RELATIONS BETWEEN HIGHER ALGEBRAIC K-THEORIES

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In this paper we outline the construction of a sequence of natural transformations

$$K_i^Q \to K_i^{BN} \to K_i^V \to K_i^{K-V}$$

between the higher algebraic K-theories K_i^Q of [Q], K_i^{BN} of [W], K_i^V of [V] and K_i^{K-V} of [K-V] and [G]. The composition $K_i^Q \to K_i^{K-V}$ is the map constructed by Gersten [G].

Let G be a discrete group. Let NG be the simplicial set in [S] such that the geometric realization $|NG|$ is the classifying space BG of G. Let ΔG denote the contractible simplicial set whose k-simplices are (k+1)-tuples $(g_0,\ldots,g_k)$ which the face and degeneracy operators being deletion and insertion. The geometric realization $|\Delta G|$ shall be denoted by EG. The map of simplicial sets $\Delta G \to NG$ given by

$$(g_0,\ldots,g_k) \to (g_0^{-1}g_1,\ldots,g_{k-1}^{-1}g_k)$$

induces as in [S] the classifying fibration for G:

$$G \to EG \to BG$$

For any coset $\alpha\cdot H$ of a subgroup H of G let $E_{\alpha\cdot H} = |\Delta_{\alpha\cdot H}|$ where $\Delta_{\alpha\cdot H}$ is the subcomplex of ΔG whose k-simplices are (k+1)-tuples $(g_0,g_1,\ldots,g_k)$ with $g_i \in \alpha\cdot H$ for $0 \leqslant i \leqslant k$.

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Let $\{H_i\}$ be a collection of subgroups of G. Let $\Delta\{G,\alpha\cdot H_i\}$ denote the bisimplicial set whose (k,ℓ)-simplices are of the form

$$(\alpha_0\cdot H_0 \subset \ldots \subset \alpha_k\cdot H_k;\ (g_0,\ldots,g_\ell))$$

where $g_i \in \alpha_0\cdot H_0$ for $0 \leqslant i \leqslant \ell$. The boundary and degeneracy maps in each factor are the usual deletions and insertions. Let $\Delta^v\{G,\alpha\cdot H_i\}$ be the simplicial space obtained by vertical realization: in dimension k, $\Delta^v\{G,\alpha\cdot H_i\}$ is the disjoint union the spaces

$$(\alpha_0\cdot H_0 \subset \ldots \subset \alpha_k\cdot H_k) \times E_{\alpha_0\cdot H_0}$$

Then as for any bi-simplical set

$$|\text{diag}\ \Delta\{G,\alpha\cdot H_i\}| \cong |\Delta^v\{G,\alpha\cdot H_i\}| .$$

We shall let $E\{G,\alpha\cdot H_i\}$ denote $|\text{diag}\ \Delta\{G,\alpha\cdot H_i\}|$. The reader is alerted to the slight notational difference between this and the $E(X,X_i)$ of [W]. Let $N\{H_i\}$ be the bi-simplicial set whose (k,ℓ)-simplices are of the form

$$(H_0 \subset \ldots \subset H_k;\ (g_1,\ldots,g_\ell))$$

where $g_i \in H_0$ for $1 \leqslant i \leqslant \ell$. The boundary and degeneracy operators in the first factor are deletion and insertion, the ones in the second factor are those of NH_0. Let $N^v\{H_i\}$ be the vertical realization of $N\{H_i\}$; this is a simplicial space which in dimension k is the disjoint union of the spaces

$$\left(H_0 \subset \ldots \subset H_k\right) \times BH_0 .$$

We also have

$$|\ \text{diag}\ N\{H_i\}| \simeq |N^v\{H_i\}| .$$

Let $B\{H_i\}$ denote $|\text{diag}\ N\{H_i\}|$.

There is a commutative diagram

$$(*)\qquad \begin{array}{ccc} E\{G,\alpha\cdot H_i\} & \longrightarrow & EG \\ \downarrow & & \downarrow \\ B\{H_i\} & \longrightarrow & BG \end{array}$$

induced by the following diagram of maps on the level of k-simplices:

$$\begin{array}{ccc} (\alpha_0\cdot H_0 \subset \cdots \subset \alpha_k\cdot H_k;\ (g_0,\ldots,g_k)) & \longrightarrow & (g_0,\ldots,g_k) \\ \downarrow & & \downarrow \\ (H_0 \subset \cdots \subset H_k;\ (g_0^{-1}g_1,\ldots,g_{k-1}^{-1}g_k)) & \longrightarrow & (g_0^{-1}g_1,\ldots,g_{k-1}^{-1}g_k) \ . \end{array}$$

Lemma 1. The sequence

$$E\{G,\alpha\cdot H_i\} \to B\{H_i\} \to BG$$

is a homotopy fibration.

Proof. The simplicial set $\Delta\{G,\alpha\cdot H_i\}$ is the pull back of the maps $\Delta G \to NG$ and $N\{H_i\} \to NG$. Since realization of simplicial sets commutes with pullbacks (*) is a cartesian square. In particular $E\{G,\alpha\cdot H_i\} \to B\{H_i\}$ is a fibration with discrete fiber G. Since EG is contractible the "nine-lemma" of [T; Appendix, §6] shows $E\{G,\alpha\cdot H_i\}$ is homotopy equivalent to the homotopy fiber of $B\{H_i\} \to BG$.

We recall briefly the definitions of K_i^{BN} and K_i^V. See [V] and [W]. Let A be an associative ring with unity and A^∞ be the standard right A-module with basis $e_1,e_2,\ldots,e_n,\ldots$. A semi-standard flag $P = \{P_1 \subset \cdots \subset P_k\}$ in A^∞ is a sequence of free submodules such that for some $n \geqslant 1$ each subspace P_i is spanned by a finite subset of $\{e_1,\ldots,e_n\}$ and such that $P_k = A^n$. We say

$P' = \{P_1' \subset \ldots \subset P_\ell'\}$ refines P, written $P \leqslant P'$, provided there is an increasing sequence $n_1 < \ldots < n_k$ with $P_i = P'_{n_i}$ for $1 \leqslant i \leqslant k$. If P is a flag with $P_k = A^n$ let $U_P \subset GL_n(A) \subset GL(A)$ be the subgroup of elements of the form $I + N$ where I is the identity and N is an $n \times n$-matrix satisfying $N(P_i) \subset P_{i-1}$ and $N(P_1) = 0$. Then let $\widehat{GL}(A) = E\{(GL(A), \alpha \cdot U_P\}$ and for $i \geqslant 1$ let

$$K_i^{BN}(A) = \pi_{i-1}\widehat{GL}(A).$$

As in (*) we have a cartesian square

$$(\dagger) \qquad \begin{array}{ccc} E\{GL(A), \alpha \cdot U_P\} & \longrightarrow & EGL(A) \\ \downarrow & & \downarrow \\ B\{U_P\} & \longrightarrow & BGL(A) \end{array}$$

and a homotopy fibration

$$\widehat{GL}(A) \to B\{U_P\} \to BGL(A).$$

Let $U \subset GL(A)$ be the subgroup of upper triangular matrices with ones on the diagonal. Let p denote any finite permutation matrix. Let $GL^V(A) \subset EGL(A)$ be the realization of the subcomplex of $\Delta GL(A)$ whose k-simplices consist of (k+1)-tuples $(g_0, \ldots, g_k)$ such that there is some coset $\alpha \cdot pUp^{-1}$ containing g_i for $0 \leqslant i \leqslant k$. Then in [V] Volodin defines

$$K_i^V(A) = \pi_{i-1} GL^V(A).$$

There is a cartesian square

$$\begin{array}{ccc} GL^V(A) & \longrightarrow & EGL(A) \\ \downarrow & & \downarrow \\ \bigcup_p BpUp^{-1} & \longrightarrow & BGL(A) \end{array}$$

where the space $\bigcup_p BpUp^{-1}$ denotes the union of the subspaces $BpUp^{-1}$ in $BGL(A)$. As in Lemma 1 we have a homotopy fibration

$$GL^V(A) \longrightarrow \bigcup_p BpUp^{-1} \longrightarrow BGL(A)$$

The natural transformation

(1) $$K_i^{BN} \to K_i^V$$

is induced by the correspondence

$$(\alpha_0 \cdot U_{P_0} \subset \ldots \subset \alpha_k \cdot U_{P_k};\ g_0, \ldots, g_k) \to (g_0, \ldots, g_k).$$

It can be shown that the direct sum of matrices induces on $\widehat{GL}(A)$ and $GL^V(A)$ the structure of a homotopy associative and commutative H-space and furthermore we have

$$K_1^{Bass}(A) = K_1^{BN}(A) = K_1^V(A)$$

and

$$K_2^{Milnor}(A) = K_2^{BN}(A) = K_2^V(A).$$

See [V] and [W].

<u>Theorem A</u>. The fundamental groups of $B\{U_P\}$ and $\bigcup_p BpUp^{-1}$ are perfect and the sequences

$$\widehat{GL}(A) \to B\{U_P\}^+ \to BGL(A)^+$$

and

$$GL^V(A) \longrightarrow (\cup BpUp^{-1})^+ \longrightarrow BGL(A)^+$$

are homotopy fibrations.

As a consequence there are maps

$$\Omega BGL(A)^+ \longrightarrow \widehat{GL}(A)$$

and

$$\Omega BGL(A)^+ \longrightarrow GL^V(A)$$

which induce natural homomorphisms

(2) $$K_i^Q(A) = \pi_{i-1}(\Omega BGL(A)^+) \to \pi_{i-1}(\widehat{GL}(A)) = K_i^{BN}(A)$$

and

(3) $$K_i^Q(A) = \pi_{i-1}(\Omega BGL(A)^+) \to \pi_{i-1}(GL^V(A)) = K_i^V(A).$$

In fact (3) is the composition of (1) and (2). It will be shown in [W'] that $H_*(B\{U_p\};Z)$ vanishs for any ring. Hence $B\{U_p\}^+$ is contractible and Theorem A shows that (2) is an isomorphism. I think the methods of [W'] will also show $H_*(\bigcup_p BpUp^{-1};Z) = 0$ and therefore (3) is probably an isomorphism too.

Finally we construct the natural transformation $K_i^{BN} \to K_i^{K-V}$. A similar construction works for K_i^V. Recall the definition of K_i^{K-V} as given in [G]. Let $A_* = \{A_n\}$ be the simplicial ring where

$$A_n = A[t_0,t_1,\ldots,t_n] \big/ \; t_0 + t_1 + \ldots + t_n = 1$$

and the face and degeneracy operators $\partial_i : A_n \to A_{n-1}$ and $s_i : A_n \to A_{n+1}$ are given by

$$\partial_i(t_\ell) = \begin{cases} t_\ell, & \ell < i \\ 0, & \ell = i \\ t_{\ell-1}, & i < \ell \end{cases} \quad \text{and} \quad s_i(t_\ell) = \begin{cases} t_\ell, & \ell < i \\ t_\ell + t_{\ell+1}, & \ell = i \\ t_{\ell+1}, & i < \ell \end{cases}$$

Let $GL(A)_* = \{GL(A_n)\}$ be the corresponding simplicial group. Then as in [G]

$$K_i^{K-V}(A) = \pi_{i-1}|GL(A_*)| = \pi_i BGL(A_*)$$

where $BGL(A_*)$ is the realization of the diagonal in the bisimplicial set $NGL(A_*)$ whose (k,ℓ)-simplices are ℓ-tuples $(g_1,\dots,g_\ell)$ with $g_i \in GL(A_k)$ for $1 \leqslant i \leqslant \ell$. The vertical face and degeneracies are those of $NGL(A_k)$ and the horizontal face and degeneracies come from the simplicial ring A_*. Let $\Delta GL(A_*)$ be the bisimplicial set whose (k,ℓ)-simplices are $(\ell+1)$-tuples $(g_0,\dots,g_\ell)$ where $g_i \in GL(A_k)$ for $1 \leqslant i \leqslant \ell$. Let $EGL(A_*)$ be the realization of the diagonal. Then there is a homotopy fibration

$$|GL(A_*)| \to EGL(A_*) \to BGL(A_*).$$

There is a similar fibration for $BU_P(A_*)$.

We want to get a commutative square like (†) for the simplicial ring A_*. For any ring A we let $U_P(A)$ denote U_P for the ring A. Let $N\{U_P(A_*)\}$ be the bi-simplicial set whose (k,ℓ) simplices are of the form

$$(U_{P_0}(A_\ell) \subset \dots \subset U_{P_k}(A_\ell);\ (g_1,\dots,g_\ell))$$

where $g_i \in U_{P_0}(A_\ell)$ for $1 \leqslant i \leqslant \ell$. Let $\Delta\{GL(A_*),\alpha\cdot U_P(A_*)\}$ be the bi-simplicial set whose (k,ℓ) simplices are of the form

$$(\alpha_0 \cdot U_{P_0}(A_\ell) \subset \dots \subset \alpha_k \cdot U_{P_k}(A_\ell);\ (g_0,\dots,g_\ell))$$

where $g_i \in \alpha_0 \cdot U_{P_0}(A_\ell)$ for $0 \leqslant i \leqslant \ell$. Let

$$B\{U_P(A_*)\} = |\mathrm{diag}\ N\{U_P(A_*)\}|$$

and

$$E\{GL(A_*),\alpha\cdot U_p(A_*)\} = |\text{diag}\ \Delta\{GL(A_*),\alpha\cdot U_p(A_*)\} \quad .$$

Then the following is a pullback square:

$$(**) \qquad \begin{array}{ccc} E\{GL(A_*),\alpha\cdot U_p(A_*)\} & \longrightarrow & EGL(A_*) \\ \downarrow & & \downarrow \\ B\{U_p(A_*)\} & \longrightarrow & BGL(A_*) \end{array}$$

Let $\widehat{GL}(A_*) = E\{GL(A_*),\alpha\cdot U_p(A_*)\}$.

<u>Lemma 2</u>. $\widehat{GL}(A_*) \cong |GL(A_*)|$.

<u>Proof</u>. As in Lemma 1 the homotopy fiber of $B\{U_p(A_*)\} \to BGL(A_*)$ is $\widehat{GL}(A_*)$. Hence it suffices to prove $B\{U_p(A_*)\}$ is contractible. This space has the homotopy type of the realization of the simplicial space $N^v\{U_p(A_*)\}$ which in dimension k is the disjoint union of the

$$(P_0 \leqslant \dots \leqslant P_k) \times BU_p(A_*).$$

Since A_* is an acyclic simplicial ring each $BU_p(A_*)$ is contractible; thus up to homotopy type we are looking at the nerve of the partially ordered set of semi-standard flags in A^∞ which is contractible.

Now let A denote the constant simplicial ring where $A_n = A$ for $n \geqslant 0$ and all the face and degeneracy operators are the identity. Then there is a cartesian square like (**) but in this case we are still, up to homotopy type, just working with the square (†) when A is not considered as a simplicial ring. The natural inclusion of simplicial rings $A \to A_*$ induces a homotopy commutative diagram of homotopy fibrations

$$
\begin{array}{ccccc}
\widehat{GL}(A) & \longrightarrow & B\{U_P(A)\}^+ & \longrightarrow & BGL(A)^+ \\
\downarrow & & \downarrow & & \downarrow \\
\widehat{GL}(A_*) & \longrightarrow & B\{U_P(A_*)\} & \longrightarrow & BGL(A_*)
\end{array} .
$$

This then gives the homomorphism

$$(4) \qquad K_i^{BN}(A) = \pi_{i-1}\widehat{GL}(A) \longrightarrow \pi_{i-1}|GL(A_*)| = K_i^{K-V}(A).$$

Putting (1), (2), and the K_i^V anlogue of (4) together gives the sequence of natural transformations

$$(5) \qquad K_i^Q \to K_i^{BN} \to K_i^V \to K_i^{K-V}$$

whose composition is the map defined by Gersten [G]. When A is left regular Quillen [Q] has shown this composition to be an isomorphism.

Corollary B. If A is left regular K_i^{BN} and K_i^V contain K_i^Q as a direct summand.

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