The relative Fargues–Fontaine curve
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There are two primary goals of this talk:

(1) Define Y_S and the relative Fargues–Fontaine curve $X_S = Y_S/\phi^{\mathbb{Z}}$ for an arbitrary perfectoid space S over $\mathbb{F}_p$. These will be adic spaces over $\operatorname{Spa}\mathbb{Q}_p := \operatorname{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ which, in the special case $S = \operatorname{Spa}(\mathbb{C}_p^\flat, \mathcal{O}_{\mathbb{C}_p}^\flat)$, reduce to the adic spaces Y^{ad} and X^{ad} which appeared in Colmez' talk.

(2) Relate Y_S to untiltings of S and describe how the formula

$$\text{“}Y_S = S \times \operatorname{Spa}\mathbb{Q}_p\text{”}$$

can be made precise using diamonds.

We mention that, by picking an auxiliary local (or perfectoid) field E, one may more generally construct $Y_{S,E}$ and $X_{S,E}$; in this talk we are implicitly restricting entirely to the case $E = \mathbb{Q}_p$.

For further details and references we refer the reader primarily to Caraiani–Scholze [1, §3.3] and Fargues [3, §1.1–1.3] [4, §1.1–1.4].

1. Constructing Y_S and X_S

1.1. **Case of affinoid perfectoid S.** We begin by constructing Y_S and X_S in the case that $S := \operatorname{Spa}(R, R^+)$ is affinoid perfectoid over $\mathbb{F}_p$; fix a pseudo-uniformiser $\pi \in R$. Set $\mathbb{A} := W(R^+)$, which is equipped with the $\langle p, [\pi]\rangle$-adic topology, and define a preadic $\operatorname{Spa}\mathbb{Q}_p$-space

$$Y_{(R,R^+)} := \operatorname{Spa}(\mathbb{A}, \mathbb{A}) \setminus V(p[\pi]).$$

Concretely, a point of $Y_{(R,R^+)}$ is a continuous absolute value $|\cdot| : \mathbb{A} \to \Gamma \cup \{0\}$ which satisfies $|a| \le 1$, for all $a \in \mathbb{A}$, and $|p[\pi]| \ne 0$; it follows from this latter condition that the vanishing ideal of $|\cdot|$ is not open in $\mathbb{A}$, i.e., $Y_{(R,R^+)}$ is an *analytic* preadic space, and that moreover the *radius function*

$$\delta : Y_{(R,R^+)} \longrightarrow (0,1), \quad (|\cdot|, \Gamma) \mapsto p^{-\sup\{r/s \in \mathbb{Q}_{\ge 0} : |[\pi]|^r \ge |p|^s\}}$$

(“the closest point to $|p|$ on the positive real line spanned by $|[\pi]|$”) is a well-defined, continuous map. We may therefore introduce, for any closed interval $I \subset (0,1)$, the associated *annulus*

$$Y_{(R,R^+)} \overset{\text{open}}{\supseteq} Y^I_{(R,R^+)} := \text{the interior of the preimage } \delta^{-1}(I),$$

which can be shown, in the case that $I = [p^{-r/s}, p^{-r'/s'}]$ for $r, s, r', s' \in \mathbb{N}$, to be the rational subdomain of $\operatorname{Spa}(\mathbb{A}, \mathbb{A})$ consisting of those points $|\cdot|$ for which $|[\pi]|^r \le |p|^s$ and $|[\pi]|^{r'} \ge |p|^{s'}$. Clearly therefore $Y_{(R,R^+)}$ is the filtered increasing union, over all closed intervals $I \subset (0,1)$, of the associated annuli.

It can be shown that $Y_{(R,R^+)}$ is sheafy, i.e., an adic space. To do this one picks a perfectoid field $E/\mathbb{Q}_p$ and checks that $Y^I_{(R,R^+)} \times_{\operatorname{Spa}\mathbb{Q}_p} \operatorname{Spa} E$ is affinoid perfectoid, hence sheafy by Scholze or Kedlaya–Liu. In other words $Y^I_{(R,R^+)}$ is *preperfectoid*, and hence is also sheafy; see [2, §2.2] for further details and references. It then follows immediately from the description of $Y_{(R,R^+)}$ as a union of annuli that it is also sheafy.

1.2. **The quotient by the Frobenius.** The usual Witt vector Frobenius ϕ on $\mathbb{A}$ induces a Frobenius action ϕ on $Y_{(R,R^+)}$ which satisfies $\delta(\phi(y)) = \delta(y)^{1/p}$ for all $y \in Y_{(R,R^+)}$. It follows that this latter action is proper and totally discontinuous, whence

$$X_{(R,R^+)} := Y_{(R,R^+)}/\phi^{\mathbb{Z}}$$

is a well-defined adic space over $\operatorname{Spa}\mathbb{Q}_p$ and $Y_{(R,R^+)} \to X_{(R,R^+)}$ is an open quotient map. Moreover, if $I = [a,b] \subset (0,1)$ is an interval satisfying $b^p < a \leq b < a^{1/p}$, then $Y^I_{(R,R^+)}$ is disjoint from $\phi^n(Y^I_{(R,R^+)})$ for all $0 \neq n \in \mathbb{Z}$, and so this quotient map sends $Y^I_{(R,R^+)}$ isomorphically to an open subspace of $X_{(R,R^+)}$. In short, sufficiently thin annuli provide an explicit affinoid open cover of $X_{(R,R^+)}$.

1.3. **The case of general S.** For any closed interval $I \subset (0,1)$ and suitable elements $f_1, \ldots, f_n, g \in R^+$, it is not hard to check that there is a natural identification between

$$Y^I_{(R,R^+)}\left\langle \frac{[f_1],\ldots,[f_n]}{[g]} \right\rangle \qquad \text{and} \qquad Y^I_{(R\langle \frac{f_1,\ldots,f_n}{g}\rangle, R\langle \frac{f_1,\ldots,f_n}{g}\rangle^+)},$$

where the left is a rational subdomain of $Y^I_{(R,R^+)}$ and the right is $Y^I_{(-,-)}$ of a localisation of the pair (R,R^+). It is therefore straightforward to glue along rational subdomains in order to define Y_S and the *relative Fargues–Fontaine curve* $X_S := Y_S/\phi^{\mathbb{Z}}$ for an arbitrary perfectoid space S over $\mathbb{F}_p$.

1.4. **The map θ.** In the case in which S is the tilt $S^{\sharp\flat}$ of some fixed perfectoid space $S^\sharp$ over $\operatorname{Spa}\mathbb{Q}_p$, there is an induced closed immersion $\theta : S^\sharp \hookrightarrow Y_S$ which is locally given by Fontaine's map $\theta : W(R^+) \to R^{\sharp+}$ arising from the universal property of Witt vectors. Remarkably, the composition $S^\sharp \overset{\theta}{\hookrightarrow} Y_S \to X_S$ is still a closed embedding: indeed, we may assume that $S = \operatorname{Spa}(R,R^+)$ and $S^\sharp = \operatorname{Spa}(R^\sharp, R^{\sharp+})$ are affinoid perfectoid, in which case the kernel of Fontaine's map is generated by a *degree one primitive element*, i.e., an element $\xi \in \mathbb{A}$ of the form $\xi = [\pi] + pu$ where $\pi \in R$ is a pseudo-uniformiser and $u \in \mathbb{A}^\times$; it follows easily that the closed immersion $\theta : \operatorname{Spa}(R^\sharp, R^{\sharp+}) \hookrightarrow Y_{(R,R^+)}$ factors through the annulus associated to the interval $[p^{-1}, p^{-1}]$, which as explained in 1.2 maps isomorphically to an open subspace of $X_{(R,R^+)}$.

2. Diamonds and untilting

If X is an analytic adic space over $\operatorname{Spa}(\mathbb{Z}_p, \mathbb{Z}_p)$, then there is an associated presheaf

$$X^\diamond : \mathrm{Perf}_{\mathbb{F}_p} \longrightarrow \mathrm{Sets}, \qquad T \mapsto \text{untilts over } X \text{ of } T,$$

where the right side is more precisely defined to the set of pairs, up to the obvious notion of an isomorphism of pairs, $(T^\sharp, \iota)$ where $T^\sharp$ is a perfectoid space over X and $\iota : T^{\sharp\flat} \overset{\simeq}{\to} T$. If X is itself a perfectoid space, then the equivalence of categories between perfectoid spaces over $X^\flat$ and perfectoid spaces over X implies that $X^\diamond$

canonical identifies with the representable presheaf $\mathrm{Hom}(-, X^\flat)$; as a special case, if X is a perfectoid space over $\mathbb{F}_p$ then $X^\diamond$ identifies with $\mathrm{Hom}(-, X)$.

An important result (though not strictly necessary for the talk) is that $X^\diamond$ is a sheaf for the pro-étale topology on $\mathrm{Perf}_{\mathbb{F}_p}$, and even a diamond (recall from Hellmann's talk that diamonds are a full subcategory – informally the pro-étale quotients of representable objects – of pro-étale sheaves on $\mathrm{Perf}_{\mathbb{F}_p}$). Informally, this is proved by picking a perfectoid cover $\{U_i\}_i$ of X in the pro-étale topology and then noting that $X^\diamond$ is a pro-étale quotient of $\bigsqcup_i U_i^\diamond = \bigsqcup_i \mathrm{Hom}(-, U_i^\flat)$.

We may now state the two main results of the talk; let S be a perfectoid space over $\mathbb{F}_p$. Firstly, there is a natural isomorphism of diamonds (equivalently, of pro-étale sheaves on $\mathrm{Perf}_{\mathbb{F}_p}$)

$$Y_S^\diamond \cong S^\diamond \times \mathrm{Spa}\,\mathbb{Q}_p^\diamond,$$

which gives a precise meaning to the sense in which Y_S is the product of S and $\mathrm{Spa}\,\mathbb{Q}_p$. Secondly, the following four collections are in canonical bijection with one another:

(I) Sections of the projection $Y_S^\diamond \to S^\diamond$.
(II) Maps $S^\diamond \to \mathrm{Spa}\,\mathbb{Q}_p^\diamond$.
(III) Untilts in characteristic zero (i.e., over $\mathrm{Spa}\,\mathbb{Q}_p$) of S.
(IV) Closed immersions into Y_S defined locally by a degree one primitive element.

Concerning proofs, we restrict ourselves here to the briefest sketch. The isomorphism in the product formula is given, for each test object $T \in \mathrm{Perf}_{\mathbb{F}_p}$, by

$$\mathrm{Hom}(T, S) \times \mathrm{Spa}\,\mathbb{Q}_p^\diamond(T) \longrightarrow Y_S^\diamond(T), \qquad (f, (T^\sharp, \iota)) \mapsto (T^\sharp, \iota),$$

where the $T^\sharp$ on the right is viewed as a perfectoid space over Y_S via the composition

$$T^\sharp \overset{\theta}{\hookrightarrow} Y_{T^{\sharp\flat}} \overset{\iota}{\cong} Y_T \xrightarrow{f} Y_S.$$

This is shown to be a bijection using the universal nature of Fontaine's map. Meanwhile, (I) and (II) trivially correspond since $Y_S^\diamond \cong S^\diamond \times \mathrm{Spa}\,\mathbb{Q}_p$; secondly, (II) and (III) correspond by the Yoneda Lemma; thirdly, (III) and (IV) correspond thanks to the converse of an assertion in 1.4, namely that each degree one primitive element $\xi \in \mathbb{A}$ gives rise to an untilt $\mathbb{A}/\xi[\frac{1}{p}]$ of R.

The two main results of the previous paragraph have obvious analogues in which Y_S is replaced by X_S, untilts are taken modulo Frobenius equivalence, and $S^\diamond$ is replaced by $S^\diamond/\phi^{\mathbb{Z}}$, though these were unfortunately not covered in the talk.

References

[1] A. Caraiani and P. Scholze, *On the generic part of the cohomology of compact unitary Shimura varieties*, `arXiv:1511.02418`, (2015).

[2] L. Fargues, *Quelques résultats et conjectures concernant la courbe*, Astérisque, (2015), pp. 325–374. In *De la géométrie algébrique aux formes automorphes (I)*.

[3] L. Fargues, *Géométrisation de la correspondance de Langlands locale*, Preprint, (2016).

[4] L. Fargues, *Geometrization of the local Langlands correspondence: an overview*, `arXiv:1602.00999`, (2016).