 0$ , and suppose that  $\{K_{n+1}^{(i)}(rY, Y_{\text{red}})\}_r = 0$  for all  $i \leq n$ . Then*

- (i) *the canonical map of pro abelian groups  $K_{n+1}(X, Y_{\text{red}}) \rightarrow \{K_{n+1}(rY, Y_{\text{red}})\}_r$  is surjective;*
- (ii) *assertion (2)<sub>n</sub> is true, i.e.,  $K_n(A, I) \rightarrow K_n(A/I^r, I/I^r) \oplus K_n(X, Y_{\text{red}})$  is injective for  $r \gg 0$ .*

*Proof.* (i): From the long exact sequence

$$\cdots \longrightarrow K_{n+1}(X, Y_{\text{red}}) \longrightarrow \{K_{n+1}(rY, Y_{\text{red}})\}_r \xrightarrow{\{\partial_r\}_r} \{K_n(X, rY)\}_r \longrightarrow \cdots,$$

we see that it is necessary and sufficient to prove that the boundary map  $\{\partial_r\}_r$  is zero.

The square (2.1) satisfies the pro Mayer–Vietoris property by the results recalled in Section 2, and so the canonical map  $\{K_n(A, I^r)\}_r \rightarrow \{K_n(X, rY)\}_r$  is an isomorphism. Passing to Adams eigenspaces yields isomorphisms  $\{K_n^{(i)}(A, I^r)\}_r \rightarrow \{K_n^{(i)}(X, rY)\}_r$ . The surjectivity of this latter map means that if we fix any  $s \geq 1$ , then there exists  $s' \geq s$  such that

$$\text{Im}(K_n^{(i)}(X, s'Y) \rightarrow K_n^{(i)}(X, sY)) \subseteq \text{Im}(K_n^{(i)}(A, I^s) \rightarrow K_n^{(i)}(X, sY)).$$

It follows from Lemma 4.1 that the right, and hence the left, side is a torsion group of bounded exponent if  $i > n$ .

Since  $rY$  is a nilpotent thickening of  $Y_{\text{red}}$ , the relative  $K$ -group  $K_{n+1}(rY, Y_{\text{red}})$  is a  $\mathbb{Q}$ -vector space which decomposes as a direct sum  $\bigoplus_{i=0}^{n+1+\dim Y} K_{n+1}^{(i)}(rY, Y_{\text{red}})$ . Our hypothesis is that this decomposition, as a pro abelian group over  $r \geq 1$ , has no

component in degrees  $i \leq n$ . Hence there exists  $s'' \geq s'$  such that the canonical map  $K_{n+1}(s''Y, Y_{\text{red}}) \rightarrow K_{n+1}(s'Y, Y_{\text{red}})$  has image in  $\bigoplus_{i=n+1}^{n+1+\dim Y} K_{n+1}^{(i)}(s'Y, Y_{\text{red}})$ .

Assembling the conclusions of the two previous paragraphs, and noting that the boundary maps  $\partial_r$  respect the Adams operators, we see that the image of the composition

$$K_{n+1}(s''Y, Y_{\text{red}}) \longrightarrow K_{n+1}(s'Y, Y_{\text{red}}) \xrightarrow{\partial_{s'}} K_n(X, s'Y) \longrightarrow K_n(X, sY)$$

is both divisible (being the image of a  $\mathbb{Q}$ -vector space) and a torsion group of bounded exponent; hence the image is zero. This means that the map  $\{\partial_r\}_r$  of pro abelian groups is zero. Claim (ii) follows from (i) and the Mayer–Vietoris sequence (2.4).  $\square$

The following is the first main result of this note, proving assertion  $(2)_n$  in certain cases:

**Theorem 4.3.** *Assume either that  $n = 2$ , or that  $Y_{\text{red}}$  is regular and  $n \geq 2$ . Then the canonical sequence of pro abelian groups*

$$0 \longrightarrow K_n(A, I) \rightarrow \{K_n(A/I^r, I/I^r)\}_r \oplus K_n(X, Y_{\text{red}}) \longrightarrow \{K_n(rY, Y_{\text{red}})\}_r \longrightarrow 0$$

*is short exact.*

*Proof.* If  $Y_{\text{red}}$  is regular then it follows from Proposition 3.2 and the Goodwillie isomorphism  $K_{n+1}^{(i+1)}(rY, Y_{\text{red}}) \cong HC_n^{(i)}(rY, Y_{\text{red}})$  that  $\{K_{n+1}^{(i+1)}(rY, Y_{\text{red}})\}_r$  vanishes for  $i < n$ . From Lemma 4.2, it then follows that (2.4) breaks into short exact sequences in positive degrees, as required.

Dropping the regularity hypothesis on  $Y_{\text{red}}$ , it follows from the resolution of singularities (specifically, since we are not imposing any finite type hypotheses, from [12, Thm. 2.3.6]) that there is an abstract blow-up square

$$\begin{array}{ccc} Y' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

in which  $X'$  is regular and  $Y'_{\text{red}}$  is a normal crossing divisor on  $X$ . Repeating the argument of the previous paragraph with  $X, Y$  replaced by  $X', Y'$ , Proposition 3.2 replaced by Proposition 3.5, and  $n$  replaced by 2, it follows that  $K_2(A, I) \rightarrow K_2(A/I^r, I/I^r) \oplus K_2(X', Y'_{\text{red}})$  is injective for  $r \gg 1$ . Hence  $K_2(A, I) \rightarrow K_2(A/I^r, I/I^r) \oplus K_2(X, Y_{\text{red}})$  is certainly injective. The desired short exact sequence now follows from (2.4) and assertion  $(2)_1$ , which was proved in Example 2.2.  $\square$

**4.2. Proof of Theorem 1.1.** Let  $k$  be a field of characteristic zero and  $Y \hookrightarrow \mathbb{P}_k^N$  a smooth projective variety. Let  $C \hookrightarrow \mathbb{A}_k^{N+1}$  be the cone over  $Y$ , and let  $(A, \mathfrak{m})$  be the local ring at the unique singular point of  $C$ . In the course of proving our second main result, we will see that assertions  $(1)_n$ – $(4)_n$  are all true for the usual resolution of  $A$ :

**Theorem 4.4.** *For any  $n \geq 0$ , the map*

$$K_n(A) \longrightarrow K_n(A/\mathfrak{m}^r) \oplus K_n(\text{Spec}(A) \setminus \{\mathfrak{m}\})$$

*is injective for  $r \gg 0$ .*

*Proof.* Let  $\check{C} \rightarrow C$  be the usual resolution of singularities, which is a line bundle over  $Y$  in such a way that the zero section  $\sigma : Y \hookrightarrow \check{C}$  is exactly the exceptional fibre of  $\check{C} \rightarrow C$  over the singular point of  $C$ .

Let  $X = \check{C} \times_C \text{Spec}(A)$  be the associated resolution of singularities of  $\text{Spec}(A)$ . Diagrammatically we have the following Cartesian squares:

$$\begin{array}{ccccc} X \times_{\text{Spec}(A)} \text{Spec}(A/\mathfrak{m}) & \longrightarrow & X & \longrightarrow & \check{C} \\ \downarrow & & \downarrow & & \downarrow \\ \text{Spec}(A/\mathfrak{m}) & \longrightarrow & \text{Spec}(A) & \longrightarrow & C \end{array}$$

in which  $(X \times_{\text{Spec}(A)} \text{Spec}(A/\mathfrak{m}))_{\text{red}} = Y$  and the left square and outer rectangle are abstract blow-up squares. The left square, which is the usual resolution square for  $A$ , satisfies  $(2)_n$  by Theorem 4.3.

Since the canonical inclusion  $Y \hookrightarrow X$  is split by  $X \rightarrow \check{C} \xrightarrow{\pi} Y$ , where  $\pi$  denotes the line bundle structure map, we see that  $K_{n+1}(X) \rightarrow K_{n+1}(Y)$  is surjective. So it follows from Lemma 2.1(iii) that the left square satisfies  $(1)_n$ . Since we wish to prove  $(3)_n$ , it is now enough by Lemma 2.1(ii) to prove that the left square satisfies  $(4)_n$ , which we will do by showing that  $K_n(X) \rightarrow K_n(Y) \oplus K_n(X \setminus Y)$  is injective.

Suppose that  $\alpha \in K_n(X)$  dies in  $K_n(X \setminus Y)$ . By comparing the localisation sequences

$$\begin{array}{ccccc} K_n(Y) & \xrightarrow{\sigma_*} & K_n(\check{C}) & \longrightarrow & K_n(\check{C} \setminus Y) \\ \parallel & & \downarrow & & \downarrow \\ K_n(Y) & \xrightarrow{\sigma_*} & K_n(X) & \longrightarrow & K_n(X \setminus Y), \end{array}$$

we see that  $\alpha$  is in the image of  $K_n(\check{C}) \rightarrow K_n(X)$ . But the composition  $K_n(\check{C}) \rightarrow K_n(X) \rightarrow K_n(Y)$  is an isomorphism since  $\check{C}$  is a line bundle over  $Y$ ; so if we now also assume that  $\alpha$  dies in  $K_n(Y)$ , then it follows that  $\alpha = 0$ . This completes the proof.  $\square$

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