

RELATIONS BETWEEN ALGEBRAIC K -THEORY AND HERMITIAN K -THEORY

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Communicated by E.M. Friedlander

Received 1 November 1983

This paper is a continuation of [4] where we computed the homology groups with coefficients of the infinite orthogonal and symplectic groups of an algebraically closed field F of characteristic $\neq 2$ and 0.

Since we have also proved in [4] that these homology groups depend only on the characteristic of F (if it is different from 2), in order to deal with the case of characteristic zero, it is sufficient to compute this homology when $F = \mathbb{C}$. More precisely, with the notations of [3] and [4], we shall prove the following statement:

Theorem 1. *The obvious topological group maps $O(\mathbb{C}) \rightarrow O(\mathbb{C})^{\text{top}}$, $Sp(\mathbb{C}) \rightarrow Sp(\mathbb{C})^{\text{top}}$ and $U(\mathbb{C}) \rightarrow U(\mathbb{C})^{\text{top}}$ induce isomorphisms*

$$H_i(BO(\mathbb{C}); \mathbb{Z}/q) \approx H_i(BO(\mathbb{C})^{\text{top}}; \mathbb{Z}/q),$$

$$H_i(BSp(\mathbb{C}); \mathbb{Z}/q) \approx H_i(BSp(\mathbb{C})^{\text{top}}; \mathbb{Z}/q),$$

$$H_i(BU(\mathbb{C}); \mathbb{Z}/q) \approx H_i(BU(\mathbb{C})^{\text{top}}; \mathbb{Z}/q).$$

In this statement, $U(\mathbb{C})$ is ${}_1O(\mathbb{C})$ with \mathbb{C} provided with the complex conjugation involution [3] and BG^{top} denotes in general the classifying space of the group G with its usual topology.

This theorem is proved below along the same lines as the theorem proved in [4], using essentially the fundamental theorem of Hermitian K -theory [3] and Gabber-Suslin's theorem [7]:

$$K_i(\mathbb{C}; \mathbb{Z}/q) \approx K_i^{\text{top}}(\mathbb{C}; \mathbb{Z}/q)$$

In fact, since we have also $K_i(\mathbb{R}; \mathbb{Z}/q) \approx K_i^{\text{top}}(\mathbb{R}; \mathbb{Z}/q)$ according to [7], we can prove in an analogous way the following theorem:

Theorem 2. *The topological group maps $O(\mathbb{R}) \rightarrow O(\mathbb{R})^{\text{top}}$ and $Sp(\mathbb{R}) \rightarrow Sp(\mathbb{R})^{\text{top}}$ induce isomorphisms*

$$H_i(\mathbf{BO}(\mathbb{R}); \mathbb{Z}/q) \approx H_i(\mathbf{BO}(\mathbb{R})^{\text{top}}; \mathbb{Z}/q),$$

$$H_i(\mathbf{BSp}(\mathbb{R}); \mathbb{Z}/q) \approx H_i(\mathbf{BSp}(\mathbb{R})^{\text{top}}; \mathbb{Z}/q).$$

If we put together Theorems 1 and 2 and some results of K. Vogtmann on the stability of the homology of these classical groups [8], [9], we get the following interesting corollary:

Corollary. *Let $F = \mathbb{R}$ or \mathbb{C} . Then we have the following homology isomorphisms:*

$$H_i(\mathbf{BO}_{2n}(F); \mathbb{Z}/q) \approx H_i(\mathbf{BO}_{2n}(F)^{\text{top}}; \mathbb{Z}/q) \quad \text{for } n \geq 3i,$$

$$H_i(\mathbf{BSp}_{2n}(F); \mathbb{Z}/q) \approx H_i(\mathbf{BSp}_{2n}(F)^{\text{top}}; \mathbb{Z}/q) \quad \text{for } n \geq 3i + 3.$$

The theorems and the corollary give more evidence for the Friedlander–Milnor conjecture which is still open:

$$H_i(\mathbf{BG}; \mathbb{Z}/q) \approx H_i(\mathbf{BG}^{\text{top}}; \mathbb{Z}/q) \quad \text{for any Lie group } G.$$

Now the Theorems 1 and 2 are consequences of a theorem of a more general nature which we shall use elsewhere [5]:

Theorem 3. *Let A be a Banach algebra with involution. Then*

$${}_eW_1(A) \approx {}_eW_1^{\text{top}}(A), \quad K_1(A; \mathbb{Z}/q) \approx K_1^{\text{top}}(A; \mathbb{Z}/q)$$

and

$${}_eL_1(A; \mathbb{Z}/q) \approx {}_eL_1^{\text{top}}(A; \mathbb{Z}/q).$$

Let us assume moreover that $K_i(A; \mathbb{Z}/q) \approx K_i^{\text{top}}(A; \mathbb{Z}/q)$ for $2 \leq i \leq N$; then

$${}_eL_i(A; \mathbb{Z}/q) \approx {}_eL_i^{\text{top}}(A; \mathbb{Z}/q) \quad \text{for } 2 \leq i \leq N.$$

Proof. For simplicity's sake, let us drop the letter A in the notations: we shall write K_i for $K_i(A)$, K_i^{top} for $K_i^{\text{top}}(A)$, etc. Following [4] we shall also write $\bar{K}_i, {}_e\bar{L}_i, \dots$ for the groups $K_i(A; \mathbb{Z}/q), {}_eL_i(A; \mathbb{Z}/q), \dots$

It is clear that the maps ${}_eL_1 \rightarrow {}_eL_1^{\text{top}}$ and ${}_eW_1 \xrightarrow{\gamma} {}_eW_1^{\text{top}}$ are surjective. Let $\alpha \in {}_eW_1$ be such that $\gamma(\alpha) = 0$. The argument used in Milnor's book [6, p. 58] shows that α is represented by a product of hyperbolic matrices in ${}_eO_{n,n}(A)$ of the form

$$\alpha_i = \begin{pmatrix} 1 + a_i & b_i \\ c_i & 1 + d_i \end{pmatrix}$$

where a_i, b_i, c_i, d_i are $n \times n$ matrices close to 0 (for the Banach algebra topology). Now the argument used in [2, p. 405] shows that α_i is a product of hyperbolic matrices and ε -elementary matrices. Hence $\alpha = 0$ and ${}_eW_1 \approx {}_eW_1^{\text{top}}$.

We have two exact sequences

$$\begin{array}{ccccccccc}
 {}_\varepsilon L_1 & \xrightarrow{\times q} & {}_\varepsilon L_1 & \longrightarrow & {}_\varepsilon \bar{L}_1 & \longrightarrow & {}_\varepsilon L_0 & \xrightarrow{\times q} & {}_\varepsilon L_0 \\
 \downarrow \sigma & & \downarrow \sigma & & \downarrow & & \downarrow & & \downarrow \\
 {}_\varepsilon L_1^{\text{top}} & \xrightarrow{\times q} & {}_\varepsilon L_1^{\text{top}} & \longrightarrow & {}_\varepsilon \bar{L}_1^{\text{top}} & \longrightarrow & {}_\varepsilon L_0 & \xrightarrow{\times q} & {}_\varepsilon L_0
 \end{array}$$

Since the kernel of σ is generated by matrices of the form $1 + \beta$ where β is close to 0, this kernel is q -divisible (consider $\sqrt[q]{1 + \beta}$). Therefore by diagram chasing, ${}_\varepsilon \bar{L}_1 \approx {}_\varepsilon \bar{L}_1^{\text{top}}$. The same argument shows that $\bar{K}_1 \approx \bar{K}_1^{\text{top}}$.

With the notations of [3] and [4], simple diagram chasing in the diagrams

$$\begin{array}{ccccccccc}
 K_1 & \longrightarrow & {}_\varepsilon L_1 & \longrightarrow & {}_\varepsilon U_0 & \longrightarrow & K_0 & \longrightarrow & {}_\varepsilon L_0 \\
 \downarrow & & \downarrow & & \downarrow & & \parallel & & \parallel \\
 K_1^{\text{top}} & \longrightarrow & {}_\varepsilon L_1^{\text{top}} & \longrightarrow & {}_\varepsilon U_0^{\text{top}} & \longrightarrow & K_0 & \longrightarrow & {}_\varepsilon L_0 \\
 {}_\varepsilon L_1 & \longrightarrow & K_1 & \longrightarrow & {}_\varepsilon V_0 & \longrightarrow & {}_\varepsilon L_0 & \longrightarrow & K_0 \\
 \downarrow & & \downarrow & & \downarrow & & \parallel & & \parallel \\
 {}_\varepsilon L_1^{\text{top}} & \longrightarrow & K_1^{\text{top}} & \longrightarrow & {}_\varepsilon V_0^{\text{top}} & \longrightarrow & {}_\varepsilon L_0 & \longrightarrow & K_0
 \end{array}$$

shows that ${}_\varepsilon U_0 \approx {}_\varepsilon U_0^{\text{top}}$ and that the map ${}_\varepsilon V_0 \rightarrow {}_\varepsilon V_0^{\text{top}}$ is surjective with q -divisible kernel. Let us consider the diagram of exact sequences

$$\begin{array}{ccccccccc}
 {}_\varepsilon U_1 & \xrightarrow{\times q} & {}_\varepsilon U_1 & \longrightarrow & {}_\varepsilon \bar{U}_1 & \longrightarrow & {}_\varepsilon U_0 & \xrightarrow{\times q} & {}_\varepsilon U_0 \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \approx & & \downarrow \approx \\
 {}_\varepsilon U_1^{\text{top}} & \xrightarrow{\times q} & {}_\varepsilon U_1^{\text{top}} & \longrightarrow & {}_\varepsilon \bar{U}_1^{\text{top}} & \longrightarrow & {}_\varepsilon U_0^{\text{top}} & \xrightarrow{\times q} & {}_\varepsilon U_0^{\text{top}}
 \end{array}$$

According to the fundamental theorem of Hermitian K-theory ([2] and [3]), ${}_\varepsilon U_1 \approx -{}_\varepsilon V_0$ and ${}_\varepsilon U_1^{\text{top}} \approx -{}_\varepsilon V_0^{\text{top}}$. Therefore, the map ${}_\varepsilon U_1 \rightarrow {}_\varepsilon U_1^{\text{top}}$ is surjective with q -divisible kernel and the map ${}_\varepsilon \bar{U}_1 \rightarrow {}_\varepsilon \bar{U}_1^{\text{top}}$ is an isomorphism. Using the fundamental theorem of Hermitian K-theory again (the mod q version), we also have an isomorphism ${}_\varepsilon \bar{V}_0 \approx {}_\varepsilon \bar{V}_0^{\text{top}}$ for any ε .

Finally, if $\bar{K}_2 \approx \bar{K}_2^{\text{top}}$, the diagram of exact sequences

$$\begin{array}{ccccccccc}
 {}_\varepsilon \bar{L}_2 & \longrightarrow & \bar{K}_2 & \longrightarrow & {}_\varepsilon \bar{V}_1 & \longrightarrow & {}_\varepsilon \bar{L}_1 & \longrightarrow & \bar{K}_1 \\
 \downarrow & & \downarrow \approx & & \downarrow & & \downarrow \approx & & \downarrow \approx \\
 {}_\varepsilon \bar{L}_2^{\text{top}} & \longrightarrow & \bar{K}_2^{\text{top}} & \longrightarrow & {}_\varepsilon \bar{V}_1^{\text{top}} & \longrightarrow & {}_\varepsilon \bar{L}_1^{\text{top}} & \longrightarrow & \bar{K}_1^{\text{top}}
 \end{array}$$

implies the surjectivity of the map ${}_{\varepsilon}\bar{V}_1 \rightarrow {}_{\varepsilon}\bar{V}_1^{\text{top}}$.

For a fixed A and any ε , let us call (H_i) and (H'_i) the following hypotheses:

$$(H_i) \left\{ \begin{array}{l} {}_{\varepsilon}\bar{L}_i(A) \rightarrow {}_{\varepsilon}\bar{L}_i^{\text{top}}(A) \text{ is an isomorphism} \\ {}_{\varepsilon}\bar{V}_{i-1}(A) \rightarrow {}_{\varepsilon}\bar{V}_{i-1}^{\text{top}}(A) \text{ is an isomorphism} \\ {}_{\varepsilon}\bar{V}_i(A) \rightarrow {}_{\varepsilon}\bar{V}_i^{\text{top}}(A) \text{ is an epimorphism.} \end{array} \right\} (H'_i)$$

According to what we have just proved, (H_1) is satisfied if $N \geq 2$. Therefore, it is sufficient to prove that $(H_i) \Rightarrow (H_{i+1})$ if $1 \leq i < N-1$ and that $(H_i) \Rightarrow (H'_{i+1})$ if $1 \leq i \leq N-1$.

Since ${}_{\varepsilon}\bar{U}_j \approx {}_{-\varepsilon}\bar{V}_{j-1}$, and ${}_{\varepsilon}\bar{U}_j^{\text{top}} \approx {}_{-\varepsilon}\bar{V}_{j-1}^{\text{top}}$, we have the following commutative diagram with three isomorphisms and one epimorphism

$$\begin{array}{ccccccccc} {}_{\varepsilon}\bar{U}_{i+1} & \longrightarrow & \bar{K}_{i+1} & \longrightarrow & {}_{\varepsilon}\bar{L}_{i+1} & \longrightarrow & {}_{\varepsilon}\bar{U}_i & \longrightarrow & \bar{K}_i \\ \downarrow & & \downarrow \approx & & \downarrow & & \downarrow \approx & & \downarrow \approx \\ {}_{\varepsilon}\bar{U}_{i+1}^{\text{top}} & \longrightarrow & \bar{K}_{i+1}^{\text{top}} & \longrightarrow & {}_{\varepsilon}\bar{L}_{i+1}^{\text{top}} & \longrightarrow & {}_{\varepsilon}\bar{U}_i^{\text{top}} & \longrightarrow & \bar{K}_i^{\text{top}} \end{array}$$

if $1 \leq i \leq N-1$. Therefore ${}_{\varepsilon}\bar{L}_{i+1} \approx {}_{\varepsilon}\bar{L}_{i+1}^{\text{top}}$ by the Five Lemma.

In the same way, assuming again (H_i) for $1 \leq i < N-1$, the commutative diagram of exact sequences

$$\begin{array}{ccccccccc} {}_{\varepsilon}\bar{L}_{i+1} & \longrightarrow & \bar{K}_{i+1} & \longrightarrow & {}_{\varepsilon}\bar{V}_i & \longrightarrow & {}_{\varepsilon}\bar{L}_i & \longrightarrow & \bar{K}_i \\ \downarrow & & \downarrow \approx & & \downarrow & & \downarrow \approx & & \downarrow \approx \\ {}_{\varepsilon}\bar{L}_{i+1}^{\text{top}} & \longrightarrow & \bar{K}_{i+1}^{\text{top}} & \longrightarrow & {}_{\varepsilon}\bar{V}_{i+1}^{\text{top}} & \longrightarrow & {}_{\varepsilon}\bar{L}_{i+1}^{\text{top}} & \longrightarrow & \bar{K}_{i+1} \end{array}$$

implies ${}_{\varepsilon}\bar{V}_i \approx {}_{\varepsilon}\bar{V}_i^{\text{top}}$. Finally, if $1 \leq i < N-1$, the diagram of exact sequences

$$\begin{array}{ccccccccc} {}_{\varepsilon}\bar{L}_{i+2} & \longrightarrow & \bar{K}_{i+2} & \longrightarrow & {}_{\varepsilon}\bar{V}_{i+1} & \longrightarrow & {}_{\varepsilon}\bar{L}_{i+1} & \longrightarrow & \bar{K}_{i+1} \\ \downarrow & & \downarrow \approx & & \downarrow & & \downarrow \approx & & \downarrow \approx \\ {}_{\varepsilon}\bar{L}_{i+2}^{\text{top}} & \longrightarrow & \bar{K}_{i+2}^{\text{top}} & \longrightarrow & {}_{\varepsilon}\bar{V}_{i+1}^{\text{top}} & \longrightarrow & {}_{\varepsilon}\bar{L}_{i+1}^{\text{top}} & \longrightarrow & \bar{K}_{i+1}^{\text{top}} \end{array}$$

implies the surjectivity of the map ${}_{\varepsilon}\bar{V}_{i+1} \rightarrow {}_{\varepsilon}\bar{V}_{i+1}^{\text{top}}$.

This proof of Theorem 3 is an illustration of a principle which roughly states that a general theorem for $K_i(A; \mathbb{Z}/q)$ implies a general theorem for ${}_{\varepsilon}L_i(A; \mathbb{Z}/q)$. As an other example we offer the following theorem:

Theorem 4. *Let $A \rightarrow B$ be a map of rings with involution such that*

$$K_0(A) \rightarrow K_0(B), \quad {}_{\varepsilon}L_0(A) \rightarrow {}_{\varepsilon}L_0(B), \quad {}_{\varepsilon}W_1(A) \rightarrow {}_{\varepsilon}W_1(B)$$

are isomorphisms, $K_1(A) \rightarrow K_1(B)$ and ${}_e L_1(A) \rightarrow {}_e L_1(B)$ are surjective with q -divisible kernels (for all ε). Then

$${}_e L_1(A; \mathbb{Z}/q) \approx {}_e L_1(B; \mathbb{Z}/q) \quad \text{and} \quad K_1(A; \mathbb{Z}/q) \approx K_1(B; \mathbb{Z}/q).$$

Let us assume moreover that $K_i(A; \mathbb{Z}/q) \approx K_i(B; \mathbb{Z}/q)$ for all i such that $2 \leq i \leq N$ and if q is even that there exists an element λ of A such that $\lambda + \bar{\lambda} = 1$. Then

$${}_e L_i(A; \mathbb{Z}/q) \approx {}_e L_i(B; \mathbb{Z}/q) \quad \text{and} \quad H_i({}_e O(A); \mathbb{Z}/q) \approx H_i({}_e O(B); \mathbb{Z}/q)$$

for any ε and $1 \leq i \leq N$.

The proof of this theorem is completely analogous to that of Theorem 3: we only have to replace K_i^{top} , ${}_e L_i^{\text{top}}$, ..., by $K_i(B)$, ${}_e L_i(B)$. According to Gabber and Suslin ([1] and [7]), the hypotheses of the theorem are fulfilled when $B = A/I$ where (A, I) is a Henselian pair with $1/q \in A$ such that I is invariant by the involution and such that there exists $\lambda \in A$ with $\lambda + \bar{\lambda} = 1$ (if q is even). For instance,

$${}_e L_i(\hat{\mathbb{Z}}_p; \mathbb{Z}/p) \approx {}_e L_i(\mathbb{Z}/p; \mathbb{Z}/q) \quad \text{with } p \neq 2 \quad \text{if } q \text{ is even.}$$

Note. Very recently, J.F. Jardine [10] has proved analogous results using also [2], [3] and [7]. In particular, he has proved Theorem 1 for $O(\mathbb{C})$ and $Sp(\mathbb{C})$ and a corollary of Theorem 4 for a Henselian pair (A, I) where $B = A/I$ is a field.

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