The relative Fargues–Fontaine curve

MATTHEW MORROW

There are two primary goals of this talk:

1. Define $Y_S$ and the relative Fargues–Fontaine curve $X_S = Y_S/\mathcal{O}^Z$ for an arbitrary perfectoid space $S$ over $\mathbb{F}_p$. These will be adic spaces over $\text{Spa} \mathbb{Q}_p := \text{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ which, in the special case $S = \text{Spa}(\mathbb{C}^p_\wp, \mathcal{O}^2_{\mathbb{C}^2_\wp})$, reduce to the adic spaces $Y^{\text{ad}}$ and $X^{\text{ad}}$ which appeared in Colmez’ talk.

2. Relate $Y_S$ to untiltings of $S$ and describe how the formula

$$ Y_S = S \times \text{Spa} \mathbb{Q}_p $$

can be made precise using diamonds.

We mention that, by picking an auxiliary local (or perfectoid) field $E$, one may more generally construct $Y_{S,E}$ and $X_{S,E}$; in this talk we are implicitly restricting entirely to the case $E = \mathbb{Q}_p$.

For further details and references we refer the reader primarily to Caraiani–Scholze [1, §3.3] and Fargues [3, §1.1–1.3] [4, §1.1–1.4].

1. Constructing $Y_S$ and $X_S$

1.1. Case of affinoid perfectoid $S$. We begin by constructing $Y_S$ and $X_S$ in the case that $S := \text{Spa}(R, R^+)$ is affinoid perfectoid over $\mathbb{F}_p$; fix a pseudo-uniformiser $\pi \in R$. Set $\mathbb{A} := W(R^+)$, which is equipped with the $(p,[\pi])$-adic topology, and define a preadic $\text{Spa} \mathbb{Q}_p$-space

$$ Y_{(R,R^+)} := \text{Spa}(\mathbb{A}, \mathbb{A}) \setminus V(p[\pi]). $$

Concretely, a point of $Y_{(R,R^+)}$ is a continuous absolute value $| \cdot | : \mathbb{A} \to \Gamma \cup \{0\}$ which satisfies $|a| \leq 1$, for all $a \in \mathbb{A}$, and $|p[\pi]| \neq 0$; it follows from this latter condition that the vanishing ideal of $| \cdot |$ is not open in $\mathbb{A}$, i.e., $Y_{(R,R^+)}$ is an analytic preadic space, and that moreover the radius function

$$ \delta : Y_{(R,R^+)} \to (0,1), \quad |\cdot|, \Gamma \mapsto p^{-\sup\{r/s \in \mathbb{Q}_{>0} : |\pi|^r \geq |p|^s\}} $$

(“the closest point to $[p]$ on the positive real line spanned by $|[\pi]|$”) is a well-defined, continuous map. We may therefore introduce, for any closed interval $I \subset (0,1)$, the associated **annulus**

$$ Y_{(R,R^+)}^{\text{open}} \ni Y_{(R,R^+)}^I := \text{the interior of the preimage $\delta^{-1}(I)$}, $$

which can be shown, in the case that $I = [p^{-r/s}, p^{-r'/s'}]$ for $r, s, r', s' \in \mathbb{N}$, to be the rational subdomain of $\text{Spa}(\mathbb{A}, \mathbb{A})$ consisting of those points $| \cdot |$ for which $[|\pi]|^r \leq |p|^s$ and $[|\pi]|^{r'} \geq |p|^{s'}$. Clearly therefore $Y_{(R,R^+)}^I$ is the filtered increasing union, over all closed intervals $I \subset (0,1)$, of the associated annuli.

It can be shown that $Y_{(R,R^+)}^I$ is sheafy, i.e., an adic space. To do this one picks a perfectoid field $E/\mathbb{Q}_p$ and checks that $Y_{(R,R^+)}^I \times_{\text{Spa} \mathbb{Q}_p} \text{Spa} E$ is affinoid perfectoid, hence sheafy by Scholze or Kedlaya–Liu. In other words $Y_{(R,R^+)}^I$ is **preperfectoid**, and hence is also sheafy; see [2, §2.2] for further details and references.
follows immediately from the description of $Y_{(R,R^+)}$ as a union of annuli that it is also sheafy.

1.2. The quotient by the Frobenius. The usual Witt vector Frobenius $\phi$ on $\mathbb{A}$ induces a Frobenius action $\phi$ on $Y_{(R,R^+)}$ which satisfies $\delta(\phi(y)) = \delta(y)^{1/p}$ for all $y \in Y_{(R,R^+)}$. It follows that this latter action is proper and totally discontinuous, whence

$$X_{(R,R^+)} := Y_{(R,R^+)}/\mathbb{Z}$$

is a well-defined adic space over $\text{Spa}(\mathbb{Q}_p)$, and $Y_{(R,R^+)} \to X_{(R,R^+)}$ is an open quotient map. Moreover, if $I = [a,b] \subset (0,1)$ is an interval satisfying $b^p < a < b^{1/p}$, then $Y^I_{(R,R^+)}$ is disjoint from $\phi^n(Y^I_{(R,R^+)})$ for all $0 \neq n \in \mathbb{Z}$, and so this quotient map sends $Y^I_{(R,R^+)}$ isomorphically to an open subspace of $X_{(R,R^+)}$. In short, sufficiently thin annuli provide an explicit affinoid open cover of $X_{(R,R^+)}$.

1.3. The case of general $S$. For any closed interval $I \subset (0,1)$ and suitable elements $f_1, \ldots, f_n, g \in R^+$, it is not hard to check that there is a natural identification between

$$Y^I_{(R,R^+)}\left(\langle f_1, \ldots, f_n \rangle_{[g]}\right) \quad \text{and} \quad Y^I_{(R,R^+)}\left(\langle \frac{f_1}{g}, \ldots, \frac{f_n}{g} \rangle_{[g]}^\times\right),$$

where the left is a rational subdomain of $Y^I_{(R,R^+)}$ and the right is $Y^I_{(R,R^+)}$ of a localisation of the pair $(R,R^+)$. It is therefore straightforward to glue along rational subdomains in order to define $Y_S$ and the relative Fargues–Fontaine curve

$$X_S := Y_S/\mathbb{Z}$$

for an arbitrary perfectoid space $S$ over $\mathbb{F}_p$.

1.4. The map $\theta$. In the case in which $S$ is the tilt $S^p$ of some fixed perfectoid space $S^d$ over $\text{Spa}(\mathbb{Q}_p)$, there is an induced closed immersion $\theta : S^d \hookrightarrow Y_S$ which is locally given by Fontaine’s map $\theta : W(R^+) \to R^{\mathbb{Z}^+}$ arising from the universal property of Witt vectors. Remarkably, the composition $S^d \xrightarrow{\theta} Y_S \to X_S$ is still a closed embedding: indeed, we may assume that $S = \text{Spa}(R,R^+)$ and $S^d = \text{Spa}(R^d,R^{\mathbb{Z}^+})$ are affinoid perfectoid, in which case the kernel of Fontaine’s map is generated by a degree one primitive element, i.e., an element $\xi \in \mathbb{A}$ of the form $\xi = [p] + pu$ where $p \in R$ is a pseudo-uniformiser and $u \in \mathbb{A}^\times$; it follows easily that the closed immersion $\theta : \text{Spa}(R^d,R^{\mathbb{Z}^+}) \to Y_{(R,R^+)}$ factors through the annulus associated to the interval $[p^{-1},p^{-1}]$, which as explained in 1.2 maps isomorphically to an open subspace of $X_{(R,R^+)}$.

2. Diamonds and untilting

If $X$ is an analytic adic space over $\text{Spa}(\mathbb{Z}_p,\mathbb{Z}_p)$, then there is an associated presheaf

$$X^\circ : \text{Perf}_{\mathbb{Z}_p} \longrightarrow \text{Sets}, \quad T \mapsto \text{untillts over } X \text{ of } T,$$

where the right side is more precisely defined to the set of pairs, up to the obvious notion of an isomorphism of pairs, $(T^d, \iota)$ where $T^d$ is a perfectoid space over $X$ and $\iota : T^p \xrightarrow{\sim} T$. If $X$ is itself a perfectoid space, then the equivalence of categories between perfectoid spaces over $X^\circ$ and perfectoid spaces over $X$ implies that $X^\circ$
canonical identifies with the representable presheaf $\text{Hom}(-, X^\ominus)$; as a special case, if $X$ is a perfectoid space over $\mathbb{F}_p$ then $X^\ominus$ identifies with $\text{Hom}(-, X)$.

An important result (though not strictly necessary for the talk) is that $X^\ominus$ is a sheaf for the pro-étale topology on $\text{Perf}_{\mathbb{F}_p}$, and even a diamond (recall from Hellmann’s talk that diamonds are a full subcategory – informally the pro-étale quotients of representable objects – of pro-étale sheaves on $\text{Perf}_{\mathbb{F}_p}$). Informally, this is proved by picking a perfectoid cover $\{U_i\}$ of $X$ in the pro-étale topology and then noting that $X^\ominus$ is a pro-étale quotient of $\bigsqcup U_i^\ominus = \bigsqcup \text{Hom}(-, U_i^\flat)$.

We may now state the two main results of the talk; let $S$ be a perfectoid space over $\mathbb{F}_p$. Firstly, there is a natural isomorphism of diamonds (equivalently, of pro-étale sheaves on $\text{Perf}_{\mathbb{F}_p}$)

$$Y_S^\ominus \cong S^\ominus \times \text{Spa} Q_p^\ominus,$$

which gives a precise meaning to the sense in which $Y_S$ is the product of $S$ and $\text{Spa} Q_p$. Secondly, the following four collections are in canonical bijection with one another:

(I) Sections of the projection $Y_S^\ominus \to S^\ominus$.

(II) Maps $S^\ominus \to \text{Spa} Q_p^\ominus$.

(III) Untilts in characteristic zero (i.e., over $\text{Spa} Q_p$) of $S$.

(IV) Closed immersions into $Y_S$ defined locally by a degree one primitive element.

Concerning proofs, we restrict ourselves here to the briefest sketch. The isomorphism in the product formula is given, for each test object $T \in \text{Perf}_{\mathbb{F}_p}$, by

$$\text{Hom}(T, S) \times \text{Spa} Q_p^\ominus(T) \to Y_S^\ominus(T), \quad (f, (T^\sharp, \iota)) \mapsto (T^\sharp, \iota),$$

where the $T^\sharp$ on the right is viewed as a perfectoid space over $Y_S$ via the composition

$$T^\sharp \overset{\vartheta}{\to} Y_{T^\vartheta} \overset{\iota}{\cong} Y_T \overset{f}{\to} Y_S.$$

This is shown to be a bijection using the universal nature of Fontaine’s map. Meanwhile, (I) and (II) trivially correspond since $Y_S^\ominus \cong S^\ominus \times \text{Spa} Q_p$; secondly, (II) and (III) correspond by the Yoneda Lemma; thirdly, (III) and (IV) correspond thanks to the converse of an assertion in 1.4, namely that each degree one primitive element $\xi \in \mathbb{A}$ gives rise to an untilt $\mathbb{A}/\xi[1/p]$ of $R$.

The two main results of the previous paragraph have obvious analogues in which $Y_S$ is replaced by $X_S$, untilts are taken modulo Frobenius equivalence, and $S^\ominus$ is replaced by $S^\ominus/\phi^\omega$, though these were unfortunately not covered in the talk.

References


