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Braiding of differential forms and homotopy types

Max KAROUBI

Mathématiques, UMR 7586 du CNRS, case 7012, Université Paris-7, 2, place Jussieu, 75251 Paris cedex 05, France
E-mail: karoubi@math.jussieu.fr

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Abstract. Let k be an arbitrary commutative ring. We associate fonctorially to any simplicial set X a differential graded algebra $\widehat{\mathcal{W}}^*(X)$ with a *globally defined braiding*, which is an improvement of a previous work [3,4]. If $k = \mathbf{Z}$ and with some mild finiteness conditions on X, we show that the quasi-isomorphisms class of $\widehat{\mathcal{W}}^*(X)$ as a *braided* differential graded algebra determines the p-adic homotopy type of X for all the prime numbers p, and also the rational homotopy type. As in [3,4], the proof uses some recent results of M.A. Mandell [5,6]. © 2000 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Tressage des formes différentielles et types d'homotopie

Résumé. *Soit k un anneau commutatif arbitraire. Nous associons de manière fonctorielle à un ensemble simplicial X une algèbre graduée $\widehat{\mathcal{W}}^*(X)$ avec un tressage défini de manière globale, ce qui améliore les conclusions d'un travail précédent* [3,4]. *Si $k = \mathbf{Z}$ et sous des hypothèses de finitude raisonnables sur X, nous montrons que la classe de quasi-isomorphisme de l'algèbre différentielle graduée tressée*[1] $\widehat{\mathcal{W}}^*(X)$ *détermine le type d'homotopie p-adique de X pour tout nombre premier p, ainsi que le type d'homotopie rationnel. Comme dans* [3,4], *la démonstration utilise quelques résultats récents de M.A. Mandell* [5,6]. © 2000 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

1. Formes différentielles quantiques sur la droite affine

Soit k un anneau commutatif arbitraire. Nous désignons par Λ l'algèbre du type $\sum_{r=-\infty}^{s} \alpha_r q^r$, $\alpha_r \in k$, munie de la valuation définie par le plus petit entier r tel que $\alpha_{-r} \neq 0$ (et de la topologie ultramétrique associée). Soit $\mathcal{W}^0(t)$ la sous-algèbre de $\Lambda[[t]]$ constituée des séries formelles $f(t) = \sum_{n=0}^{\infty} a_n t^n$, $a_n \in \Lambda$, telles que, pour tout $s \geqslant 0$, les coefficients $a_n q^{sn}$ tendent vers 0 lorsque $n \to \infty$ et vérifient l'identité[2] :

$$(\mathrm{E}_0) \qquad \sum_{n=0}^{\infty} a_n q^{sn} = 0.$$

Note présentée par Alain CONNES.

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La somme directe $\mathcal{W}^*(t) = \mathcal{W}^0(t) \oplus \mathcal{W}^1(t)$ est alors une algèbre différentielle graduée [3] pour la différentielle quantique introduite dans [3], munie d'un tressage R prolongeant celui défini dans l'algèbre des polynômes et ses différentielles [4]. À titre d'exemple, nous avons ainsi la formule suivante :

$$R\big(\omega(t) \otimes f(t)\big) = f(qt) \otimes \otimes(t), \quad \text{pour } f \in \mathcal{W}^0(t) \text{ et } \omega \in \mathcal{W}^1(t).$$

La cohomologie de cette ADG est concentrée en degré 0 et isomorphe à Λ.

2. Formes différentielles quantiques sur un ensemble simplicial

Si $(x_0, \ldots, x_r)$ sont $r+1$ indéterminées, nous définissons $\mathcal{W}^*(x_0, \ldots, x_r)$ comme le produit tensoriel gradué $\mathcal{W}^*(x_0) \otimes \cdots \otimes \mathcal{W}^*(x_r)$. De même, si Δ_r désigne le simplexe standard de dimension r, nous définissons $\mathcal{W}^*(\Delta_r)$ comme l'égalisateur des deux flèches évidentes :

$$\prod_i \mathcal{W}^*(x_0, \ldots, \widehat{x_i}, \ldots, x_r) \rightrightarrows \prod_{i<j} \mathcal{W}^*(x_0, \ldots, \widehat{x_i}, \ldots, \widehat{x_j}, \ldots, x_r).$$

THÉORÈME 2.1. – *Les $\mathcal{W}^*(\Delta_r)$ définissent une ADG* semi-simpliciale *avec des opérateurs face définis par les évaluations $x_i = 1$. Pour chaque r, sa cohomologie est concentrée en degré 0 et isomorphe à Λ* (lemme de Poincaré). *Enfin, le tressage sur les formes différentielles de la droite affine induit un tressage*

$$\mathcal{W}^*(x_0, \ldots, x_r) \otimes \mathcal{W}^*(x_0, \ldots, x_s) \longrightarrow \mathcal{W}^*(x_0, \ldots, x_s) \otimes \mathcal{W}^*(x_0, \ldots, x_r),$$

d'où on déduit un tressage « bisimplicial » en un sens évident

$$\mathcal{W}^*(\Delta_r) \otimes \mathcal{W}^*(\Delta_s) \longrightarrow \mathcal{W}^*(\Delta_s) \otimes \mathcal{W}^*(\Delta_r).$$

Soit maintenant X un ensemble simplicial quelconque et soit $\mathrm{C}_\bullet = \mathrm{C}_\bullet(X)$, le Λ-module *cosimplicial* des cochaînes usuelles sur X. Nous définissons alors l'ADG des formes différentielles quantiques $\mathcal{W}^*(X)$ sur X comme la « Λ-réalisation » à la Bousfield–Kan entre $\mathrm{C}_\bullet(X)$ et le Λ-module *semi-simplicial* $\mathcal{W}^*(\Delta_\bullet)$, que nous noterons $\mathcal{W}^*(X) = \mathrm{C}_\bullet(X) \nabla \mathcal{W}^*(\Delta_\bullet)$ comme dans [3].

THÉORÈME 2.2. – *La structure tressée bisimpliciale des $\mathcal{W}^*(\Delta_\bullet)$ induit un tressage* global *sur l'algèbre différentielle graduée $\mathcal{W}^*(X)$. Son algèbre de cohomologie est naturellement isomorphe à la cohomologie usuelle de X à coefficients dans Λ.*

Comme dans la note précédente [3], nous pouvons définir aussi une version « stabilisée » de $\mathcal{W}^*(X)$ en remplaçant $\mathcal{W}^*(\Delta_r)$ par $\mathrm{colim}_n \mathcal{W}^*(\Delta_r)^{\otimes n}$, notée $\widehat{\mathcal{W}}^*(\Delta_r)$. L'ADG tressée correspondante $\mathrm{C}_\bullet(X) \nabla \widehat{\mathcal{W}}^*(\Delta_\bullet)$ est notée $\widehat{\mathcal{W}}^*(X)$. Elle est dite *spéciale* dans le sens où le noyau symétrique [4] de $\widehat{\mathcal{W}}^*(X)^{\otimes n}$ est quasi isomorphe à $\widehat{\mathcal{W}}^*(X)^{\otimes n}$.

3. Description algébrique du type d'homotopie en termes d'ADG tressées

Une application (entre d'autres du même type) de ce qui précède est le théorème suivant :

THÉORÈME 3.1. – *Soient X et Y deux ensembles simpliciaux connexes, nilpotents, et de type fini* [5]. *Supposons qu'il existe une suite en zigzag de quasi-isomorphismes d'ADG* tressées *spéciales* (*avec* $k = \mathbb{Z}$)

$$\widehat{\mathcal{W}}^*(X) \longrightarrow A \longleftarrow B \longrightarrow \cdots \longleftarrow \widehat{\mathcal{W}}^*(Y).$$

Alors X et Y ont le même type d'homotopie rationnel et le même type d'homotopie p-adique.

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1. Quantum differential forms on the affine line

1.1. Let k be a commutative ring. We define an extension Λ of k as the algebra of formal power series $\sum_{r=-\infty}^{s} \alpha_r q^r$, $\alpha_r \in k$, with the valuation defined by the least integer r such that $\alpha_{-r} \neq 0$ (and the associated ultrametric topology). Note that all the polynomials $n_q = 1 + q + \cdots + q^n$ are invertible in Λ. Let $\Omega^*(t)$ be the differential graded algebra [6] defined by $\Omega^0(t) = \Lambda[t]$, $\Omega^1(t) = \Lambda[t]\,\mathrm{d}t$, the differential $\mathrm{d} : \Omega^0(t) \to \Omega^1(t)$ being given by the formula $\mathrm{d}(t^n) = n_q t^{n-1}$. The multiplication in $\Omega^*(t)$ is described by the following rule:

$$[u \cdot \mathrm{d}v] \cdot w = u\bar{w} \cdot \mathrm{d}v$$

(the endomorphism $w \mapsto \bar{w}$ of the algebra $\Lambda[t]$ is defined by $f(t) \mapsto \bar{f}(t) = f(qt)$, as in [4]).

We have shown in [4] that $\Omega^*(t)$ is a braided DGA, with the following braiding on homogeneous elements (u and v being chosen as elements of $\Omega^0(t)$):

$$R(u \otimes v) = v \otimes u, \quad R(u\mathrm{d}v \otimes w) = \bar{w} \otimes u\mathrm{d}v,$$
$$R(u \otimes v\mathrm{d}w) = v\mathrm{d}w \otimes u + v(w - \bar{w}) \otimes \mathrm{d}u, \quad R(u\mathrm{d}v \otimes w\mathrm{d}t) = -\bar{w}\bar{t} \otimes u\mathrm{d}v.$$

1.2. Let $\varepsilon : \Omega^*(t) \to \Lambda$ be the augmentation defined by putting $t = 1$. The "natural" formula $R \cdot (\varepsilon \otimes 1) = (1 \otimes \varepsilon) \cdot R$ is then false. This prevented us to define in [3] a *global* braiding on $\Omega^*(X)$, the DGA of quantum differential forms on the simplicial set X of *polynomial type*, constructed from $\Omega^*(t)$, as a building block.

In order to avoid this difficulty, we consider now the algebra $\mathcal{W}^0(t)$ of formal *power series* $f(t) = \sum_{n=0}^{\infty} a_n t^n$, $a_n \in \Lambda$, such that, for all $s \geqslant 0$, $a_n q^{sn} \to 0$ when $n \to \infty$ and satisfy [7] the identity

$$\sum_{n=0}^{\infty} a_n q^{sn} = \sum_{n=0}^{\infty} a_n. \tag{E$_0$}$$

In the same spirit, we consider the Λ-module $\mathcal{W}^1(t)$ of formal differential forms of degree one $\omega(t) = \sum_{n=0}^{\infty} b_n t^n\,\mathrm{d}t$ such that, for all $s \geqslant 0$, $b_n q^{ns} \to 0$ when $n \to \infty$ and satisfy the identity

$$\sum_{n=0}^{\infty} b_n q^{sn} = 0. \tag{E$_1$}$$

It is not difficult to see that $\mathcal{W}^1(t)$ consists of formal differentials of elements in $\mathcal{W}^0(t)$ and that $\mathcal{W}^*(t) = \mathcal{W}^0(t) \oplus \mathcal{W}^1(t)$ is a DGA with a cohomology concentrated in degree 0, isomorphic to Λ. Moreover, $\mathcal{W}^*(t)$ is a braided DGA with formulas of the same type as above. For instance, we have $R(\omega(t) \otimes f(t)) = f(qt) \otimes \omega(t)$.

THÉORÈME 1.3. – *Let $\varepsilon : \mathcal{W}^*(t) \to \Lambda$ be the augmentation defined by either $t = 0$ or $t = 1$. Then we have the following commutative diagrams*:

$$\begin{array}{ccc} \mathcal{W}^*(t) \otimes \mathcal{W}^*(t) & \xrightarrow{R} & \mathcal{W}^*(t) \otimes \mathcal{W}^*(t) \\ \downarrow{\scriptstyle 1\otimes\varepsilon} & & \downarrow{\scriptstyle \varepsilon\otimes 1} \\ \mathcal{W}^*(t) \otimes \Lambda & \xrightarrow{R} & \Lambda \otimes \mathcal{W}^*(t) \end{array} \qquad \begin{array}{ccc} \mathcal{W}^*(t) \otimes \mathcal{W}^*(t) & \xrightarrow{R} & \mathcal{W}^*(t) \otimes \mathcal{W}^*(t) \\ \downarrow{\scriptstyle \varepsilon\otimes 1} & & \downarrow{\scriptstyle 1\otimes\varepsilon} \\ \mathcal{W}^*(t)\Lambda \otimes \mathcal{W}^*(t) & \xrightarrow{R} & \mathcal{W}^*(t) \otimes \Lambda \end{array}$$

(Note that the morphism R on the bottom row is just the flip.)

2. Quantum differential forms on simplices and simplicial sets

2.1. If $(x_0,\dots,x_r)$ are $(r+1)$ indeterminates, we define $\mathcal{W}^*(x_0,\dots,x_r)$ as the graded tensor product $\mathcal{W}^*(x_0)\otimes\cdots\otimes\mathcal{W}^*(x_r)$. Since the cohomology of each $\mathcal{W}^*(x_i)$ is concentrated in degree 0, the same property is true for the Λ-module $\mathcal{W}^*(x_0,\dots,x_r)$. If Δ_r is the standard r-simplex, we now define $\mathcal{W}^*(\Delta_r)$ as the kernel of the difference of the two obvious morphisms

$$\prod_i \mathcal{W}^*(x_0,\dots,\widehat{x_i},\dots,x_r) \rightrightarrows \prod_{i<j}\mathcal{W}^*(x_0,\dots,\widehat{x_i},\dots,\widehat{x_j},\dots,x_r).$$

THÉORÈME 2.2. – *The* $\mathcal{W}^*(\Delta_r)$ *defines a* semi-simplicial *differential graded algebra with the face maps induced by the evaluations* $x_i=1$. *Its cohomology is concentrated in degree* 0 *and isomorphic to* Λ (Poincaré's lemma). *Moreover, the obvious "braiding" obtained by suitable tensor products of the braiding* R *defined in* 1.2:

$$\mathcal{W}^*(x_0,\dots,x_r)\otimes\mathcal{W}^*(x_0,\dots,x_s)\longrightarrow\mathcal{W}^*(x_0,\dots,x_s)\otimes\mathcal{W}^*(x_0,\dots,x_r)$$

induces by a bisimplicial *braiding*

$$\mathcal{W}^*(\Delta_r)\otimes\mathcal{W}^*(\Delta_s)\longrightarrow\mathcal{W}^*(\Delta_s)\otimes\mathcal{W}^*(\Delta_r).$$

2.3. In order to define $\mathcal{W}^*(X)$ for any simplicial set X, we need a general construction on *semi*-simplicial and cosimplicial modules which we recall briefly [8]. The forgetful functor from simplicial Λ-modules to semi-simplicial Λ-modules $\mathrm{S}_\bullet$ admits a left adjoint L and it is well known that the natural morphism from $\mathrm{S}_\bullet$ to $L\mathrm{S}_\bullet$ is an homology isomorphism [8]. Dually, if $\mathrm{C}_\bullet$ is a semi-cosimplicial Λ-module, we can define its associated right adjoint cosimplicial Λ-module $R\mathrm{C}_\bullet$ and the morphism $R\mathrm{C}_\bullet\to\mathrm{C}_\bullet$ is a cohomology isomorphism.

Now, let $\mathrm{C}_\bullet$ be a *cosimplicial* Λ-module and $\mathrm{S}_\bullet$ a *semi-simplicial* Λ-module. We define the "Λ-realization" $\mathrm{C}_\bullet\nabla\mathrm{S}_\bullet$ as the following Λ-module: we take the direct sum of the various tensor products $C_n\otimes S_n$ and then its quotient by the usual identifications (using only faces) [9]. If M is any Λ-module, we have the following identities between various Hom's in different categories:

$$\begin{aligned}\mathrm{Hom}(\mathrm{C}_\bullet\nabla\mathrm{S}_\bullet,M)&\approx\mathrm{Hom}\big(\mathrm{C}_\bullet,\mathrm{Hom}(\mathrm{S}_\bullet,M)\big)\approx\mathrm{Hom}\big(\mathrm{C}_\bullet,R[\mathrm{Hom}(\mathrm{S}_\bullet,M)]\big)\\&\approx\mathrm{Hom}\big(\mathrm{C}_\bullet,\mathrm{Hom}(L\mathrm{S}_\bullet,M)\big)\approx\mathrm{Hom}(\mathrm{C}_\bullet,\nabla L\mathrm{S}_\bullet,M),\end{aligned}$$

where $\mathrm{C}_\bullet\nabla L\mathrm{S}_\bullet$ takes now into account the degeneracies. These identities show in particular that the obvious map $\mathrm{C}_\bullet\nabla\mathrm{S}_\bullet\to\mathrm{C}_\bullet\nabla L\mathrm{S}_\bullet$ is an isomorphism of Λ-modules.

2.4. We apply these general considerations to the cases $\mathrm{C}_\bullet=\mathrm{C}_\bullet(X)$, the usual *cochain* complex on a simplicial set X viewed as a *cosimplicial* Λ-module [10] and for $\mathrm{S}_\bullet$ the *semi-simplicial* differential graded algebra $\mathcal{W}^*(\Delta_\bullet)$ introduced above. With this data, using the considerations in 2.3, we define $\mathcal{W}^*(X)$ by the following formula:

$$\mathcal{W}^*(X)=\mathrm{C}_\bullet(X)\nabla\mathcal{W}^*(\Delta_\bullet)\cong\mathrm{C}_\bullet(X)\nabla L\big(\mathcal{W}^*(\Delta_\bullet)\big).$$

THÉORÈME 2.5. – *The bisimplicial braided DGA structure on the* $\mathcal{W}^*(\Delta_\bullet)$ *induces a globally defined braided DGA structure on* $\mathcal{W}^*(X)$. *Its cohomology is naturally isomorphic to the cohomology of* X *with coefficients in* Λ *with its usual multiplicative structure.*

Sketch of the proof. – For each r, the *chain* complex $\mathcal{W}^r(\Delta_\bullet)$ is shown to be acyclic by choosing an homotopy operator $\mathcal{W}^r(\Delta_{n-1}) \to \mathcal{W}^r(\Delta_n)$ defined by the multiplication with the function $1 - V(x_n)$, V being defined in the footnote 7. On the other hand, for each n, Poincaré's lemma is true for $\mathcal{W}^*(\Delta_n)$. Therefore, the method described by H. Cartan in [2] shows that $\mathcal{W}^*(X)$ defines the usual Λ-cohomology with the correct multiplicative structure. Moreover, since the braiding

$$\mathcal{W}^*(\Delta_\bullet) \otimes \mathcal{W}^*(\Delta_\bullet) \xrightarrow{R} \mathcal{W}^*(\Delta_\bullet) \otimes \mathcal{W}^*(\Delta_\bullet)$$

is bisimplicial with respect to the face operators, it induces a braiding on $\mathcal{W}^*(X) = \mathrm{C}_\bullet \nabla \mathcal{W}^*(\Delta_\bullet)$ as expected.

2.6. As a matter of fact, we can do much better: the functor $X \mapsto \mathcal{W}^*(X)$ defines a "braiding category" in the sense that there is a braiding

$$B_{X,Y} : \mathcal{W}^*(X) \otimes \mathcal{W}^*(Y) \longrightarrow \mathcal{W}^*(Y) \otimes \mathcal{W}^*(X)$$

defined for any two spaces X and Y, which satisfy obvious axioms, similar to the ones for braided DGA's.

2.7. As a concrete example, le X be a *finite* simplicial complex, imbedded in Δ^r for some r. This means that X is the union S_P of faces associated to some finite subsets P of $\{0, 1, \ldots, r\}$. Since each face P can be identified with Δ_s, for suitable s, Sullivan's definition [7] may be used here to define $\mathcal{W}^*(X)$ by gluing the $\mathcal{W}^*(P)$ along the various intersections $P \cap P'$. Let now Y be another subcomplex of Δ^r. The novel feature is then the braiding $B_{X,Y}$ above which is induced by "local" braiding

$$\mathcal{W}^*(P) \otimes \mathcal{W}^*(Q) \longrightarrow \mathcal{W}^*(Q) \otimes \mathcal{W}^*(P),$$

where P is a face of X and Q a face of Y. The compatibility of R with the face operators insure that R is globally defined.

2.8. As in the previous Note [3], we could define as well a "stabilized" version of $\mathcal{W}^*(X)$, replacing $\mathcal{W}^*(\Delta_r)$ by $\mathrm{colim}_n \mathcal{W}^*(\Delta_r)^{\otimes n}$, denoted by $\widehat{\mathcal{W}}^*(\Delta_r)$. We write $\widehat{\mathcal{W}}^*(X)$ for the corresponding braided differential graded algebra $\mathrm{C}_\bullet(X)\nabla\widehat{\mathcal{W}}^*(\Delta_r)$. It is a *special* braided DGA with the definition given in [4]: the symmetric kernel of $\widehat{\mathcal{W}}^*(X)^{\otimes n}$ is quasi-isomorphic to $\widehat{\mathcal{W}}^*(X)^{\otimes n}$.

THÉORÈME 2.9. – *The symmetric kernels of the $\widehat{\mathcal{W}}^*(X)^{\otimes n}$ define a neo-algebra which is quasi-isomorphic to the neo-algebra* [11] $\widehat{\Omega}^*(X)$ *defined in* [3].

3. Algebraic description of homotopy types via braided DGA's

The following theorem is an application (among others of the same type) of the previous considerations. Its proof is based on the main result of [3,5], and [6].

THEOREM 3.1. – *Let X and Y be two connected simplicial sets, nilpotent and of finite type* [12]. *Let us suppose there exists a zigzag sequence of quasi-isomorphisms of special* braided *DGA's* (*with $k = \mathbb{Z}$*)

$$\widehat{\mathcal{W}}^*(X) \longrightarrow A \longleftarrow B \longrightarrow \cdots \longleftarrow \widehat{\mathcal{W}}^*(Y).$$

Then X and Y have the same rational homotopy type and the same p-adic homotopy type for all p.

3.2. The braided structure of $\mathcal{W}^*(X)$ is not arbitrary: it is "commutative" in the braided sense, which means that R commutes with the multiplication of $\mathcal{W}^*(X)$. Therefore, we can "iterate" the bar-construction and obtain a complex which cohomolgy is the cohomology of the corresponding iterated loop space of X, as in [4]. This gives a method to relate homotopy groups to some kind of homological algebra linked with braided structures. We shall come back to this subject on another occasion.

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1 Plus pécisément de type « spécial », *cf.* § 2.

2 Un exemple fondamental (dû à Euler) est décrit dans la note 7, en bas de page.

3 ADG en bref.

4 *Voir* [4].

5 *Voir* la note 12 de bas de page.

6 called DGA in short.

7 The series defined by the Euler product $V(t) = \prod_{s=0}^{\infty}(1 - q^{-s}t)$ is a well-known example of an element of $\mathcal{W}^0(t)$. Note that this product is also $e_q(\lambda t)$, $\lambda = q/(1-q)$, where e_q is the quantum exponential defined by $e_q(u) = \sum u^n/n!$. Here, $n!$ denotes the "quantum factorial" defined as the product of the polynomials $(q^r - 1)/(q-1)$, where r runs from 1 to n.

8 We follow here the terminology of [8].

9 This construction is essentially due to A. Bousfield and D. Kan [1].

10 We could as well take the Godement flabby *cosimplicial* resolution of a sheaf of Λ-commutative algebras over *any* topological space X.

11 More precisely, in the definition of [3], one has to replace k by the algebra Λ.

12 This means that its Postnikov tower is choosen such that each fiber is of the type $K(G,n)$, where G is a finitely generated abelian group.

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