Some problems and conjectures in algebraic K-theory

by Max Karoubi

I. DEFINITIONS. Let A be a ring with unit. We denote by K_n (A) the group K_n of the ring A as defined by Quillen. If A is provided by an involution $a \longmapsto \bar{a}$, we denote by ${}_\varepsilon O_{n,n}(A)$ the group of "ε- orthogonal matrices", i.e. the multiplicative group of 2n × 2n matrices of the form

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

such that $M.M^* = M^*.M = I$ where

$$M^* = \begin{pmatrix} {}^t\bar{d} & \varepsilon\,{}^t\bar{b} \\ \varepsilon\,{}^t\bar{c} & {}^t\bar{a} \end{pmatrix}$$

and where ε is an element of the center of A such that $\varepsilon.\bar{\varepsilon} = 1$. Let ${}_\varepsilon O(A) = \varinjlim {}_\varepsilon O_{n,n}(A)$ and let $B_{{}_\varepsilon O(A)}$ be the classifying space of ${}_\varepsilon O(A)$ (For simplicity we assume from now that 2 is invertible in A). Following Quillen, one may add 2 and 3 celle to $B_{{}_\varepsilon O(A)}$ in order to make $\pi_1(B_{{}_\varepsilon O(A)})$ abelian without changing the homology of $B_{{}_\varepsilon O(A)}$. In that way one obtains a space $B^+_{O(A)}$ which homotopy groups are called ${}_\varepsilon L_n(A)$. One has natural homomorphisms $L_n(A) \longrightarrow K_n(A)$ and $K_n(A) \longrightarrow {}_\varepsilon L_n(A)$ induced by the forgetful functor and the hyperbolic functor respectively. The kernel (resp. the cokernel) of this homomorphism is called ${}_\varepsilon L'_n(A)$ (resp. ${}_\varepsilon W_n(A)$). On the other hand the maps $B^+_{GL(A)} \longrightarrow B^+_{{}_\varepsilon O(A)}$ and $B^+_{{}_\varepsilon O(A)} \longrightarrow B^+_{GL(A)}$ have homotopic fibers called ${}_\varepsilon\mathcal{U}(A)$ and ${}_\varepsilon\mathcal{V}(A)$. Let

Then $\Lambda \subset \Theta$ by sending x to $(t^{a_o},\ldots,t^{a_n})$. Here $\alpha_\Lambda = (\prod_{j\neq i} (1-t^{a_j-a_i}))$; i.e. the i-th co-ordinate of α_Λ is $\prod_{j\neq i} (1-t^{a_j-a_i})$.

(4) Describe the ideal in Λ that annihilates Γ/Λ .

(5) Petrie remarks that he has some interesting examples of $\Lambda \subset \Gamma \subset \Theta$ with $\Lambda \neq \Gamma$ and with the annihilator of Γ/Λ the principal ideal in Λ generated by $\Phi_{pq}(t) \cdot 1$, p and q prime.

(2) In addition to the above assumptions, assume Λ , Γ , Θ all have non-singular R-valued bilinear forms

$\langle\ ,\ \rangle_\Lambda$ $\langle\ ,\ \rangle_\Gamma$, $\langle\ ,\ \rangle_\Theta$, so that there are elements $\alpha_\Lambda \in \Lambda$, $\alpha_\Gamma \in \Gamma$, with

$$\langle x,y \rangle_\Lambda = \mathrm{Id}\left(\frac{x\cdot y}{\alpha_\Lambda}\right)$$

$$\langle z,w \rangle_\Gamma = \mathrm{Id}\left(\frac{z\cdot w}{\alpha_\Gamma}\right)$$

$$\langle u,v \rangle = \mathrm{Id}\,(u\cdot v)$$

where $\mathrm{Id} : \Theta \to R$ is defined by

$$\mathrm{Id}\,(\phi_o,\ldots,\phi_n) = \phi_o+\ldots+\phi_n \ .$$

One can assume α_Λ and α_Γ have the form

$t^{\lambda_i} \prod_{j=1}^{n} (1-t^{x_{ij}})$ in Θ , x_{ij} integers, $j=1,2,\ldots,n$, $i=o,\ldots,n$.

(3) Petrie points out that an important starting point is the order

$$\Lambda = \frac{R[x]}{\left(\prod_{i=o}^{n}\right)(x-t^{a_i}))} \quad , \ a_i \text{ distinct integers.}$$

An (unexpected) positive solution would also solve 2. In Cappell's paper in the splitting principle in these proceedings, he gives a unitary nil-group as the obstruction to the general splitting principal. So another version of 14 would be the question of whether or not this group is trivial. As noted above, (3.2.2) and (3.2.3) can be established geometrically over a ring R, $Z \subset R \subset Q$, with $1/2 \in R$, without the square-root closed condition.

17. We conclude with a problem of Petrie on orders over $Q[t,t^{-1}]$ and bilinear forms.

(1) Let $R = Q[t,t^{-1}]$, F its field of fractions, $\Theta = \prod_{r=o}^{n} R$, $(n+1)$ copies of R. Then Θ is an R-order in $\Theta \otimes_R F$. Fix an order $\Lambda \subset \Theta$ and clasisfy all orders Γ such that

(a) $\Lambda \subset \Gamma$

(b) All orders Λ, Γ, Θ are closed under the Adams operations on Θ defined by

$$\psi^k(\phi_o(t), \phi_1(t), \ldots, \phi_n(t)) = (\phi_o(t^k), \phi_1(t^k), \ldots, \phi_n(t^k)) .$$

and

(c) $\Lambda_\zeta = \Gamma_\zeta$, $\zeta = (\Phi_{p^r}(t))$ the prime ideal in R defined by the cyclictomic polynomial $\Phi_{p^r}(t)$, p prime.

${}_\varepsilon V_n(A) = \pi_n(\mathcal{V}(A))$ and ${}_\varepsilon U_n(A) = \pi_n(\mathcal{U}(A))$, $n > 0$. For $n = 0$ the definitions have to be modified in an obvious way.

II. PROBLEMS AND CONJECTURES.

1. Let Z_2 acts on $GL(A)$ by the formula

$$\alpha \longmapsto ({}^t\bar{\alpha})^{-1}$$

where ${}^t\bar{\alpha}$ is the conjugate of the transpose of A. Then $\mathbf{Z}_2$ acts also on $B_{GL(A)}$, $B^+_{GL(A)}$ and $K_n(A)$. Let $k_n^{odd}(A) = H^{odd}(\mathbf{Z}_2; K_n(A))$ and $k_n^{ev}(A) = H^{ev}(\mathbf{Z}_2; K_n(A))$. Then I conjecture that $k_n(A) \approx k_n(A[x])$ with $k_n = k_n^{ev}$ or k_n^{odd}. By the fundamental theorem in hermitian K-theory (see the "clock exact sequence" below) this problem is equivalent to ${}_\varepsilon L_n^{\bullet}(A) \approx {}_\varepsilon L_n^{\bullet}(A[x])$ and ${}_\varepsilon W_n(A) \approx {}_\varepsilon W_n(A[x])$ (this is true for $n = 0$). More generally let A be a ring with an antiautomorphism σ such that $\sigma^{2k} = 0$. Then $\mathbf{Z}_{2k}$ acts on $K_n(A)$ and one may ask about the homotopy invariance of the group $H^i(\mathbf{Z}_{2k}; K_n(A))$, at least if $(2k)!$ is invertible in A.

2. Shaneson, Wall, Novikov, Ranicki and others have proved that ${}_\varepsilon L_n^{\bullet}(A_z) \approx {}_\varepsilon L_n^{\bullet}(A) \oplus {}_\varepsilon L_{n-1}^{\bullet}(A)$ and ${}_\varepsilon W_n(A_z) \approx {}_\varepsilon W_n(A) \oplus {}_\varepsilon W_{n-1}(A)$ for $n = 0,1,2$ where A_z is the ring of Laurent polynomials $A[z, z^{-1}]$ with the involution $a\,z \longmapsto \bar{a}\,z^{-1}$. The natural conjecture is that these statements must be true for arbitrary n.

3. The conjecture 2 is related to the following: one conjectures an exact sequence

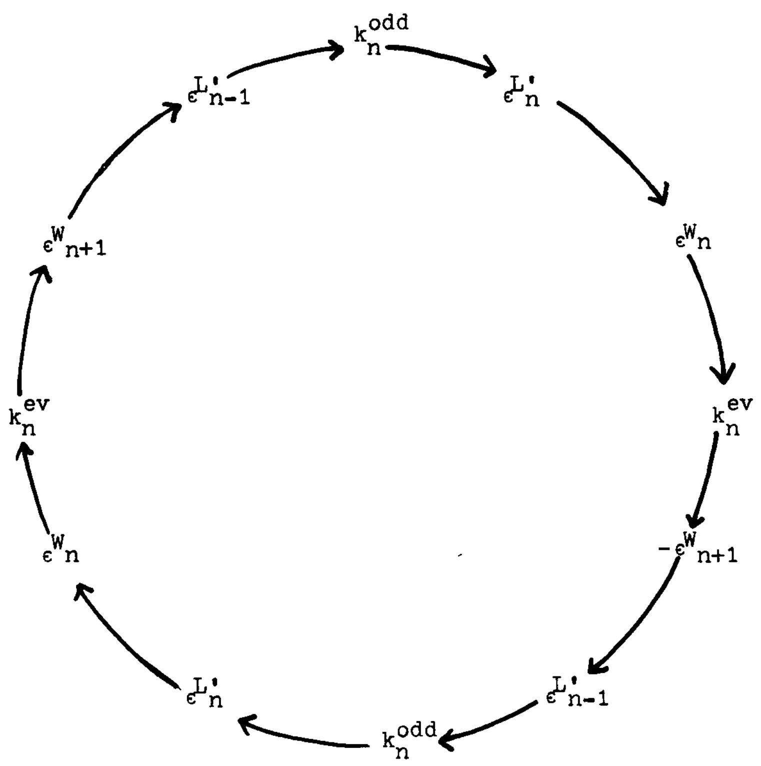
$$0 \longrightarrow K_n(A) \longrightarrow K_n(A[z]) + K_n(A[z^{-1}]) \longrightarrow K_n(A_z) \longrightarrow K_{n-1}(A) \longrightarrow 0$$

where the last map (the "index map") is induced by the canonical homomorphism from A_z to SA, the suspension of the ring A (one has then to use the isomorphism $K_n(SA) \approx K_{n-1}(A)$ proved by Wagoner and Gersten). More precisely this exact sequence implies the isomorphisms

$$k_n^{ev}(A_z) \approx k_n^{ev}(A) \oplus k_{n-1}^{odd}(A)$$

$$k_n^{odd}(A_z) \approx k_n^{odd}(A) \oplus k_{n-1}^{ev}(A)$$

(a weaker form of conjecture 3). Then conjecture 2 follows from the five lemma applied to the "clock exact sequence" which is analogous to the Rothenberg exact sequence:



4. The conjecture 3 can be generalized in many directions. One of them is to conjecture a Mayer-Vietoris type of exact

sequences

$${}_{\epsilon}L'_n(C) \longrightarrow {}_{\epsilon}L'_n(A) \oplus {}_{\epsilon}L'_n(B) \longrightarrow {}_{\epsilon}L'_n(A \underset{C}{*} B) \longrightarrow$$

$${}_{\epsilon}L'_{n-1}(C) \longrightarrow$$

$${}_{\epsilon}W_n(C) \longrightarrow {}_{\epsilon}W_n(A) \oplus {}_{\epsilon}W_n(B) \longrightarrow {}_{\epsilon}W_n(A \underset{C}{*} B) \longrightarrow$$

$$W_{n-1}(C)$$

(for n = 1,2 this seems to have been proved by Cappell).

5. Using hermitian K-theory and Quillen's recent results one can prove that $K_3(Z)$ is finite and $\# K_3(Z) \geq 48$ (this contradicts a recent conjecture of Lichtenbaum). The problem now is to compute exactly $K_3(\mathbb{Z})$ and ${}_{\epsilon}L_3(\mathbb{Z}')$ where where $\mathbb{Z}' = \mathbb{Z}[\frac{1}{2}]$. Is the homomorphism $\pi_n^s \longrightarrow K_n(\mathbb{Z})$ injective?

6. If 2 is invertible in A one can prove an isomorphism between the theories ${}_{\epsilon}V_n(A)$ and ${}_{-\epsilon}U_{n+1}(A)$. If A is arbitrary the problem is to find reasonable definitions of ${}_{\epsilon}L_n$, ${}_{\epsilon}U_n$, ${}_{\epsilon}V_n$,... such that we have this theorem again.

7. I offer the following problem: Find a theory $S_n(A)$ for any commutative ring A such that

1) $S_n(A) \otimes Z[\frac{1}{2}] \approx W_n(A) \otimes Z[\frac{1}{2}]$

2) For any exact sequence of rings

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

one has a long exact sequence

$$S_{n+1}(A) \longrightarrow S_{n+1}(A'') \longrightarrow S_n(A') \longrightarrow S_n(A) \longrightarrow S_n(A'')$$

3) If A is the ring of real continuous functions functions on X, then $S_n(A) \approx KO^{-n}(X)$.

8. Find a K-theoretical proof of Quillen's results about the K-theory of a finite field and of Borel's results about the K-theory of number fields.

9. Clifford algebras play an important role in topographical K-theory. On the other hand the periodicity theorems in algebraic K-theory don't use Clifford algebras. What is the exact role of Clifford algebras in both theories? Is it possible to find a unified approach for the periodicity theorems in the topological and algebraic contexts using Clifford algebras?

10. Volodin has defined new K-groups of a ring A (called $\overset{V}{K}_n(A)$ here). It remains to prove that $\overset{V}{K}_n(A) \approx K_n(A)$ (this seems to have been done by Wasserstein and Wagoner). Anyway, it looks reasonable to define in the same manner groups ${}_\varepsilon\overset{V}{L}_n(A)$, ${}_\varepsilon\overset{V}{U}_n(A)$, ${}_\varepsilon\overset{V}{V}_n(A)$. Is it true then that ${}_\varepsilon L_n(A) \approx {}_\varepsilon\overset{V}{L}_n(A)$, ${}_\varepsilon V_n(A) \approx {}_\varepsilon V_n^V(A)$, ${}_\varepsilon U_n(A) \approx {}_\varepsilon U_n^V(A)$?