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Cyclic Homology and Characteristic Classes of Bundles with Additional Structures.

An Informal Report

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This short paper is just a report on some work relating cyclic homology and characteristic classes. The proofs and more details may be found in the references at the end of this present redaction. I hope that the informal style adopted here will help a better understanding.

O. Dictionary. The functor which associates to any C^∞ vector bundle V over a manifold X its space of sections $E = \Gamma(V)$ is in fact an equivalence of categories

$$\mathcal{E}(X) \longrightarrow \mathcal{P}(A)$$

Here $\mathcal{E}(X)$ is the category of vector bundles over X and $\mathcal{P}(A)$ is the category of finitely generated projective A-modules with $A = C^\infty(X)$ (Serre-Swan's theorem). To a large extent, the classical Chern-Weil theory on $\mathcal{E}(X)$ can be extended in a Chern-Weil theory on $\mathcal{P}(A)$ for any k-algebra A where k is an arbitrary commutative ring(*) (but A not necessarily commutative). One of the purposes of cyclic homology is to accomplish this goal [C] [K1]. In particular, the generalization of the classical Chern character

$$K(X) \longrightarrow H^{\text{even}}(X)$$

will be

$$K_0(A) \longrightarrow H_{\text{even}}(A)$$

where H_{even} is essentially cyclic homology. The following dictionary gives more examples of translating a geometrical concept into an algebraic one.

Space X	Ring A
Locally compact space X	C^*-algebra
Manifold $X+$ ordinary differential calculus in $\Omega^* X$	Ring $A+$ differential graded algebra $\Omega_*(A)$ with $A = \Omega_0 A$ and A dense subalgebra of a C^*-algebra.
Integration of smooth forms $\int$	Graded trace on $\Omega_*(A)$
Differential forms $\Omega^* X$	Hochschild homology $H_*(A, A)$
De Rham cohomology of forms $H^*(X)$	Cyclic homology $H_*(A)$ or $HC_*(A)$
Vector bundle V	Finitely generated projective module E
Connexion $D : \Gamma(V) \to \Gamma(V \otimes T^* X)$	Connexion $D : E \to E \bigotimes_A \Omega_1 A$
Matrix $\Gamma_{ij}^k =< D_{e_i} e_j,\ dx^k >$	Matrix $\Gamma_{ij} \in M_n(\Omega_1 A)$
Curvature $R = D^2 = d\Gamma + \frac{1}{2}[\Gamma, \Gamma] = d\Gamma + \Gamma^2$	same

(*) We are assuming $\mathbb{Q} \subset k$ for sake of simplicity—see [K3] in general.

Topological K-theory $K^0(X) = K(X)$	Algebraic K-theory $K_0(A) = K(A)$
Topological K-theory $K^{-n}(X) = \widetilde{K}(S^n X^+)$	Topological K-theory $K_n^{\text{top}}(A)$ $(= \pi_{n-1}(GL(A)))$ if A is a Fréchet algebra
Bundles with additional structures	Modules with additional structures
Multiplicative K-theory $MK(X)^{(*)}$	Multiplicative K-theory $MK(A)$

1. The Chern character for the group K_0

Let A be a ring with a unit. We want to define a "Chern character"

$$K_0(A) \longrightarrow H_*(A)$$

where $H_*(A)$ is some kind of "homology theory" associated to A of a De Rham type. We start with a differential graded algebra $\Omega_*(A)$

$$0 \to A = \Omega_0 A \xrightarrow{d} \Omega_1 A \xrightarrow{d} \Omega_2 A \to \cdots$$

(think of $\Omega_* A = \Omega^* X$, usual De Rham complex, if $A = C^\infty X$). If $w_n \in \Omega_n A$, $w_p \in \Omega_p A$, define their graded commutator $[w_n, w_p] = w_n w_p - (-1)^{np} w_p w_n$. Denote by $\overline{\Omega}_* A$ the quotient of $\Omega_* A$ by the k-module generated by graded commutators. The *non commutative De Rham homology* of A (or rather $\Omega_* A$) is the homology of the complex (cf. [K1])

$$0 \to \overline{\Omega}_0 A \xrightarrow{d} \overline{\Omega}_1 A \xrightarrow{d} \overline{\Omega}_2 A \xrightarrow{d} \cdots$$

which we shall denote by $H_*(A)$. Now the Chern character

$$\text{ch} : K_0(A) \longrightarrow H_{2*}(A)$$

can be defined in two ways:

a) If $E = \text{Im}(e)$ with $e^2 = e \in M_r(A)$, then we put $\text{ch}_n(E) = \frac{1}{n!}\text{Trace}(e(de)^{2n}) \in H_{2n}(A)$. This is the simplest definition ([K1], [K3], [C]).

b) A *connexion* on E is a k-linear map

$$D : E \otimes_A \Omega_* A \longrightarrow E \otimes_A \Omega_{*+1}(A)$$

with the Leibnitz rule

$$D(s.w) = D(s).w + (-1)^{\deg(s)} s.dw$$

Its *curvature* $R = D^2$ is A-linear and we put $\text{ch}_n(E)$ (also denoted by $\text{ch}_n(D)$) $= \frac{1}{n!}\text{Trace}(R^n) \in H_{2n}(A)$ because the homology class is independent of the choice

(*) The letter M stands for "multiplicative". We give up the notation $\mathcal{H}$ used in [K3] for typographical reasons.

of D. In this algebraic setting there is a *universal* example $\Omega_0 A = A$, $\Omega_1 A = \operatorname{Ker}(A \otimes A \to A)$ which is a A-bimodule, $\Omega_n A = \Omega_1 A \bigotimes_A \Omega_1 A \bigotimes_A \cdots \bigotimes_A \Omega_1 A$ (n factors); $d : A \to \Omega_1 A$ defined by $d(x) = 1 \otimes x - x \otimes 1$, etc....

Theorem: $H_n A = \operatorname{Ker}\left(HC_n(A) \xrightarrow{B} H_{n+1}(A, A)\right)$ *where* $HC_n(A)$ *is the cyclic homology of A. Connes and* $H_n(A, A)$ *is the Hochschild homology*(*).

2. The Chern character for the groups K_1 and K_i.

If $\alpha \in GL_r(A)$, define

$$\operatorname{ch}_n(\alpha) = \frac{(n-1)!}{(2n-1)!} \operatorname{Trace}(\alpha^{-1} d\, \alpha)^{2n-1}$$

This induces a character ($l = n - 1$)

$$K_1(A) \longrightarrow H_{1+2l}(A) \subset HC_{1+2l}(A)$$

More generally, one can define character maps (for $l \geq 0$)

$$K_i(A) \longrightarrow H_{i+2l}(A)$$

where K_i are the Quillen K-groups. This connects the homology of the group $GL(A)$ and the Lie algebra homology of $gl(A)$. In fact, one has $\operatorname{Prim}(H_*(gl(A))) = HC_{*-1}(A)$ (Loday-Quillen-Feigan-Tsygan theorem: cf. [LQ]).

If A is a Banach (or Fréchet) algebra, these higher Chern characters can be extended to topological K-theory through commutative diagrams

$$\begin{array}{ccc} K_i(A) & \longrightarrow & H_{i+2l}(A) \\ \downarrow & \nearrow & \\ K_i^{\text{top}}(A) & & \end{array} \qquad \begin{array}{ccc} K_i^{\text{top}}(A) & \longrightarrow & H_{i+2l}(A) \\ \beta \downarrow & & \downarrow S \\ K_{i+2}^{\text{top}}(A) & \longrightarrow & H_{i+2l-2}(A) \end{array}$$

where β is Bott periodicity and S is the periodicity map of cyclic homology (coming from the periodicity of the homology of finite cyclic groups).

3. "Relative K-theory"—Borel regulators.

Although the characters defined in §2 detect some part of algebraic or topological K-theory, this is not the full story!

Examples.

a) The simplest case of failure is the determinant map $K_1(\mathbf{C}) \to \mathbf{C}^*$. This is not covered completely by §2.

(*) Strictly speaking, one has to consider *reduced* cyclic homology in the statement of this theorem (cf. [C], [K3]).