Topology and its Applications 25 (1987) 179-184
North-Holland

K-THEORY FOR SPHERICAL SPACE FORMS

Peter B. GILKEY*
Mathematics Department, University of Oregon, Eugene, OR 97403, USA

Max KAROUBI**
UER de Mathematique, Universite Paris VII, 75251 Paris, France

Received 11 January 1986

We relate the representation theory of the fundamental group of a spherical space form to the corresponding real, complex, and quaternionic K-theory and show that except for exceptional Z_2 factors coming from the K-theory of a sphere, the complete K-theory arises from the representation theory. The main result (Theorem 1) has been considered as 'folklore' for some years and has been proved or has been of interest for particular cases in the work of various people (Fujii [3, 4], Gilkey [5, 6], Mahammed [9, 10], Pitt [11], Yasuo [13]). This result is also intimately linked with the study of equivariant cobordism (Bahri [2]). However, this result has never, to our knowledge, appeared in this generality in the literature.

AMS (MOS) Subj. Class.: Primary 55N15; secondary 19L47

spherical space form — topological K-theory
representation theory — Clifford algebras

Let G be a compact Lie group and let $\tau: G\to O(k)$ be a fixed point free representation. We shall assume $k\geq 3$. Let $M_\tau = S^{k-1}/\tau(G)$. If k is *odd*, then either $|G|=1$ and $M_\tau = S^{k-1}$ or $|G|=2$ and $M_\tau = RP^{k-1}$. As both these spaces are well understood, we shall assume $k=2j\geq 4$ is even henceforth.

If G is finite, then M_τ is a compact Riemannian manifold of constant sectional curvature 1 and all such manifolds arise in this fashion; such a manifold is called a spherical space form. We are interested primarily in the case $|G|$ finite. We include results in the more general case since there are no additional technical difficulties.

Let $\mathrm{K}(M_\tau)$, $\mathrm{KO}(M_\tau)$, $\mathrm{KSp}(M_\tau)$ denote the complex, real, and quaternionic or symplectic K-theory. Let $\mathrm{R}(G)$, $\mathrm{RO}(G)$, $\mathrm{RSp}(G)$ denote the corresponding representation groups. If ρ is a representation of G, let V_ρ denote the bundle over M_τ corresponding to ρ. The map $\rho\to V_\rho$ defines morphisms

$$\mathrm{R}(G)\xrightarrow{\theta_c}\mathrm{K}(M_\tau),\ \mathrm{RO}(G)\xrightarrow{\theta_r}\mathrm{KO}(M_\tau),\ \mathrm{RSp}(G)\xrightarrow{\theta_h}\mathrm{KSp}(M_\tau).$$

* Research partially supported by NSF contract DMS-8414528.
** Research partially supported by NSF contract DMS-8502351.

0166-8641/87/$3.50

It is well known that the map $\mathrm{R}(G)\to\mathrm{K}(M_\tau)$ is surjective (see for example Atiyah [1] if G is cyclic). This is the starting point for the results of Gilkey [5] relating the eta invariant to $\mathrm{K}(M_\tau)$ for spherical space forms. The following theorem is the natural generalization of Atiyah's result and plays the corresponding role in the analysis of KO and KSp for lens spaces (see Gilkey [6]).

Theorem 1. *Let G be a compact Lie group and let $\tau: G\to O(k)$ be a fixed point free representation where $k\geq 4$ is even. Let $M_\tau = S^{k-1}/\tau(G)$ then:*

(a) *If G has positive dimension or if $|G|$ is even, then θ is surjective—i.e. we have short exact sequences*

$$\mathrm{R}(G)\xrightarrow{\theta_c}\mathrm{K}(M_\tau)\to 0,\qquad \mathrm{RO}(G)\xrightarrow{\theta_r}\mathrm{KO}(M_\tau)\to 0,$$

$$\mathrm{RSp}(G)\xrightarrow{\theta_h}\mathrm{KSp}(M_\tau)\to 0.$$

(b) *If $|G|$ is odd, then there are split short exact sequences*

$$\mathrm{R}(G)\xrightarrow{\theta_c}\mathrm{K}(M_\tau)\xrightarrow{\alpha_c^*}\tilde{\mathrm{K}}(S^{k-1})\to 0,$$

$$\mathrm{RO}(G)\xrightarrow{\theta_r}\mathrm{KO}(\mathrm{M}_\tau)\xrightarrow{\alpha_r^*}\tilde{\mathrm{KO}}(S^{k-1})\to 0,$$

$$\mathrm{RSp}(G)\xrightarrow{\theta_h}\mathrm{KSp}(M_\tau)\xrightarrow{\alpha_h^*}\widetilde{\mathrm{KSp}}(S^{k-1})\to 0,$$

where the maps α^* are induced by the projection $S^{k-1}\xrightarrow{\alpha}M_\tau$.

Remark. Since $\tilde{\mathrm{K}}(S^{k-1})=0$ if k is even, we see that $\mathrm{R}(G)\to\mathrm{K}(M_\tau)$ is always surjective. We recall

$$\tilde{\mathrm{KO}}(S^{k-1})=\begin{Bmatrix} Z_2 & \text{if } k\equiv 2(8)\\ 0 & \text{if } k \text{ is even otherwise}\end{Bmatrix}$$

$$\widetilde{\mathrm{KSp}}(S^{k-1})=\begin{Bmatrix} Z_2 & \text{if } k\equiv 6(8)\\ 0 & \text{if } k \text{ is even otherwise}\end{Bmatrix},$$

so θ has cokernel Z_2 only in these exceptional cases. Let $|G|$ be odd and let

$$\alpha_*^r:\ \tilde{\mathrm{KO}}(S^{k-1})\to\tilde{\mathrm{KO}}(M_\tau),\qquad \alpha_*^h:\ \widetilde{\mathrm{KSp}}(S^{k-1})\to\widetilde{\mathrm{KSp}}(M_\tau)$$

be the Gysin homomorphisms. Since $\alpha^*\alpha_*$ is multiplication by $|G|$, this is an isomorphism on $\tilde{\mathrm{KO}}(S^{k-1})$ and $\widetilde{\mathrm{KSp}}(S^{k-1})$. We let α_* provide the splitting.

The remainder of this paper is devoted to the proof of Theorem 1. For ease of exposition, we shall restrict primarily to the case of KO. We will indicate what changes must be made for KSp as appropriate. We recall that Bott periodicity provides an isomorphism between KO^n and KSp^{n+4} where n is defined modulo 8.

Let $V=R^k$ with the G-action provided by τ. Let B denote the unit ball and let S denote the unit sphere of V with the usual Euclidean inner product. These are G invariant closed subsets. Consider the long exact sequence of the pair (B, S):

$$\mathrm{KO}_G(B)\to \mathrm{KO}_G(S)\to \mathrm{KO}_G^1(B, S)\to \mathrm{KO}_G^1(B). \tag{1.1}$$

There is a natural isomorphism between $\mathrm{KO}_G(B)$ and $\mathrm{RO}(G)$ and between $\mathrm{KO}_G(S)$ and $\mathrm{KO}(M_\tau)$. Furthermore $\mathrm{KO}_G^1(B)=\mathrm{KO}^{-7}(G)=0$. (In the symplectic case we use the fact that $\mathrm{KSp}_G^1(B)= =\mathrm{KO}_G^5(B)=\mathrm{KO}^{-3}(G)=0$. We refer to [7, p. 228, Prop. 2.3.14] for details. Thus we can rewrite the sequence (1.1) in the form:

$$\mathrm{RO}(G)\to \mathrm{KO}(M_\tau)\to \mathrm{KO}_G^1(B, S)\to 0. \tag{1.2}$$

We shall prove Theorem 1 by studying $\mathrm{KO}_G^1(B, S)$. At this point it is convenient to introduce a new functor. Let W be a real vector space with a positive definite inner product. Let Clif(W) denote the real Clifford algebra. This is the universal algebra generated by W subject to the relations $w_1\cdot w_2+w_2\cdot w_1=2(w_1, w_2)$. Assume a G-action on W and let $E(G, W)$ be the category of real vector spaces E with actions of G and Clif(W) on E. We assume as a compatibility condition that $g\cdot(w\cdot e)=(g\cdot w)\cdot(g\cdot e)$. Let G act trivially on R and let $\phi: E(G, V\oplus R)\to E(G, V)$ forget the action of R on E. This gives rise to a long exact sequence

$$\begin{aligned}&\mathrm{KO}^n(E(G, V\oplus R))\to \mathrm{KO}^n(E(G, V))\to \mathrm{KO}^{n+1}(\phi)\\ &\quad\to \mathrm{KO}^{n+1}(E(G, V\oplus R))\to \mathrm{KO}^{n+1}(E(G, V))\end{aligned} \tag{1.3}$$

The major technical result we shall need and which lies at the heart of the matter is the following Theorem.

Theorem 2 [8, p. 240]. *With the notation established above,* $\mathrm{KO}_G^n(B, S)=\mathrm{KO}^n(\phi)$.

This reduces the problem of computing $\mathrm{KO}_G^1(B, S)$ to the more algebraic problem of computing $\mathrm{KO}^1(\phi)$, or in the symplectic case of computing $\mathrm{KSp}_G^1(B, S)=\mathrm{KO}_G^5(B, S)=\mathrm{KO}^5(\phi)$. Since $E(G, V\oplus R)$ is a finite dimensional Banach category, it follows that $\mathrm{KO}^1(E(G, V\oplus R))=0$. We refer to [8] for details. This yields the short exact sequence

$$\mathrm{KO}(E(G, V\oplus 1))\to \mathrm{KO}(E(G, V))\to \mathrm{KO}^1(\phi)\to 0. \tag{1.4}$$

(In the symplectic case one uses the analogous fact that $\mathrm{KO}^5(E(G, V\oplus 1))=0$).

We prove Theorem 1(a) as follows. If $|G|$ is even, then there exists $g_0\in G$ of order 2. Similarly if G has positive dimension, let T be a maximal torus and let $g_0\in T$ be of order 2. Since τ is fixed point free, it is faithful. Since $\tau(g_0)^2=I$, $\tau(g_0)$ has eigenvalues ± 1. Since τ is fixed point free, the eigenvalue 1 does not occur so $\tau(g_0)=-I$. As τ is faithful, this implies g_0 is unique so there is only one element of order 2 in G. This shows g_0 is central. Let $E\in E(G, V)$ and let $x\in R$ be the basis element. To extend E to $E(G, V\oplus R)$, we must define a compatible action by x. We define $x\cdot e=g_0\cdot e$. Then $x^2\cdot e=g_0^2\cdot e=e$ since $g_0^2=1$. Furthermore,

$$g\cdot(x\cdot e)=g\cdot(g_0\cdot e)=g_0\cdot g\cdot e=x\cdot g\cdot e=(g\cdot x)\cdot(g\cdot e)$$

since G acts trivially on R. Finally,

$$x \cdot v \cdot e = g_0 \cdot (v \cdot e) = (g_0 \cdot v)(g_0 \cdot e) = -v \cdot (g_0 \cdot e) = -v \cdot x \cdot e$$

since $g_0 \cdot v = -v$. This gives a splitting to $\phi : E(G, V \oplus R) \to E(G, V)$ and proves that ϕ is surjective. This shows $\mathrm{KO}^1(\phi) = 0$ and completes the proof of Theorem 1(a). We note that a similar argument shows the category $E(G, W \oplus R)$ is isomorphic to $E(G, W) \times E(G, W)$ and that ϕ is the functor $(E, F) \to E \oplus F$.

We complete the proof of Theorem 1 by considering the case $|G|$ is *odd.* As k is even, τ is conjugate to a unitary representation $\tau : G \to U(j) \subseteq \mathrm{SO}(2j)$; we refer to Wolf [12] for details (this also follows from Lemma 5 as we shall see shortly). Therefore, we can lift τ to a spin^c representation $\hat{\tau}_c$. If $\gamma = \det(\hat{\tau}_c)$ is the associated $U(1)$ representation, then γ has odd order so $\gamma^{-1/2}$ is well defined. By multiplying $\hat{\tau}_c$ by $\gamma^{-1/2}$ we can lift τ to a spin representation $\hat{\tau} : G \to \mathrm{Clif}(V)$ such that $g \cdot v = \hat{\tau}(g) \cdot v \cdot \hat{\tau}(g)^{-1}$. Since $|G|$ is odd, this lifting is unique. (One can also use a cohomological argument since the obstruction to lifting a representation from $\mathrm{SO}(2j)$ to $\mathrm{Spin}(2j)$ lies in $H^2(G; Z_2)$ which is zero since $|G|$ is odd).

We use the spin representation to untwist the action.

Lemma 3. *Let G be a spin action on W. Let $[W]$ denote W with a trivial G-action. Then there is a natural equivalence of categories between $E(G, W)$ and $E(G, [W])$.*

Proof. Let the spin action be $\hat{\tau}(g)$. Let $E \in E(G, W)$ and define a new action of G on E by $g * e = \hat{\tau}(g)^{-1} \cdot g \cdot e$ where $\hat{\tau}(g) \in \mathrm{Clif}(W)$. Let $\tilde{E}$ denote E with this new action. We note $1 * e = e$ and compute:

$$\begin{aligned} h * (g * e) &= \hat{\tau}(h)^{-1} \cdot h \cdot (\hat{\tau}(g)^{-1} \cdot g \cdot e) = \hat{\tau}(h)^{-1} \cdot h(\hat{\tau}(g)^{-1}) \cdot hg \cdot e \\ &= \hat{\tau}(h)^{-1} \cdot \hat{\tau}(h) \cdot \hat{\tau}(g)^{-1} \cdot \tau(h)^{-1} \cdot hg \cdot e = \hat{\tau}(g)^{-1} \cdot \hat{\tau}(h)^{-1} \cdot hg \cdot e \\ &= \hat{\tau}(hg)^{-1} \cdot hg \cdot e = (hg) * e \end{aligned}$$

which shows that $*$ is an action of G. We check the compatibility condition:

$$\begin{aligned} h * (v \cdot e) &= \hat{\tau}(h)^{-1} \cdot h \cdot v \cdot e = \hat{\tau}(h)^{-1} \cdot (h \cdot v) \cdot (h \cdot e) \\ &= \hat{\tau}(h)^{-1} \cdot \hat{\tau}(h) \cdot v \cdot \hat{\tau}(h)^{-1} \cdot h \cdot e = v \cdot (h * e), \end{aligned}$$

so that $\tilde{E} \in E(G, [W])$. Conversely, let $\tilde{E} \in E(G, [W])$ and let $g * e$ denote the action. Define $g \cdot e = \hat{\tau}(g) \cdot (g * e)$. Let E denote the resulting object. We compute

$$\begin{aligned} h \cdot (g \cdot e) &= \hat{\tau}(h) \cdot h * \hat{\tau}(g) \cdot g * e \\ &= \hat{\tau}(h) \cdot \hat{\tau}(g) \cdot h * g * e = \hat{\tau}(hg) \cdot (hg) * e = (hg) \cdot e, \end{aligned}$$

so this gives an action of G on W. We check the compatibility condition:

$$\begin{aligned} h \cdot v \cdot e &= \hat{\tau}(h) \cdot h * v \cdot e = \hat{\tau}(h) \cdot v \cdot h * e \\ &= \hat{\tau}(h) \cdot v \cdot \hat{\tau}(h)^{-1} \cdot \hat{\tau}(h) \cdot h * e = (h \cdot v) \cdot (h \cdot e), \end{aligned}$$

so $E \in E(G, W)$. The operations $E \to \tilde{E}$ and $\tilde{E} \to E$ are inverses of each other and morphisms are preserved. This provides the desired equivalence of categories. $\square$

Thus to compute $\mathrm{KO}^1(\phi)$ or $\mathrm{KSp}^1(\phi)$, we may safely replace V by $[V]$. We study the short exact sequence

$$\mathrm{KO}(E(G,[V]\oplus R)) \to \mathrm{KO}(E(G,[V])) \to \mathrm{KO}^1(\phi) \to 0. \tag{1.5}$$

Suppose first $k \equiv 0(8)$. Let $v = 2^{k/2}$ so that Clif(V) is naturally isomorphic to the ring of $v \times v$ real matrices $M_v(R)$. Then Clif($V \oplus R$) $= M_v(R) \times M_v(R)$ so that $E(G,[V]\oplus R) = E(G,[V]) \times E(G,[V])$ and $\phi(E, F) = E \oplus F$. Thus again ϕ has a natural splitting so $\mathrm{KO}^1(\phi) = \mathrm{KO}^5(\phi) = 0$. If $k \equiv 4(8)$, the analysis is similar if we replace $M_v(R)$ by $M_v(H)$ where H denotes the quaternions. This completes the proof of Theorem 1 in these instances. (If $|G|$ is even, then τ lifts to a spinor representation if $k \equiv 0(4)$ so (a) also follows using this argument for this case).

If $k \equiv 2(8)$, then Clif(V) is Morita equivalent to R and Clif($V \oplus R$) is Morita equivalent to C and we get the sequence

$$\mathrm{R}(G) \to \mathrm{RO}(G) \to \mathrm{KO}^1(\phi) \to 0$$

while if $k \equiv 6(8)$, Clif(V) is Morita equivalent to H and Clif($V \oplus R$) is Morita equivalent to C and we get the sequence

$$\mathrm{R}(G) \to \mathrm{RSp}(G) \to \mathrm{KO}^1(\phi) \to 0.$$

To analyze $\mathrm{KSp}^1(\phi) = \mathrm{KO}^5(\phi)$, we interchange R and H. (We note that this step fails if $|G|$ is even since τ does not lift to a spin representation if $k \equiv 2(4)$ in general. This proves the following lemma.

Lemma 4. *Let $|G|$ be odd.*

(a) *If $k \equiv 2(8)$ then $\mathrm{KO}^1(\phi) = \mathrm{RO}(G)/\mathrm{R}(G)$ and $\mathrm{KSp}^1(\phi) = \mathrm{RSp}(G)/\mathrm{R}(G)$.*

(b) *If $k \equiv 6(8)$ then $\mathrm{KO}^1(\phi) = \mathrm{RSp}(G)/\mathrm{R}(G)$ and $\mathrm{KSp}^1(\phi) = \mathrm{RO}(G)/\mathrm{R}(G)$.*

This reduces the proof of Theorem 1 to a calculation involving the representation groups. We consider the following well known morphisms $r: \mathrm{R}(G) \to \mathrm{RO}(G)$, $h: \mathrm{R}(G) \to \mathrm{RSp}(G)$, $c: \mathrm{RO}(G) \to \mathrm{R}(G)$, $c: \mathrm{RSp}(G) \to \mathrm{R}(G)$. In particular $(c \cdot r)(\rho) = (c \cdot h)(\rho) = \rho + \rho^*$, ρ^* being the dual representation of ρ.

Let now ρ be an irreducible real non-trivial representation. Decompose $c(\rho)$ as $\rho_1 + \cdots + \rho_n$ where the ρ_i are irreducible complex representations. Then $r(c(\rho)) = r(\rho_1) + \cdots + r(\rho_n) = 2\rho$. Therefore $n = 2$ and $\rho_1 + \rho_2 = (\rho_1 + \rho_2)^* = \rho_1^* + \rho_2^*$. Hence, $\rho_2 = \rho_1^*$ or $\rho_1 = \rho_2^*$. In the first case, $\rho = r(\rho_1)$ because $c(\rho) = c(r(\rho_1)) = \rho_1 + \rho_1^*$. In the second case, by the lemma below, ρ_1 and ρ_2 are trivial representations, which is impossible according to the initial hypothesis. This shows that $\mathrm{RO}(G)/\mathrm{R}(G) = Z_2$, the factor of Z_2 coming from the odd dimensional trivial real representation. In the same way, one can prove that $\mathrm{RSp}(G)/\mathrm{R}(G) = 0$ because all trivial symplectic representations are of the form $h(\rho)$. As Lemma 4 will then produce exactly the correct Z_2 factors, the proof of the theorem will then be complete. It remains to prove this last lemma.

Lemma 5. *Let $|G|$ be odd and let G admit a fixed point free representation into $O(k)$. Let $\rho \in \mathrm{R}(G)$ be non-trivial and irreducible, then $\rho \neq \rho^*$.*

Proof. Suppose first that $G = Z_n = \{\lambda \in C: \lambda^n = 1\}$ is cyclic. Let $\rho_s(\lambda) = \lambda^s$ parametrize the irreducible unitary representations of G. Then $\rho_s = \rho_s^*$ implies $2s \equiv 0(n)$. Since n is *odd*, we conclude $s \equiv 0(n)$ so ρ_s is the trivial representation. This completes the proof in this case. More generally, we use Wolf [12] to argue that G is metacyclic—i.e. is a semi-direct product

$$0 \to H \to G \to G/H \to 0$$

where H and G/H are cyclic. Let ρ be an irreducible representation of G with complex vector space E such that $\rho = \rho^*$. We are going to show ρ is trivial. Decompose $\rho|_H$ as $c \cdot \rho_0 + \sum_\nu c_\nu(\rho_\nu + \rho_\nu^*)$ where ρ_0 is the trivial representation. Since $\dim(E)$ divides the order of G, it is odd; therefore $c \neq 0$ and $E_0 = \{x \mid h \cdot x = x \; \forall x \in H\}$ is non-trivial. Since H is normal, E_0 is G-invariant and $E_0 = E$ since ρ is irreducible. Therefore, we can extend ρ to G/H with $\rho = \rho^*$. Since G/H is cyclic, ρ is trivial on G/H and the lemma is proved. □

Remark. If G is *any* finite group of odd order then one can prove with the same argument (using the Feit–Thompson theorem) that if $\rho \in \mathrm{R}(G)$ is an irreducible representation with $\rho = \rho^*$, then ρ is trivial. As a consequence $\mathrm{RO}(G)/\mathrm{R}(G) = Z_2$ so every real even dimensional representation of G admits a complex structure.

References

[1] M.F. Atiyah, K-theory (W.A. Benjamin, New York, 1967).

[2] T. Bahri and P. Gilkey, The eta invariant, Pin^c bordism, and equivariant Spin^c bordism for cyclic 2-groups, Pacific J. (1986) to appear.

[3] K. Fujii, On the KO-ring of S^{4n+3}/H_m, Hiroshima Math. J. 4 (1974) 459–475.

[4] K. Fujii, T. Kobayashi, K. Shimomura and M. Sugawara, KO groups of lens spaces modulo powers of two, Hiroshima Math. J. 8 (1978) 468–489.

[5] P.B. Gilkey, The eta invariant and the K-theory of odd dimensional spherical space forms, Inventiones Mathematicae 76 (1984) 421–453.

[6] P.B. Gilkey, $\widetilde{\mathrm{KO}}$ for odd dimensional lens spaces, Math. Zeitschrift (1987) to appear.

[7] M. Karoubi, Algebras de Clifford et K-theorie, Ann. Sci. Ec. Norm (1968) 161–270.

[8] M. Karoubi, Sur la K-theorie equivariante, Springer Lecture Notes 136 (Springer Verlag, Berlin, 1970) 187–253

[9] N. Mahammed, K-theorie des formes spheriques, These Univ. Sci. Tech. Lille, 1975.

[10] N. Mahammed, P. Piccinni and U. Suter, Some applications of topological K-theory, North-Holland Math. Studies 45 (North-Holland, Amsterdam, 1980).

[11] D. Pitt, Proc. L.M.S. 26 (1973) 1–18.

[12] J.A. Wolf, Spaces Of Constant Curvature (Publish or Perish Press, Delaware, 5th ed., 1984).

[13] M. Yasuo, γ-dimension and products of lens spaces, Mem. Fac. Sci, Kyushu Univ. Ser. A 31 (1977) 113–126.