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Stabilizing and commuting cochains

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Abstract. As it is well known in K-theory, stabilization of matrices enables them to commute "up to homotopy". The purpose of this short paper is to describe an analogous philosophy for cochains on a space. It is in fact a direct application of a paper of Henri Cartan [1], together with a new idea of stabilization for cochains, similar to matrices. The application below may be also deduced from a paper of J. Halperin and J. Stasheff [2] by a quite different method. This paper is part of a joint project with P. Baum about the cohomology of homogeneous spaces. Since it has some independent interest, it might be useful to present it on its own right. © 2001 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Stabilisation et commutation des cochaînes

Résumé. *Comme il est bien connu en K-théorie, la stabilisation des matrices permet de les faire commuter « à homotopie près ». Dans cette Note, nous décrivons une philosophie analogue pour les cochaînes sur un espace. Celle-ci est en fait une conséquence directe d'un article de Henri Cartan* [1] *et d'une nouvelle idée de stabilisation des cochaînes, analogue à celle de la stabilisation des matrices. Nous donnons aussi une application qui peut être déduite également d'un article de J. Halperin et J. Stasheff* [2] *par une méthode entièrement différente. Cet article fait partie d'un projet de recherche avec P. Baum sur la cohomologie des espaces homogènes. Puisqu'il a un intérêt en lui-même, nous avons préféré le publier indépendamment de cet objectif.* © 2001 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

THEOREM. – *Let k be a commutative ring, X an arbitrary space and $\mathrm{C}^*(X)$ the differential graded algebra* (*DGA*) *of k-cochains on X. Then there exists a functorially defined DGA $\widehat{\mathrm{C}}^*(X)$ and a DGA-quasi isomorphism $\mathrm{C}^*(X) \to \widehat{\mathrm{C}}^*(X)$ with the following property. For any countable sequence of elements $\{x_i\}$ in the cohomology $\mathrm{H}^*(X)$* (*with k-coefficients*)*, we can find cochain representatives $\widehat{x}_i$ of the x_i in $\widehat{\mathrm{C}}^*(X)$ such that the $\widehat{x}_i$'s commute with each other* (*in the graded sense*)*. The DGA $\widehat{\mathrm{C}}^*(X)$ is called the "stabilization" of $\mathrm{C}^*(X)$.*

Proof. – Without loss of generality we may assume that X is a simplicial set. Let us consider the free k-module $\mathrm{C}^r(\Delta_m)$ with basis the maps from $[r]$ to $[m]$, where $[p]$ denotes in general the finite set $\{0, \ldots, p\}$. It is well known [1] that the $\mathrm{C}^*(\Delta_\natural)$ define a simplicial DGA where $*$ denotes the degree and $\natural$ the simplicial dimension. Moreover, $\mathrm{C}^*(X)$ is quasi-isomorphic to the k-module $\mathrm{Mor}\,(X_\natural, \mathrm{C}^*(\Delta_\natural))$ of simplicial maps

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from $X_\natural$ to $\mathrm{C}^*(\Delta_\natural)$. The essential properties of the simplicial differential graded algebra $\mathrm{C}^*(\Delta)$ used in [1] are the following:

1. POINCARÉ'S LEMMA. – *The complex*

$$0 \longrightarrow \mathrm{C}^0(\Delta_m) \longrightarrow \mathrm{C}^1(\Delta_m) \longrightarrow \mathrm{C}^2(\Delta_m) \longrightarrow \cdots$$

has trivial cohomology, except in degree 0*, where we find* k (*as a trivial simplicial module*).

2. EXTENSION'S LEMMA. – *For a fixed* r*, the homotopy groups of the simplicial module* $\mathrm{C}^r(\Delta_\natural)$ *are reduced to* 0.

Now, the main idea of the proof (already used in [3]) is to *stabilize* the simplicial DGA $\mathrm{C}^*(\Delta_\natural)$ by introducing $\widehat{\mathrm{C}}^*(\Delta_\natural)$ as the following inductive limit

$$\widehat{\mathrm{C}}^*(\Delta_m) = \mathrm{colim}_n\big[\mathrm{C}^*(\Delta_m)\big]^{\otimes n},$$

where the map from $\big[\mathrm{C}^*(\Delta_m)\big]^{\otimes n}$ to $\big[\mathrm{C}^*(\Delta_m)\big]^{\otimes (n+1)}$ is given by $\omega \mapsto \omega \otimes 1$. The Künneth spectral sequence shows that Poincaré's lemma is still true for $\widehat{\mathrm{C}}^*(\Delta_m)$. Moreover, from the Eilenberg–Zilber theorem, the (simplicial) homotopy groups of $\widehat{\mathrm{C}}^*(\Delta_\natural)$ are the homology groups of the chain complex $\mathrm{colim}_n\big[\mathrm{C}^*(\Delta_\natural)\big]^{\otimes n}$, where the homology differential is defined by the alternating sum of the face maps. By the Künneth spectral sequence again, we see that the extension lemma is also true for the stabilized differential graded algebra $\widehat{\mathrm{C}}^*(\Delta_\natural)$. Therefore, the cochain complex

$$\widehat{\mathrm{C}}^*(X) = \mathrm{Mor}\big(X_\natural, \widehat{\mathrm{C}}^*(\Delta_\natural)\big)$$

is naturally quasi-isomorphic to $\mathrm{C}^*(X) = \mathrm{Mor}\big(X_\natural, \mathrm{C}^*(\Delta_\natural)\big)$.

More precisely, we have obvious quasi-isomorphisms of simplicial DGA's

$$\alpha_i : \mathrm{C}^*(\Delta_\natural) \longrightarrow \widehat{\mathrm{C}}^*(\Delta_\natural)$$

defined by $\alpha_i(\omega) = 1 \otimes \cdots \otimes \omega \otimes \cdots$, where ω is located at the ith spot. Each of them induces a specific quasi-isomorphism $\overline{\alpha}_i$ between $\mathrm{C}^*(X)$ and $\widehat{\mathrm{C}}^*(X)$. By the very definition of $\widehat{\mathrm{C}}^*(X)$, the elements in the image of $\overline{\alpha}_i$ commute with those in the image of $\overline{\alpha}_j$ for $i \neq j$. The remaining part of the proof is now as follows: let $y_1, y_2, \ldots$ be cochain representatives of the cohomology classes $x_1, x_2, \ldots$ of the statement in the theorem. We define now $\widehat{x}_i$ as $\overline{\alpha}_i(y_i)$. It is clear that the $\widehat{x}_i$'s commute with each other.

APPLICATION. – *Let* X *be a space which cohomology* $\mathrm{H}^*(X)$ *is a polynomial algebra with a countable set of generators, say*

$$\mathrm{H}^*(X) \cong k[x_1, \ldots, x_n, \ldots].$$

Then the differential graded algebra $\mathrm{C}^*(X)$ *is related to the cohomology algebra* $\mathrm{H}^*(X)$ *— viewed as a DGA with* 0 *differential — by a zigzag sequence of two quasi-isomorphisms.* [1]

Proof. – According to the theorem, there exists a morphism $\mathrm{H}^*(X) \to \widehat{\mathrm{C}}^*(X)$ of DGA's, sending the x_i's to the $\widehat{x}_i$'s. It is clearly the quasi-isomorphism requested.

[1] This statement is obvious if the cohomology is a polynomial algebra with *one* generator. Therefore, there is no contradiction with the existence of non-trivial Steenrod operations.

References

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