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M. KAROUBI

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TORSION OF THE WITT GROUP

by

M. KAROUBI

The purpose of this paper is to give an elementary proof of the following theorem (well known if  $A$  is a field).

Theorem. Let  $A$  be a commutative ring and let  $\Gamma(A)$  be the subring (= subgroup) of the Witt ring  $W(A)$  generated by the classes of projective modules of rank one. Then the torsion of  $\Gamma(A)$  is 2-primary.

Proof. Let  $L(A)$  be the Grothendieck group of the category of non degenerated bilinear  $A$ -modules. Let  $x = [L_1 \oplus \dots \oplus L_n] \in L(A)$  where  $L_i$  are projective of rank one and let us assume that the class of  $px$  in  $W(A)$  is zero,  $p$  being an odd prime. We want to show that the class of  $x$  in  $W(A)$  is equal to 0. We need the following lemma :

Lemma. Let  $\Gamma_0$  be the subring of  $W(A)$  generated by the  $\langle L_i \rangle$  and let  $y \in \Gamma_0$  such that  $py = 0$  in  $W(A)$  with  $p$  an odd prime. Then  $y \in p\Gamma_0$ .

Proof of the lemma. Let  $y = \langle R_1 \oplus \dots \oplus R_m \rangle \in \Gamma_0$  where the  $R_i$  are projectives of rank one and monomial of the  $L_i$  and  $L_i$ . Let  $\bar{\Gamma}_0$  be the subring of  $L(A)$  generated by the  $L_i$  and  $L_i$  and  $\bar{y}$  be the class of  $R = R_1 \oplus \dots \oplus R_m$  in  $L(A)$ . Following Grothendieck we write

$$\lambda_t(\bar{y}) = 1 + t \lambda^1(\bar{y}) + \dots + t^m \lambda^m(\bar{y}) \in L(A)[t] \quad (\text{note that } \lambda^1(\bar{y}) \in \bar{\Gamma}_0).$$

Since  $\lambda_t(u+v) = \lambda_t(u) \lambda_t(v)$  according to the general properties of the exterior powers, we have

$$\lambda_t(p\bar{y}) = (\lambda_t(\bar{y}))^p = 1 + t^p \lambda^1(\bar{y})^p + \dots + t^{mp} \lambda^m(\bar{y})^p \text{ mod. } p \bar{\Gamma}_0.$$

Moreover,

$$\begin{aligned} \lambda^1(\bar{y})^p &= [R_1 \oplus \dots \oplus R_m]^p = [R_1^p \oplus \dots \oplus R_m^p] = [R_1 \oplus \dots \oplus R_m] = \\ &= \bar{y} \text{ mod. } p \bar{\Gamma}_0, \end{aligned}$$

because  $[R_i]^2 = 1$ . It follows from this computation that  $\lambda^p(p\bar{y}) = \bar{y} \text{ mod. } p \bar{\Gamma}_0$ .

Since  $p\bar{y}$  is stably metabolic and since  $p$  is odd,  $\lambda^p(p\bar{y})$  is stably metabolic.

Hence  $y = 0 \text{ mod. } p \Gamma_0$ .

Proof of the theorem (followed). Since  $[L_i]^2 = 1$ ,  $\Gamma_0$  is a finitely generated  $\mathbb{Z}$ -module. From the lemma it follows that the  $p$ -torsion of is zero if  $p$  is odd. Hence the torsion of  $\Gamma_0$  is 2-primary which implies  $x = 0$  as required.

Part of these considerations can be generalized for rings with involution. Of course we have not necessarily  $[L]^2 = 1$  if  $L$  is projective of rank one (except if  $A$  is local). However, we can consider the subring  $\Gamma^q(A)$  of  $W(A)$  generated by the classes of projective modules of rank one such that  $[L]^q = 1$  (see the example below). Then I claim that the torsion of  $\Gamma^q(A)$  is  $2q$ -primary (i.e.  $px = 0$  implies  $x = 0$  if  $p$  is prime to  $2q$ ). The proof is along the same lines as the proof of the first theorem. If we write  $x = \langle L_1 \oplus \dots \oplus L_n \rangle$  we can consider the subring  $\Gamma_0^q$  of  $\Gamma^q(A)$  generated by the  $L_i$  and the subring  $\bar{\Gamma}_0^q$  of  $L(A)$  generated by the  $L_i$  and  $L_i^-$ . Let  $\alpha$  be an integer such that  $p^\alpha - 1$  is divisible by  $q$  (for instance the Euler indicator). Then, with the notations of the lemma we

have  $\lambda_t(p^\alpha \bar{y}) = 1 + t^{p^\alpha} \lambda^1(\bar{y})^{p^\alpha} + \dots \pmod{p \bar{\Gamma}_0^q}$ . Hence

$$\lambda^{p^\alpha}(p^\alpha \bar{y}) = \lambda^1(\bar{y})^{p^\alpha} = [R_1^{p^\alpha} \oplus \dots \oplus R_m^{p^\alpha}] = [R_1 \oplus \dots \oplus R_m] = \bar{y} \pmod{p \bar{\Gamma}}$$

(because  $R_i^q = 1$ ). Therefore the  $p$ -torsion of  $\Gamma_0^q$  is  $p$ -divisible which implies  $x = 0$ .

Example. Let  $A$  be the ring of complex continuous functions on the lens space  $X = S^{2n+1}/Z_q$  where  $S^{2n+1}$  is the  $2n+1$ -dimensional sphere imbedded in  $\mathbb{C}^{n+1}$ ,  $Z_q$  acting by the action of  $q^{\text{th}}$  roots of the unity. If we provide  $A$  with the complex conjugation involution, the Witt ring  $W(A)$  can be identified with the complex  $K$ -theory  $K_C(X)$  of the space  $X$  (this is true for any compact space  $X$ ). This complex  $K$ -theory is generated by the trivial bundles and by the line bundle  $L = S^{2n-1} \times C$ . If we put  $t = \langle L \rangle$  we have in fact  $W(A) = Z[t]/I$  where  $I$  is

the ideal generated by the polynomials  $t^q - 1$  and  $(t-1)^n$ . Hence

$$W(A) = \Gamma^q(A) = Z \oplus T$$

where  $T$  is a torsion group which is  $q$ -primary.

Remark. If we consider the ring  $B$  of real continuous functions on  $X$ , it is not hard to show that  $W(B) \otimes Z[\frac{1}{2}]$  is isomorphic to the invariant part of  $W(A) \otimes Z[\frac{1}{2}]$  by the action of  $Z_2$  acting by  $t \rightarrow t^{-1} = t^{q-1}$ . Hence  $W(B)$  can have arbitrary torsion (not just 2-torsion).

U.E.R. de Mathématiques  
Tour 45-55 5<sup>e</sup> étage  
Université PARIS VII  
2, Place Jussieu

75230 PARIS CEDEX 05  
FRANCE