

Homology of the Infinite Orthogonal and Symplectic Groups over Algebraically Closed Fields

An Appendix to the Paper of A. Suslin

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As a consequence of Suslin’s result and the fundamental theorem of Hermitian K -theory [3], we shall prove the following statement:

Theorem. *Let k be an algebraically closed field and k_0 the algebraic closure of the prime field. Then, the obvious homomorphisms*

$$O_{2n}(k_0) \rightarrow O_{2n}(k) \quad \text{and} \quad Sp_{2n}(k_0) \rightarrow Sp_{2n}(k)$$

induce isomorphisms

$$\begin{aligned} \varinjlim H_*(O_{2n}(k_0); \mathbf{Z}/q) &\xrightarrow{\cong} \varinjlim H_*(O_{2n}(k); \mathbf{Z}/q) \\ \varinjlim H_*(Sp_{2n}(k_0); \mathbf{Z}/q) &\xrightarrow{\cong} \varinjlim H_*(Sp_{2n}(k); \mathbf{Z}/q) \end{aligned}$$

where q is arbitrary if $\text{char}(k) \neq 2$ and q is odd if $\text{char}(k) = 2$.

Proof. Let F denote either k or k_0 . Here $Sp_{2n}(F)$ is the usual symplectic group and $O_{2n}(F)$ the orthogonal group of F^{2n} provided with the “hyperbolic” quadratic form

$$(x_1, \dots, x_n, y_1, \dots, y_n) \mapsto \sum_{i=1}^n x_i y_i.$$

Therefore $O(F) = \varinjlim O_{2n}(F)$ and $Sp(F) = \varinjlim Sp_{2n}(F)$ are just the groups denoted by ${}_1O(F)$ and ${}_1Sp(F)$ in [2]. If $\text{char}(F) \neq 2$, since $\sqrt{-1} \in F$, $O(F)$ may be considered as the direct limit of the orthogonal groups of F^n provided with the quadratic form defined by the sum of the squares of the coordinates. Since homology commutes with direct limits, we have to prove isomorphisms

$$H_*(O(k); \mathbf{Z}/q) \approx H_*(O(k_0); \mathbf{Z}/q) \quad \text{and} \quad H_*(Sp(k); \mathbf{Z}/q) \approx H_*(Sp(k_0); \mathbf{Z}/q).$$

Let us recall now some basic definitions and results of [2, 3] adapted to this situation. The Hermitian K -theory spectrum ${}_\epsilon \mathcal{L}(F)$, with $\epsilon = \pm 1$, is defined as follows ([2], p. 212–218):

$$\begin{aligned} {}_\epsilon \mathcal{L}_n(F) &= \Omega^n({}_\epsilon L_0(F) \times B_\epsilon O(F)^+) && \text{for } n \geq 0 \\ {}_\epsilon \mathcal{L}_n(F) &= {}_\epsilon L_0(S^{-n}F) \times B_\epsilon O(S^{-n}F)^+ && \text{for } n < 0. \end{aligned}$$

The Hermitian “ L -groups” are given by the formulas

$${}_eL_n(F) = \pi_0({}_e\mathcal{L}_n(F)), \quad {}_eL_n(F; \mathbf{Z}/q) = \pi_0({}_e\mathcal{L}_n(F); \mathbf{Z}/q) = \pi_i({}_e\mathcal{L}_{n-i}(F); \mathbf{Z}/q)$$

(the last equality may be taken as a definition for $i \geq 2$). For example ${}_{-1}L_0(F) \approx {}_1L_0(F) \approx \mathbf{Z}$ (this is just the Grothendieck group of the category of F -vector spaces provided with a non degenerate skew symmetric or quadratic form). We have also ${}_{-1}L_1(F) \approx H_1(Sp(F); \mathbf{Z}) = O$ and

$${}_1L_1(F) \approx H_1(O(F); \mathbf{Z}) \approx \mathbf{Z}/2.$$

The Quillen K -theory spectrum $\mathcal{K}(F)$ is described in an analogous way ([2], p. 212–218):

$$\begin{aligned} \mathcal{K}_n(F) &= \Omega^n(K_0(F) \times BGL(F)^+) && \text{for } n \geq 0, \\ \mathcal{K}_n(F) &= K_0(S^{-n}F) \times BGL(S^{-n}F)^+ && \text{for } n < 0. \end{aligned}$$

Note that $K_n(F) = \pi_0(\mathcal{K}_n(F))$ and $K_n(F; \mathbf{Z}/q) \approx \pi_i(\mathcal{K}_{n-i}(F); \mathbf{Z}/q)$ for $i \geq 2$. Note that $K_n(F)$ (resp. ${}_eL_n(F)$) are for $n > 0$ the homotopy groups of an infinite loop space which homology is precisely that of $GL(F)$ (resp. ${}_eO(F)$). Note finally that for $n < 0$, $K_n(F) = 0$ ([4], p. 73), but ${}_eL_n(F) \neq 0$ in general (compare with [3], p. 278).

Finally, there are maps of spectra

$$\mathcal{K}_n(F) \rightarrow {}_e\mathcal{L}_n(F) \quad \text{and} \quad {}_e\mathcal{L}_n(F) \rightarrow \mathcal{K}_n(F)$$

induced by the hyperbolic and the forgetful functor respectively. If we denote by ${}_e\mathcal{U}_n(F)$ and ${}_e\mathcal{V}_n(F)$ the homotopy fibers of these two maps, then the fundamental theorem of [3] implies the existence of an homotopy equivalence

$${}_e\mathcal{V}_n(F) \approx \Omega_{-e}\mathcal{U}_n(F)$$

if $\text{char}(F) \neq 2$ and

$${}_e\mathcal{V}_n(F)_{(2)} \sim \Omega_{-e}\mathcal{U}_n(F)_{(2)}$$

if $\text{char}(F) = 2$ ([3], p. 260; cf. [2], p. 253 for the definition of $X_{(2)}$ when X is an H -space). Therefore, if we define ${}_eV_n(F)$ (resp. ${}_eU_n(F)$) as $\pi_0({}_e\mathcal{V}_n(F))$ (resp. $\pi_0({}_e\mathcal{U}_n(F))$), then we have an isomorphism ${}_eV_n(F) \approx {}_{-e}U_{n+1}(F)$ if $\text{char}(F) \neq 2$ and ${}_eV_n(F) \otimes \mathbf{Z}[\frac{1}{2}] \approx {}_{-e}U_{n+1}(F) \otimes \mathbf{Z}[\frac{1}{2}]$ if $\text{char}(F) = 2$. In the same way as above we may introduce U and V groups with coefficients in \mathbf{Z}/q and then we have four types of exact sequences:

$$\begin{aligned} K_{n+1}(F) &\rightarrow {}_eL_{n+1}(F) \rightarrow {}_eU_n(F) \rightarrow K_n(F) \rightarrow {}_eL_n(F) \\ {}_eL_{n+1}(F) &\rightarrow K_{n+1}(F) \rightarrow {}_eV_n(F) \rightarrow {}_eL_n(F) \rightarrow K_n(F) \\ \bar{K}_{n+1}(F) &\rightarrow {}_e\bar{L}_{n+1}(F) \rightarrow {}_e\bar{U}_n(F) \rightarrow \bar{K}_n(F) \rightarrow {}_e\bar{L}_n(F) \\ {}_e\bar{L}_{n+1}(F) &\rightarrow \bar{K}_{n+1}(F) \rightarrow {}_e\bar{V}_n(F) \rightarrow {}_e\bar{L}_n(F) \rightarrow \bar{K}_n(F) \end{aligned}$$

where, for short, we denote by \bar{K} , \bar{L} , \bar{U} and \bar{V} the corresponding K , L , U and V groups with coefficients in \mathbf{Z}/q . Note that ${}_e\bar{V}_n(F) \approx {}_{-e}\bar{U}_{n+1}(F)$ (q is assumed odd if $\text{char}(F) = 2$). We are going to prove in three steps that ${}_e\bar{L}_n(k) \approx {}_e\bar{L}_n(k_0)$, i.e. $\pi_n({}_e\mathcal{L}(k); \mathbf{Z}/q) \approx \pi_n({}_e\mathcal{L}(k_0); \mathbf{Z}/q)$ for all values of n . The Hurewicz theorem mod. q will then imply the theorem of this appendix.

Step 1. The groups ${}_{\varepsilon}\bar{L}_n(k)$ and ${}_{\varepsilon}\bar{L}_n(k_0)$ are isomorphic for $n=0$ and 1. The exact sequence (E_n) :

$${}_{\varepsilon}L_{n+1}(F) \xrightarrow{\times q} {}_{\varepsilon}L_{n+1}(F) \rightarrow {}_{\varepsilon}\bar{L}_{n+1}(F) \rightarrow {}_{\varepsilon}L_n(F) \xrightarrow{\times q} {}_{\varepsilon}L_n(F)$$

written for $n=0$, shows that ${}_{\varepsilon}\bar{L}_1(F) \approx \text{Coker } {}_{\varepsilon}L_1(F) \xrightarrow{\times q} {}_{\varepsilon}L_1(F)$, therefore is 0 if q is odd or $\varepsilon = -1$ and is $\mathbf{Z}/2$ if q is even and $\varepsilon = 1$. For the group ${}_{\varepsilon}\bar{L}_0(F)$ let us distinguish two cases according to the characteristic of F .

If $\text{char}(F) \neq 2$, we write the two exact sequences

$$\begin{array}{ccccccccc} {}_{\varepsilon}L_0(F) & \longrightarrow & K_0(F) & \longrightarrow & {}_{\varepsilon}V_{-1}(F) & \longrightarrow & {}_{\varepsilon}L_{-1}(F) & \longrightarrow & K_{-1}(F) \\ & & & & \Downarrow & & & & \\ & & & & & & & & \\ K_1(F) & \longrightarrow & {}_{-1}L_1(F) & \longrightarrow & {}_{-1}U_0(F) & \longrightarrow & K_0(F) & \longrightarrow & {}_{-1}L_0(F) \end{array}$$

Let α be the map ${}_{\varepsilon}L_0(F) \rightarrow K_0(F)$. Since ${}_{-1}L_1(F) = 0$, the second line in the diagram above shows that $0 = {}_{-1}U_0(F) \approx {}_{+1}V_{-1}(F) \approx \text{Coker } {}_1\alpha$. Since $K_{-1}(F) = 0$, we also get ${}_1L_{-1}(F) = 0$. In the same way, since ${}_{-1}V_{-1}(F) \approx {}_1U_0(F) \approx \mathbf{Z}/2$ and $\text{Coker } {}_{-1}\alpha = \mathbf{Z}/2$, we have also ${}_{-1}L_{-1}(F) = 0$. Therefore, the exact sequence (E_{-1}) above will imply ${}_{\varepsilon}\bar{L}_0(F) \approx \text{Coker } {}_{\varepsilon}L_0(F) \xrightarrow{\times q} {}_{\varepsilon}L_0(F) \approx \mathbf{Z}/q$.

If $\text{char}(F) = 2$, let us denote $\tilde{K}, \tilde{L}, \tilde{U}, \tilde{V}$ the groups K, L, U and V tensorized by $\mathbf{Z}[\frac{1}{2}]$. Then the same arguments as above show that ${}_{\varepsilon}\tilde{L}_{-1}(F) = 0$ since ${}_{\varepsilon}\tilde{V}_{-1}(F) = {}_{-1}\tilde{U}_0(F) = \text{Coker } {}_{\varepsilon}\tilde{\alpha} = 0$. Therefore ${}_{\varepsilon}\bar{L}_0(F) = \mathbf{Z}/q$ for q odd.

Step 2. If $\text{char}(k) \neq 2$, ${}_{\varepsilon}L_n(k) \approx {}_{\varepsilon}L_n(k_0)$ and ${}_{\varepsilon}\bar{L}_n(k) \approx {}_{\varepsilon}\bar{L}_n(k_0)$ for $n < 0$.
If $\text{char}(k) = 2$, ${}_{\varepsilon}\bar{L}_n(k) \approx {}_{\varepsilon}\bar{L}_n(k_0)$ for $n < 0$.

This step is not really necessary for the proof of our main theorem. It is just added for the sake of completeness. If $\text{char}(F) \neq 2$ we have already proved in Step 1 that ${}_{\varepsilon}L_{-1}(F) = 0$. Now if $p > 2$,

$$\begin{array}{l} {}_{\varepsilon}L_{-p}(F) \approx {}_{\varepsilon}V_{-p}(F) \approx {}_{-1}\varepsilon U_{-p+1}(F) \approx {}_{-1}\varepsilon L_{-p+2}(F), \\ {}_{\varepsilon}L_{-2}(F) \approx {}_{\varepsilon}V_{-2}(F) \approx {}_{-1}\varepsilon U_{-1}(F) \approx \text{Coker}(K_0(F) \rightarrow {}_{-1}\varepsilon L_0(F)). \end{array}$$

Therefore ${}_{\varepsilon}L_{-p}(k) \approx {}_{\varepsilon}L_{-p}(k_0)$ by induction on p . The exact sequence (E_n) , written for negative values of n , will then imply ${}_{\varepsilon}\bar{L}_{-p}(k) \approx {}_{\varepsilon}\bar{L}_{-p}(k_0)$. If $\text{char}(k) = 2$, the arguments are completely analogous when replacing the groups K, L, U and V by $\tilde{K}, \tilde{L}, \tilde{U}$ and \tilde{V} .

Step 3. The isomorphism ${}_{\varepsilon}\bar{L}_{n-1}(k) \approx {}_{\varepsilon}\bar{L}_{n-1}(k_0)$ implies the isomorphism

$${}_{-1}\varepsilon \bar{L}_{n+1}(k) \approx {}_{-1}\varepsilon \bar{L}_{n+1}(k_0).$$

This step, together with Step 1 will of course finish the proof of the theorem. Let us write the two exact sequences

$$\begin{array}{ccccccccc} {}_{\varepsilon}\bar{L}_n(k) & \longrightarrow & \bar{K}_n(k) & \longrightarrow & {}_{\varepsilon}\bar{V}_{n-1}(k) & \longrightarrow & {}_{\varepsilon}\bar{L}_{n-1}(k) & \longrightarrow & \bar{K}_{n-1}(k) \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \approx & & \uparrow \\ {}_{\varepsilon}\bar{L}_n(k_0) & \longrightarrow & \bar{K}_n(k_0) & \longrightarrow & {}_{\varepsilon}\bar{V}_{n-1}(k_0) & \longrightarrow & {}_{\varepsilon}\bar{L}_{n-1}(k_0) & \longrightarrow & \bar{K}_{n-1}(k_0) \end{array}$$

The main theorem of Suslin implies that the maps $K_n(k_0) \rightarrow K_n(k)$ and $K_{n-1}(k_0) \rightarrow K_{n-1}(k)$ are isomorphisms. If, at the beginning of his proof of the main theorem, we take for R the functor ${}_{\varepsilon}\bar{V}_{n-1}$, we see that the map ${}_{\varepsilon}\bar{V}_{n-1}(k_0) \rightarrow {}_{\varepsilon}\bar{V}_{n-1}(k)$ is injective. The surjectivity of this map is obvious by diagram chasing. Using the fundamental theorem of Hermitian K -theory, we deduce that ${}_{-\varepsilon}\bar{U}_n(k_0) \approx {}_{-\varepsilon}\bar{U}_n(k)$. We have now two other exact sequences

$$\begin{array}{ccccccccc}
 {}_{-\varepsilon}\bar{U}_{n+1}(k) & \longrightarrow & \bar{K}_{n+1}(k) & \longrightarrow & {}_{-\varepsilon}\bar{L}_{n+1}(k) & \longrightarrow & {}_{-\varepsilon}\bar{U}_n(k) & \longrightarrow & \bar{K}_n(k) \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow \approx & & \uparrow \\
 {}_{-\varepsilon}\bar{U}_{n+1}(k_0) & \longrightarrow & K_{n+1}(k_0) & \longrightarrow & {}_{-\varepsilon}\bar{L}_{n+1}(k_0) & \longrightarrow & {}_{-\varepsilon}\bar{U}_n(k_0) & \longrightarrow & \bar{K}_n(k_0)
 \end{array}$$

and again ${}_{-\varepsilon}\bar{L}_{n+1}(k_0) \approx {}_{-\varepsilon}\bar{L}_{n+1}(k)$ by the same argument as above (take for R the functor ${}_{-\varepsilon}\bar{L}_{n+1}$).

Corollary. *Let us assume $\text{char}(k) > 0$. Then:*

$$H_*(O(k); \mathbf{Z}/q) \approx H_*(Sp(k); \mathbf{Z}/q) \approx \mathbf{Z}/q[p_1, p_2, \dots]$$

for q odd and $\text{char}(k) \nmid q$, where p_i are homology classes in dimension $4i$.

$$H_*(O(k); \mathbf{Z}/q) = H_*(Sp(k); \mathbf{Z}/q) = \mathbf{Z}/q$$

if q is odd and is a power of the characteristic.

$$H_*(O(k); \mathbf{Z}/2) \approx \mathbf{Z}/2[w_1, w_2, \dots]$$

where $w_i \in H_i(O(k); \mathbf{Z}/2)$ if $\text{char}(k) \neq 2$.

$$H_*(Sp(k); \mathbf{Z}/2) \approx \mathbf{Z}/2[p_1, p_2, \dots]$$

where $p_i \in H_{4i}(Sp(k); \mathbf{Z}/2)$ if $\text{char}(k) \neq 2$.

The corollary follows from the main theorem of this appendix and the computation of $H_*(O(k_0))$ and $H_*(Sp(k_0))$ made by E. Friedländer [1]. The work of K. Vogtman [5] on the stability of the homology of the orthogonal group enables us to write explicitly some homology groups $H_i(O_{2n}(k); \mathbf{Z}/q)$. More precisely, $H_i(O_{2n}(k); \mathbf{Z}/q) \approx H_i(O(k); \mathbf{Z}/q)$ if $n \geq 3i$.

References

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