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Continuity of the norm map on Milnor K-theory

by

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Abstract

The norm map on the Milnor K-groups of a finite extension of complete, discrete valuation fields is continuous with respect to the unit group filtrations. The only proof in the literature, due to K. Kato, uses semi-global methods. Here we present an elementary algebraic proof.

Key Words: Milnor K-theory, norm map, higher local class field theory.
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1. Introduction

The Milnor K-groups $K_m(F)$ of a complete, discrete valuation field are equipped with a decreasing unit group filtration $U^i K_m(F)$, $i \geq 1$, and a theorem of K. Kato states that this filtration is compatible with the norm map on Milnor K-theory, in the following sense:

Theorem 1.1 *[8, Prop. 2] Let L/F be a finite extension of complete, discrete valuation fields. Then*

$$N_{L/F}(U^{ei} K_m(L)) \subseteq U^i K_m(F)$$

for all $i \geq 1$, where $e = e(L/F)$ is the ramification degree of the extension.

Kato's proof relies on the use of two-dimensional local rings and Weierstrass preparation. The purpose of this article is to present a relatively short proof which feels more 'in the spirit of Milnor K-theory'. Since the norm maps on Milnor K-theory remain one of the more unpleasant features for newcomers to the theory, I hope that it is useful to have a new proof available in the literature.

We begin with a short review of Milnor K-theory to fix notation. Lemma 2.3 and corollary 2.4 show that $N_{L/F}(U^1 K_m(L)) \subseteq U^1 K_m(F)$, which is not needed for the main result but which seemed worth including. The main proof starts with lemma 2.5, and the reader may wish to start there, before the paper closes with some remarks.

We finish this introduction with a general discussion of the norm maps on Milnor K-theory, their history and applications, and the importance of Kato's result.

The norm map on Milnor K_2 (as well as K_0 and K_1, via identification with Quillen K-theory) was first defined by J. Milnor [12, §14]. This norm map was essential for Tate's study [15] of torsion in K_2. In particular, Tate showed that if F is a local field containing a root of unity ζ of order n, then every element $\xi \in K_2(F)$ such that $n\xi = 0$ has the form $\xi = \{\zeta, a\}$ for some $a \in F^\times$. He also conjectured that if F is a local field with roots of unity $\mu \subseteq F^\times$, then the Hilbert symbol $K_2(F)_{\text{tors}} \to \mu$ is an isomorphism; using a descent result of Tate from the aforementioned paper, this conjecture was later settled in the positive by A. Merkurjev [11].

The norm maps for Milnor K_n, all $n \geq 0$, were then introduced by J. Tate and H. Bass [1, Chap. 1, §5], but they could not show in general that their definition was independent of a choice of generators for L/F. The well-definedness was established by K. Kato [6, §1.7]; Kato distributed his result at the 1980 Oberwolfach Algebraic K-theory conference, and A. Suslin [14] immediately used it to prove that the Milnor K-theory of a field embeds into the Quillen K-theory, at least modulo torsion. Moreover, the norm maps played the same role in Kato's higher local class field theory [5, 6, 7] that the usual norm map does in classical local class field theory: given a finite abelian extension L/F of n-dimensional local fields, there is a canonical isomorphism

$$K_m(F)/N_{L/F}\,K_m(L) \xrightarrow{\simeq} \mathrm{Gal}(L/F).$$

This highlights the arithmetic importance of calculating the image of the norm map, which sadly remains difficult.

Turning now to the particular case of a complete, discrete valuation field F and the unit group filtration $U^i K_m(F)$, $i \geq 1$, on $K_m(F)$, we have already remarked that theorem 1.1 is due to Kato. It implies that the norm map induces $N_{L/F} : \widehat{K_m(L)} \to \widehat{K_m(F)}$, where

$$\widehat{K_m(L)} = \varprojlim_i K_m(L)/U^i K_m(L)$$

is the completion of $K_m(L)$ with respect to its unit group filtration. This was used in Kato's [8] development of his residue homomorphism on Milnor K-theory, which has played an important role in the theory of explicit reciprocity laws [3] [10]. Theorem 1.1 also allows the norm map to be more closely analysed via its behaviour on the graded pieces $U^i K_m(L)/U^{i+1} K_m(L)$, which are quotients of spaces of differential forms [9, §3.1] (this reference explains the importance of understanding the Milnor K-groups of complete discrete valuation fields in arithmetic geometry); surprisingly, this description of the graded pieces does not seem to offer any way to prove theorem 1.1.

1.1. Acknowledgments

I am very grateful to the referee for suggestions on improving the article, and to K. Kato for directing me towards T. Fukaya's aforementioned article.

2. Continuity of the norm map

Let F be a field and let $K_m(F) = K_m^M(F)$ be its m^{th} Milnor K-group [12] (more detailed introductions may be found in [2, Chap. IX], [4, Chap. 7]):

$$K_m(F) = F^{\times \otimes m} / \langle a_1 \otimes \cdots \otimes a_m : a_i + a_j = 1 \text{ for some } i \neq j \rangle$$

(we adopt the notation K_m since we will not use any Quillen K-theory). Elements of $K_m(F)$ are sums of symbols $\{a_1, \ldots, a_m\}$, $a_i \in F^\times$, and concatenation of symbols makes $K_*(F) = \bigoplus_{m \geq 0} K_m(F)$ into a graded commutative ring. We will need:

Lemma 2.1 *Let $L = F(\alpha)$ be a finite extension of F generated by a single element. Then $K_m(L)$ is generated by symbols of the form*

$$\{a_1, \ldots, a_{s-1}, g_s(\alpha), \ldots, g_m(\alpha)\}$$

with $1 \leq s \leq m+1$, $a_i \in F^\times$, and $g_j(X) \in F[X]$ monic irreducible polynomials such that $0 < \deg g_s < \cdots < \deg g_m < |L : F|$.

Proof: This is well-known; see e.g. [1, Chap. 1, Cor. 5.3] or [2, Chap. IX, §2.5 Cor. 1]. □

Suppose ν is a discrete valuation on F with residue field $\overline{F}$. The unit group $U_\nu K_m(F)$ of $K_m(F)$ is the subgroup generated by symbols $\{u_1, \ldots, u_m\}$ with $\nu(u_j) = 0$ for all j. For any $i \geq 1$, the higher unit group $U_\nu^i K_m(F)$ is the subgroup generated by symbols $\{u, a_1, \ldots, a_{m-1}\}$ with $\nu(u-1) \geq i$ and $a_j \in F^\times$ for all j. There are exact sequences

$$0 \longrightarrow U_\nu K_m(F) \longrightarrow K_m(F) \xrightarrow{\partial_\nu} K_{m-1}(\overline{F}) \longrightarrow 0$$

$$0 \longrightarrow U_\nu^1 K_m(F) \longrightarrow U_\nu K_m(F) \xrightarrow{\partial'_\nu} K_m(\overline{F}) \longrightarrow 0$$

where the border homomorphisms ∂_ν, ∂'_ν are characterised by

$$\partial_\nu \{u_1, \ldots, u_{m-1}, t\} = \{\overline{u}_1, \ldots, \overline{u}_{m-1}\}$$

$$\partial'_\nu \{u_1, \ldots, u_m\} = \{\overline{u}_1, \ldots, \overline{u}_m\}$$

for all u_j with $\nu(u_j) = 0$ and all t with $\nu(t) = 1$.

If L/F is a finite extension then there is a norm homomorphism $N_{L/F} : K_m(L) \to K_m(F)$ satisfying the following properties [1, Chap. 1, §5] [2, Chap. IX, §3] [6, §1.7]:

1. Linearity: $N_{L/F} : K_*(L) \to K_*(F)$ is a homomorphism of left $K_*(F)$-modules.
2. Transitivity: If M is an intermediate extension then $N_{L/F} = N_{M/F} N_{L/M}$.
3. Reciprocity: $\sum_\nu N_{F(\nu)/F} \partial_\nu : K_m(F(X)) \to K_{m-1}(F)$ is zero, where ν ranges over all discrete valuations on $F(X)$ which are trivial on F and $F(\nu)$ denotes the residue field of $F(X)$ at ν.

Remark 2.2 If $\nu = \nu_f$ is the discrete valuation on $F(X)$ associated to a monic irreducible polynomial $f(X)$, and $g_1,\dots,g_m \in F[X]$ are coprime to f, then $\nu(g_j) = 0$ for each j and so the characterising property of ∂_ν implies

$$\partial_\nu(\{g_1(X),\dots,g_m(X), f(X)\}) = \{g_1(\overline{X}),\dots,g_m(\overline{X})\},$$

where $\overline{X}$ is the image of X in $F(\nu)$. Also note that these conditions imply

$$\partial_\nu(\{g_1(X),\dots,g_{m-1}(X)\}) = 0.$$

Apart from the discrete valuations on $F(X)$ of the form ν_f, for $f(X)$ a monic irreducible polynomial, there is also the valuation at infinity, ν_∞, characterised by $\nu_\infty(X^{-1}) = 1$.

Now let F be a complete, discrete valuation field, with valuation ν, integers $\mathcal{O}_F$, prime ideal $\mathfrak{p}_F$, residue field $\overline{F}$, and unit group filtration $U_F \supseteq U_F^1 \supseteq U_F^2 \supseteq \dots$. Since ν is the unique (normalised) discrete valuation on F, there is no harm in writing $UK_m(F) = U_\nu K_m(F)$, $\partial = \partial_F = \partial_\nu$, etc. If L is a finite extension of F then the following diagram commutes [6, Lem. 16]:

$$\begin{array}{ccc} K_m(L) & \xrightarrow{\partial_L} & K_{m-1}(\overline{L}) \\ {\scriptstyle N_{L/F}}\downarrow & & \downarrow{\scriptstyle N_{\overline{L}/\overline{F}}} \\ K_m(F) & \xrightarrow[\partial_F]{} & K_{m-1}(\overline{F}). \end{array}$$

Since $\operatorname{Ker}\partial_F = UK_m(F)$ and similarly for L, we see that $N_{L/F}(UK_m(L)) \subseteq UK_m(F)$.

Now we begin proving some results.

Lemma 2.3 *Let L/F be a finite extension of complete, discrete valuation fields. Then the following diagram commutes:*

$$\begin{array}{ccc} UK_m(L) & \xrightarrow{\partial'_L} & K_m(\overline{L}) \\ {\scriptstyle N_{L/F}}\downarrow & & \downarrow{\scriptstyle e(L/F)N_{\overline{L}/\overline{F}}} \\ UK_m(F) & \xrightarrow[\partial'_F]{} & K_m(\overline{F}), \end{array}$$

where $e(L/F)$ is the ramification degree of L/F.

Proof: Let t be a uniformiser for F, and consider the following diagram of homomorphisms:

$$\begin{tikzcd}
UK_m(L) \arrow[r, "\times\{t\}"] \arrow[d, "N_{L/F}"'] \arrow[rr, bend left, "e(L/F)\partial'_L"] & K_{m+1}(L) \arrow[r, "\partial_L"] \arrow[d, "N_{L/F}"] & K_m(\overline{L}) \arrow[d, "N_{\overline{L}/\overline{F}}"] \\
UK_m(F) \arrow[r, "\times\{t\}"] \arrow[rr, bend right, "\partial'_F"'] & K_{m+1}(F) \arrow[r, "\partial_F"] & K_m(\overline{F})
\end{tikzcd}$$

where $\times\{t\}$ denotes right multiplication by the symbol $\{t\}$.

By the paragraph preceding the lemma, $N_{L/F}$ does take $UK_m(L)$ to $UK_m(F)$ and the inner right square commutes. The inner left square commutes by linearity of the norm map. The two outer 'triangles' commute by the characterising properties of the border maps. Hence the outer 'square' of the diagram commutes, which, after moving $e(L/F)$, is the desired result. □

Corollary 2.4 *Let L/F be a finite extension of complete, discrete valuation fields. Then $N_{L/F}(U^1K_m(L)) \subseteq U^1K_m(F)$.*

Proof: Immediate from the previous lemma and the identity $U^1K_m(F) = \operatorname{Ker}\partial'_F$ (and similarly for L). □

A lemma is required before the main proof:

Lemma 2.5 *Let F be a field, let $c \in F^\times$, and let $g_1,\dots,g_{m-1}$ be distinct, monic irreducible polynomials over F, none equal to $X - c^{-1}$. For $j = 1,\dots,m-1$, let α_j be a root of g_j and set $M_j = F(\alpha_j)$. Then*

$$\sum_{j=1}^{m-1}(-1)^{m-1-j} N_{M_j/F}(\{1-c\alpha_j, g_1(\alpha_j),\dots,g_{j-1}(\alpha_j), g_{j+1}(\alpha_j),\dots,g_{m-1}(\alpha_j)\})$$
$$+(-1)^{m-1}\{g_1(c^{-1}),\dots,g_{m-1}(c^{-1})\}$$
$$-\left(\prod_{j=1}^{m-1}\deg g_j\right)\{c,-1,\dots,-1\}$$

is zero in $K_{m-1}(F)$.

Proof: Set

$$\zeta := \{1-cX, g_1(X), \dots, g_{m-1}(X)\} \in K_m(F(X)).$$

Let ν be a discrete valuation on $F(X)$ which is trivial on F; then remark 2.2 implies $\partial_\nu(\zeta) = 0$ unless ν is one of the following valuations:

$$\nu_{X-c^{-1}}, \nu_{g_1}, \dots, \nu_{g_{m-1}}, \nu_\infty.$$

We now calculate $\partial_\nu(\zeta)$ for each of these valuations ν:

1. $\nu = \nu_{X-c^{-1}}$. Remark 2.2 implies
$$\partial_{\nu_{X-c^{-1}}}(\zeta) = (-1)^{m-1}\{g_1(c^{-1}), \dots, g_{m-1}(c^{-1})\} \in K_{m-1}(F).$$

2. $\nu = \nu_{g_j}$. The same remark implies
$$\begin{aligned}&\partial_{\nu_{g_j}}(\zeta) = \\ &\quad (-1)^{m-1-j}\{1-c\alpha_j, g_1(\alpha_j), \dots, g_{j-1}(\alpha_j), g_{j+1}(\alpha_j), \dots, g_{m-1}(\alpha_j)\} \\ &\quad \in K_{m-1}(M_j),\end{aligned}$$
where we identify M_j with $F(\nu_{g_j}) = F[X]/\langle g_j\rangle$ and α_j with X mod g_j.

3. $\nu = \nu_\infty$. For any monic polynomial $g(X)$, it is clear that $X^{-\deg g} g(X)$ is a principal unit with respect to ν_∞. Since any symbol containing a principal unit is killed by the border map ∂_{ν_∞} (since $U^1_{\nu_\infty} K_m(F(X)) \subseteq U_{\nu_\infty} K_m(F(X)) = \operatorname{Ker}\partial_{\nu_\infty}$), we deduce that
$$\begin{aligned}\partial_{\nu_\infty}(\zeta) &= \partial_{\nu_\infty}(\{-c, X^{\deg g_1}, \dots, X^{\deg g_{m-1}}\}) \\ &\quad + \partial_{\nu_\infty}(\{X, X^{\deg g_1}, \dots, X^{\deg g_{m-1}}\}) \\ &= \left(\prod_{j=1}^{m-1} \deg g_j\right) \partial_{\nu_\infty}(\{-c, -1, \dots, -1, X\}) \\ &\quad + \left(\prod_{j=1}^{m-1} \deg g_j\right) \partial_{\nu_\infty}(\{-1, -1, \dots, -1, X\}) \\ &= -\left(\prod_{j=1}^{m-1} \deg g_j\right) \{c, -1, \dots, -1\} \in K_{m-1}(F).\end{aligned}$$

Now apply the reciprocity law for the norm map to ζ using identities 1 – 3 to complete the proof. □

We reach the main theorem:

(Re)proof of theorem 1.1: For any fixed m, we will prove the result by induction on the degree $d = |L : F|$ since it is trivial when $L = F$. If M is a subextension strictly between L and F, and the result is true for the two extensions L/M and M/F, then the result holds for L/F. Therefore we may also assume that no subextension strictly between L and F exists.

We start with the classical case $m = 1$ for the sake of completeness. A typical element of U_L^{ei} can be written in the form $1 - c\alpha$, where $c \in \mathfrak{p}_F^i$ and $\alpha \in \mathcal{O}_L$. If $c = 0$ or $\alpha \in \mathcal{O}_F$, then $N_{L/F}(1 - c\alpha) = (1 - c\alpha)^{|L:F|}$ and the result is clear. So we may assume $c \neq 0$ and $\alpha \notin \mathcal{O}_F$, whence $F(\alpha)$ is a strict extension of F and so $F(\alpha) = L$. Let $f(X) = X^d + a_{d-1}X^{d-1} + \cdots + a_0$ be the minimal polynomial of α over F; note that $a_j \in \mathcal{O}_F$ for all j since $\alpha \in \mathcal{O}_L$. Then the minimal polynomial of $1 - c\alpha$ over F is $(-c)^d f(c^{-1}(1 - X))$; since $1 - c\alpha$ generates L, the constant term of this polynomial is exactly $(-1)^d N_{L/F}(1 - c\alpha)$. Therefore this norm is

$$c^d f(c^{-1}) = 1 + a_{d-1}c + \cdots + a_0 c^d,$$

which belongs to U_F^i since $c \in \mathfrak{p}_F^i$, as required.

We now proceed inductively on m, so suppose $m > 1$; as explained above, we may assume that L/F has no proper subextensions. Any element of $U^{ei}K_m(L)$ is a sum of symbols of the form $\{1 - c\alpha\}\xi$, with $\xi \in K_{m-1}(L)$, $c \in \mathfrak{p}_F^i$, and $\alpha \in \mathcal{O}_L$. If $c = 0$ or $\alpha \in \mathcal{O}_F$, then

$$N_{L/F}(\{1 - c\alpha\}\xi) = \{1 - c\alpha\}N_{L/F}(\xi)$$

by linearity of the norm map, and $1 - c\alpha \in U_F^i$, so there is nothing more to show. Else, as in the case $m = 1$, we have $L = F(\alpha)$. Lemma 2.1 now implies that ξ is a sum of symbols of the form

$$\{a_1, \ldots, a_{s-1}, g_s(\alpha), \ldots, g_{m-1}(\alpha)\}$$

with $1 \le s \le m$, $a_1, \ldots, a_{s-1} \in F^\times$, $g_j(X) \in F[X]$ monic irreducible polynomials, and $0 < \deg g_s < \cdots < \deg g_{m-1} < |L : F|$. Clearly it is enough to assume that ξ is such a symbol, and we now treat two easy cases. Firstly, if $s > 1$, then

$$\begin{aligned} &N_{L/F}(\{1 - c\alpha, a_1, \ldots, a_{s-1}, g_s(\alpha), \ldots, g_{m-1}(\alpha)\}) \\ &\qquad = -\{a_1\}N_{L/F}(\{1 - c\alpha, a_2, \ldots, a_{s-1}, g_s(\alpha), \ldots, g_{m-1}(\alpha)\}), \end{aligned}$$

which the inductive hypothesis on m implies belongs to $U^i K_m(F)$. Secondly, if $s = 1$ but $g_1(X) = X - c^{-1}$, then

$$\begin{aligned} \{1 - c\alpha, g_1(\alpha), \ldots, g_{m-1}(\alpha)\} &= \{1 - c\alpha, -c^{-1}(1 - c\alpha), g_2(\alpha), \ldots, g_{m-1}(\alpha)\} \\ &= \{1 - c\alpha, -c^{-1}, g_2(\alpha), \ldots, g_{m-1}(\alpha)\} \\ &\quad + \{1 - c\alpha, -1, g_2(\alpha), \ldots, g_{m-1}(\alpha)\}, \end{aligned}$$

and the norm of this belongs to $U^i K_m(F)$ by the case $s > 1$ which we just considered.

Therefore the proof will be complete if we can show that $N_{L/F}(\zeta) \in U^i K_m(F)$ when

$$\zeta = \{1 - c\alpha, g_1(\alpha), \dots, g_{m-1}(\alpha)\},$$

where $g_j(X) \in F[X]$ are monic irreducible polynomials with $0 < \deg g_1 < \cdots < \deg g_{m-1} < |L : F|$ and $g_1(X) \neq X - c^{-1}$; we will do this by explicitly computing its norm. Set

$$\widetilde{\zeta} = \{1 - cX, g_1(X), \dots, g_{m-1}(X), f(X)\} \in K_{m+1}(F(X)),$$

where $f(X) \in \mathcal{O}_F[X]$ is the minimal polynomial of α. Let ν be a discrete valuation on $F(X)$ which is trivial on F. By remark 2.2 (and similarly to the previous lemma), $\partial_\nu(\widetilde{\zeta}) = 0$ unless ν is one of the following valuations:

$$\nu_{X-c^{-1}}, \nu_{g_1}, \dots, \nu_{g_{m-1}}, \nu_f, \nu_\infty.$$

When ν is one of these valuations, we now calculate $\partial_\nu(\widetilde{\zeta})$ as in the previous lemma:

1. $\nu = \nu_{X-c^{-1}}$. Then

$$\partial_{\nu_{X-c^{-1}}}(\widetilde{\zeta}) = (-1)^m \{g_1(c^{-1}), \dots, g_{m-1}(c^{-1}), f(c^{-1})\} \in K_m(F).$$

2. $\nu = \nu_{g_j}$. Then

$$\begin{aligned}&\partial_{\nu_{g_j}}(\widetilde{\zeta}) = \\ &\quad (-1)^{m-j}\{1 - c\alpha_j, g_1(\alpha_j), \dots, g_{j-1}(\alpha_j), g_{j+1}(\alpha_j), \dots, g_{m-1}(\alpha_j), f(\alpha_j)\} \\ &\quad \in K_m(M_j),\end{aligned}$$

   where $M_j := F(\nu_{g_j}) = F[X]/\langle g_j \rangle$ and $\alpha_j := X \bmod g_j$. We denote this symbol in $K_m(M_j)$ by $(-1)^{m-j}\zeta_j$ to ease notation.

3. $\nu = \nu_f$. Then

$$\partial_{\nu_f}(\widetilde{\zeta}) = \zeta \in K_m(L),$$

   identifying L with $F(\nu_f) = F[X]/\langle f \rangle$ and α with $X \bmod f$. This is exactly why we are studying $\widetilde{\zeta}$.

4. $\nu = \nu_\infty$. Then, just as in the previous lemma,

$$\begin{aligned}\partial_{\nu_\infty}(\widetilde{\zeta}) &= -d\left(\prod_{j=1}^{m-1}\deg g_j\right)\{c, \underbrace{-1,\dots,-1}_{m-1\text{ times}}\}\\ &= -\left(\prod_{j=1}^{m-1}\deg g_j\right)\{c, \underbrace{-1,\dots,-1}_{m-2\text{ times}}, c^{-d}\} \in K_m(F).\end{aligned}$$

where the second line follows from the first via the identity $\{c,-1\} = \{c,c^{-1}\}$.

From the reciprocity law for the norm map we now obtain from 1 – 4 the identity

$$\begin{aligned}-N_{L/F}(\zeta) = &\sum_{j=1}^{m-1}(-1)^{m-j}N_{M_j/F}(\zeta_j)\\ &+(-1)^m\{g_1(c^{-1}),\dots,g_{m-1}(c^{-1}),f(c^{-1})\}\\ &-\left(\prod_{j=1}^{m-1}\deg g_j\right)\{c,-1,\dots,-1,c^{-d}\}.\end{aligned}$$

As we already observed in the case $m = 1$, $c^d f(c^{-1})$ belongs to U_F^i; therefore

$$\{g_1(c^{-1}),\dots,g_{m-1}(c^{-1}),f(c^{-1})\} \equiv \{g_1(c^{-1}),\dots,g_{m-1}(c^{-1}),c^{-d}\} \mod U^i K_m(F).$$

Similarly, we claim that $\{1-c\alpha_j, f(\alpha_j)\} \equiv \{1-c\alpha_j, c^{-d}\} \mod U^{e(M_j/F)i}K_2(M_j)$ for each $j = 1,\dots,m-1$; for convenience, fix j and write $\alpha = \alpha_j$, $M = M_j$. If $\alpha \in \mathcal{O}_M$ then $1 - c\alpha \in U_M^{e(M/F)i}$ so the claim is clear. Else $\alpha^{-1} \in \mathfrak{p}_M$ and so we write $\alpha^{-d} f(\alpha) = 1 - A\alpha^{-1}$, where

$$A := -(a_{d-1} + a_{d-2}\alpha^{-1} + \cdots + a_0\alpha^{-d+1}) \in \mathcal{O}_M$$

(using the same notation for f as in the case $m = 1$), and then use the identity

$$\{1-c\alpha, \alpha^{-d} f(\alpha)\} = -\left\{(1-c\alpha)A\alpha^{-1}, \underbrace{1+\frac{Ac}{1-A\alpha^{-1}}}_{(\dagger)}\right\}$$

(see footnote[1]). Since (†) belongs to $U_M^{e(M/F)i}$, we get $\{1 - c\alpha, \alpha^{-d} f(\alpha)\} \in U^{e(M/F)i}K_2(M)$. But the Steinberg identity implies $\{1 - c\alpha, \alpha^d\} = \{1 - c\alpha, c^{-d}\}$,

[1] $\{1-x,1-y\} = -\{\frac{1}{1-x}, x(1-y)\} = -\{\frac{1-x(1-y)}{1-x}, x(1-y)\} = \{x(1-y), 1+\frac{xy}{1-x}\}$.

proving the claim. The claim implies that

$$\zeta_j \equiv \{1-c\alpha_j, g_1(\alpha_j),\ldots,g_{j-1}(\alpha_j),g_{j+1}(\alpha_j),\ldots,g_{m-1}(\alpha_j),c^{-d}\} \mod U^{e(M_j/F)i} K_m(M_j)$$

for $j=1,\ldots,m-1$.

Applying the inductive hypothesis, which is valid since each extension M_j/F has degree $< |L:F|$, now reveals that

$$\begin{aligned} -N_{L/F}(\zeta) \equiv{}& \\ &\sum_{j=1}^{m-1}(-1)^{m-j} N_{M_j/F}(\{1-c\alpha_j, g_1(\alpha_j),\ldots,g_{j-1}(\alpha_j),g_{j+1}(\alpha_j),\ldots,g_{m-1}(\alpha_j),c^{-d}\}) \\ &+(-1)^m\{g_1(c^{-1}),\ldots,g_{m-1}(c^{-1}),c^{-d}\} \\ &-\left(\prod_{j=1}^{m-1}\deg g_j\right)\{c,-1,\ldots,-1,c^{-d}\} \mod U^i K_m(F). \end{aligned}$$

All that remains to do is to multiply the identity of the previous lemma on the right by $\{c^{-d}\}$; hence $N_{L/F}(\zeta)\equiv 0 \mod U^i K_m(F)$, which completes the proof. □

Remark 2.6 The key step of the proof where completeness was required was at the beginning when $m=1$: it ensures that $\mathcal{O}_L$ is the integral closure of $\mathcal{O}_F$ and therefore the minimal polynomial $f(X)$ of α has coefficients in $\mathcal{O}_F$. This result, and therefore the theorem, remains true for a finite extension L/F of Henselian, discrete valuation fields.

Remark 2.7 I am indebted to the referee for suggesting that I look at S. Rosset and J. Tate's paper [13], which contains the following reciprocity law: If $f(X), g(X)$ are distinct, monic irreducible polynomials over a field F, neither equal to X, and with roots y, x respectively, then

$$N_{F(x)/F}(\{x, f(x)\}) = N_{F(y)/F}(\{y, g(0)^{-1}g(y)\}) - \{(-1)^{\deg g} g(0)^{-1}, (-1)^{\deg f}\}. \tag{RT}$$

(This is an unravelling of their formula $\left(\frac{f}{g}\right)=\left(\frac{g^*}{f}\right)-\{c(g^*),c(f)\}$). This can be quickly deduced by applying the argument of lemma 2.5 to $\zeta=\{X,g(X),f(X)\}$.

Alternatively, this reciprocity law can be obtained directly, though somewhat tediously, from lemma 2.5 (with $m=3$) by changing variables $X \rightsquigarrow 1-cX$ as follows. Set $g_1(X)=(-c)^{-\deg f} f(1-cX)$ and $g_2(X)=(-c)^{-\deg g} g(1-cX)$. These are distinct, monic irreducible polynomials, neither equal to $X-c^{-1}$, and with respective roots $\alpha_1 = c^{-1}(1-y)$ and $\alpha_2 = c^{-1}(1-x)$; so lemma 2.5 says that

$$\begin{aligned} -N_{F(\alpha_1)/F}(\{1-c\alpha_1, g_2(\alpha_1)\}) + N_{F(\alpha_2)/F}(\{1-c\alpha_2, g_1(\alpha_2)\}) \\ +\{g_1(c^{-1}), g_2(c^{-1})\} - \deg g_1 \deg g_2 \{c,-1\} = 0 \end{aligned}$$

in $K_2(F)$. That is,

$$\begin{aligned} -N_{F(y)/F}(\{y,(-c)^{-\deg g} g(y)\}) + N_{F(x)/F}(\{x,(-c)^{-\deg f} f(x)\}) \\ +\{(-c)^{-\deg f} f(0), (-c)^{-\deg g} g(0)\} - \deg f \deg g \{c,-1\} = 0. \end{aligned} \quad (1)$$

The straightforward identities

$$\begin{aligned} N_{F(y)/F}(\{y,(-c)^{-\deg g}\}) &= \{(-1)^{\deg f} f(0), (-c)^{-\deg g}\} \\ N_{F(x)/F}(\{x,(-c)^{-\deg f}\}) &= \{(-1)^{\deg g} g(0), (-c)^{-\deg f}\} \\ \{(-c)^{-\deg f} f(0), (-c)^{-\deg g} g(0)\} &= \{(-1)^{-\deg f} f(0), (-c)^{-\deg g}\} \\ &\quad + \{c^{-\deg f}, (-c)^{-\deg g}\} \\ &\quad + \{(-c)^{-\deg f}, (-1)^{\deg g} g(0)\} \\ &\quad - \{(-c)^{-\deg f}, (-1)^{\deg g}\} + \{f(0), g(0)\} \end{aligned}$$

transform (1) to

$$\begin{aligned} -N_{F(y)/F}(\{y, g(y)\}) + N_{F(x)/F}(\{x, f(x)\}) \\ +\{c^{-\deg f}, (-c)^{-\deg g}\} - \{(-c)^{-\deg f}, (-1)^{\deg g}\} \\ +\{f(0), g(0)\} - \deg f \deg g \{c,-1\} = 0. \end{aligned} \quad (2)$$

The terms in this expression involving c are

$$\deg f \deg g (\{c,-c\} + \{-c,-1\} - \{c,-1\}) = \deg f \deg g \{-1,-1\},$$

so that (2) becomes

$$\begin{aligned} -N_{F(y)/F}(\{y, g(y)\}) + N_{F(x)/F}(\{x, f(x)\}) \\ +\deg f \deg g \{-1,-1\} + \{f(0), g(0)\} = 0. \end{aligned} \quad (3)$$

Finally use $N_{F(y)/F}(\{y, g(0)^{-1}\}) = \{(-1)^{\deg f} f(0), g(0)^{-1}\}$ to turn (3) into (RT).

I would not be surprised if lemma 2.5 were to have applications beyond the proof of theorem 1.1, but I do not know of any at present.

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