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A Descent Theorem in Topological K-Theory

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Abstract. We prove the Lichtenbaum–Quillen conjecture in the topological context: in other words, real K-theory can be deduced from complex K-theory via the usual descent spectral sequence. More precise results are proved, however, and new applications are stated. The main ingredients in the proof are Atiyah's *KR*-theory and the definition of higher K-groups via Clifford algebras.

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1. Introduction

Let A be a real Banach algebra. We define its 'K-theory space' $\mathcal{K}(A)$ as the direct limit

$$\operatorname*{colim}_n \ \mathrm{Proj}_n(A^{2n}),$$

where $\mathrm{Proj}_n(A^{2n})$ denotes the space of projection operators in A^{2n} with the $2n \times 2n$ matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

as the base point. This space is homotopically equivalent (noncanonically*) to the product

$$\mathcal{K}(A) = K_0(A) \times \ \mathrm{BGL}\,(A),$$

where $\mathrm{GL}(A)$ and $\mathrm{BGL}(A)$ have the usual topology and $K_0(A)$ has the discrete topology. Therefore, its homotopy groups are the topological K-groups of the Banach algebra A, which are periodic of period 8 (Bott). If A can be provided with a complex structure, these homotopy groups are in fact periodic of period 2.

2. If $A' = A \otimes_{\mathbf{R}} \mathbf{C}$ is the complexification of A, the group $G = \mathbf{Z}/2$ acts on $\mathcal{K}(A')$ by complex conjugation (we keep $G = \mathbf{Z}/2$ through out the paper) and we have a natural map

* As it was pointed out to me by B. Kahn.

$$\sigma : \mathcal{K}(A) \longrightarrow \mathcal{K}(A')^{hG}.$$

Here, Y^{hG} generally denotes the homotopy fixed point set of the G-space Y. More precisely, Y^{hG} is the space of (continuous) sections of the Borel fibration

$$\begin{array}{c} EG \times_G Y \\ \downarrow \\ BG. \end{array}$$

It is easy to see that Y^{hG} is also the space of equivariant maps $EG \longrightarrow Y$. The purpose of this paper is to prove the following 'descent theorem'*.

3. THEOREM. *The map* $\sigma : \mathcal{K}(A) \longrightarrow \mathcal{K}(A')^{hG}$ *defined above is a homotopy equivalence.*

3.1. APPLICATIONS

One application of this theorem we have in mind is the Baum–Connes conjecture in the real case (this will be included in a forthcoming paper 'BKR' by Paul Baum, John Roe, and the author). The classical Baum–Connes conjecture (in the complex case) states that the index map

$$\mu(\Gamma) : K_j^{\Gamma}(\underline{E}\Gamma) \longrightarrow K_j(C_r^*(\Gamma))$$

is an isomorphism for a discrete (countable) group Γ (where $j = 0, 1$ mod. 2). In the BKR paper, we show, using Theorem 3 as one of the essential ingredients, that if $\mu(\Gamma)$ is an isomorphism, the real analog

$$\mu_{\mathbf{R}}(\Gamma) : KO_j^{\Gamma}(\underline{E}\Gamma) \longrightarrow K_j(C_r^*(\Gamma, \mathbf{R}))$$

is also an isomorphism for $j = 0, 1, \ldots, 7$, mod. 8.

Another application is a comparison theorem between algebraic K-theory and *KR*-theory of real varieties (related to the Quillen–Lichtenbaum conjecture for such varieties). This will appear in a forthcoming paper by Charles Weibel and the author.

4. The first step of the proof of Theorem 3 is to remark the following basic fact: if Y is the product $X \times X$ with the action of G switching the factors, Y^{hG} may be identified with the space of maps from *EG* to X, a space which is homotopically equivalent to X, since *EG* is contractible. Therefore, the map $\sigma : X \longrightarrow (X \times X)^{hG}$ is a homotopy equivalence. From this remark, we immediately deduce the following proposition:

* This theorem is well known if A is the Banach algebra of real numbers: cf. [2], Lemma 3.5, for instance. Other proofs have been given by J. Lannes (unpublished) and B. Kahn in a joint work with Hinda Hamraoui (in preparation). On the other hand, the statement seems new if A is the algebra of quaternions.

5. PROPOSITION. *The theorem is true if A is the underlying real algebra of a complex Banach algebra.*

Proof. It is well known and easy to see that the homomorphism $A' = A \otimes_{\mathbf{R}} \mathbf{C} \longrightarrow A \times A$ defined by $a \otimes z \mapsto (az, a\bar{z})$ is an isomorphism, the complex conjugation switching the factors. Therefore, the space $\mathcal{K}(A')$ is just the product $\mathcal{K}(A) \times \mathcal{K}(A)$ and the proposition follows from the considerations in Section 4.

The paper is now devoted to reducing the theorem to this easy case, using in an essential way, the *KR*-theory of Atiyah [1]. The main ingredient in the proof is mentioned at the end of Section 7.

6. *KR*-Theory of Banach Algebras

Let us consider a compact space X provided with an involution $x \mapsto \bar{x}$, following the notation used by Atiyah [1]. We write $A(X)$ for the Banach algebra of continuous functions $f\colon X \longrightarrow A'$ such that $f(\bar{x}) = \overline{f(x)}$, the complex conjugate of $f(x)$. If A is the field of real numbers and X an arbitrary G-space, it is easy to see that the K-theory of $A(X)$ is isomorphic to Atiyah's *KR(X)*. If X is a locally compact space, we may extend this definition by choosing $A(X)$ to be the space of continuous functions (with values in A') which go to 0 when x goes to ∞ (with the same conjugation condition). Note that if the involution on X is trivial, $A(X)$ is just the usual Banach algebra of continuous functions on X with values in A. On the other hand, if X is a space with 2 points which are switched by the involution, $A(X)$ is isomorphic to A'.

7. The Role of Clifford Algebras

In general, we define $S^{p,q}$ (resp. $D^{p,q}$) as the sphere (resp. the ball) of $\mathbf{R}^{p+q}$ with the involution induced by $(x_1, \ldots, x_p, y_1, \ldots, y_q) \mapsto (-x_1, \ldots, -x_p, y_1, \ldots, y_q)$ on $\mathbf{R}^{p+q}$. For $p > q$, the locally compact space $S^{p,0} - S^{q,0}$ is G-homeomorphic to $S^{p-q,0} \times \mathbf{R}^{q,0}$ and we therefore have the following exact sequence of Banach algebras:

$$0 \longrightarrow A(S^{p-q,0})(\mathbf{R}^{q,0}) \longrightarrow A(S^{p,0}) \longrightarrow A(S^{q,0}) \longrightarrow 0.$$

On the other hand, using the Clifford algebra definition of the higher K-groups, it has been proved in [3] that for any Banach algebra with unit Λ, we have natural isomorphisms

$$K^{p,0}(\Lambda) \cong K(\Lambda(\mathbf{R}^{p,0})) \cong K^p(\Lambda) \cong K(\Lambda(\mathbf{R}^{0,8k-p}))$$

for k large enough. In this formula $K^{p,q}(\Lambda)$ denotes in general the Grothendieck group of the 'restriction of scalars' function $\mathcal{P}(C^{p,q+1} \otimes \Lambda) \longrightarrow \mathcal{P}(C^{p,q} \otimes \Lambda)$. Here $C^{p,q}$ is the standard Clifford algebra of $\mathbf{R}^{p+q}$ provided with the quadratic form

$$-(x_1)^2 - \cdots - (x_p)^2 + (x_{p+1})^2 + \cdots + (x_{p+q})^2$$

and $\mathcal{P}(B)$ is the category of finitely generated projective B-modules.

In more modern and accurate homotopical terms, one may say alternatively that the homotopy fiber of the map

$$\mathcal{K}(C^{p,1} \otimes \Lambda) \longrightarrow \mathcal{K}(C^{p,0} \otimes \Lambda)$$

is also the homotopy fiber $\mathcal{F}$ of the map

$$\mathcal{K}(\Lambda) \cong \mathcal{K}(\Lambda(D^{p,0}) \longrightarrow \mathcal{K}(\Lambda(S^{p,0})).$$

One basic theorem proved in [3], Section 3.4, is the following: there is a natural homotopy equivalence $\Omega(\mathcal{K}(\Lambda(S^{p,0})) \cong \mathcal{F} \times \Omega(\mathcal{K}(\Lambda))$ if $p \geqslant 3$. In other words, using Bott periodicity, we have a homotopy equivalence

$$\mathcal{K}(\Lambda(S^{p,0})) \cong \mathcal{K}(\Lambda) \times \Omega^{8k-p-1}(\mathcal{K}(\Lambda)) \cong \mathcal{K}(\Lambda) \times \Omega^{-p-1}(\mathcal{K}(\Lambda))$$

if $p \geqslant 3$ and $8k \geqslant p+1$ (with a slight abuse of notations, we write $\Omega^{-n}(\mathcal{K}(\Lambda))$ for $\Omega^{-n+8k}(\mathcal{K}(\Lambda))$, k large enough). More precisely, we have a homotopy fibration

$$\Omega^{-p}(\mathcal{K}(\Lambda)) \longrightarrow \mathcal{K}(\Lambda(D^{p,0})) \longrightarrow \mathcal{K}(\Lambda(S^{p,0}))$$

and the first arrow is induced by the cup-product with η^p, where η is the genererator of $\pi_1(\mathcal{K}(\mathbf{R})) \cong \mathbf{Z}/2$. It is well known that $\eta^p = 0$ when $p \geqslant 3$ and, therefore, the fisrt arrow is null-homotopic in this case. What is proved in [3], Section 3.4, is slightly more precise: there is a natural splitting $\mathcal{K}(\Lambda(S^{p,0})) \longrightarrow \mathcal{K}(\Lambda(D^{p,0}))$ from which we deduce the homotopy decomposition of $\mathcal{K}(\Lambda(S^{p,0}))$ mentioned above.

8. Proof of the Descent Theorem

We first prove by induction on $p (1 \leqslant p \leqslant 3)$ that σ induces a homotopy equivalence

$$\sigma_p \colon \mathcal{K}(A(S^{p,0})) \longrightarrow \mathcal{K}(A'(S^{p,0}))^{hG}.$$

In this notation $A'(Z)$ simply means the algebra of continuous functions on Z with values in A': $A'(Z)$ is of course the complexification of $A(Z)$. For $p = 1$, this has already been shown in Section 5 since $A(S^{1,0}) \cong A'$ and $(A')' \cong A' \times A'$. For $p = 2$, we have the following commutative diagram of homotopy fibrations (cf. Section 7):

$$\begin{array}{ccccc}
\mathcal{K}(A(S^{1,0})(\mathbf{R}^{1,0})) & \longrightarrow & \mathcal{K}(A(S^{2,0})) & \longrightarrow & \mathcal{K}(A(S^{1,0})) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{K}(A'(S^{1,0})(\mathbf{R}^{1,0}))^{hG} & \longrightarrow & \mathcal{K}(A'(S^{2,0}))^{hG} & \longrightarrow & \mathcal{K}(A'(S^{1,0}))^{hG}.
\end{array}$$

Since the two extreme vertical maps are homotopy equivalences for any Banach algebra A, it follows that the second vertical map is also a homotopy equivalence.

For $p = 3$, the same argument shows that σ_3 is also a homotopy equivalence.

Now, according to Section 7 again, we have a commutative diagram of *canonically split* homotopy fibrations

$$\begin{array}{ccccccccc} * & \longrightarrow & \mathcal{K}(A) & \longrightarrow & \mathcal{K}(A(S^{3,0})) & \longrightarrow & \Omega^4(\mathcal{K}(A)) & \longrightarrow & * \\ & & \downarrow & & \downarrow & & \downarrow & & \\ * & \longrightarrow & \mathcal{K}(A')^{hG} & \longrightarrow & \mathcal{K}(A'(S^{3,0}))^{hG} & \longrightarrow & \Omega^4(\mathcal{K}(A'))^{hG} & \longrightarrow & *. \end{array}$$

Since the middle map is a homotopy equivalence by our previous induction, the descent theorem follows immediately.

9. From the previous considerations, we can also extract a spectral sequence converging to $K^{p+q}(A(S^{3,0})) = K^{p+q}(A) \oplus K^{p+q+4}(A)$. This may be done by general methods of equivariant cohomology. More precisely, using the elementary ideas developed in [4] Section 1, one can prove that the Grothendieck group $K(A(X))$ is naturally isomorphic to the group of homotopy classes of G-equivariant maps $X \longrightarrow \mathcal{K}(A')$. In other words, the K-theory space $\mathcal{K}(A(X))$ may be identified – up to homotopy – with the function space of G-equivariant maps $X \longrightarrow \mathcal{K}(A')$. In particular, one has the homotopy equivalences

$$\mathcal{K}(A) \times \Omega^4(\mathcal{K}(A)) \cong \mathcal{K}(A(S^{3,0})) \cong \{S^{3,0}, \mathcal{K}(A')\}^G,$$

where $\{\,,\,\}$ means function space. Since $\{S^{3,0}, \mathcal{K}(A')\}^G$ is also the space of sections of the Borel fibration

$$S^{3,0} \times_G \mathcal{K}(A') \longrightarrow S^{3,0}/G = RP^2,$$

one has the following theorem.

10. THEOREM. *There is a spectral sequence with* E_2 *term* $H^p(RP_2; K^q(A'))$ *converging to* $K^{p+q}(A) \oplus K^{p+q+4}(A)$.

In this spectral sequence, we have, of course, $0 \leqslant p \leqslant 2$ and q defined mod 4. There is at most one nonzero differential which is $d_2\colon H^0(RP^2; K^q(A')) \longrightarrow H^2(RP^2; K^{q-1}(A'))$, where the groups $K^*(A')$ define a local coefficient system over RP^2.

11. EXAMPLE. If we come back to Atiyah's *KR*-theory of a space X, one has therefore a spectral sequence with E_2 term $H^p(RP^2; KU^q(X))$ converging to $KR^{p+q}(X) \oplus KR^{p+q+4}(X)$.

12. *Remark.* Let $\mathcal{KST}(A)$ be the homotopy fiber of the map $1-t\colon \mathcal{K}(A') \longrightarrow \mathcal{K}(A')$, where t is the complex conjugation. As it was essentially shown by Atiyah (see also [4], p. 178), one has a homotopy fibration

$$\mathcal{K}(A') \longrightarrow \mathcal{K}(A) \times \mathcal{K}(A'') \longrightarrow \mathcal{KST}(A).$$

The associated homotopy exact sequence essentially gives rise to the same information as the spectral sequence above.

13. Generalization

The previous considerations may be used in a variety of contexts (for example, to prove the Thom isomorphism in *real* K-theory, starting from the Thom isomorphism in complex K-theory) and it might be useful for the future to extract its main ideas. For this, we should consider two functors from Banach algebras to spaces, called, for instance, $F(A)$ and $G(A)$ (in our example, $\mathcal{K}(A)$ and $\mathcal{K}(A')^{h\mathbf{Z}/2}$, respectively), and a natural transformation $\alpha\colon F(A)\longrightarrow G(A)$.

The claim is now the following: under suitable hypothesis, if $F(A') \cong G(A')$ by this natural transformation, then $F(A) \cong G(A)$ (isomorphisms are taken in the homotopy category).

Following our previous arguments (Sections 7 and 8), we see by inspection that it is enough to verify the following three conditions:

(1) F and G satisfy the Mayer–Vietoris axiom (Cartesian squares of Banach algebras give rise by F and G to homotopy Cartesian squares).
(2) $F(A(S^{1,0}))$ (resp. $G(A(S^{1,0}))$) is naturally isomorphic to $F(A')$ [resp. $G(A')$] in a way compatible with α.
(3) For p large enough, and in a way compatible with α, $F(A)$ (resp. $G(A)$) is a natural direct summand of $F(A(S^{p,0}))$ (resp. $G(A(S^{p,0}))$) through the map induced by the obvious ring homomorphism $A\longrightarrow A(S^{p,0})$.

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