Two questions on pro Milnor K-theory in characteristic p

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The goal of this Oberwolfach Report[1] is to present one of the main results of the preprint [2], which concerns the (continuous) K-theory and p-adic motivic cohomology of formal schemes in characteristic p, and use it to motivate some curious problems concerning the Milnor K-theory of truncated polynomial algebras in characteristic p.

We fix throughout $A := R[[t_1, \dots, t_c]]$, where R is a regular $\mathbb{F}_p$-algebra which is assumed to be finitely generated over its subring of p^{th}-powers, and we set $I := (t_1, \dots, t_c) \subset A$. We will explain the interest in resolving the following two questions:

(I) Is the inverse system

$$p\text{-torsion in the relative Milnor } K\text{-group } K_n^M(A/I^s, I/I^s)$$

Mittag-Leffler zero (a.k.a. essentially zero) as $s \to \infty$?

(II) Is the "pro excison" square of pro abelian groups

$$\begin{array}{ccc} \{K_n^M(R[[x,y]]/(x^s y^s))\}_s & \longrightarrow & \{K_n^M(R[[x,y]]/(x^s))\}_s \\ \downarrow & & \downarrow \\ \{K_n^M(R[[x,y]]/(y^s))\}_s & \longrightarrow & \{K_n^M(R[[x,y]]/(x^s, y^s))\}_s \end{array}$$

bicartesian (perhaps only modulo p^r for any $r \geq 1$)?

1. Pro Geisser–Levine and Bloch–Kato–Gabber theorem

If B is any $\mathbb{F}_p$-algebra, then we may consider the natural homomorphisms

$$K_n(B)/p^r \longleftarrow K_n^M(B)/p^r \xrightarrow{\text{dlog}[\cdot]} W_r\Omega^n_{B,\log},$$

where $W_r\Omega^n_{B,\log}$ (also denoted by $\nu_r^n(B)$ in the literature) is the subgroup of the Hodge–Witt group $W_r\Omega^n_B$ consisting of elements which can be written étale locally as sums of dlog forms, and the map $\text{dlog}[\cdot]$ is given by $\{b_1, \dots, b_n\} \mapsto \text{dlog}[b_1] \cdots \text{dlog}[b_n]$ as usual. If B is regular and local (e.g., $B = A$ or R) then both of these homomorphisms are known to be isomorphisms: this reduces, via Gersten sequences, to the case that B is a field, in which case the leftwards isomorphism is due to Geisser and Levine [1], who also proved that $K_n(B)$ is p-torsion-free, and the rightwards isomorphism is the Bloch–Kato–Gabber theorem (see [2, Thm. 5.1] for more details and references; also, to avoid issues caused by finite residue fields, we use Kerz–Gabber's improved Milnor K-theory throughout). On the other hand, if B is not regular (e.g., $B = A/I^s$ for $s > 1$) then neither homomorphism is typically an isomorphism; nevertheless, it is shown in [2] that the obstructions to having an isomorphism $K_n(A/I^s)/p^r \cong W_r\Omega^n_{A/I^s,\log}$ are Mittag-Leffler zero as $s \to \infty$:

[1] *Algebraic K-theory and Motivic Cohomology*, 26 June – 2 July 2016.

Theorem 1.1. *The homomorphisms of pro abelian groups*

$$\{K_n(A/I^s)/p^r\}_s \longleftarrow \{K_n^M(A/I^s)/p^r\}_s \xrightarrow{\text{dlog}[\cdot]} \{W_r\Omega^n_{A/I^s,log}\}_s$$

are surjective and have the same kernel, thereby inducing an isomorphism

$$\{K_n(A/I^s)/p^r\}_s \xrightarrow{\simeq} \{W_r\Omega^n_{A/I^s,log}\}_s.$$

Moreover, the pro abelian group $\{K_n(A/I^s)\}_s$ is p-torsion-free.

2. The open problems (I) and (II)

In the case that $c = 1$ (i.e., $A = R[[t]]$), Theorem 1.1 can be improved: the stated homomorphisms of pro abelian groups are not merely surjective but even isomorphisms, i.e., $\{K_n^M(R[t]/(t^s))/p^r\}_s \xrightarrow{\simeq} \{K_n(R[t]/(t^s))/p^r\}_s$. The goal of the remainder of this note is to discuss the general case $c \geq 1$, which we find a curious problem reflecting the fact that the Milnor K-theory of multi-variable truncated polynomial algebras has not been studied very much.

In fact, using Theorem 1.1, the Bloch–Kato–Gabber and Geisser–Levine theorems, and the fact that $K_n(A/I^s, I/I^s)$ and $K_n^M(A/I^s, I/I^s)$ are p-power torsion, it is not hard to prove that the following conditions are equivalent for any fixed $n \geq 0$:

(1) The homomorphisms of Theorem 1.1 are isomorphisms, i.e.,

$$\{K_n^M(A/I^s)/p^r\}_s \xrightarrow{\simeq} \{K_n(A/I^s)/p^r\}_s,$$

for all $r \geq 1$.

(2) The pro abelian group

$$\{p\text{-torsion in } K_n^M(A/I^s, I/I^s)\}_s$$

is zero, i.e., a positive answer to question (I).

(3) The map of pro abelian relative K-groups

$$\{K_n^M(A/I^s, I/I^s)\}_s \longrightarrow \{K_n(A/I^s, I/I^s)\}_s$$

is injective.

(4) The square of pro abelian groups

$$\begin{array}{ccc} \{K_n^M(A/I^s)\}_s & \longrightarrow & \{K_n(A/I^s)\}_s \\ \downarrow & & \downarrow \\ K_n^M(A/I) & \longrightarrow & K_n(A/I) \end{array}$$

is bicartesian, which (since the vertical arrows are surjective) means equivalently that the map in (3) is an isomorphism.[2]

[2]Condition (4) should perhaps be viewed as an infinitesimal analogue for truncated polynomial algebras of Beilinson's conjecture that $K_n^{\text{ind}} := \text{coker}(K_n^M \to K_n)$ is torsion for regular local $\mathbb{F}_p$-algebras, since it implies that $\{K_n^{\text{ind}}(A/I^s)\}_s \xrightarrow{\simeq} K_n^{\text{ind}}(A/I)$.

We conjecture that these equivalent properties hold for all $c \geq 1$. Note that the kernel of the map in (3) is killed by p^N, where N is the p-adic valuation of $(n-1)!$; this is a trivial consequence of the existence of the Chern class from K_n to K_n^M and the fact that $K_n^M(A/I^s, I/I^s)$ is p-power torsion. The same arguments used to prove that (1)–(4) are equivalent then reveal that (1)–(4) are at least true up to obstructions killed by p^N. Moreover, as already stated:

Proposition 2.1. *The equivalent properties (1)–(4) are true if $c = 1$.*

Proof. The proof is a really a recent result of Rülling–Saito [3, Thm. 4.13], who construct an isomorphism of pro-abelian groups

$$\gamma_n : \{\mathbb{W}_s\Omega_R^{n-1}\}_s \stackrel{\simeq}{\to} \{K_n^M(R[t]/(t^s), (t)/(t^s))\}_s,$$

which is a version for Milnor K-theory of the Bloch–Deligne–Illusie comparison between the de Rham–Witt complex and curves on K-theory. Since the left side has no p-torsion by Illusie, the same is true of the right side and thus property (2) holds. □

Combining the proposition with Theorem 1.1 leads to a log/exp isomorphism between the relative part of $W_r\Omega^n_{R[t]/(t^s),\log}$ and the big Hodge–Witt groups of R, which we state explicitly here as it may be of interest:

Corollary 2.2. *The map of pro abelian groups*

$$\mathrm{dlog}[\cdot] \circ \gamma_n : \{\mathbb{W}_s\Omega_R^{n-1}/p^r\}_s \longrightarrow \{\mathrm{Ker}(W_r\Omega^n_{R[t]/(t^s),log} \to W_r\Omega^n_{R,log})\}_s$$

is an isomorphism.

Finally we discuss the origin of question (II). Theorem 1.1 holds in greater generality than we have stated here: for example, the ideal I may be replaced by any ideal generated by a collection of monomials in the variables $t_1, \ldots, t_c$. This general case is reduced to Theorem 1.1 by using pro excision for algebraic K-theory and the logarithmic Hodge–Witt groups. To then extend assertions (1)–(4) to this more general monomial case, a (seemingly new) pro excision property for Milnor K-theory is required, for which question (II) is the most elementary case.

References

[1] Geisser, T., and Levine, M. The K-theory of fields in characteristic p. *Invent. Math. 139*, 3 (2000), 459–493.

[2] Morrow, M. The K-theory and logarithmic Hodge–Witt sheaves of formal schemes in characteristic p. `1512.04703` (2015).

[3] Rülling, K., and Saito, S. Higher Chow groups with modulus and relative Milnor K-theory. `arXiv:1504.02669` (2015).