A note on higher direct images in crystalline cohomology

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Abstract

We check that Ogus' convergent F-isocrystal, associated to a proper smooth morphism of smooth varieties in characteristic p, is precisely the higher direct image in crystalline cohomology. We also show independently that these higher direct images satisfy base change and Hard Lefschetz theorems.

CONTENTS

1 INTRODUCTION

Let $f : X \to S$ be a proper smooth morphism of smooth varieties over a perfect field k of characteristic p, whose ring of Witt vectors will be denoted by $W = W(k)$, and let $i \geq 0$. In this short note we check various properties about the higher direct image (in crystalline cohomology) $R^i f_{\mathrm{crys}*}\mathcal{O}_{X/W}$. In particular we show directly, using only the classical machinery of crystalline cohomology, that it (or rather, its isogeny class) is naturally an F-isocrystal on $(S/W)_{\mathrm{crys}}$ which satisfies base change and Hard Lefschetz theorems.

In Section 6 we show that the isogeny class of $R^i f_{\mathrm{crys}*}\mathcal{O}_{X/W}$ is nothing other than the convergent F-isocrystal $R^i f_*\mathcal{O}_{X/K}$ constructed by Ogus [8]; we refer to Section 6 for the precise assertion. The reader who is primarily interested in this identification may skip Sections 4 and 5; in fact, once this identification is established the results of Sections 4 and 5 may be recovered from [8], but our proofs have the advantage, as mentioned above, of using only classical crystalline cohomology. Strangely, this identification has not previously been checked in the literature, though it was asserted without proof by Trihan–Matsuda [7, Corol. 3].

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2 Crystalline cohomology and isocrystals

We first introduce some notation and terminology. Given a smooth variety X over k, the crystalline site $(X/W)_{\mathrm{crys}}$ we use is the one whose objects are data $(U \hookrightarrow T, \gamma)$, where: U is an open subscheme of X; $U \hookrightarrow T$ is a nilpotent closed embedding of W-schemes; and γ is a divided power structure on the ideal sheaf $\mathrm{Ker}(\mathcal{O}_T \twoheadrightarrow \mathcal{O}_U)$ of $\mathcal{O}_T$ which is required to be compatible with the canonical pd-structure on pW, i.e., it is required to satisfy $\gamma^n(p) = p^n/n!$ for all $n \geq 0$. The structure sheaf $\mathcal{O}_{X/W}$ on $(X/W)_{\mathrm{crys}}$ is given by $(U \hookrightarrow T, \gamma) \mapsto \Gamma_{\mathrm{Zar}}(T, \mathcal{O}_T)$, and we denote by $\mathrm{Mod}(X/W)$ the category of $\mathcal{O}_{X/W}$-modules on $(X/W)_{\mathrm{crys}}$. Given $\mathcal{E} \in \mathrm{Mod}(X/W)$ and an object $(U \hookrightarrow T, \gamma) \in (X/W)_{\mathrm{crys}}$, the associated $\mathcal{O}_T$-module (the "value" in most of the literature) on T_{Zar} is denoted by $\mathcal{E}_T$; then $\mathcal{E}$ is called a crystal[1] if and only if each $\mathcal{E}_T$ is a coherent $\mathcal{O}_T$-module and, for every morphism $h : (U' \hookrightarrow T', \gamma') \to (U \hookrightarrow T, \gamma)$ in $(X/W)_{\mathrm{crys}}$, the canonical base change map $h^*\mathcal{E}_T \to \mathcal{E}_{T'}$ of coherent $\mathcal{O}_{T'}$-modules is an isomorphism. Thus crystals form a full subcategory[2] $\mathrm{Cr}(X/W)$ of $\mathrm{Mod}(X/W)$.

More generally, the theory of the previous paragraphs applies whenever W is replaced by, for example, any Noetherian, p-adically complete, flat W-algebra $\mathcal{A}$ and X is replaced by a smooth scheme over $\mathcal{A}/p$.

We denote by $\mathrm{Mod}(X/W) \otimes \mathbb{Q}_p$ and $\mathrm{Isoc}(X/W) := \mathrm{Cr}(X/W) \otimes \mathbb{Q}_p$ the isogeny categories obtained by replacing Hom by $\mathrm{Hom} \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{p}]$; the image of an object $\mathcal{E}$ will be denoted by $\mathcal{E} \otimes \mathbb{Q}_p$ and called the *isogeny class* of $\mathcal{E}$. It is convenient, and hopefully not misleading, to say that an object of $\mathrm{Mod}(X/W) \otimes \mathbb{Q}_p$ "is" an isocrystal if and only if it belongs to the essential image of the fully faithful inclusion $\mathrm{Isoc}(X/W) \to \mathrm{Mod}(X/W) \otimes \mathbb{Q}_p$.

A morphism $f : X \to S$ of smooth k-varieties induces a morphism of ringed topoi

$$f : ((X/W)_{\mathrm{crys}}, \mathcal{O}_{X/W}) \longrightarrow ((S/W)_{\mathrm{crys}}, \mathcal{O}_{S/W}),$$

with associated adjoint pair denoted by

$$\begin{aligned} f_{\mathrm{crys}*} &: \mathrm{Mod}(X/W) \longrightarrow \mathrm{Mod}(S/W), \quad \mathcal{E} \mapsto f_{\mathrm{crys}*}\mathcal{E}, \\ f^*_{\mathrm{crys}} &: \mathrm{Mod}(S/W) \longrightarrow \mathrm{Mod}(X/W), \quad \mathcal{E} \mapsto f^{-1}_{\mathrm{crys}}\mathcal{E} \otimes_{f^{-1}\mathcal{O}_{S/W}} \mathcal{O}_{X/W} \end{aligned}$$

Here $(f_{\mathrm{crys}*}, f^{-1}_{\mathrm{crys}}) : (X/W)_{\mathrm{crys}} \to (S/W)_{\mathrm{crys}}$ is the usual morphism of topoi of sheaves of sets. The resulting higher direct images, computed in terms of resolutions by injective $\mathcal{O}_{X/W}$-modules, are denoted by $R^i f_{\mathrm{crys}*} : \mathrm{Mod}(X/W) \longrightarrow \mathrm{Mod}(S/W)$.

Our first main result is the following:

Proposition 2.1. *Let $f : X \to S$ be a proper smooth morphism of smooth varieties over k. Then the object $R^i f_{crys*}\mathcal{O}_{X/W} \otimes \mathbb{Q}_p \in \mathrm{Mod}(X/W) \otimes \mathbb{Q}_p$ is an isocrystal.*

Proof. Given $\mathcal{E} \in \mathrm{Mod}(S/W)$, the question of whether $\mathcal{E} \otimes \mathbb{Q}_p$ is an isocrystal is Zariski local on S, by [9, Lem. 0.7.5], so we may suppose that $S = \mathrm{Spec}\, A$ is affine. Since a morphism in a $\mathbb{Z}_p$-linear abelian category becomes an isomorphism in the isogeny category if and only if it had kernel and cokernel killed by a power of p (i.e., an *isogeny*), the proposition reduces to the following lemma: ☐

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[1] From the general point of view of ringed topi, a crystal is exactly a quasi-coherent $\mathcal{O}_{X/W}$-module of finite presentation, by [1, Prop. IV.1.1.3].

[2] Irrelevant warning: Although both categories are abelian (crystals by [1, Prop. IV.1.7.6]), the inclusion functor $\mathrm{Cr}(X/W) \to \mathrm{Mod}(X/W)$ does not preserve kernels.

Lemma 2.2. *Let $f : X \to S$ be a proper smooth morphism of smooth varieties over k; assume that $S = \operatorname{Spec} A$ is affine. Then there exists $\mathcal{E}^i_{X/S} \in \mathrm{Cr}(S/W)$ and a morphism of $\mathcal{O}_{S/W}$-modules $\mathcal{E}^i_{X/S} \to R^i f_{crys*}\mathcal{O}_{X/W}$ whose kernel and cokernel are killed by a power of p.*

To prove Lemma 2.2 we briefly recall the standard description of crystals on a smooth affine base:

Lemma 2.3. *Suppose that $\operatorname{Spec} A$ is a smooth affine variety over k, and let $\mathcal{A}$ be a p-adically complete, formally smooth W-algebra lifting A. Then the following categories are equivalent:*

- $\mathrm{Cr}(\operatorname{Spec} A/W)$*;*
- *finitely generated $\mathcal{A}$-modules with HPD-stratification;*
- *finitely generated $\mathcal{A}$-modules with a topologically quasi-nilpotent integrable connection.*

Proof. To explain “HPD-stratification” in this context, let $\mathcal{A}(1)$ and $\mathcal{A}(2)$ be the p-adic completions of the pd-envelopes of the “multiplication followed by mod p” maps $\mathcal{A}\widehat{\otimes}_W\mathcal{A} \to A$ and $\mathcal{A}\widehat{\otimes}_W\mathcal{A}\widehat{\otimes}_W\mathcal{A} \to A$, and let

$$p_1, p_2 : \mathcal{A} \to \mathcal{A}(1), \qquad \mu : \mathcal{A}(1) \to \mathcal{A}, \qquad p_{12}, p_{13}, p_{22} : \mathcal{A}(1) \to \mathcal{A}(2)$$

be the maps induced by the obvious projections and multiplication. Standard properties of pd-envelopes imply that $\mathcal{A}(1)/p^r$ is the pd-envelope of the “multiplication followed by mod p” map $\mathcal{A}/p^r \otimes_{W_r} \mathcal{A}/p^r \to A$, and similarly for $\mathcal{A}(2)$, for any $r \geq 1$; moreover, each map $p_i : \mathcal{A} \to \mathcal{A}(1)$ is flat by [3, Corol. 3.35], and similarly for the maps $\mathcal{A}(1) \to \mathcal{A}(2)$.

An HPD-stratification on a finitely generated $\mathcal{A}$-module M is by definition an isomorphism $\varepsilon : M \otimes_{\mathcal{A},p_1} \mathcal{A}(1) \xrightarrow{\sim} M \otimes_{\mathcal{A},p_2} \mathcal{A}(1)$ of $\mathcal{A}(1)$-modules which satisfies the usual cocycle conditions, namely that $\mu^*(\varepsilon) : M \to M$ is the identity and that $p_{12}^*(\varepsilon) \circ p_{23}^*(\varepsilon) = p_{13}^*(\varepsilon)$.

A finitely generated $\mathcal{A}$-module M with HPD-stratification ε induces a crystal $\mathcal{E}(M,\varepsilon)$ as follows: for each object $(U \hookrightarrow T, \gamma)$ in $(\operatorname{Spec} A/W)_{\mathrm{crys}}$, the formal smoothness of $W \to \mathcal{A}$ implies the existence of a map of W-schemes $g_T : T \to \operatorname{Spec} \mathcal{A}$ lifting the inclusion $U \subseteq \operatorname{Spec} A$, whence $\mathcal{E}(M,\varepsilon)_T := g^*M$ is a coherent $\mathcal{O}_T$-module. Given a morphism $h : (U_2 \hookrightarrow T_2, \gamma_2) \to (U_1 \hookrightarrow T_1, \gamma_1)$ in $(X/W)_{\mathrm{crys}}$, the universal property of $\mathcal{A}(1)$ implies the existence of a unique morphism $g_{T_1,T_2} : T_2 \to \operatorname{Spec} \mathcal{A}(1)$ which is compatible with the pd structures on both sides and satisfies

$$p_1 \circ g_{T_1,T_2} = g_{T_1} \circ h \text{ and } p_2 \circ g_{T_1,T_2} = g_{T_2} : T_2 \to \operatorname{Spec} \mathcal{A}$$

Then $g^*_{T_1,T_2}(\varepsilon)$ defines an isomorphism $h^*g^*_{T_1}M \xrightarrow{\sim} g^*_{T_2}M$ of $\mathcal{O}_{T_2}$-modules, i.e., $h^*\mathcal{E}(M,\varepsilon)_{T_2} \xrightarrow{\sim} \mathcal{E}(M,\varepsilon)_{T_1}$. The cocycle conditions on ε are exactly designed so that these isomorphisms give the association $(U \hookrightarrow T, \gamma) \mapsto \mathcal{E}(M,\varepsilon)_T$ the structure of a coherent $\mathcal{O}_{\operatorname{Spec} A/W}$-module, even a crystal, which up to isomorphism does not depend on the chosen maps $\{g_T : (U \hookrightarrow T, \gamma) \in (\operatorname{Spec} A/W)_{\mathrm{crys}}\}$.

In the other direction, to a crystal $\mathcal{E}$ we associate the finitely generated $\mathcal{A}$-module $\mathcal{E}(\mathcal{A}) := \varprojlim_r \mathcal{E}(\operatorname{Spec} A \hookrightarrow \operatorname{Spec} \mathcal{A}/p^r, \gamma_r)$, where γ_r is the canonical pd-structure on the ideal $p\mathcal{A}/p^r\mathcal{A}$, satisfying $\gamma_r^n(p) = p^n/n!$ for all $n \geq 0$. The HPD-structure on $\mathcal{E}(\mathcal{A})$ is given by $\varprojlim_r$ of the base change isomorphisms (where we omit writing all the canonical pd-structures):

$$\begin{tikzcd}
\mathcal{E}(\operatorname{Spec} A \hookrightarrow \operatorname{Spec} \mathcal{A}/p^r) \otimes_{\mathcal{A}/p^r, p_1} \mathcal{A}(1)/p^r \arrow[d, "\cong"'] \\
\mathcal{E}(\operatorname{Spec} A \overset{\mu}{\hookrightarrow} \operatorname{Spec} \mathcal{A}(1)/p^r) \\
\mathcal{E}(\operatorname{Spec} A \hookrightarrow \operatorname{Spec} \mathcal{A}/p^r) \otimes_{\mathcal{A}/p^r, p_2} \mathcal{A}(1)/p^r \arrow[u, "\cong"]
\end{tikzcd}$$

The cocycle conditions follow from similar base change identities involving $\mathcal{A}(2)/p^r$.

Next, starting with a finitely generated $\mathcal{A}$-module M with HPD-stratification ε, one may construct a connection $\nabla : M \to M \otimes_{\mathcal{A}} \Omega^1_{\mathcal{A}/W}$ (the latter object is the $\mathcal{A}$-module of continuous relative Kähler differentials) as the composition

$$M \to M \otimes_{\mathcal{A},p_1} \operatorname{Ker}\mu \to M \otimes_{\mathcal{A}} \Omega^1_{\mathcal{A}/W}, \qquad m \mapsto (m \otimes 1) - (\mathrm{id} \otimes \mathrm{sw})(\varepsilon(m \otimes 1)).$$

Here $\mathrm{sw} : \mathcal{A}(1) \to \mathcal{A}(1)$ is induced by the "swap map" $\mathcal{A} \otimes_W \mathcal{A} \to \mathcal{A} \otimes_W \mathcal{A}$, and the map $\operatorname{Ker}\mu \to \Omega^1_{\mathcal{A}/W}$ is induced by the isomorphism

$$\Omega^1_{\mathcal{A}/W} \xrightarrow{\simeq} \operatorname{Ker}\mu/(\operatorname{Ker}\mu)^{[2]}, \qquad a\,db \mapsto a \otimes b - b \otimes a$$

where $^{[2]}$ denotes the divided square of a pd-ideal. The cocyle conditions satisfied by ε formally imply that ∇ is integrable. We will not require the definition of "topologically quasi-nilpotent", nor that this construction gives an equivalence of categories. $\square$

Corollary 2.4. *Maintain the set-up of the previous lemma, and let* $\mathcal{E} \in \mathrm{Cr}(\operatorname{Spec} A/W)$*. Then* $\mathcal{E}(\mathcal{A})[\frac{1}{p}]$ *is a finite projective module over* $\mathcal{A}[\frac{1}{p}]$*.*

Proof. By the standard result that modules equipped with an integrable connection over a characteristic zero regular ring with enough differential operators are flat, this follows from the third category of the previous lemma.[3] $\square$

Proof of Lemma 2.2. Write $S = \operatorname{Spec} A$ and let $\mathcal{A}$ be a p-adically complete, formally smooth W-algebra lifting A, as in the previous lemma and corollary.

Base change for crystalline cohomology [3, Thm. 7.8] implies that the canonical base change maps

$$R\Gamma_{\mathrm{crys}}(X/\mathcal{A}) \overset{\mathbb{L}}{\otimes}_{\mathcal{A},p_1} \mathcal{A}(1) \longrightarrow R\Gamma_{\mathrm{crys}}(X/\mathcal{A}(1)) \longleftarrow R\Gamma_{\mathrm{crys}}(X/\mathcal{A}) \overset{\mathbb{L}}{\otimes}_{\mathcal{A},p_2} \mathcal{A}(1)$$

are quasi-isomorphisms in $D(\mathcal{A}(1))$ (to be precise, the cited base change theorem yields isomorphism after applying $\overset{\mathbb{L}}{\otimes}_{\mathbb{Z}} \mathbb{Z}/p^n$ for any $n \geq 0$, but the desired result is the obtained

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[3] It seems that the range of validity of this corollary and the previous lemma has not been explored in the literature. Both results likely remain valid whenever $\mathcal{A}$ is a regular, p-adically complete, W-algebra for which $\mathcal{A}/p\mathcal{A}$ is regular and F-finite (i.e., finitely generated over its subring of p^{th}-powers). Indeed, these hypotheses imply that $\Omega^1_{\mathcal{A}/W}$ is a finite projective $\mathcal{A}$-module and that there are enough differential operators.

by taking Rlim_n and appealing to [3, Thm. 7.24]). Taking cohomology and using the flatness of $p_1, p_2 : \mathcal{A} \to \mathcal{A}(1)$ (as mentioned at the start of the previous lemma), it follows that

$$\varepsilon : H^i_{\mathrm{crys}}(X/\mathcal{A}) \otimes_{\mathcal{A},p_1} \mathcal{A}(1) \cong H^i_{\mathrm{crys}}(X/\mathcal{A}(1)) \cong H^i_{\mathrm{crys}}(X/\mathcal{A}) \otimes_{\mathcal{A},p_2} \mathcal{A}(1),$$

which defines an HPD-stratification ε on $H^i_{\mathrm{crys}}(X/\mathcal{A})$ (that ε satisfies the cocycle conditions is an easy consequence of base change isomorphisms involving $\mathcal{A}(2)$).

As explained in the previous lemma, there is an associated crystal

$$\mathcal{E}^i_{X/S} := \mathcal{E}(H^i_{\mathrm{crys}}(X'/\mathcal{A}), \varepsilon)$$

on X whose value on an object $(U \hookrightarrow T, \gamma)$ is given by $g_T^* H^i_{\mathrm{crys}}(X/\mathcal{A})$ for any chosen map of W-schemes $g_T : T \to \operatorname{Spec} \mathcal{A}$ lifting the inclusion $U \subseteq \operatorname{Spec} A$. Using the base change quasi-isomorphism

$$Lg_T^* R\Gamma_{\mathrm{crys}}(X/\mathcal{A}) \xrightarrow{\sim} (Rf_{\mathrm{crys}*}\mathcal{O}_{X/W})_T$$

of complexes of $\mathcal{O}_T$-modules (c.f., [3, Corol. 7.11]), we may define a map of coherent $\mathcal{O}_T$-modules

$$(\mathcal{E}^i_{X/S})_T = g_T^* H^i_{\mathrm{crys}}(X/\mathcal{A}) \longrightarrow H^i(Lg_T^* R\Gamma_{\mathrm{crys}}(X/\mathcal{A})) \xrightarrow{\simeq} (R^i f_{\mathrm{crys}*}\mathcal{O}_{X/W})_T. \qquad (\dagger)$$

That these maps assemble to give a morphism of $\mathcal{O}_{X/W}$-modules $\mathcal{E}^i_{X/S} \to R^i f_{\mathrm{crys}*}\mathcal{O}_{X/W}$ is a straightforward consequence of the fact that the HPD structure on $\mathcal{E}^i_{X/S}$ was defined using base change.

To complete the proof we must show the existence of a large power of p (independent of the object $(U \hookrightarrow T, \gamma)$) which kills the kernel and cokernel of (†). The obstruction to (†) being an isomorphism comes from the higher Tors $L_j g_T^* H^{i+j}_{\mathrm{crys}}(X/\mathcal{A})$ for $j > 0$. Since $R\Gamma_{\mathrm{crys}}(X/\mathcal{A})$ is bounded by [3, Thm. 7.4.3], it is enough to prove the following claim for each fixed $i \geq 0$ and $j > 0$: there exists a power of p which kills $L_j g_T^* H^i_{\mathrm{crys}}(X/\mathcal{A})$ for every object $(U \hookrightarrow T, \gamma)$. But this claim is a trivial consequence of $M := H^i_{\mathrm{crys}}(X/\mathcal{A})$ being a finitely generated $\mathcal{A}$-module which becomes finite projective after inverting p (by the previous corollary). For example, first pick a finite length resolution $P_\bullet \to M$ by finite projective $\mathcal{A}$-modules; then the fact that $M[\frac{1}{p}]$ is finite projective over $\mathcal{A}[\frac{1}{p}]$ implies that there exists a section $\sigma : M[\frac{1}{p}][0] \to P^\bullet[\frac{1}{p}]$ to the augmentation and that the composition $P^\bullet[\frac{1}{p}] \to M[\frac{1}{p}][0] \xrightarrow{\sigma} P^\bullet[\frac{1}{p}]$ is homotopic to the identity; clearing denominators yields a morphism $\overline{\sigma} : M[0] \to P^\bullet$ such that $M[0] \xrightarrow{\overline{\sigma}} P^\bullet \to M[0]$ is multiplication by a power of p and $P^\bullet \to M[0] \xrightarrow{\overline{\sigma}} P^\bullet$ is homotopic to multiplication by a power of p, say p^N; it follows at once that $L^j g_T^* M$ is killed by p^N for all $j > 0$. This completes the proof. □

This completes the proof of Proposition 2.1, i.e., that $R^i f_{\mathrm{crys}*}\mathcal{O}_{X/W} \otimes \mathbb{Q}_p$ is an isocrystal on $(S/W)_{\mathrm{crys}}$ for any proper smooth morphism $f : X \to S$ of smooth k-varieties.

3 UPGRADING $R^i f_{\mathrm{CRYS}*}\mathcal{O}_{X/W} \otimes \mathbb{Q}_p$ TO AN F-ISOCRYSTAL

Now we consider base change properties and upgrade $R^i f_{\mathrm{crys}*}\mathcal{O}_{X/W} \otimes \mathbb{Q}_p$ to have the structure of an F-isocrystal.

Remark 3.1 (Pull-backs)**.** Suppose that k' is another perfect field of characteristic p and that $\overline{\sigma} : k \to k'$ is a homomorphism, with induced map on Witt vectors denoted by $\sigma : W' := W(k') \to W$. Let S' be a smooth k'-variety, and $j : S' \to S$ a morphism over $\overline{\sigma}$. To eliminate any ambiguity about pull-backs, we mention again that j^*_{crys} denotes the pull-back functor associated to j as a morphism of ringed topoi $((S'/W')_{\mathrm{crys}}, \mathcal{O}_{S'/W'}) \to ((S/W)_{\mathrm{crys}}, \mathcal{O}_{S/W})$.

Let $\mathcal{E}$ be a crystal on $(S/W)_{\mathrm{crys}}$ and recall the following consequences of [1, Corol. IV.1.2.4]. Firstly, $j^*_{\mathrm{crys}}\mathcal{E}$ is a crystal on $(S'/W')_{\mathrm{crys}}$. Secondly, if $S = \operatorname{Spec} A$ and $S' = \operatorname{Spec} A'$ are affine, with p-adically complete, formally smooth W (resp. W')-algebra lifts $\widetilde{j} : \mathcal{A} \to \mathcal{A}'$ over σ, and $\mathcal{E}$ is represented by the finitely generated $\mathcal{A}$-module $\mathcal{E}(\mathcal{A})$ with HPD-stratification, then $g^*_{\mathrm{crys}}(\mathcal{E})$ is represented by the finitely generated $\mathcal{A}'$-module $\mathcal{E}(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{A}'$ with the obvious HPD-stratification obtained by base change; indeed, this is an easy consequence of the isomorphisms

$$\mathcal{E}(\operatorname{Spec} A \hookrightarrow \operatorname{Spec} \mathcal{A}/p^r) \otimes_{\mathcal{A}/p^r} \mathcal{A}'/p^r \overset{\simeq}{\to} (g^*_{\mathrm{crys}}\mathcal{E})(\operatorname{Spec} A' \hookrightarrow \operatorname{Spec} \mathcal{A}'/p^r)$$

(where we omit writing the canonical pd-structures) for $r \geq 1$.

Proposition 3.2. *Let $f : X \to S$ be a proper smooth morphism of smooth varieties over k.*

(1) With $j : S' \to S$ as in the previous remark, let

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "j'"'] & S' \arrow[r] \arrow[d, "j"] & \operatorname{Spec} k' \arrow[r] \arrow[d, "\overline{\sigma}"] & \operatorname{Spec} W' \arrow[d, "\sigma"] \\ X \arrow[r, "f"'] & S \arrow[r] & \operatorname{Spec} k \arrow[r] & \operatorname{Spec} W \end{tikzcd}$$

be the resulting commutative diagram of schemes in which the left square is cartesian. Then the base change morphism

$$j^*_{crys} R^i f_{crys*}\mathcal{O}_{X/W} \longrightarrow R^i f'_{crys*}\mathcal{O}_{X'/W'}$$

in $\operatorname{Mod}(S'/W')$ becomes an isomorphism of isocrystals in $\operatorname{Mod}(S'/W') \otimes \mathbb{Q}_p$;

(2) Denoting by $F_S : S \to S$ the absolute Frobenius, the natural morphisms

$$F^*_{S\,crys} R^i f_{crys*}\mathcal{O}_{X/W} \longrightarrow R^i f^{(p/S)}_{crys*}\mathcal{O}_{X^{(p/S)}/W} \xrightarrow{F^*_{X/S}} R^i f_{crys*}\mathcal{O}_{X/W}$$

of $\mathcal{O}_{X/W}$-modules become isomorphisms of isocrystals in $\operatorname{Mod}(X/W) \otimes \mathbb{Q}_p$, thereby giving $R^i f_{crys*}\mathcal{O}_{X/W} \otimes \mathbb{Q}_p$ the structure of an F-isocrystal on $(S/W)_{crys}$.

Proof. To prove (1) we may suppose that $S = \operatorname{Spec} A$ and $S' = \operatorname{Spec} A'$ are affine, with p-adically complete, formally smooth W (resp. W')-algebra lifts $\widetilde{j} : \mathcal{A} \to \mathcal{A}'$ over σ. According to the previous remark, and the definition of the crystals $\mathcal{E}^i_{X/S}$, $\mathcal{E}^i_{X'/S'}$ in the proof of Lemma 2.2, we must show that the canonical base change map

$$H^i_{\mathrm{crys}}(X/\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{A}' \longrightarrow H^i_{\mathrm{crys}}(X'/\mathcal{A}')$$

has kernel and cokernel killed by a power of p. But this follows from the exact same Tor vanishing estimates at the end of the proof of Lemma 2.2.

In (2) we are using the standard notation for the pull-back $f^{(p/S)} : X^{(p/S)} \to S$ of $f : X \to S$ along $F_S : S \to S$. The first arrow is then the base change morphism, which is an isogeny by part (1), while the second arrow is induced by the relative Frobenius $F_{X/S}$; it is also an isogeny, by [4, Thm. 1.9]. □

4 FIBRES OF $R^i f_{\text{CRYS}*}\mathcal{O}_{X/W} \otimes \mathbb{Q}_p$

If $\mathcal{E}$ is a crystal in $(S/W)_{\text{crys}}$, and $j : y = \operatorname{Spec} k' \hookrightarrow S$ is a closed point (here k' is a finite extension of k), then the pull-back $j^*_{\text{crys}}\mathcal{E}$ is a crystal in $(y/W')_{\text{crys}}$ by Remark 3.1, which may be identified with the finitely generated W'-module $\mathcal{E}_y := H^0_{\text{crys}}(y/W', j^*_{\text{crys}}\mathcal{E})$.

By naturality of the Frobenius and isogeny category, if $\mathcal{E} \otimes \mathbb{Q}_p$ is an F-isocrystal on $(S/W)_{\text{crys}}$ then $(\mathcal{E} \otimes \mathbb{Q}_p)_y := \mathcal{E}_y[\frac{1}{p}]$ inherits the structure of an F-isocrystal on $(y/W')_{\text{crys}}$, i.e., a finitely generated $W'[\frac{1}{p}]$-vector-space equipped with an isomorphism with its Frobenius pull-back (we will often say "F-isocrystal over W'").

Proposition 4.1. *Let $f : X \to S$ be a proper smooth morphism of smooth varieties over k, and let $y = \operatorname{Spec} k' \hookrightarrow S$ be a closed point. Then there exists a natural base change isomorphism*

$$(R^i f_{crys*}\mathcal{O}_{X/W} \otimes \mathbb{Q}_p)_y \overset{\simeq}{\to} H^i_{crys}(X_y/W')[\tfrac{1}{p}]$$

of F-isocrystals over W'.

Proof. This is a special case of the base change isomorphism of Proposition 3.2(1) (which is clearly compatible with the F-isocrystal structure given in part (2) of the same proposition). □

It is worth noting here, since it will be required in a moment, that a global section of an isocrystal is determined by its value at a point:

Lemma 4.2. *Let S be a smooth connected variety over k, and fix a closed point $y = \operatorname{Spec} k' \hookrightarrow S$. Then the functor*

$$\operatorname{Isoc}(S/W) \longrightarrow \text{fin. gen. } W'[\tfrac{1}{p}]\text{-mods}, \qquad \mathcal{E} \otimes \mathbb{Q}_p \mapsto \mathcal{E}_y[\tfrac{1}{p}]$$

is faithful.

Proof. Using internal Homs, it is sufficient to prove that the canonical map on global sections $H^0_{\text{crys}}(S/W, \mathcal{E}) \to \mathcal{E}_y$ has kernel killed by a power of p; moreover, by picking a finite affine open cover of S it is easy to reduce to the case that $S = \operatorname{Spec} A$ is affine. Let $\mathcal{A}$ be a p-adically complete, formally smooth W-algebra lifting A, and lift the k'-point $A \to k'$ to a W'-point $\mathcal{A} \to W'$ using formal smoothness; denote the kernel of the latter map by I.

Let $M = \mathcal{E}(\mathcal{A})$ be the finitely generated $\mathcal{A}$-module with HPD-stratification ε corresponding to the crystal $\mathcal{E}$, and $m \in M$ an element satisfying $\varepsilon(m \otimes 1) = m \otimes 1$ in $M \otimes_{\mathcal{A}_{p_2}} \mathcal{A}(1)$, i.e., a global section of $\mathcal{E}$ on $(S/W)_{\text{crys}}$. Since $\mathcal{E}_y = M \otimes_{\mathcal{A}} W'$ by Remark 3.1, the lemma reduces to proving the following: assuming that $m \in IM$ we must show that m is killed by a power of p.

Let $\mathcal{B}$ be the I-adic completion of $\mathcal{A}$, whence formal étaleness of $W \to W'$ implies that the map of W-algebras $\mathcal{B} \to \mathcal{B}/I\mathcal{B} = \mathcal{A}/I = W'$ has a section $\alpha : W' \to \mathcal{B}$. We will use the two (very different) maps from $\mathcal{A}$ to $\mathcal{B}$:

$$g_1 : \mathcal{A} \hookrightarrow \mathcal{B}, \qquad g_2 : \mathcal{A} \hookrightarrow \mathcal{B} \twoheadrightarrow \mathcal{B}/I\mathcal{B} = W' \overset{\alpha}{\hookrightarrow} \mathcal{B}.$$

Note that after composition with $\mathcal{B} \twoheadrightarrow W' \twoheadrightarrow k'$, both g_1 and g_2 become the map $\mathcal{A} \twoheadrightarrow A \to k'$. Therefore, if we let $\mathcal{D}$ be the p-adic completion of the pd-envelope of the ideal $I\mathcal{B} \subseteq \mathcal{B}$, and let $j : \mathcal{B} \to \mathcal{D}$ denote the canonical map, then it follows from the universal property of $\mathcal{A}(1)$ that there is a unique map $g_{1,2} : \mathcal{A}(1) \to \mathcal{D}$ which satisfies

$$g_{1,2} \circ p_i = j \circ g_i : \mathcal{A} \to \mathcal{D}$$

for $i = 1, 2$ (it may be helpful to the reader to note at this point that we are making essentially the same argument as the third paragraph of the proof of Lemma 2.3).

Then ε induces an isomorphism $M \otimes_{\mathcal{A},g_1} \mathcal{D} \overset{\simeq}{\to} M \otimes_{\mathcal{A},g_2} \mathcal{D}$ which sends $m \otimes 1$ to $m \otimes 1$ (this crucially uses that m is a global section). But the element $m \otimes 1 \in M \otimes_{\mathcal{A},g_2} \mathcal{D}$ is zero since $m \in IM$, and so we deduce that the element $m \otimes 1 \in M \otimes_{\mathcal{A},g_1} \mathcal{D}$ is also zero. To deduce that this implies that m is killed by a power of p (and so complete the proof), we must show that $\mathrm{Ker}(M \to M \otimes_{\mathcal{A},g_1} \mathcal{D})$ is killed by a power of p; but M is a direct summand of a free $\mathcal{A}$-module up a power of p, by Corollary 2.4, and so it is sufficient to show that $j \circ g_1$ is injective. Since g_1 is injective, it remains only to note that the canonical map $j : \mathcal{B} \to \mathcal{D}$ is also injective, which can be proved in various ways, for example as follows: firstly, since $\mathcal{B}$ is $I\mathcal{B}$-adically complete and regular, and $\mathcal{B}/I\mathcal{B} = W'$ is local and regular, it follows that $\mathcal{B}$ is local and that $I\mathcal{B}$ is generated by a regular sequence $t_1, \dots, t_d$; so there is a non-canonical isomorphism $\mathcal{B} \cong W'[[\underline{T}]] := W'[[T_1, \dots, T_d]]$, $t_i \mapsto T_i$ of W'-algebras, using which we may define an injective map $\underline{p} : \mathcal{B} \to W'[[\underline{T}]]$, $t_i \mapsto pT_i$; this map $\underline{p}$ factors through j since $W'[[\underline{T}]]$ is p-adically complete and its ideal generated by p has divided powers; it follows that j is also injective. ☐

5 HARD LEFSCHETZ

Let $f : X \to S$ be a projective smooth morphism of smooth varieties over k, let L be a line bundle on X which is relatively ample with respect to f, and let $u := c_1(L) \in H^2_{\mathrm{crys}}(X/W)$ denote its crystalline Chern class (which is defined without inverting p in the crystalline cohomology by, e.g., [4, §3.1]). Also denote by u the induced cup product map of $\mathcal{O}_{S/W}$-modules $u \cup - : R^i f_{\mathrm{crys}*}\mathcal{O}_{X/W} \to R^{i+2} f_{\mathrm{crys}*}\mathcal{O}_{X/W}$; as further explanation, this cup product map results from taking the image of $c_1(L)$ in $H^0_{\mathrm{crys}}(S/W, R^2 f_{\mathrm{crys}*}\mathcal{O}_{X/W})$ and then using the graded $\mathcal{O}_{S/W}$-algebra structure on $\bigoplus_{i \geq 0} R^i f_{\mathrm{crys}*}\mathcal{O}_{X/W}$.

Proposition 5.1. *Under the set-up of the previous paragraph, assume in addition that f has pure relative dimension $d \geq 0$. Then the induced morphism of isocrystals*

$$u^i : R^{d-i} f_{crys*}\mathcal{O}_{X/W} \otimes \mathbb{Q}_p \to R^{d+i} f_{crys*}\mathcal{O}_{X/W} \otimes \mathbb{Q}_p$$

on $(S/W)_{crys}$ is an isomorphism.

Remark 5.2 (Classical Hard Lefschetz for crystalline cohomology)**.** Consider first the case that $S = \operatorname{Spec} k$, so that X is a smooth, projective variety over k, of pure dimension d, and L is an ample line bundle on X with $u := c_1(L) \in H^2_{\text{crys}}(X/W)$; then the previous proposition is the assertion that the induced cup product map

$$u^i : H^{d-i}_{\text{crys}}(X/W)[\tfrac{1}{p}] \longrightarrow H^{d+i}_{\text{crys}}(X/W)[\tfrac{1}{p}] \tag{1}$$

is an isomorphism. This is precisely the classical Hard Lefschetz theorem for crystalline cohomology, whose proof we now recall.

Assuming in addition that $L = \mathcal{O}(D)$ for some smooth hyperplane section D of X, that X is geometrically connected over k, and that k is finite, isomorphism (1) follows from the ℓ-adic case, as explained in [6]. We now explain how to reduce the more general assertion above to this special case. Firstly, we may assume that X is geometrically connected over k by replacing k by $H^0(X, \mathcal{O}_X)$; then we may assume k is finite by a standard spreading out argument [5, §3.8]; thirdly we may assume L is very ample by replacing L by L^m, as this merely replaces u by mu; and finally we may assume $L = \mathcal{O}(D)$ for some smooth hyperplane section D of X, by passing to a finite extension of k after which $L = i^*\mathcal{O}(1)$ for some closed embedding $i : X \hookrightarrow \mathbb{P}^N_k$ such that there exists a hyperplane in $\mathbb{P}^N_k$ having smooth intersection with X.

Proof of Proposition 5.1. We may suppose that S is connected, and so apply the lemma and proposition of Section 4 to reduce to showing that

$$c_1(L|_{X_y})^i : H^{d-i}_{\text{crys}}(X_y/W)[\tfrac{1}{p}] \longrightarrow H^{d+i}_{\text{crys}}(X_y/W)[\tfrac{1}{p}]$$

(the i^{th} iterate of the cup product with the Chern class of the ample line bundle $L|_{X_y}$ on X_y) is an isomorphism, where we have picked a k-rational point $y \in S$. This is indeed an isomorphism by the classical Hard Lefschetz isomorphism (1) of the previous remark. □

6 COMPARING $R^i f_{\text{CRYS}*}\mathcal{O}_{X/W} \otimes \mathbb{Q}_p$ TO $R^i f_*\mathcal{O}_{X/K}$

We now recall some results concerning (p-adically) convergent isocrystals from [8], restricting to the case in which S is a smooth variety over k since this is our only case of interest. An *enlargement* of S is a pair (T, z_T) where T is a p-adic formal scheme which is flat and (topologically) of finite type over W, and $z_T : (T \times_W k)_{\text{red}} \to S$ is a morphism of k-varieties; a *p-adic enlargement* is almost the same, except that z_T is instead a morphism $T \times_W k \to S$. A morphism between (p-adic) enlargements is defined in the obvious way, and there is a functor from the category of p-adic enlargements to that of enlargements given by sending (T, z_T) to $(T, (T \times_W k)_{\text{red}} \to T \times_W k \xrightarrow{z_T} S)$.

If T is a p-adic formal scheme which is flat and of finite type over W, then we follow Ogus and denote by $\operatorname{Coh}(\mathcal{O}_T \otimes K)$ the isogeny category $\operatorname{Coh} \mathcal{O}_T \otimes \mathbb{Q}_p$ associated to the category of coherent $\mathcal{O}_T$-modules (here $K = W(k)[\frac{1}{p}]$, though it is appearing only as a piece of notation).

A *convergent isocrystal* $\mathcal{E}$ on S is the following: firstly, for each enlargement (T, z_T) of S, a given object $\mathcal{E}_T \in \operatorname{Coh}(\mathcal{O}_T \otimes K)$; secondly, for each morphism $g : (T', z_{T'}) \to (T, z_T)$ of enlargements, a given isomorphism $g^*\mathcal{E}_T \overset{\simeq}{\to} \mathcal{E}_{T'}$ in $\operatorname{Coh}(\mathcal{O}_T \otimes K)$; thirdly, these isomorphisms are required to satisfy the obvious cocycle relation in the presence of a third morphism of enlargements. A *p-adically convergent isocrystal* is defined in the same way, but restricting to p-adic enlargements.

There are restriction functors

$$\begin{tikzcd}
\mathrm{Cr}(S/W) \arrow[r, "\rho"] & \mathrm{Isoc}^{(p)}(S/W) & \mathrm{Isoc}^{(1)}(S/W) \arrow[l] \arrow[ll, bend left]
\end{tikzcd}$$

where the middle and right terms are our notation for (p-adically) convergent isocrystals. The right arrow is obtained by simply pulling back the convergent crystal to the category of p-adic enlargements. The bendy arrow was constructed by Ogus in [9, §7]. The left arrow associates a p-adically convergent isocrystal $\rho(\mathcal{E})$ to a crystal $\mathcal{E}$ as follows [8, Ex. 2.7.3]: for any p-adic enlargement (T, z_T), we let $\rho(\mathcal{E})_T$ be the isogeny class of the coherent $\mathcal{O}_T$-module

$$T \supseteq \operatorname{Spf} \mathcal{A} \mapsto (z^*_{T,\mathcal{A}\,\mathrm{crys}}\mathcal{E})(\mathcal{A}).$$

Here $\operatorname{Spf}\mathcal{A} \subseteq T$ is an affine open (so $\mathcal{A}$ is a p-adically complete, flat W-algebra), we write $z_{T,\mathcal{A}} : \operatorname{Spec}\mathcal{A}/p \subseteq T \times_W k \xrightarrow{z_T}$ for the indicated composition, $z^*_{T,\mathcal{A}\,\mathrm{crys}}\mathcal{E} \in \mathrm{Cr}((\operatorname{Spec}\mathcal{A}/p)/\mathcal{A})$ is the resulting pull-back of the crystal (a generalisation of Remark 3.1), and finally

$$(z^*_{T,\mathcal{A}\,\mathrm{crys}}\mathcal{E})(\mathcal{A}) := \varprojlim_r (z^*_{T,\mathcal{A}\,\mathrm{crys}}\mathcal{E})(\operatorname{Spec}\mathcal{A}/p \hookrightarrow \operatorname{Spec}\mathcal{A}/p^r, \text{can. pd. str.})$$

are the "sections" of $z^*_{T,\mathcal{A}\,\mathrm{crys}}\mathcal{E}$ on $\mathcal{A}$, in the sense of (a generalisation of) Lemma 2.3.

By the universal property of the isogeny category, ρ extends to $\rho : \mathrm{Isoc}(S/W) \otimes \mathbb{Q}_p \to \mathrm{Isoc}^{(p)}(S/W)$. By a slight abuse of notation we will also allow ourselves to apply ρ to any object of $\mathrm{Mod}(S/W) \otimes \mathbb{Q}_p$ which is an isocrystal, i.e., isomorphic to an object of $\mathrm{Isoc}(S/W)$ (to make this precise, one should perhaps introduce a category of modules $\mathrm{Mod}^{(p)}(S/W)$, containing $\mathrm{Isoc}^{(p)}(S/W)$ as a full subcategory, and note that ρ extends to a functor $\rho : \mathrm{Mod}(S/W) \otimes \mathbb{Q}_p \to \mathrm{Mod}^{(p)}(S/W)$; then ρ will take isocrystals to the essential image of the fully faithful inclusion $\mathrm{Isoc}^{(p)}(S/W) \to \mathrm{Mod}^{(p)}(S/W)$).

Proposition 6.1. *Let $f : X \to S$ be a proper smooth morphism of smooth varieties over k. For any p-adic enlargement (T, z_T) of S, the coherent $\mathcal{O}_T \otimes K$-module $\rho(R^i f_{crys*}\mathcal{O}_{X/W} \otimes \mathbb{Q}_p)_T$ is naturally isomorphic to the isogeny class of the coherent $\mathcal{O}_T$-module*

$$T \supseteq \operatorname{Spf}\mathcal{A} \mapsto H^i_{crys}(X \times_S (\operatorname{Spec}\mathcal{A}/p)/\mathcal{A}).$$

Proof. This the indicated $\mathcal{O}_T$-module is really a well-defined coherent $\mathcal{O}_T$-module is an easy consequence of base change and finite generation for crystalline cohomology.

The proof of Proposition 3.2 worked in much greater generality than stated, in particular for the diagram

$$\begin{tikzcd}
X \times_S \operatorname{Spec}\mathcal{A}/p \arrow[r, "f'"] \arrow[d, "j'"'] & \operatorname{Spec}\mathcal{A}/p \arrow[r, "="] \arrow[d, "z_{T,\mathcal{A}}"] & \operatorname{Spec}\mathcal{A}/p \arrow[r] \arrow[d] & \operatorname{Spec}\mathcal{A} \arrow[d] \\
X \arrow[r, "f"'] & S \arrow[r] & \operatorname{Spec} k \arrow[r] & \operatorname{Spec} W
\end{tikzcd}$$

thereby showing that the canonical base change map

$$(z^*_{T,\mathcal{A}\,\mathrm{crys}} R^i f_{\mathrm{crys}*}\mathcal{O}_{X/W})(\mathcal{A}) \longrightarrow H^i_{\mathrm{crys}}(X \times_S (\operatorname{Spec}\mathcal{A}/p)/\mathcal{A})$$

has kernel and cokernel killed by a power of p, which is exactly the desired assertion. $\square$

The restriction functors we introduced above obviously respect the additional data of the structure of an F-isocrystal, thereby giving rise to restriction functors

$$\begin{tikzcd}\text{F-Isoc}(S/W) \arrow[r, "\rho"] & \text{F-Isoc}^{(p)}(S/W) & \text{F-Isoc}^{(1)}(S/W) \arrow[l] \arrow[ll, bend left]\end{tikzcd}$$

(the notation should be clear). These functors are equivalences by Dwork's trick, c.f., [8, Prop2.18] [2, Thm. 2.4.2], and in the statement of the next proposition we use these equivalences to identify the categories:

Corollary 6.2. *With notation as in the previous proposition, the F-isocrystal $R^i f_{crys*}\mathcal{O}_{X/W} \otimes \mathbb{Q}_p$ is naturally isomorphic to the convergent F-isocrystal $R^i f_* \mathcal{O}_{X/K}$ constructed by Ogus [8, Thm. 3.1].*

In particular, the global sections of $R^i f_* \mathcal{O}_{X/K}$ are equal to $H^0_{crys}(S/W, R^i f_{crys*}\mathcal{O}_{X/W})[\frac{1}{p}]$.

Proof. The previous proposition shows that $\rho(R^i f_{\text{crys}*}\mathcal{O}_{X/W} \otimes \mathbb{Q}_p)$ has the characterising property of $R^i f_* \mathcal{O}_{X/K}$ as a p-adically convergent crystal, while the F-isocrystal structures on each are defined in exactly the same way, namely by base change along the absolute Frobenius of S followed by the relative Frobenius isomorphism (the reader should directly compare the proof of Proposition 3.2(2) with that of [8, Thm. 3.7]). The assertion about global sections then follows from [9, Bottom of pg. 160]. □

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