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HIGHER ALGEBRAIC K-THEORIES

BY

D. ANDERSON, M. KAROUBI AND J. WAGONER⁽¹⁾

ABSTRACT. A homotopy fibration is established relating the Volodin or BN -pair definition of algebraic K -theory to the theory defined by Quillen.

In [2] we outlined the construction of natural homomorphisms

$$K_*^Q \rightarrow K_*^{BN} \rightarrow K_*^V \rightarrow K_*^{KV}$$

between higher algebraic K -theories K_*^Q of [10] and [11], K_*^{BN} of [17], K_*^V of [16], and K_*^{KV} of [7] and [8]. This was one of the steps in proving the various definitions of higher K -theory are equivalent. It turns out they all agree—including the theory K_*^S of [14], [5], and [8]—provided one restricts to the category of regular rings when using K_*^{KV} . See [1], [2], [5], [8] and [18]. The purpose of this paper is to prove the following theorem, announced in [2], which yields the construction of $K_*^Q \rightarrow K_*^{BN}$.

THEOREM. *For any associative ring with identity A*

$$GL^{BN}(A) \rightarrow B\{U_F\}^+ \rightarrow BGL(A)^+$$

is a homotopy fibration.

For the reader's convenience and because the presentation of the BN -pair K -theory K_*^{BN} used here is slightly different from that of [17], we shall briefly recall the definition of GL^{BN} and $B\{U_F\}$ in the first section.

1. Preliminaries. Let $\{H_\alpha\}$ be the collection of hyperplanes in n -dimensional euclidean space R^n given by the condition $\alpha = 0$ where $\alpha = e_i - e_j$, $i \neq j$, is a linear root. Here e_i is the i th coordinate function. This determines a stratification of R^n whose strata F we call facettes as in [3]. By definition a facette of codimension k is a component of the complement in the union of the k -fold intersections of the H_α of the subset consisting of the union of the $(k + 1)$ -fold intersections. Let P^n be the set of facettes of R^n partially ordered by the condition that $F < G$ iff $F \subset \bar{G}$. We shall also let P^n denote the simplicial complex whose k -simplices are $(k + 1)$ -tuples $(F_0 < \cdots < F_k)$ where $F_i \in P^n$. P^n is a piecewise linear triangulation of the standard $(n - 1)$ -simplex. The stabilization map $R^n \rightarrow R^{n+1}$ defined by

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$$(*) \quad (x_1, \dots, x_n) \rightarrow (x_1, \dots, x_n, x_n)$$

takes each facette F to a facette F' and preserves the relation " $<$ ". Thus we can consider P^n as a subset (or subcomplex) of P^{n+1} and we let $P^\infty = \bigcup_n P^n$. If $F \in P^n$, let $U_F \subset GL(n, A)$ be the subgroup generated by the elementary matrices $e_{ij}(\lambda)$ where $e_i - e_j > 0$ on F and $\lambda \in A$. Note that if $F \in P^\infty$ lies in P^n , then $U_F \subset GL(\infty, A)$ is the direct limit $U_F \rightarrow U_{F'} \rightarrow U_{F''} \rightarrow \dots$.

Now for $1 \leq n \leq \infty$ let B_n be the realization of the simplicial space which in dimension $k \geq 0$ is the disjoint union of the spaces $(F_0 < \dots < F_k) \times BU_{F_0}$ where $F_i \in P^n$. Then $B_\infty = \lim_{n \rightarrow \infty} B_n$ and by definition we let $B\{U_F\} = B_\infty$. The inclusions $BU_F \subset BGL(n, A)$ induce a map

$$B_n \rightarrow BGL(n, A)$$

for $1 \leq n \leq \infty$. Recall from [2] that $\pi_1 B\{U_F\} = St(A)$ and $\pi_1 B\{U_F\} \rightarrow \pi_1 BE(A)$ is just $St(A) \rightarrow E(A)$.

If $\alpha \cdot U_F$ and $\beta \cdot U_G$ are two left cosets in $GL(n, A)$ define

$$\alpha \cdot U_F < \beta \cdot U_G$$

to mean $F < G$ and $\alpha \cdot U_F \subset \beta \cdot U_G$. For $2 \leq n \leq \infty$ define G_n (resp. E_n) to be the simplicial complex where k -simplices are $(k+1)$ -tuples

$$(\alpha_0 \cdot U_{F_0} < \dots < \alpha_k \cdot U_{F_k})$$

where $F_i \in P^n$ and $\alpha_i \in GL(n, A)$, respectively $\alpha_i \in E(n, A) =$ the subgroup of elementary matrices. We have $G_\infty = \text{ind lim}_n G_n$ and $E_\infty = \text{ind lim}_n E_n$ and by definition we set

$$GL^{BN}(A) = G_\infty \text{ and } E^{BN}(A) = E_\infty.$$

The group $GL(n, A)$ acts on G_n by left multiplication: $\alpha \cdot (\alpha_0 \cdot U_{F_0} < \dots < \alpha_k \cdot U_{F_k}) = (\alpha\alpha_0 \cdot U_{F_0} < \dots < \alpha\alpha_k \cdot U_{F_k})$ and this restricts to an action of $E(n, A)$ on E_n . Moreover, $\pi_0 GL^{BN}(A) = K_1(A)$ and $GL^{BN}(A) = K_1(A) \times E^{BN}(A)$. See [17].

Now let $G = E(A)$ and define $E\{\alpha \cdot U_F\}$ to be the pullback of the diagram

$$\begin{array}{ccc} E\{\alpha \cdot U_F\} & \overset{i}{\dashrightarrow} & EG \\ \downarrow & & \downarrow \pi \\ B\{U_F\} & \xrightarrow{j} & BG \end{array}$$

Here we let EG be the realization of the simplicial set whose k -simplices are $(k + 1)$ -tuples (g_0, \dots, g_k) . The universal principal G -bundle $\pi: EG \rightarrow BG$ is defined by

$$\pi(g_0, \dots, g_k) = (g_0^{-1}g_1, \dots, g_{k-1}^{-1}g_k).$$

See [2]. Let $E(\alpha \cdot U_F) \subset EG$ denote the contractible subcomplex whose k -simplices are those $(k + 1)$ -tuples for which $g_i \in \alpha \cdot U_F$ for $0 \leq i \leq k$. Then $E\{\alpha \cdot U_F\}$ is the realization of the simplicial space which in dimension $k \geq 0$ is the disjoint union of the spaces

$$(\alpha_0 \cdot U_{F_0} < \dots < \alpha_k \cdot U_{F_k}) \times E(\alpha_0 \cdot U_{F_0})$$

where $F_i \in P^\infty$. Hence by [10, Lemma for Theorem B] the natural map $E\{\alpha \cdot U_F\} \rightarrow E^{BN}(A)$ is a homotopy equivalence. Since EG is contractible the "nine-lemma" [15] implies

$$E\{\alpha \cdot U_F\} \rightarrow B\{U_F\} \rightarrow BG$$

is a homotopy fibration and so

$$E^{BN}(A) \rightarrow B\{U_F\} \rightarrow BG$$

is a homotopy fibration. Similarly letting $G = GL(A)$ there is a homotopy fibration

$$GL^{BN}(A) \rightarrow B\{U_F\} \rightarrow BGL(A).$$

We must show this remains a fibration when the "plus-construction" is performed on the second and third spaces. Since the universal cover of $BGL(A)^+$ is $BE(A)^+$ and since $\pi_1 BGL(A)^+ = K_1(A)$, to prove the main theorem it suffices to prove

THEOREM 1. *For any associative ring with identity A*

$$E^{BN}(A) \rightarrow B\{U_F\}^+ \rightarrow BE(A)^+$$

is a homotopy fibration.

The idea of the proof is to consider the diagram

$$\begin{array}{ccccc} E^{BN}(A) & \longrightarrow & B\{U_F\} & \longrightarrow & BE(A) \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & B\{U_F\}^+ & \xrightarrow{j} & BE(A)^+ \end{array}$$

where X is the homotopy theoretic fiber of the map j . Suppose we can verify that:

(I) $E^{BN}(A)$ is a connected H -space such that $E(A) = \pi_1 BE(A)$ acts trivially on $H_*(E^{BN}(A))$, and

(II) X is a connected H -space.

Note that $BE(A)^+$ is simply connected and so its fundamental group acts trivially on $H_*(X)$. Then since the "plus-construction" preserves homology, the Comparison Theorem for the spectral sequence of a fibration [9] implies $E^{BN}(A) \rightarrow X$ is a homology equivalence. Hence it is a homotopy equivalence by [4, Lemma 6.2]. Condition (I) will be established in §2 and §3; (II) will be shown in §4 by seeing that $B\{U_F\}^+ \rightarrow BE(A)^+$ is an H -map and so by the proof of Theorem 2 of [13] its homotopy fiber is an H -space.

For convenience we state the following lemma of [18, Lemma 3.3]. Let K denote a partially ordered set and also the corresponding simplicial complex. Let $f: K \rightarrow P^n$ be a map of partially ordered sets. Thus for each vertex v of K , $f(v)$ is a facette of R^n . Now let $g: K \rightarrow P^n$ be any map of sets (not necessarily order preserving). Give $K \times I$ the standard triangulation as a partially ordered set where $I = \{0, 1\}$ with $0 < 1$. Together f and g define a map

$$\{\text{vertices of } K \times I\} \rightarrow P^n$$

which does not necessarily preserve order except on $K \times 0$.

LEMMA. *There is a triangulation $(K \times I)'$ of the simplicial complex $K \times I$ as a partially ordered set which refines the standard triangulation of $K \times I$ leaving $K \times 0$ unchanged, and there is an order preserving map $w: (K \times I)' \rightarrow P^n$ of the vertices of this new triangulation such that*

(a) $w|_{K \times 0} = f$;

(b) if v is a vertex of $K \times 1$ in the new and also in the old triangulation, then $w(v) = g(v)$;

(c) if $g: K \rightarrow P^n$ is order preserving, then $K \times 1$ with the new triangulation is just a copy of K ;

(d) if $\sigma = (v_0 < \dots < v_k)$ is a simplex of the standard triangulation of $K \times I$, v is a vertex in σ of the new triangulation, and $e_{ij}(\lambda)$ lies in $U_{w(v_i)}$ for $0 \leq s \leq k$, then

$$e_{ij}(\lambda) \in U_{w(v)}.$$

2. H -space structure on $E^{BN}(A)$. In this section we show the direct sum homomorphism

$$\alpha \oplus \beta = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$

from $E(m, A) \times E(n, A)$ to $E(m + n, A)$ induces an H -space structure on $E^{BN}(A)$. Compare [16].

For this it will be convenient to describe facettes $F \in P^n$ in terms of partitions of the set $\{e_1, \dots, e_n\}$ of standard dual basis vectors for R^n . We write

$$F = X_1 | X_2 | \dots | X_r$$

to mean that F is determined by the conditions

$$\begin{aligned} e_i - e_j &= 0 && \text{if } e_i, e_j \in X_\alpha, \\ e_i - e_j &> 0 && \text{if } e_i \in X_\alpha, e_j \in X_\beta, \text{ and } \alpha < \beta. \end{aligned}$$

If $n = \infty$ we require that each X_α is finite for $1 \leq \alpha < r$. Let $m, n < \infty$ and let $F = X_1 | \dots | X_r$ and $G = Y_1 | \dots | Y_s$ lie in P^m and P^n respectively. Define

$$F \oplus G = X_1 | \dots | X_r | Y'_1 | \dots | Y'_s$$

where Y'_j is obtained from Y_j by adding n to the indices of the e_i to get a subset of $\{e_{m+1}, \dots, e_{m+n}\}$. If $F_1 < F_2$ and $G_1 < G_2$, then $F_1 \oplus G_1 < F_2 \oplus G_2$.

We shall let $\Delta \in P^n$ be the diagonal facetted defined by setting all $e_i - e_j = 0$. There is another stabilization map $F \rightarrow F \oplus \Delta$ from P^m to P^{m+n} which is not quite the same as n repetitions $F \rightarrow F^{(n)}$ of (*) of §1. However, $F^{(n)} < F \oplus \Delta$ for all $F \in P^m$ and if $\alpha \cdot U_F < \beta \cdot U_G$, then there is a commutative square

$$\begin{array}{ccc} (\alpha \oplus 1) \cdot U_{F^{(n)}} < (\alpha \oplus 1) \cdot U_{G^{(n)}} & & \\ \wedge & & \wedge \\ (\alpha \oplus 1) \cdot U_{F \oplus \Delta} < (\alpha \oplus 1) \cdot U_{G \oplus \Delta} & & \end{array}$$

which shows [12] that the two stabilization maps $E_m \rightarrow E_{m+n}$ defined respectively by

$$\alpha \cdot U_F \rightarrow (\alpha \oplus 1) \cdot U_{F^{(n)}} \quad \text{and} \quad \alpha \cdot U_F \rightarrow (\alpha \oplus 1) \cdot U_{F \oplus \Delta}$$

are homotopic. Note that the second stabilization does not take the base point U_Δ of E_m to the base point U_Δ of E_{m+n} . However, consider the contractible complex P^m as embedded in E_m by the correspondence $F \rightarrow U_F$. Then stabilization $E_m \rightarrow E_{m+n}$ via $F \rightarrow F \oplus \Delta$ takes P^m to P^{m+n} and hence determines a base point preserving map well defined up to base point preserving homotopy. From now on in this section we use the second stabilization.

If $\alpha_0 \cdot U_{F_0} < \alpha_1 \cdot U_{F_1}$ in E_m and $\beta_0 \cdot U_{G_0} < \beta_1 \cdot U_{G_1}$ and E_n , then $(\alpha_0 \oplus \beta_0) \cdot U_{F_0 \oplus G_0} < (\alpha_1 \oplus \beta_1) \cdot U_{F_1 \oplus G_1}$ in E_{m+n} . This gives a map

$$\gamma_{m,n}: E_m \times E_n \rightarrow E_{m+n}$$

which does not preserve base point; but since $\gamma_{m,n}(P^n \times P^n) \subset P^{m+n}$, it does determine a base point preserving map well defined up to base point preserving homotopy.

PROPOSITION 2. *The diagrams ($n \geq 2$)*

$$(**) \quad \begin{array}{ccc} E_n \times E_n & \xrightarrow{\quad} & E_{2n} \times E_{2n} \\ \downarrow \gamma_{n,n} & & \downarrow \gamma_{2n,2n} \\ E_{2n} & \xrightarrow{\quad} & E_{4n} \end{array}$$

are commutative up to base point preserving homotopy and give rise to an H -space structure on $E^{BN}(A)$.

PROOF OF PROPOSITION 2. The left-hand restriction map $\Phi = \gamma_{n,n}: E_n \times U_\Delta \rightarrow E_{2n}$ is defined by $\alpha \cdot U_F \rightarrow (\alpha \oplus 1) \cdot U_{F \oplus \Delta}$. The main step will be to show that Φ is homotopic by a base point preserving homotopy to the right-hand restriction $\Psi = \gamma_{n,n}: U_\Delta \times E_n \rightarrow E_{2n}$, which is given by the correspondence $\alpha \cdot U_F \rightarrow (1 \oplus \alpha) \cdot U_{\Delta \oplus F}$. The two maps $E_n \times E_n \rightarrow E_{4n}$ of (***) are induced by the homomorphisms

$$(a) \quad (\alpha, \beta) \rightarrow \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$(b) \quad (\alpha, \beta) \rightarrow \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The homotopy commutativity of (***) is obtained by applying essentially the same argument for $\Phi \sim \Psi$ to the "second and third rows and columns." Finally, the $\gamma_{n,n}$ are telescoped together to give the H -space structure on $E^{HN}(A)$.

The proof that $\Phi \sim \Psi$ will be based on the matrix identities

$$\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix},$$

$$\begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix} = \begin{pmatrix} 1 & \alpha^{-1} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -\alpha & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & \alpha^{-1} \\ 0 & 1 \end{pmatrix} \\ \cdot \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

together with the following six commutative squares wherein if $x, y,$ and z are matrices, then $x \rightarrow^z y$ means $y = x \cdot z$:

$$\begin{array}{l} \text{(i)} \quad \begin{pmatrix} 1 & \alpha^{-1} \\ 0 & 1 \end{pmatrix} \begin{matrix} \downarrow \\ \downarrow \end{matrix} \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \xrightarrow{\begin{pmatrix} \delta & 0 \\ 0 & 1 \end{pmatrix}} \begin{pmatrix} \alpha\delta & 0 \\ 0 & 1 \end{pmatrix} \begin{matrix} \downarrow \\ \downarrow \end{matrix} \begin{pmatrix} 1 & \delta^{-1}\alpha^{-1} \\ 0 & 1 \end{pmatrix} \\ \text{(ii)} \quad \begin{pmatrix} 1 & 0 \\ -\alpha & 1 \end{pmatrix} \begin{matrix} \downarrow \\ \downarrow \end{matrix} \begin{pmatrix} \alpha & 1 \\ 0 & 1 \end{pmatrix} \xrightarrow{\begin{pmatrix} \delta & 0 \\ 0 & 1 \end{pmatrix}} \begin{pmatrix} \alpha\delta & 1 \\ 0 & 1 \end{pmatrix} \begin{matrix} \downarrow \\ \downarrow \end{matrix} \begin{pmatrix} 1 & 0 \\ -\alpha\delta & 1 \end{pmatrix} \\ \text{(iii)} \quad \begin{pmatrix} 1 & \alpha^{-1} \\ 0 & 1 \end{pmatrix} \begin{matrix} \downarrow \\ \downarrow \end{matrix} \begin{pmatrix} 0 & 1 \\ -\alpha & 1 \end{pmatrix} \xrightarrow{\begin{pmatrix} \delta & 0 \\ 0 & 1 \end{pmatrix}} \begin{pmatrix} 0 & 1 \\ -\alpha\delta & 1 \end{pmatrix} \begin{matrix} \downarrow \\ \downarrow \end{matrix} \begin{pmatrix} 1 & \delta^{-1}\alpha^{-1} \\ 0 & 1 \end{pmatrix} \\ \text{(iv)} \quad \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{matrix} \downarrow \\ \downarrow \end{matrix} \begin{pmatrix} 0 & 1 \\ -\alpha & \alpha \end{pmatrix} \xrightarrow{\begin{pmatrix} \delta & 1-\delta \\ 0 & 1 \end{pmatrix}} \begin{pmatrix} 0 & 1 \\ -\alpha\delta & \alpha\delta \end{pmatrix} \begin{matrix} \downarrow \\ \downarrow \end{matrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \\ \text{(v)} \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{matrix} \downarrow \\ \downarrow \end{matrix} \begin{pmatrix} 1 & 1 \\ 0 & \alpha \end{pmatrix} \xrightarrow{\begin{pmatrix} 1 & 1-\delta \\ 0 & \delta \end{pmatrix}} \begin{pmatrix} 1 & 1 \\ 0 & \alpha\delta \end{pmatrix} \begin{matrix} \downarrow \\ \downarrow \end{matrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \\ \text{(vi)} \quad \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{matrix} \downarrow \\ \downarrow \end{matrix} \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix}} \begin{pmatrix} 1 & 0 \\ 0 & \alpha\delta \end{pmatrix} \begin{matrix} \downarrow \\ \downarrow \end{matrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \end{array}$$

Step 1. Consider the correspondence

$$(1) \quad \alpha \cdot U_F \rightarrow \begin{pmatrix} \alpha & 1 \\ 0 & 1 \end{pmatrix} \cdot U_{F \oplus \Delta}.$$

The commutative square (i) shows this is order preserving and hence defines a simplicial map $\Phi_1: E_n \rightarrow E_{2n}$. But

$$\begin{pmatrix} 1 & \alpha^{-1} \\ 0 & 1 \end{pmatrix} \in U_{F \oplus \Delta}$$

so therefore

$$\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \cdot U_{F \oplus \Delta} = \begin{pmatrix} \alpha & 1 \\ 0 & 1 \end{pmatrix} \cdot U_{F \oplus \Delta}.$$

Hence $\Phi_1 = \Phi$.

Step 2. Before applying (ii) a preliminary homotopy of Φ_1 must be made. For each $F = X_1|X_2|\cdots|X_r$ in P^n let $\Delta \alpha F = \{e_{n+1}, \dots, e_{2n}\}|X_1|X_2|\cdots|X_r$. If $F < G$ then, $\Delta \alpha F < \Delta \alpha G$; and if $\delta \in U_F$, then $\delta \oplus 1 \in U_{\Delta \alpha F}$. Hence the correspondence

$$(1.5) \quad \alpha \cdot U_F \rightarrow \begin{pmatrix} \alpha & 1 \\ 0 & 1 \end{pmatrix} \cdot U_{\Delta \alpha F}$$

preserves order and defines a simplicial map $\Phi_{1.5}: E_n \rightarrow E_{2n}$. We claim that Φ_1 and $\Phi_{1.5}$ are homotopic by a base point preserving homotopy: Let g_1 and g_2 be the two order preserving maps from P^n to P^{2n} defined respectively by $g_1(F) = F \oplus \Delta$ and $g_2(F) = \Delta \alpha F$. Apply Lemma of §1 to find a subdivision $(P^n \times I)'$ of $P^n \times I$ and an order preserving map $w: (P^n \times I)' \rightarrow P^{2n}$ satisfying conditions (a) through (d). Now consider the simplicial map $\pi: E_n \rightarrow P^n$ which takes the vertex $\alpha \cdot U_F$ to the vertex F . This is nondegenerate on simplices. Similarly, if we give $E_n \times I$ and $P^n \times I$ the standard triangulations, then the natural simplicial map $\pi \times 1: E_n \times I \rightarrow P^n \times I$ is also nondegenerate on each simplex. Hence the subdivision $(P^n \times I)'$ of $P^n \times I$ induces a subdivision $(E_n \times I)'$ of $E_n \times I$. Now let $\sigma = (\alpha_0 \cdot U_{F_0} < \cdots < \alpha_k \cdot U_{F_k})$ be a simplex of E_n and let ν be a vertex in the standard triangulation of $\pi(\sigma) \times I$ in $P^n \times I$. Then $w(\nu) = F \oplus \Delta$ or $w(\nu) = \Delta \alpha F$ where F is one of the F_i . Since $\delta \oplus 1 \in U_{F \oplus \Delta}$ and $\delta \oplus 1 \in U_{\Delta \alpha F}$ for each $\delta \in U_{F_0}$, condition (d) of the Lemma shows that $\delta \oplus 1 \in U_{w(\nu)}$ for any vertex ν of the new triangulation of $\pi(\sigma) \times I$. Now let u be any vertex in the new triangulation of $\sigma \times I$ and let $\nu = (\pi \times 1)(u)$. Define

$$\Omega(u) = \begin{pmatrix} \alpha_0 & 1 \\ 0 & 1 \end{pmatrix} \cdot U_{w(\nu)}.$$

The above remarks show $\Omega(u)$ is independent of the representative α_0 of the class $\alpha_0 \cdot U_{F_0}$ and we get a simplicial map $\Omega: (E_n \times I)' \rightarrow E_{2n}$ which is the

required homotopy between Φ_1 and $\Phi_{1.5}$.

Now consider the correspondence

$$(1.5') \quad \alpha \cdot U_F \rightarrow \begin{pmatrix} 0 & 1 \\ -\alpha & 1 \end{pmatrix} \cdot U_{\Delta \alpha F}.$$

The commutative square (ii) shows this is order preserving and induces a simplicial map $\Phi'_{1.5}: E_n \rightarrow E_{2n}$ such that $\Phi'_{1.5} = \Phi_{1.5}$ because $\begin{pmatrix} 1 & 0 \\ -\alpha & 1 \end{pmatrix} \in U_{\Delta \alpha F}$ for all $F \in P^n$. Arguing as above, we see that $\Phi'_{1.5}$ is homotopic to the simplicial map $\Phi_2: E_n \rightarrow E_{2n}$ defined by the correspondence

$$(2) \quad \alpha \cdot U_F \rightarrow \begin{pmatrix} 0 & 1 \\ -\alpha & 1 \end{pmatrix} \cdot U_{F \oplus \Delta}.$$

Step 3. Consider the map

$$(3) \quad \alpha \cdot U_F \rightarrow \begin{pmatrix} 0 & 1 \\ -\alpha & 0 \end{pmatrix} \cdot U_{F \oplus \Delta}.$$

The square (iii) shows this is order preserving and we get a simplicial map $\Phi_3: E_n \rightarrow E_{2n}$ which agrees with Φ_2 because

$$\begin{pmatrix} 1 & \alpha^{-1} \\ 0 & 1 \end{pmatrix} \in U_{F \oplus \Delta} \quad \text{for all } F \in P^n.$$

Step 4. Square (iv) shows that the correspondence

$$(4) \quad \alpha \cdot U_F \rightarrow \begin{pmatrix} 0 & 1 \\ -\alpha & \alpha \end{pmatrix} \cdot U_{F \oplus \Delta}$$

defines an order preserving simplicial map $\Phi_4: E_n \rightarrow E_{2n}$ which agrees with Φ_3 .

Step 5. As in Step 2 it is first necessary to deform Φ_4 by a homotopy before using (v). Recall [18] that if $F \in P^n$ is of the form $F = X_1|X_2|\cdots|X_r$, then $F \square F \in P^{2n}$ is defined as

$$F \square F = X'_1|X_1|X'_2|X_2|\cdots|X'_r|X_r,$$

where $X'_i \subset \{e_{n+1}, \dots, e_{2n}\}$ is obtained from X_i by adding n to the indices of the $e_j \in X_i$. Consider the two maps $g_1, g_2: \{\text{vertices of } P^n\} \rightarrow P^{2n}$ defined by $g_1(F) = F \oplus \Delta$ and $g_2(F) = F \square F$. The map g_1 is order preserving but g_2 is not! Apply the Lemma of §1 to construct a simplicial map $w: (P^n \times I)' \rightarrow P^{2n}$ satisfying (a) through (d). As in Step 2, let $\sigma = (\alpha_0 \cdot U_{F_0} < \cdots < \alpha_k \cdot U_{F_k})$ be a simplex of E_n and let ν be a vertex in the standard triangulation of $\pi(\sigma) \times I$ in $P^n \times I$. Then $w(\nu) = F \oplus \Delta$ or $w(\nu) = F \square F$ where F is one of the F_i . Now for each $\delta \in U_{F_0}$, the matrix $\begin{pmatrix} \delta & 1-\delta \\ 0 & 1 \end{pmatrix}$ lies in $U_{F \oplus \Delta}$ and also in $U_{F \square F}$. Hence

condition (d) of the Lemma shows that $\begin{pmatrix} \delta & 1-\delta \\ 0 & 1 \end{pmatrix} \in U_{w(v)}$ for each vertex v of the new triangulation of $\pi(\sigma) \times I$ in $(P^n \times I)'$. For any vertex u in the new triangulation of $\sigma \times I$ in $(E_n \times I)'$ let $v = (\pi \times 1)(u)$ and define

$$\Omega(u) = \begin{pmatrix} 0 & 1 \\ -\alpha_0 & \alpha_0 \end{pmatrix} \cdot U_{w(v)}.$$

Then the preceding remarks show $\Omega(u)$ is independent of the representative α_0 of $\alpha_0 \cdot U_{F_0}$ and we get a simplicial map $\Omega: (E_n \times I)' \rightarrow E_{2n}$. Let $\Phi'_4 = \Omega|(E_n \times 1)'$. For each vertex v of $\pi(\sigma) \times 1$ in the standard triangulation, the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ belongs to $U_{w(v)}$. Hence by (d) of the Lemma this matrix belongs to $U_{w(v)}$ for any vertex v of $(\pi(\sigma) \times 1)'$. Therefore for any vertex u in $(\sigma \times 1)$ we have

$$\Phi'_4(u) = \begin{pmatrix} 1 & 1 \\ 0 & \alpha_0 \end{pmatrix} \cdot U_{w(v)}$$

where $v = (\pi \times 1)(u)$. See (v). Since $\begin{pmatrix} 1 & 1-\delta \\ 0 & \delta \end{pmatrix} \in U_{w(v)}$ for every $\delta \in U_{F_0}$, this new formula for Φ'_4 is independent of the choice of representative α_0 of $\alpha_0 \cdot U_{F_0}$ by (v). Since $\begin{pmatrix} 1 & 1-\delta \\ 0 & \delta \end{pmatrix}$ belongs to $U_{\Delta \oplus F}$ and to $U_{F \square F}$ for $\delta \in U_{F_0}$ and $F = F_0, \dots, F_k$, we can construct as above a homotopy between Φ'_4 and $\Phi_5: E_n \rightarrow E_{2n}$ defined by the order preserving correspondence

$$\alpha \cdot U_F \rightarrow \begin{pmatrix} 1 & 1 \\ 0 & \alpha \end{pmatrix} \cdot U_{\Delta \oplus F}.$$

Step 6. Finally (vi) shows that Φ_5 is the same as Ψ , which is defined by

$$\alpha \cdot U_F \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} \cdot U_{\Delta \oplus F},$$

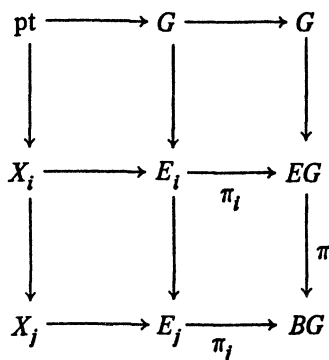
because $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ lies in each $U_{\Delta \oplus F}$.

The homotopy between Φ and Ψ in Steps 1 through 6 does not keep the base point fixed. However, it can be deformed to one which does, because $\Delta \alpha \Delta = \Delta \square \Delta$ and hence U_Δ is deformed along a path of the form $\gamma * \gamma^{-1}$ where γ is the path traced out by U_Δ during the first three steps of the argument. Q.E.D.

3. Action on the fiber. In §1 we saw that there is a homotopy equivalence $\theta: X \rightarrow E\{\alpha \cdot U_F\} \cong E^{BN}(A)$ where X is the homotopy fiber of $B\{U_F\} \rightarrow BE(A)$.

PROPOSITION 3. *Under θ the action of $\pi_1 BE(A)$ on X can be identified up to homotopy with left multiplication of $E(A)$ on $E\{\alpha \cdot U_F\}$ which, moreover, induces the identity on $H_*(E\{\alpha \cdot U_F\})$.*

PROOF. If $f: K \rightarrow L$ is any map we convert it into an actual fibration $X_f \rightarrow E_f \xrightarrow{\pi_f} L$ as usual by letting E_f be the set of pairs (x, ω) where $x \in K$ and ω is a path in L with $\omega(1) = f(x)$. The map π_f takes (x, ω) to $\omega(0)$. The fiber X_f consists of those (x, ω) for which $\omega(0) = \text{base point of } L$. Applying this to the horizontal rows of the pullback square of §1 defining $E\{\alpha \cdot U_F\}$ gives the commutative diagram



Since G is discrete, π has the unique path lifting property. This implies E_i is homomorphic to the pullback of π_j and π . Hence all the horizontal and vertical rows are fibrations, $X_i \rightarrow X_j$ is a homeomorphism, and $X_j \simeq E_i$ because EG is contractible. A specific homotopy equivalence $\theta: X_j \rightarrow E_i$ is defined by $\theta(x, \nu) = (x, \bar{\nu}(1); \bar{\nu})$ where $\bar{\nu}$ is the unique path in EG starting at the base point and lifting ν . Now let γ be a fixed loop in BG representing $g \in \pi_1 BG$. Then

$$\theta(g \cdot (x, \nu)) = \theta(x, \gamma * \nu) = (x, \overline{\gamma * \bar{\nu}(1)}; \overline{\gamma * \bar{\nu}})$$

and

$$g \cdot \theta(x, \nu) = g \cdot (x, \bar{\nu}(1); \bar{\nu}) = (x, g \cdot \bar{\nu}(1); g \cdot \bar{\nu}).$$

It follows using the standard construction as in [6] for the universal cover of BG that these two maps are homotopic.

It remains to show the correspondence $\alpha \cdot U_F \rightarrow g\alpha \cdot U_F$ induces the identity on $H_*(E\{\alpha \cdot U_F\}) = H_*(E^{BN}(A))$. Any homology class is supported in some E_n and we can choose n large enough to have $g \in E(n, A)$. In §2 it was shown that there is a subdivision $(E_n \times I)'$ of the standard triangulation of $E_n \times I$ such $(E_n \times 0)' = E_n \times 0$ and $(E_n \times 1)' = E_n \times 1$ and there is a simplicial map $h: (E_n \times I)' \rightarrow E_{2n}$ having the property that $h_0 = h|_{E_n \times 0}$ is

$$\alpha \cdot U_F \rightarrow (\alpha \oplus 1) \cdot U_{F \oplus \Delta}$$

and $h_1 = h|_{E_n \times 1}$ is

$$\alpha \cdot U_F \rightarrow (1 \oplus \alpha) \cdot U_{\Delta \oplus F}.$$

Hence $g \cdot h_0: E_n \rightarrow E_{2n}$ defined by

$$\alpha \cdot U_F \rightarrow (g\alpha \oplus 1) \cdot U_{F \oplus \Delta}$$

is homotopic to $g \cdot h_1: E_n \rightarrow E_{2n}$ defined by

$$\alpha \cdot U_F \rightarrow (g \oplus \alpha) \cdot U_{\Delta \oplus F}.$$

Therefore it suffices to show $g \cdot h_1$ is homotopic to the map $E_n \rightarrow E_{2n}$ given by

$$\alpha \cdot U_F \rightarrow (1 \oplus \alpha) \cdot U_{\Delta \oplus F}.$$

Since $g \in E(n, A)$ is the product of elements lying in the subgroups U_F , this fact is in turn a consequence of several applications of the following: Let $G < G'$ in P^n and assume $g \in U_{G'}$. Let $x \in E(n, A)$. Then the two maps $E_n \rightarrow E_{2n}$ defined respectively by

$$\alpha \cdot U_F \rightarrow \begin{pmatrix} x & 0 \\ 0 & \alpha \end{pmatrix} \cdot U_{G \oplus F}$$

and

$$\alpha \cdot U_F \rightarrow \begin{pmatrix} xg & 0 \\ 0 & \alpha \end{pmatrix} \cdot U_{G' \oplus F}$$

are homotopic. But this is clear because

$$\begin{pmatrix} x & 0 \\ 0 & \alpha \end{pmatrix} \cdot U_{G \oplus F} < \begin{pmatrix} x & 0 \\ 0 & \alpha \end{pmatrix} \cdot U_{G' \oplus F} = \begin{pmatrix} xg & 0 \\ 0 & \alpha \end{pmatrix} \cdot U_{G' \oplus F}.$$

4. *H-space structure on $B\{U_F\}^+$.* In this section we show how direct sum of matrices gives an *H-space* structure on $B\{U_F\}^+$.

By a sheaf of spaces over a simplicial complex K we mean a collection $X = \{X_\sigma\}$, $\sigma =$ simplex of K , together with connecting maps $i_{\sigma\tau}: X_\tau \rightarrow X_\sigma$ for $\sigma < \tau$ such that $i_{\sigma\tau} \circ i_{\tau\gamma} = i_{\sigma\gamma}$ whenever $\sigma < \tau < \gamma$. The realization $|X|$ of X is the disjoint union $\coprod_{\sigma \in K} \sigma \times X_\sigma$ modulo the identification setting $(x, y) = (x', y')$ iff $x = x'$ and $y' = i_{\sigma\tau}(y)$ for $y \in X_\tau, y' \in X_\sigma$, and $\sigma < \tau$. Any simplicial subdivision K' of K induces a subdivision X' of X as follows: For τ a simplex of K' let $X'_\tau = X_\sigma$ where σ is the smallest simplex of K containing τ . If $\tau < \tau'$ and σ, σ' are the smallest simplices of K containing τ, τ' respectively, then $\sigma < \sigma'$ and we let $i_{\tau\tau'} = i_{\sigma\sigma'}$. The natural map $|X'| \rightarrow |X|$ is a homeomorphism. Define $X \times I$ to be the sheaf of spaces over $K \times I$ by setting $(X \times I)_\tau = X_\sigma$ where σ is the smallest simplex of K such that $\tau \subset \sigma \times I$. The space B_n is clearly the realization of a sheaf of spaces of the form $B U_F$ over P^n .

The direct sum homomorphism \oplus from $E(m, A) \times E(n, A) \rightarrow E(m + n, A)$ gives a family of compatible homomorphisms

$$\oplus: U_F \times U_G \rightarrow U_{F \oplus G}$$

and hence a family of compatible maps

$$\oplus: BU_F \times BU_G \rightarrow BU_{F \oplus G}$$

which fit together to give maps

$$\rho_{m,n}: B_m \times B_n \rightarrow B_{m+n}.$$

By the universal property of the “plus construction” we get base point preserving maps

$$\rho_{m,n}^+: B_m^+ \times B_n^+ \rightarrow B_{m+n}^+.$$

A word about base points: The complex P^n can be embedded in B_n by sending $x \in \sigma = (F_0 < \dots < F_k)$ to $(x, \text{base point of } BU_{F_0})$. The base point $*$ of B_n is $(\Delta, \text{base point of } BU_\Delta)$. The map $\rho_{m,n}$ does not preserve the base point but $\rho_{m,n}(P^m \times P^n) \subset P^{m+n}$. Hence $\rho_{m,n}$ determines a base point preserving map $\rho_{m,n}^+$ defined up to base point preserving homotopy. Also, the stabilization map $B_n \rightarrow B_{2n}$ given by $BU_F \rightarrow BU_{F \oplus \Delta}$ is not the standard inclusion $B_n \hookrightarrow B_{2n}$ which is given by $BU_F \rightarrow BU_{F^{(n)}}$. However, $F^{(n)} < F \oplus \Delta$ for all $F \in P^n$ implies these two maps are homotopic up to a base point preserving homotopy; so from now on we use the stabilization induced by $F \rightarrow F \oplus \Delta$.

PROPOSITION 4. For $n \geq 3$ the diagrams

$$\begin{array}{ccc} B_n^+ \times B_n^+ & \longrightarrow & B_{2n}^+ \times B_{2n}^+ \\ \downarrow \rho_{n,n}^+ & & \downarrow \rho_{2n,2n}^+ \\ B_{2n}^+ & \longrightarrow & B_{4n}^+ \end{array}$$

are homotopy commutative and give rise to an H -space structure on $B\{U_F\}^+$ in such a way that $B\{U_F\}^+ \rightarrow BE(A)^+$ is an H -map.

REMARK. It follows immediately from [13] that the homotopy theoretic fiber X of $B\{U_F\}^+ \rightarrow BE(A)^+$ is a connected H -space as required by (II) of §1.

PROOF OF PROPOSITION 4. The restriction map $\Phi = \rho_{n,n}: B_n^+ \times * \rightarrow B_{2n}^+$ comes from the inclusion $B_n \rightarrow B_{2n}$ induced by the stabilization homomorphism

$$\phi(\alpha) = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \in U_{F \oplus \Delta}.$$

The essential step in the proof is to show that the restriction map

$$\Psi = \rho_{n,n}: * \times B_n^+ \rightarrow B_{2n}^+$$

is homotopic to Φ by a base point preserving homotopy. Note that Ψ is induced by the homomorphism

$$\psi(\alpha) = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$$

satisfying $\psi(U_F) \subset U_{\Delta \oplus F}$ for each $F \in P^n$. By the universal property of the "plus construction" it suffices to show

(†) For $n \geq 3$ the maps $\Phi, \Psi: B_n \rightarrow B_{2n}^+$ are homotopic by a base point preserving homotopy.

In fact B_{2n}^+ is simply connected for $2 \leq n$ so it suffices to show Φ and Ψ are homotopic as maps into B_{2n} . To prove this we shall use the following matrix identities:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}.$$

Step 1. Consider the homomorphism $\phi_1: E(n, A) \rightarrow E(2n, A)$ given by

$$\phi_1(\alpha) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & \alpha - 1 \\ 0 & 1 \end{pmatrix}.$$

For each $F \in P^n$ the matrix $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \in U_{F \oplus \Delta}$; so $\phi_1(U_F) \subset U_{F \oplus \Delta}$. The induced maps $BU_F \rightarrow BU_{F \oplus \Delta}$ fit together to give a map $\Phi_1: B_n \rightarrow B_{2n}$. The construction of [12] gives a specific homotopy $H_F: BU_F \times I \rightarrow BU_{F \oplus \Delta}$ for each facet $F \in P^n$ between the maps induced by the conjugate homomorphisms ϕ and ϕ_1 such that $H_{F'}|BU_F \times I = H_F$ whenever $F < F'$. Hence these homotopies fit together to give a homotopy $H: B_n \times I \rightarrow B_{2n}$ from Φ to Φ_1 .

Step 2. Let $f, g: P^n \rightarrow P^{2n}$ be defined by $f(F) = F \oplus \Delta$ and $g(F) = F \square F$. The map f preserves order but g does not. Apply the lemma of §1 to find an order preserving map $w: (P^n \times I)' \rightarrow P^{2n}$ of some subdivision of the standard triangulation of $P^n \times I$ satisfying (a) through (d). This subdivision induces a subdivision $(B_n \times I)'$ of $B_n \times I$. Consider a simplex $\sigma = (F_0 < \dots < F_k)$ in P^n and let v be any vertex of a simplex in the standard triangulation of $\sigma \times I$. Then $w(v) = F \oplus \Delta$ or $w(v) = F \square F$ where F is one of the facettes

F_i The formula for ϕ_1 shows that $\phi_1(U_{F_0}) \subset U_{w(v)}$ and hence by (d) the same inclusion holds for any vertex v of $(\sigma \times I)'$. This yields homotopy $\Omega: (B_n \times I)' \rightarrow B_{2n}$ between Φ_1 and the map $\Phi'_1: (B_n \times I)' \rightarrow B_{2n}$ defined as the restriction of Ω to $(B_n \times 1)'$. Now for $F \in P^n$ the matrix $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ belongs to $U_{F \square F}$. Applying (d) of the lemma shows that $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \in U_{w(v)}$ for any vertex v of $(P^n \times 1)'$. Let $\phi_2: E(n, A) \rightarrow E(2n, A)$ be defined by

$$\phi_2(\alpha) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \alpha - 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \alpha - 1 \\ 0 & \alpha \end{pmatrix}$$

and let Φ'_2 be the induced map. More precisely, if v is a vertex of $(P^n \times 1)'$, let $\sigma = (F_0 < \dots < F_k)$ be the smallest simplex of $P^n \times 1$ such that $v \in \sigma$. Then $\phi_2(U_{F_0}) \subset U_{w(v)}$ so there is an induced map $BU_{F_0} \rightarrow BU_{w(v)}$. These fit together to give Φ'_2 . Since $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \in U_{w(v)}$ for any vertex v of $(P^n \times 1)'$, the conjugate homomorphisms ϕ_1 and ϕ_2 give rise to homotopic maps Φ'_1 and Φ'_2 .

Now let $f, g: P^n \rightarrow P^{2n}$ be defined by $f(F) = F \square F$ and $g(F) = \Delta \oplus F$. The map g preserves order but f does not. Use the lemma to construct a homotopy $w: (P^n \times I)' \rightarrow P^{2n}$ satisfying (a) through (d). Again let $\sigma = (F_0 < \dots < F_k)$ be a simplex of $P^n \times 1 \simeq P^n$ and let v be a vertex in the standard triangulation. Then $w(v) = F \square F$ or $w(v) = \Delta \oplus F$ where F is one of the F_i . The formula for ϕ_2 shows that $\phi_2(U_{F_0}) \subset U_{w(v)}$ and hence by (d) the same is true for any vertex v of $(\sigma \times I)'$. As above there is a homotopy between Φ'_2 and $\Phi_2: B_n \rightarrow B_{2n}$ which, by definition, is obtained by fitting together the maps $BU_F \rightarrow BU_{\Delta \oplus F}$ induced by ϕ_2 .

Step 3. Let $\phi_3: E(n, A) \rightarrow E(2n, A)$ be defined by

$$\phi_3(\alpha) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \alpha - 1 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}.$$

Note that $\phi_3 = \psi$. Since $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \in U_{\Delta \oplus F}$ for all $F \in P^n$ we get a homotopy between Φ_2 and Ψ as in Step 1.

To show the homotopy commutativity of the diagram in Proposition 4 one applies essentially the same arguments as above to the second and third rows and columns of the homomorphisms (a) and (b) of §2.

Finally, the maps $\rho_{n,n}: B_n^+ \times B_n^+ \rightarrow B_{2n}^+$ telescope together to give the H -space structure on $B\{U_F\}^+$. That $B\{U_F\}^+ \rightarrow BE(A)^+$ is an H -map follows immediately from the fact that the H -space structure on $BE(A)^+$ arises from the direct sum maps

$$BE(n, A) \times BE(n, A) \rightarrow BE(2n, A)$$

and that we have a strictly commutative diagram

$$\begin{array}{ccc}
 B_n \times B_n & \xrightarrow{\quad\quad\quad} & B_{2n} \\
 \downarrow & & \downarrow \\
 BE(n, A) \times BE(n, A) & \xrightarrow{\quad\quad\quad} & BE(2n, A)
 \end{array}$$

which commutes up to base point preserving homotopy when the "plus construction" is performed. Q.E.D.

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