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Algebraic and real K-theory of Real varieties

Max Karoubi[a], Charles Weibel[b,*,1,2]

[a] Institut de Mathématiques, Université Paris 7, 2, place Jussieu, 75251 Paris, cedex 05, France
[b] School of Mathematics, Institute for Advanced Study, Princeton, NJ 08540 USA

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Abstract

An algebraic variety defined over the real numbers has an associated topological space with involution, and algebraic vector bundles give rise to Real vector bundles. We show that the associated map from algebraic K-theory to Atiyah's Real K-theory is, after completion at two, an isomorphism on homotopy groups above the dimension of the variety.

Suppose that V is a quasi-projective variety, defined over the real numbers $\mathbb{R}$. Complex conjugation defines an involution on the underlying topological space $V(\mathbb{C})$ of complex points. Every algebraic vector bundle on V induces a complex vector bundle E on $V(\mathbb{C})$, and conjugation gives E the structure of a Real vector bundle in the sense of Atiyah [1]. Passing to Grothendieck groups, this induces a homomorphism α_0: $K_0(V) \to KR^0(V(\mathbb{C}))$, where KR^0 is Atiyah's Real K-theory [1]. This may be extended to natural maps α_n: $K_n(V) \to KR^{-n}(V(\mathbb{C}))$ for all n; see Section 1.

In this paper we show that the maps $K_n(V;\mathbb{Z}/m) \to KR^{-n}(V(\mathbb{C});\mathbb{Z}/m)$ are isomorphisms for all $n \geqslant \dim(V)$ and all nonsingular V, at least when $m = 2^\nu$ (see 4.8). The corresponding assertion for m odd is a special case of the Quillen–Lichtenbaum conjecture (see 4.3). Our key descent result (5.1 and/or 6.1) is a comparison of the K-theory space $\mathbf{K}(V)$ with the homotopy fixed point set of the K-theory space $\mathbf{K}(V_{\mathbb{C}})$ when V has no real points.

For curves we can do one better: the map is an isomorphism for all $n \geqslant 0$ (see 4.12), and m can be any integer (see 4.4). In the appendix, we show that we can also do better for the coordinate rings of

* Corresponding author. Tel.: +1-732-445-3999; fax: +1-732-445-5530.
E-mail addresses: karoubi@math.jussieu.fr (M. Karoubi), weibel@math.rutgers.edu (C. Weibel).
1 Current address: Rutgers University, New Brunswick, NJ 08903, USA.
2 Weibel partially supported by NSF grants.

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spheres. We prove that

$$K_n\left(\mathbb{R}[x_0,\ldots,x_d]/\left(\sum x_j^2=1\right);\mathbb{Z}/m\right)\xrightarrow{\cong}KO^{-n}(S^d;\mathbb{Z}/m),\quad n\geqslant 0,\ m=2^{\nu}.$$

The cases $n=0,1,2$ of this result were studied by Milnor [20] and others.

The key topological result we need is a Real version of the Riemann–Roch theorem: if $f: V\to Y$ is a proper map then α is compatible with the direct image map f_*. (See Theorem 3.7.) The key motivic result we need is Voevodsky's theorem [36], which implies that the Quillen–Lichtenbaum conjecture holds for complex varieties at the prime 2; see [24,29]. We will also need the Postnikov style tower of Friedlander and Suslin [11].

Our results were motivated by the calculations in [23], which in hindsight showed that if V is a smooth real curve and $m=2^{\nu}$ then $K_n(V;\mathbb{Z}/m)\cong KR^{-n}(V(\mathbb{C});\mathbb{Z}/m)$ for all $n\geqslant 0$. (See 4.12 below for $n=0$.) This result has been used by Friedlander and Walker in [12] to prove that the semi-topological K-theory of a smooth real curve agrees with its KR-theory. Using this, they provide an alternative proof of our main theorem 4.8; see [12].

1. Comparing *K* and *KR*

We begin by recalling the definition of KR of a Real space, taken from [1]. Let X be a *Real space*, i.e., a topological space with an involution (written $x\mapsto\bar{x}$). By a *Real vector bundle* over X we mean a complex vector bundle E over X which is equipped with an involution compatible with $E\to X$, such that for every $z\in\mathbb{C}$ and $e\in E$ we have $\overline{ze}=\bar{z}\cdot\bar{e}$. As with complex vector bundles, we define $KR(X)$ to be the Grothendieck group of Real vector bundles under $\oplus$. In fact the (topological) category $\mathscr{ER}(X)$ of Real vector bundles is an additive category with a fiberwise notion of short exact sequence.

Atiyah's definition is motivated by analogy with algebraic geometry. If V is an algebraic variety defined over $\mathbb{R}$, its complex points form a Real space $V(\mathbb{C})$, whose involution is induced by complex conjugation. Moreover, any algebraic vector bundle on V induces a Real vector bundle on $V(\mathbb{C})$. Thus we have an additive functor from the category $\mathbf{VB}(V)$ of algebraic vector bundles on V to $\mathscr{ER}(V(\mathbb{C}))$, and an associated map $K_0(V)\to KR(V(\mathbb{C}))$.

When X is compact, Atiyah also constructs graded cohomology groups $KR^n(X)$ having an 8-periodicity isomorphism $KR^n(X)\cong KR^{n-8}(X)$ [1, 3.10]. For $n>0$, $KR^{-n}(X)=\widetilde{KR}(\Sigma^n X_+)$, where Σ denotes suspension. There are also bigraded cohomology groups $KR^{p,q}(X)=KR(X\times B^{p,q},X\times S^{p,q})$. Here $B^{p,q}$ and $S^{p,q}$ are the unit ball and unit sphere in $\mathbb{R}^{p,q}=\mathbb{R}^q\oplus i\mathbb{R}^p$, the real vector space with involution. There is also a $(1,1)$ periodicity $KR^{p,q}(X)\cong KR^{p+1,q+1}(X)$, arising from a Bott element in $KR^{1,1}(\text{point})=KR(\mathbb{CP}^2)$, and in fact $KR^{p-q}(X)\cong KR^{p,q}(X)$ for all p and q.

We first dispense with the case in which X is a compact space with involution. Consider the Banach algebra $A(X)$ of continuous functions $f\colon X\to\mathbb{C}$ such that $f(\bar{x})=\overline{f(x)}$ for all $x\in X$. The space $\Gamma(E)$ of global sections of a Real vector bundle E has the natural structure of the complexification of a projective $A(X)$-module. Since $A(X)\otimes_{\mathbb{R}}\mathbb{C}$ is the Banach algebra $\mathbb{C}(X)$ of all continuous maps $X\to\mathbb{C}$, Swan's theorem for complex bundles on X yields an equivalence between $\mathscr{ER}(X)$ and the topological category $\mathbf{P}A(X)$ of finitely generated projective $A(X)$-modules. It is well known that $KR^{-n}(X)$ is just the topological K-theory of the Banach category $\mathscr{ER}(X)$; see [16, Ex. III.7.14] or [17]. It follows (Ex. III.7.16 of [16]) that the topological K-theory of the Banach algebra

$A(X)$ is isomorphic to Atiyah's KR-theory of X. Using this identification, the following result is proven in [18].

Theorem 1.1. *If X is a compact Real space then the spectrum $\mathbf{KR}(X) = \mathbf{K}^{\mathrm{top}}\mathcal{ER}(X)$ is homotopy equivalent to the homotopy fixed point set $\mathbf{KU}(X)^{hG}$ of the action of the group $G = \mathbb{Z}/2$ on the topological K-theory spectrum $\mathbf{KU}(X)$.*

The topological K-theory spectrum of $\mathcal{ER}(X)$ may also be constructed using Quillen's Q-construction (or even Waldhausen's $S.$ construction); $Q\mathcal{ER}(X)$ is a topological category, and if $B^{\mathrm{top}}Q\mathcal{ER}(X)$ is its topological classifying space then $\Omega B^{\mathrm{top}}Q\mathcal{ER}(X)$ is the zeroth space of the spectrum $\mathbf{KR}(X)$.

Example 1.2. When V is a projective variety over $\mathbb{R}$, its space $V(\mathbb{C})$ of complex points of V is compact. Its algebraic K-theory spectrum $\mathbf{K}(V)$ is defined by applying Quillen's ΩBQ-construction to the category $\mathbf{VB}(V)$ (equipped with the usual exact sequences).

If we take the K-theory of the exact functors $\mathbf{VB}(V) \to \mathbf{P}(A) \cong \mathcal{ER}(V(\mathbb{C}))$, where $A = A(V(\mathbb{C}))$, we obtain natural homomorphisms

$$\alpha_n\colon K_n(V) \to K_n(A) \to K_n^{\mathrm{top}}(A) \cong KR^{-n}(V(\mathbb{C})), \quad n \geqslant 0.$$

In fact, we may take the (discrete and topological) K-theory spectra $\mathbf{K}(V)$, etc. to get maps of ring spectra (see [40])

$$\alpha\colon \mathbf{K}(V) \to \mathbf{K}^{\mathrm{alg}}(A) \to \mathbf{K}^{\mathrm{top}}(A) \cong \mathbf{K}^{\mathrm{top}}\mathcal{ER}(V(\mathbb{C})) = \mathbf{KR}(V(\mathbb{C}))$$

which induce the homomorphism α_n on the nth homotopy groups. This topological construction has the advantage that we automatically obtain homomorphisms for K-theory with finite coefficients $\mathbb{Z}/m$, viz., $K_n(V;\mathbb{Z}/m) \to KR^{-n}(V(\mathbb{C});\mathbb{Z}/m)$.

Since $V(\mathbb{C})$ is also the underlying space of the complex variety $V_{\mathbb{C}} = V \times_{\mathbb{R}} \mathrm{Spec}(\mathbb{C})$, this presentation also makes it clear that we have a homotopy commutative square:

$$\begin{array}{ccc} \mathbf{K}(V) & \xrightarrow{\alpha} & \mathbf{KR}(V(\mathbb{C})) \\ \downarrow & & \downarrow \\ \mathbf{K}(V_{\mathbb{C}}) & \xrightarrow{\alpha'} & \mathbf{KU}(V(\mathbb{C})). \end{array} \tag{1.2.0}$$

Friedlander and Walker have given an independent construction of the map α in [12, 4.1].

It is sometimes useful to have an equivariant triangulation of $V(\mathbb{C})$. If Z is a closed subvariety, we would also like $Z(\mathbb{C})$ to be a subcomplex. The existence of such an equivariant triangulation of (V,Z) is a special case of a far more general result, which we learned from Mark Goresky. We believe that it was first proven by Lellmann in [19].

Real Triangulation Theorem 1.3. *Let V be a projective variety over $\mathbb{R}$, and let $\{Z_i\}$ be a finite set of closed subvarieties of V. Then there is a triangulation of $X = V(\mathbb{C})$ so that each $Z_i(\mathbb{C})$ is a closed union of simplices. Moreover complex conjugation permutes the simplices in this triangulation of X.*

Proof. Consider the quotient space $V(\mathbb{C})/G$ of $V(\mathbb{C})$ by the Galois group $G = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. It suffices to find a triangulation of $V(\mathbb{C})/G$ so that each $Z_i(\mathbb{C})/G$ is a subcomplex, since the lift of the triangulation to V will have the desired properties. Moreover, by embedding V in $\mathbb{P}^m_{\mathbb{R}}$ for some m, we may suppose that $V = \mathbb{P}^m_{\mathbb{R}}$ and $V(\mathbb{C}) = \mathbb{CP}^m$.

Embed the quotient space $\mathbb{CP}^m/G$ into real Euclidean space $\mathbb{R}^M$, so that the projection $f: \mathbb{CP}^m \to \mathbb{R}^M$ is real algebraic. (This is not hard, and is left as an exercise.) By definition, the quotients $\mathbb{CP}^m/G$ and $Z_i(\mathbb{C})/G$ are subanalytic subsets of $\mathbb{R}^M$. By the Triangulation Theorem of Hardt [14] and Hironaka [15], there is a triangulation of $\mathbb{R}^M$ so that $\mathbb{CP}^m/G$ and each $Z_i(\mathbb{C})/G$ are subcomplexes, as desired. □

In general, if L is a polyhedron and K is a closed union of simplices, then the complement $L - K$ not only has the induced triangulation by open simplices (in the sense of [15]) but also has a finite-dimensional triangulation, obtained by recursively subdividing its open simplices. Clearly this procedure works in the equivariant setting as well.

Now every affine variety, and more generally every quasi-projective variety, has the form $V = \bar{V} - Z$ for some projective variety $\bar{V}$ and some closed subvariety Z. Thus the Real Triangulation Theorem 1.3 yields the following consequence.

Corollary 1.4. *Let V be a quasi-projective variety over $\mathbb{R}$, and W a closed subvariety. Then $V(\mathbb{C})$ has a finite-dimensional equivariant triangulation. Moreover, $W(\mathbb{C})$ is a subcomplex. Finally, the one-point compactification of $V(\mathbb{C})$ is homeomorphic to an equivariant polyhedron, via a homeomorphism which identifies the one-point compactification of $W(\mathbb{C})$ with a subcomplex.*

Next, we consider the case when X is not compact, such as the Real space underlying an affine algebraic variety over $\mathbb{R}$. In this case we follow the tradition for topological K-theory, and define the groups $KR^n(X)$ in such a way that KR-theory forms an equivariant generalized cohomology theory for the group $G = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$; see [7]. As such, KR-theory is represented by a G-spectrum structure on **KU**, which is constructed as usual from Grassmannians $G_n(\mathbb{C}^m)$ with the usual conjugation involution; see [34,10, 5.3]. By definition, $KR^*(X)$ is the graded group of stable G-equivariant maps from X to **KU**.

In the case of interest to us, when X is a finite-dimensional (but possibly infinite) CW-complex with finitely many connected components, the representable KU and KR-theories of X coincide with Atiyah's geometric description of $KU^*(X)$ and $KR^*(X)$ in terms of vector bundles, exactly as in the compact case. As a matter of fact, a close look at the various definitions and theorems in this case shows that the main ingredient used is the following lemma. Recall that a Real vector bundle on X is called *trivial* if it is isomorphic to $X \times \mathbb{C}^n$ with the involution $\overline{(x, v)} \mapsto (\bar{x}, \bar{v})$.

Lemma 1.5. *Let X be a finite-dimensional CW-complex with finitely many connected components. Then:*

(a) *Every complex vector bundle on X is a summand of a trivial bundle*;
(b) *If X is a Real space, every Real vector bundle on X is a summand of a trivial bundle.*

Proof. If X is compact, cases (a) and (b) follow from the corresponding assertions about projective modules over the Banach algebras $\mathbb{C}(X)$ and $A(X)$, respectively.

In the non-compact complex case (a), we may assume that X is connected. Any complex vector bundle E over X has a Hermitian metric and a constant rank, $n = \mathrm{rank}(E)$. Since X is paracompact,

E is the pullback of the universal bundle on BU_n by a continuous map $f\colon X \to BU_n$. Since X is finite-dimensional and BU_n is the union of the Grassmannians $G_n(\mathbb{C}^m)$, f can be factored through a specific Grassmannian. But the universal bundle on the compact space $G_n(\mathbb{C}^m)$ is a summand of a trivial bundle, so the same is true of E.

The same argument works in the Real situation of (b). Each Grassmannian $G_n(\mathbb{C}^m)$, and hence BU_n, is a Real space and the universal bundle on BU_n is actually a Real vector bundle. A Real vector bundle E on X may be given an equivariant Hermitian metric, and is the pullback of the universal bundle under a Real classifying map $X \to BU_n$. □

Suppose now that X is a finite-dimensional CW-complex with finitely many components. Using Lemma 1.5, the standard arguments also show that Swan's theorem extends to this case: (a) the category of complex vector bundles over X is equivalent to the category $\mathbf{P}\,\mathbb{C}(X)$ of finitely generated projective modules over the algebra $\mathbb{C}(X)$ of continuous functions $X \to \mathbb{C}$, and (b) if X is a Real space the category $\mathscr{E}\mathscr{R}(X)$ of Real vector bundles is equivalent to the category $\mathbf{P}A(X)$ of finitely generated projective modules over the subalgebra $A(X)$ of $\mathbb{C}(X)$.

Another important result which also goes through using Lemma 1.5 is Bott periodicity for X. For instance the topological algebra $B = \mathbb{C}(X)$ satisfies $\pi_n GL(B) \cong \pi_{n+2} GL(B)$ for $n > 0$, and $K_0(B) \cong \pi_2 GL(B)$. The topological algebra $A = A(X)$ satisfies $\pi_n GL(A) \cong \pi_{n+8} GL(A)$ for $n > 0$, and $K_0(A) \cong \pi_8 GL(A)$.

Example 1.6. Let V be an affine variety, or more generally a quasi-projective variety over $\mathbb{R}$. If V is not projective then its space $V(\mathbb{C})$ of complex points is not compact. However, we can still construct natural maps $\alpha_n\colon K_n(V) \to KR^{-n}(V(\mathbb{C}))$, $n \geqslant 0$.

To construct the α_n, we use the fact that $V(\mathbb{C})$ has a compact Real subspace X_0 as a Real deformation retract. To get X_0 one embeds V in a projective variety $\bar{V}$ with complement Z, and chooses an equivariant triangulation of $(\bar{V}(\mathbb{C}), Z(\mathbb{C}))$ given by 1.4 above. We may then take $X_0 \subset \bar{V}(\mathbb{C})$ to be the complement of an equivariant regular neighborhood N of $Z(\mathbb{C})$; see [26]. The deformation retraction $V(\mathbb{C}) \to X_0$ is just the (equivariant) radial projection onto the simplicial complement of N.

Since KR^* is a G-equivariant cohomology theory, $KR^*(V(\mathbb{C})) \cong KR^*(X_0)$. As before, there is an exact functor $\mathbf{VB}(V) \to \mathbf{P}A(X_0) \cong \mathscr{E}\mathscr{R}(X_0)$. Thus we have natural homomorphisms, independent of the choice of X_0:

$$\alpha_n\colon K_n(V) \to K_n^{\mathrm{alg}} A(X_0) \to K_n^{\mathrm{top}} A(X_0) \cong KR^{-n}(X_0) \xleftarrow{\cong} KR^{-n}(V(\mathbb{C})).$$

As in Example 1.2, both homomorphisms come from a map of K-theory ring spectra, unique up to homotopy equivalence:

$$\alpha\colon \mathbf{K}(V) \to \mathbf{K}^{\mathrm{alg}} A(X_0) \to \mathbf{K}^{\mathrm{top}} A(X_0) \cong \mathbf{K}^{\mathrm{top}} \mathscr{E}\mathscr{R}(X_0) \xleftarrow{\sim} \mathbf{K}^{\mathrm{top}} \mathscr{E}\mathscr{R}(V(\mathbb{C})).$$

This map fits into a homotopy commutative square of form (1.2.0). Thus there is also a homomorphism $K_n(V;\mathbb{Z}/m) \to KR^{-n}(V(\mathbb{C});\mathbb{Z}/m)$ for K-theory with finite coefficients, compatible with the better known map $K_n(V_{\mathbb{C}};\mathbb{Z}/m) \to KU^{-n}(V(\mathbb{C});\mathbb{Z}/m)$.

Remark 1.7. Since $\mathbb{G}_m = \operatorname{Spec}(\mathbb{R}[t,t^{-1}])$ has $\mathbb{G}_m(\mathbb{C}) \simeq S^{1,1}$, the Fundamental Theorem of algebraic K-theory for $K_*(V \times \mathbb{G}_m)$ is compatible with Atiyah's periodicity theorem for $KR^*(X \times B^{1,1}, X \times S^{1,1})$.

In particular, there is a natural extension of the maps α_n to $n<0$. Of course, if $n<0$ then α_n can only be nontrivial when V is singular.

Proposition 1.8. *If X is a compact Real space with no fixed points then the groups $KR^n(X)$ are periodic of period 4 in n.*

Proof. By [1, 3.8] we have exact sequences, natural in X:

$$0 \to KR^n(X) \xrightarrow{\pi^*} KR^n(X \times S^{3,0}) \xrightarrow{\delta} KR^{4+n}(X) \to 0.$$

Let $u \in KR^0(S^{3,0})$ be such that $\delta(u)$ is a generator of $KR^4(\text{point})=\mathbb{Z}$. The cup product with u, followed by δ, defines a map $KR^n(X) \to KR^{4+n}(X)$. If X is $Y \times G$ then (up to sign) this map is the square of Bott periodicity isomorphism in $KU^*(Y) = KR^*(X)$, because the map $KU^4(\text{point}) \to KR^4(\text{point})$ is an isomorphism. The general case follows from a Mayer–Vietoris argument using the fact that X is the union of spaces of the type $Y \times \mathbb{Z}/2$. □

2. *KR*-theory with supports

Next, we need to introduce KR-theory with supports. Let X be a closed subspace of a locally compact Y, and suppose that Y has an involution mapping X to itself. That is, Y is a Real space and X is a Real subspace. For simplicity, we shall make the:

Running Assumption 2.0. The one-point compactification of (Y, X) is homeomorphic to a finite simplicial G-pair, i.e., a finite simplicial pair in which the involution permutes the simplices.

This assumption always holds for $Y = V(\mathbb{C})$ and $X = W(\mathbb{C})$ when V is a closed subvariety of a quasi-projective variety W defined over $\mathbb{R}$, by Corollary 1.4 above.

Lemma 2.1. *Under the Running Assumption 2.0, there is a compact $A \subset Y - X$ which is a Real deformation retraction. In particular, $KR^*(Y - X) \cong KR^*(A)$.*

Proof. By assumption, there is an equivariant open regular neighborhood N of $\dot{X}$ in the one-point compactification $\dot{Y}$; see [26]. Let A be the closed complement of N in $\dot{Y}$. The (equivariant) radial projection onto the simplicial complement of N is a Real deformation retraction $Y - X \to A$. By G-homotopy invariance, $KR^*(Y - X) \cong KR^*(A)$. □

Consider the topological (additive) category $\mathscr{C}_X(Y)$ of bounded chain complexes

$$0 \to E_n \xrightarrow{\sigma_n} E_{n-1} \xrightarrow{\sigma_{n-1}} \cdots \to E_m \to 0$$

of Real vector bundles on Y whose homology is supported on X. The category $\mathscr{C}_X(Y)$ has an induced notion of short exact sequence. Write $KR_X(Y)$ for the quotient of the Grothendieck group of this category by the relation $[E] = 0$ for each exact complex E.

As usual, we can define graded cohomology groups using periodicity: for positive n, we define $KR_X^{-n}(Y)$ to be $\widetilde{KR}_{\Sigma^n(X_+)}(\Sigma^n(Y_+))$, where as usual X_+ denotes the disjoint union of X with a point.

For $n=0$ this yields $KR^0_X(Y)=KR_X(Y)$. We shall call $KR^*_X(Y)$ the Real K-theory of Y with supports in X, as in [4, 1.1].

Excision Theorem 2.2. *Suppose that A is a closed Real subspace of Y, disjoint from X, and that $A\subset(Y-X)$ is a Real deformation retract. Then*

$$KR^*_X(Y)\cong KR^*(Y,A)=\widetilde{KR}^*(Y/A).$$

Proof. The proof on pp. 183–184 of [4] goes through. There is an alternative proof using the analogue $KR^0(Y,A)\cong \mathscr{L}R(Y,A)$ of [2, 2.6.1] and [2, 2.6.12], together with Lemma 1.5; we leave this as an exercise for the reader. □

Corollary 2.3. *Under the Running Assumption 2.0, there is a long exact sequence*:

$$\cdots\to KR^{n-1}(Y-X)\xrightarrow{\partial} KR^n_X(Y)\to KR^n(Y)\to KR^n(Y-X)\to\cdots$$

Proof. This is just the long exact sequence of the pair (Y,A), using Excision 2.2 to replace $KR^*(Y,A)$ with $KR^n_X(Y)$. □

Corollary 2.4. *If U is an open neighborhood of X in Y, and $\dot{U}$ is open in $\dot{Y}$, the restriction $KR^*_X(Y)\to KR^*_X(U)$ is an isomorphism.*

Proof. By assumption, there is an equivariant open regular neighborhood N of $\dot{X}$ in $\dot{Y}$ contained in U. Then the complement A of N is a Real deformation retract of $Y-X$, and $U\cap A$ is a Real deformation retract of $U-X$. Since $Y/A\cong U/(U\cap A)$, the result follows from Excision 2.2. □

If E is a Real vector bundle on Y, then E is an open neighborhood of the zero-section Y_0 in the projective bundle $\mathbb{P}(E\oplus 1)$, where “1” denotes the trivial one-dimensional Real vector bundle on Y. The complement of E is naturally isomorphic to $\mathbb{P}(E)$. We define the *Thom space* Y^E of E to be $\mathbb{P}(E\oplus 1)/\mathbb{P}(E)$; if Y is compact then Y^E is just the one-point compactification of E.

Since $A=\mathbb{P}(E)$ is a Real deformation retract of $\mathbb{P}(E\oplus 1)-Y_0$, Excision 2.2 yields:

Corollary 2.5. *Let E be a Real vector bundle on Y, and Y^E its Thom space. Then*

$$\widetilde{KR}^*(Y^E)\cong KR^*_{Y_0}(\mathbb{P}(E\oplus 1)).$$

Associated to a Real vector bundle $\pi: E\to Y$ is the Koszul–Thom class λ_E in $\widetilde{KR}^0(Y^E)\cong KR^0_{Y_0}(E)$, defined by the exterior algebra of E; see [1, 2.4], [4, 1.4] or (3.1) below. The *Thom–Gysin map* $\phi: KR^*_X(Y)\to KR^*_X(E)$ is defined by: $\phi(x)=\pi^*(x)\cup\lambda_E$.

Proposition 2.6 (Thom isomorphism). *Suppose that the one-point compactification of (Y,X) is homeomorphic to a finite simplicial G-pair. If $\pi: E\to Y$ is a Real vector bundle, the Thom–Gysin map $\phi: KR^*_X(Y)\to KR^*_X(E)$ is an isomorphism.*

Proof. Choose an equivariant metric on E so we have the closed unit ball $B(E)$, the unit sphere $S(E)$ and the closure E' of $E-B(E)$. By Lemma 2.1, there is a Real deformation retract $A\subset(Y-X)$

so that $KR^*_X(Y) \cong \widetilde{KR}^*(Y/A)$. It follows that $(\pi^{-1}A \cup E') \subset E - X$ is a Real deformation retract. By Excision, we also have

$$KR^*_X(E) \cong \widetilde{KR}^*(E/(\pi^{-1}A \cup E')) \cong \widetilde{KR}^*(B(E)/S(E) \cup B(E|_A)).$$

Thus it suffices to prove that $KR^*(Y,A) \xrightarrow{\cong} KR^*(B(E), S(E) \cup B(E|_A))$. As argued on p. 185 of [4], we may assume that X and Y are finite CW complexes. But in this case, the usual Thom isomorphism [1, 2.4] gives the isomorphism over Y and over A. We deduce the relative case from the 5-lemma applied to 2.3:

$$\begin{array}{ccccc} KR^*(Y,A) & \longrightarrow & KR^*(Y) & \longrightarrow & KR^*(A) \\ \downarrow \lambda_E & & \downarrow \lambda_E & & \downarrow \lambda_E \\ KR^*(B(E), S(E) \cup B(E|_A)) & \longrightarrow & KR^*(B(E), S(E)) & \longrightarrow & KR^*(B(E|_A), S(E|_A)). \end{array}$$ □

We conclude this section with an alternative construction of $KR^*_X(Y)$. This construction will make it easier to compare with algebraic K-theory, and will also make it obvious that the above assertions also hold for KR-theory with coefficients.

The above remarks show that $\mathscr{C}_X(Y)$ has the structure of a topological Waldhausen category. Write $\mathbf{KR}_X(Y)$ for the resulting topological K-theory space $\mathbf{K}\mathscr{C}_X(Y)$, and write $KR^*_X(Y)$ for the corresponding homotopy groups. It is well known that this recovers the group $KR^0_X(Y)$ defined above.

Theorem 2.7. *There is a natural isomorphism* $\mathbf{KR}_Y(Y) \cong \mathbf{KR}(Y)$, *and the inclusion of* $\mathscr{C}_X(Y)$ *in* $\mathscr{C}_Y(Y)$ *identifies* $\mathbf{K}\mathscr{C}_X(Y)$ *with the homotopy fiber of* $\mathbf{KR}(Y) \to \mathbf{KR}(Y - X)$.

Proof. The proof in [35, 1.11.7] of the Gillet–Grayson theorem goes through for topological exact categories, proving that the canonical inclusion of $\mathscr{ER}(Y)$ in $\mathscr{C}_Y(Y)$ induces a homotopy equivalence of K-theory spaces. This proves the first part.

To prove the second part, we say that two complexes in $\mathscr{C}_Y(Y)$ are w-equivalent if their restrictions to $Y - X$ are quasi-isomorphic. The topological version of Waldhausen's Fibration Theorem [35, 1.8.2] shows that there is a homotopy fibration sequence

$$\mathbf{K}(\mathscr{C}_X(Y)) \to \mathbf{K}(\mathscr{C}_Y(Y)) \to \mathbf{K}(w\mathscr{C}_Y(Y)).$$

Now every Real bundle on $Y - X$ is a summand of a trivial bundle by 1.5. Hence the topological version of the Approximation Theorem [35, 1.9.1] applies to prove that $\mathbf{K}(w\mathscr{C}_Y(Y))$ is homotopy equivalent to $\mathbf{K}(\mathscr{C}_{Y-X}(Y - X)) \simeq \mathbf{KR}(Y - X)$, as required. □

3. Riemann–Roch Theorem

On the algebraic side, suppose that X is a subvariety of a smooth variety Y defined over $\mathbb{R}$. Following Thomason-Trobaugh, we have the Waldhausen category of bounded chain complexes of algebraic vector bundles on Y which are acyclic on $Y - X$; we write $K_*(Y \text{ on } X)$ for the algebraic K-theory of this category (cf. [35, 3.5]). Theorem 5.1 of [35] states that this agrees with the relative term for $K_*(Y) \to K_*(Y - X)$, which for Y smooth is just $G_*(X)$.

Clearly, a bounded chain complex of algebraic vector bundles gives a topological complex of Real vector bundles on the underlying Real spaces. Thus we have a natural map

$$\mathbf{K}(Y \text{ on } X) \xrightarrow{\alpha} \mathbf{KR}_X(Y).$$

If $E \xrightarrow{\pi} Y$ is a Real vector bundle, we have a canonical section $\mathcal{O}_E \to \pi^*E$. This determines a homomorphism from the dual bundle $\mathscr{F} = (\pi^*E)\check{}$ to $\mathcal{O}_E$. The Koszul complex

$$0 \to \Lambda^d \mathscr{F} \to \cdots \to \Lambda^2 \mathscr{F} \to \mathscr{F} \to \mathcal{O}_E \to 0 \tag{3.1}$$

is exact off the zero section Y; the element of $KR^0_Y(E)$ it determines is the *Koszul–Thom* class λ_E. See [4, 1.4]; this is the dual of the λ_E of [1, p. 100], and it is called U_E in [16, p. 183].

Now suppose that $i: Y \subset Z$ is a closed embedding of C^∞ manifolds, with involution, such that the normal bundle $N \xrightarrow{p} Y$ is a Real bundle.

Lemma 3.2. *The Thom–Gysin map* $i_*: \mathbf{KR}_X(Y) \to \mathbf{KR}_X(Z)$ *is a homotopy equivalence.*

Proof. In fact, i_* is defined to be the composite of the Thom map $\mathbf{KR}_X(Y) \to \mathbf{KR}_X(N)$, which is a homotopy equivalence by 2.6, with the excision map θ^*: $\mathbf{KR}_X(Z) \xrightarrow{\cong} \mathbf{KR}_X(N)$ induced by the mapping θ: $N \hookrightarrow Z$ onto a tubular neighborhood of Y. By 2.4, θ^* is also a homotopy equivalence. □

On the algebraic side, if we have an embedding $i: Y \subset Z$ of varieties over $\mathbb{R}$ of finite Tor-dimension (e.g., Y is locally a complete intersection) then we end up in the category of bounded chain complexes of perfect $\mathcal{O}_Z$ modules, acyclic off X; this has the same higher K-theory as $K_*(Z \text{ on } X)$ by [35, 3.7]. Hence we get a direct image map i_*: $\mathbf{K}(Y \text{ on } X) \to \mathbf{K}(Z \text{ on } X)$. If Y and Z are nonsingular, both are identified with $\mathbf{G}(X)$, and i_* is a homotopy equivalence. Here is the analogue of [4, p. 166]:

Theorem 3.3. *Let* $i: Y \subset Z$ *be a closed embedding of nonsingular varieties over* $\mathbb{R}$. *Then for any closed* $X \subset Y$, *the following diagram homotopy commutes*:

$$\begin{array}{ccc}
\mathbf{K}(Y \text{ on } X) & \xrightarrow[\simeq]{i_*} & \mathbf{K}(Z \text{ on } X) \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha} \\
\mathbf{KR}_X(Y) & \xrightarrow[\simeq]{i_*} & \mathbf{KR}_X(Z).
\end{array}$$

Proof. Suppose first that $Z = \mathbb{A}(\mathscr{E}) \xrightarrow{p} Y$ is an algebraic vector bundle on Y, and i: $Y \subset Z$ is the zero section. In this case the Koszul complex (3.1) is a resolution $\Lambda^* \mathscr{F}$ of the structure sheaf $i_*\mathcal{O}_Y$ by vector bundles on Z. It follows that we can tensor any complex C of vector bundles on Y with (3.1) to get a functorial resolution $\Lambda^*\mathscr{F} \otimes_Z \pi^*(C)$ of $\pi^*(C) \otimes_Z i_*\mathcal{O}_Y = i_*(C)$ by vector bundles on Z. That is, the algebraic map i_* factors as

$$\mathbf{K}(Y \text{ on } X) \xrightarrow{\pi^*} \mathbf{K}(Z \text{ on } p^{-1}X) \xrightarrow{\cup[\Lambda^*\mathscr{F}]} \mathbf{K}(Z \text{ on } X),$$

and this is compatible with the topological Thom–Gysin map

$$\mathbf{KR}_X(Y) \xrightarrow{\pi^*} \mathbf{KR}_{p^{-1}(X)}(Z) \xrightarrow{\cup\lambda_E} \mathbf{KR}_X(Z).$$

For a general embedding $Y \subset Z$, one uses the method of deformation to the normal bundle. Let W be the blowup of $Z \times \mathbb{A}^1$ along $Y \times \{0\}$. The proof of the lemma on p. 166 of [4] goes through to construct a commutative diagram of closed embeddings of varieties over $\mathbb{R}$, whose squares are Real-transverse in the sense of loc. cit.:

$$\begin{array}{ccccc} Y & \xrightarrow{j_1} & Y \times \mathbb{A}^1 & \xleftarrow{j_0} & Y \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \bar{i}} \\ Z & \xrightarrow{k_1} & W & \xleftarrow{k_0} & N. \end{array}$$

Applying K-theory and KR-theory with supports yields two diagrams, which homotopy commute by the transverse properties. Here is the diagram for KR-theory:

$$\begin{array}{ccccc} \mathbf{KR}_{X(\mathbb{C})}(Y(\mathbb{C})) & \xleftarrow{j_1^*} & \mathbf{KR}_{X(\mathbb{C})\times\mathbb{C}}(Y(\mathbb{C}) \times \mathbb{C}) & \xrightarrow{j_0^*} & \mathbf{KR}_{X(\mathbb{C})}(Y(\mathbb{C})) \\ \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle \psi_*} & & \downarrow{\scriptstyle \bar{i}_*} \\ \mathbf{KR}_{X(\mathbb{C})}(Z(\mathbb{C})) & \xleftarrow{k_1^*} & \mathbf{KR}_{X(\mathbb{C})\times\mathbb{C}}(W(\mathbb{C}) \times \mathbb{C}) & \xrightarrow{k_0^*} & \mathbf{KR}_{X(\mathbb{C})}(N(\mathbb{C})). \end{array}$$

On the algebraic side, the top maps $\mathbf{K}(Y \text{ on } X) \simeq \mathbf{K}(Y \times \mathbb{A}^1 \text{ on } X \times \mathbb{A}^1)$ are homotopy equivalences. On the topological side, we also have $\mathbf{KR}_{X(\mathbb{C})}(Y(\mathbb{C})) \simeq \mathbf{KR}_{X(\mathbb{C})\times\mathbb{C}}(Y(\mathbb{C}) \times \mathbb{C})$. Moreover, the vertical Thom–Gysin maps are equivalences by Lemma 3.2. Now the theorem follows by a diagram chase, as on p. 168 of [4]. $\square$

Corollary 3.4. *If j: $X \subset Z$ is a closed embedding of nonsingular varieties over $\mathbb{R}$, then the following diagram homotopy commutes, and the horizontal composites are j^*.*

$$\begin{array}{ccccccc} \mathbf{K}(X) & \xrightarrow{\simeq} & \mathbf{K}(X \text{ on } X) & \xrightarrow{\simeq} & \mathbf{K}(Z \text{ on } X) & \longrightarrow & \mathbf{K}(Z) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} \\ \mathbf{KR}(X) & \xrightarrow{\simeq} & \mathbf{KR}_X(X) & \xrightarrow{\simeq} & \mathbf{KR}_X(Z) & \longrightarrow & \mathbf{KR}(Z) \end{array}$$

Proof. For the middle square, take $X = Y$ nonsingular and $X \subset Z$ in 3.3. The outer squares commute by naturality. $\square$

Proposition 3.5. *Let* $\pi : \mathbb{P}^n_{\mathbb{R}} \to \operatorname{Spec} \mathbb{R}$ *be the projection. Then*

$$\begin{array}{ccc} K_0(\mathbb{P}^n) & \xrightarrow{\alpha_0} & KR^0(\mathbb{P}^n) \\ \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} \\ K_0(\mathbb{R}) & \xrightarrow{\alpha_0} & KR^0(\text{point}) \end{array}$$

commutes.

Proof. As in [4, p. 176] we proceed by induction on n, the case $n = 0$ being trivial. Since $K_0(\mathbb{P}^n_{\mathbb{R}})$ is free abelian on $[\mathcal{O}(i)]$, $i = 0, 1, \ldots, n$, it suffices to show that $\alpha_0 \pi_* \mathcal{O}(i) = \pi_* \alpha_0 \mathcal{O}(i)$ for these i. We proceed inductively, starting with the well known formula:

$$\alpha_0 \pi_* \mathcal{O} = \alpha_0[\mathbb{R}] = 1 = \pi_*(1) = \pi_* \alpha_0(\mathcal{O}).$$

For $i > 0$, set $\mathcal{F} = \mathcal{O}_{\mathbb{P}^{n-1}}(i)$ and j: $\mathbb{P}^{n-1} \to \mathbb{P}^n$. Because we have exact sequences

$$0 \to \mathcal{O}(i-1) \to \mathcal{O}(i) \to j_* \mathcal{F} \to 0,$$

we have $[\mathcal{O}(i)] = [\mathcal{O}(i-1)] + [j_*\mathcal{F}]$ in $K_0(\mathbb{P}^n)$. But by induction on $\pi' = \pi j : \mathbb{P}^{n-1} \to$ point and 3.4 for j_*, we have

$$\alpha_0 \pi_*(j_*\mathcal{F}) = \alpha_0 \pi'_* \mathcal{F} = \pi'_* \alpha_0 \mathcal{F} = \pi_* j_* \alpha_0 \mathcal{F} = \pi_* \alpha_0 (j_* \mathcal{F}).$$

Since the formula holds for $j_*\mathcal{F}$ and $\mathcal{O}(i-1)$, it holds for $\mathcal{O}(i)$, as desired. □

Corollary 3.6. *For quasi-projective V, the following square commutes up to a map which is zero on all homotopy groups, including homotopy groups with finite coefficients.*

$$\begin{array}{ccc} \mathbf{K}(V \times \mathbb{P}^n) & \xrightarrow{\alpha} & \mathbf{KR}(V \times \mathbb{P}^n) \\ \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} \\ \mathbf{K}(V) & \xrightarrow{\alpha} & \mathbf{KR}(V) \end{array}$$

Proof. By functoriality of the K-theory product, the spectrum $\mathbf{K}(V \times \mathbb{P}^n)$ is a module spectrum for $\mathbf{K}(V)$, and the map π_* on the left is a $K(V)$-module map. Similarly, $\mathbf{KR}(V \times \mathbb{P}^n)$ is a module spectrum for $\mathbf{KR}(V)$, and the map π_* on the right is a $\mathbf{KR}(V)$-module map. Moreover, α: $\mathbf{K}(V) \to \mathbf{KR}(V)$ is a map of ring spectra (see 1.2). On homotopy groups, it is well known [25,1] that $K_*(V \times \mathbb{P}^n) \cong K_*(V) \otimes K_0(\mathbb{P}^n)$, which is a free $K_*(V)$ module, and similarly for $KR^*(V \times \mathbb{P}^n)$. From Proposition 3.5 we see that the difference $\alpha\pi - \pi\alpha$ vanishes on all homotopy groups. By obstruction theory, the difference also vanishes on $\pi_*(;\mathbb{Z}/m)$. □

Riemann–Roch Theorem 3.7. *For every proper morphism $V \xrightarrow{f} Y$ of nonsingular varieties over $\mathbb{R}$, the following square commutes up to a map which is zero on all homotopy groups, including homotopy groups with finite coefficients.*

$$\begin{array}{ccc} \mathbf{K}(V) & \xrightarrow{\alpha} & \mathbf{KR}(V) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ \mathbf{K}(Y) & \xrightarrow{\alpha} & \mathbf{KR}(Y) \end{array}$$

In particular, we get a commutative square of homotopy groups with coefficients $\mathbb{Z}/m$:

$$\begin{array}{ccc} K_n(V;\mathbb{Z}/m) & \xrightarrow{\alpha_n} & KR^{-n}(V;\mathbb{Z}/m) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ K_n(Y;\mathbb{Z}/m) & \xrightarrow{\alpha_n} & KR^{-n}(Y;\mathbb{Z}/m). \end{array}$$

Proof. Standard, as f factors as an embedding $V \to Y \times \mathbb{P}^n$ (for which 3.4 applies), followed by the projection π_* (for which 3.6 applies). □

Remark. An alternative approach to proving the Riemann–Roch theorem has been developed by Panin and Smirnov [22]. The functors KR^* from smooth varieties to abelian groups form an *oriented cohomology theory* in the sense of Panin–Smirnov [22, 3.1]: the functors are homotopy invariant—$KR^*(X) \cong KR^*(X \times \mathbb{C})$, have localization sequences by 2.3, Thom isomorphisms by 2.6, and Excision by 2.2. (The Nisnevich excision axiom follows easily from this.) Presumably it is also a ring cohomology theory in the sense of [22, 2.3].

4. The main theorem

If V is a smooth variety defined over $\mathbb{C}$, we write V_{an} for the locally compact space of complex points in V. The following fundamental theorem is a consequence of Voevodsky's theorem [36]; see [24, 4.1] and [11].

Theorem 4.1. *If V is a smooth variety defined over $\mathbb{C}$ and $m = 2^\nu$ then*

$$\alpha': K_n(V;\mathbb{Z}/m) \to KU^{-n}(V_{\mathrm{an}};\mathbb{Z}/m)$$

is an isomorphism for all $n \geqslant \dim(V) - 1$.

The corresponding assertion for any m is known as the Quillen–Lichtenbaum conjecture for complex varieties. It is expected that the Quillen–Lichtenbaum conjecture for odd m will follow from recent work of Rost and Voevodsky on norm residues.

Now suppose that V is a smooth variety defined over $\mathbb{R}$, and set $G = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Then the variety $V_{\mathbb{C}}$ is defined over $\mathbb{C}$, and the space $V(\mathbb{C}) = (V_{\mathbb{C}})_{\mathrm{an}}$ of complex points of V is a G-space with involution. The analogue of the Quillen–Lichtenbaum conjecture for varieties over $\mathbb{R}$ concerns the maps $\alpha_n: K_n(V;\mathbb{Z}/m) \to KR^{-n}(V(\mathbb{C});\mathbb{Z}/m)$, constructed in Section 1, which by (1.2.0) fit into the diagrams

$$\begin{array}{ccc} K_n(V;\mathbb{Z}/m) & \xrightarrow{\alpha_n} & KR^{-n}(V(\mathbb{C});\mathbb{Z}/m) \\ \downarrow & & \downarrow \\ K_n(V_{\mathbb{C}};\mathbb{Z}/m)^G & \xrightarrow{\alpha_n'} & KU^{-n}(V(\mathbb{C});\mathbb{Z}/m)^G. \end{array} \tag{4.1.0}$$

When V is a variety defined over $\mathbb{R}$, we shall use notation $KR_n(V)$ for $KR^{-n}(V(\mathbb{C}))$, and $KR_n(V;\mathbb{Z}/m)$ for $KR^{-n}(V(\mathbb{C});\mathbb{Z}/m)$.

Corollary 4.2. *If V is defined over $\mathbb{C}$ then α_n: $K_n(V;\mathbb{Z}/m) \cong KR_n(V;\mathbb{Z}/m)$ for all $n \geqslant \dim(V) - 1$ and all $m = 2^\nu$.*

Proof. Because $V_{\mathbb{C}} \cong V \times G$, we have $V(\mathbb{C}) \cong V_{\mathrm{an}} \times G$. Hence $KR^*(V(\mathbb{C})) \cong KU^*(V_{\mathrm{an}})$ by [1]. If $n \geqslant \dim(V) - 1$, Theorem 4.1 and (4.1.0) yield the desired result:

$$\begin{aligned} K_n(V;\mathbb{Z}/m) &\cong K_n(V_{\mathbb{C}};\mathbb{Z}/m)^G \cong KU^{-n}(V_{\mathrm{an}} \times G;\mathbb{Z}/m)^G \\ &\cong KU^{-n}(V_{\mathrm{an}};\mathbb{Z}/m) \cong KR^{-n}(V(\mathbb{C});\mathbb{Z}/m) = KR_n(V;\mathbb{Z}/m). \qquad \square \end{aligned}$$

Corollary 4.3. *Let V be a smooth variety defined over $\mathbb{R}$. If m is odd, the Quillen–Lichtenbaum conjecture for $V_{\mathbb{C}}$ implies that the maps α_n: $K_n(V;\mathbb{Z}/m) \to KR_n(V;\mathbb{Z}/m)$ are isomorphisms for all $n \geqslant \dim(V) - 1$.*

Proof. The Galois group G acts on the K-theory of $V_{\mathbb{C}}$ and the KU-theory of $V(\mathbb{C})$. Since m is odd, the usual transfer argument shows that $K_*(V;\mathbb{Z}/m) \cong K_*(V_{\mathbb{C}};\mathbb{Z}/m)^G$ and:

$$KR^*(V(\mathbb{C});\mathbb{Z}/m) \cong KU^*(V(\mathbb{C});\mathbb{Z}/m)^G \cong KU^*(V(\mathbb{C}) \times G;\mathbb{Z}/m)^G.$$

By (4.1.0), the map α_n is just the G-invariant part $K_n(V_{\mathbb{C}};\mathbb{Z}/m)^G \to KU^{-n}(V(\mathbb{C});\mathbb{Z}/m)^G$ of the map in the Quillen–Lichtenbaum conjecture for $V_{\mathbb{C}}$. □

Example 4.4. Let V be a smooth real curve. Suslin proved the Quillen–Lichtenbaum conjecture for $V_{\mathbb{C}}$ in [29] (cf. [23]). It follows from 4.3 that if m is odd we have $K_n(V;\mathbb{Z}/m) \cong KR_n(V;\mathbb{Z}/m)$ for all $n \geqslant 0$. (This also holds for m even; see 4.12 below.)

Example 4.5 (Brauer–Severi curve). Let Q denote the projective plane curve over $\mathbb{R}$ defined by $X^2+Y^2+Z^2=0$. This is the Brauer–Severi variety associated to the quaternions $\mathbb{H}$. Quillen proved in [25, p. 137] that $\mathbf{K}(Q)$ is homotopy equivalent to $\mathbf{K}(\mathbb{R}) \times \mathbf{K}(\mathbb{H})$. Suslin observed in [28, 3.5] that after completing at any prime, $\mathbf{K}(Q)$ is equivalent to $(\mathbb{Z}\times BO)\times(\mathbb{Z}\times BSp)$. Since $\mathbb{Z}\times BSp$ is homotopy equivalent to $\Omega^4 BO$ we see that the homotopy groups $K_*(Q;\mathbb{Z}/m)$ are 4-periodic, and $K_*(Q;\mathbb{Z}/m) \cong KO^*(\text{point};\mathbb{Z}/m) \oplus KO^{*+4}(\text{point};\mathbb{Z}/m)$.

On the other hand, the Real space of complex points of Q is $Q(\mathbb{C}) = S^{3,0}$, the two-sphere with antipodal involution. Atiyah computed that $KR^q(S^{3,0}) \cong KO^q \oplus KO^{q+4}$ in [1, 3.8]. Passing to finite coefficients yields the same abstract groups as $K_*(Q;\mathbb{Z}/m)$. Theorem 4.7 below shows that the map α_n is an isomorphism:

$$\alpha_n\colon K_n(Q;\mathbb{Z}/m) \cong KR_n(Q;\mathbb{Z}/m) = KR^{-n}(S^{3,0}), \qquad n \geqslant 0.$$

Lemma 4.6. *Let $K \to L$ be a G-map of G-spaces. If $\pi_n(K) \to \pi_n(L)$ is an isomorphism for all $n \geqslant d$, then the maps $\pi_n(K^{hG}) \to \pi_n(L^{hG})$ are also isomorphisms for all $n \geqslant d$.*

Proof. The Bousfield–Kan spectral sequence $E_2^{pq} = H^p(G,\pi_{-q}K) \Rightarrow \pi_{-p-q}K^{hG}$ (see [6, XI.7.1] or [32, 5.13 and 5.43]) is associated to a complete exhaustive filtration on the homotopy groups of K^{hG}; see [6, IX.5.4], [5, 7.1] or [32, 5.47]. The lemma now follows from the Comparison Theorem (see [39, 5.5.11], [5, 5.3], [32, 5.55]), between this spectral sequence and its analogue for L^{hG}. □

Theorem 4.7. *If $V(\mathbb{R}) = \emptyset$ and $m = 2^\nu$, the map $\alpha_n\colon K_n(V;\mathbb{Z}/m) \to KR_n(V;\mathbb{Z}/m)$ is an isomorphism for each $n \geqslant \dim(V)-1$. In particular, $K_n(V;\mathbb{Z}/m) \cong K_{n+4}(V;\mathbb{Z}/m)$ in this range.*

The proof of Theorem 4.7 will depend upon a technical result (Theorem 5.1), whose proof we postpone until the next section in order to not disrupt our line of attack.

Proof. Set $X = V(\mathbb{C})$ and $d = \dim(V)$. Since $X^G = \emptyset$, the groups $KR^*(X)$ and $KR^*(X;\mathbb{Z}/m)$ are 4-periodic by Proposition 1.8. The topological map $\mathbf{K}(V_{\mathbb{C}}) \to \mathbf{KU}(X)$ is G-equivariant. Taking homotopy fixed points and then homotopy groups yields a commutative diagram, whose labels we explain below:

$$\begin{array}{ccccccc}
\cdots \longrightarrow & K_n(V;\mathbb{Z}/m) & \xrightarrow{5.1} & \pi_n(\mathbf{K}(V_{\mathbb{C}})^{hG};\mathbb{Z}/m) & \longrightarrow & K_n(V_{\mathbb{C}};\mathbb{Z}/m) \\
& \downarrow & & \downarrow{\scriptstyle n\geq d-1} & & \downarrow{\scriptstyle n\geq d-1} \\
\cdots \longrightarrow & KR^{-n}(X;\mathbb{Z}/m) & \xrightarrow{\simeq} & \pi_n(\mathbf{KU}(X)^{hG};\mathbb{Z}/m) & \longrightarrow & KU^{-n}(X;\mathbb{Z}/m).
\end{array}$$

By Theorem 1.1, the lower left horizontal map is an isomorphism for all n. When $n \geqslant d-1$, the right vertical is an isomorphism by Theorem 4.1; by Lemma 4.6, the middle vertical is also an isomorphism in this range. Theorem 4.7 now follows from Theorem 5.1 below, which states that the top left map (labelled '5.1') is an isomorphism for all $n \geqslant d-1$. □

Main Theorem 4.8. *If V is a smooth variety defined over $\mathbb{R}$, and $m = 2^\nu$, then the map α_n: $K_n(V;\mathbb{Z}/m) \to KR_n(V;\mathbb{Z}/m)$ is an isomorphism for all $n \geqslant \dim(V)$.*

Proof. If Q is the Brauer–Severi curve over $\mathbb{R}$ of Example 4.5 then $V \times_{\mathbb{R}} Q$ is the Brauer–Severi variety over V associated to the constant Azumaya algebra defined by $\mathbb{H}$. Quillen proved in [25, p. 137] that the map $K_*(V) \to K_*(V \times Q)$ is a split injection, and that the splitting is given by the direct image map f_* associated to the projection f: $V \times Q \to V$.

By naturality and Riemann–Roch 3.7, the following diagram commutes:

$$\begin{array}{ccccc}
K_n(V;\mathbb{Z}/m) & \xrightarrow{f^*} & K_n(V\times Q;\mathbb{Z}/m) & \xrightarrow{f_*} & K_n(V;\mathbb{Z}/m) \\
\downarrow \alpha & & \cong\downarrow \alpha & & \downarrow \alpha \\
KR_n(V;\mathbb{Z}/m) & \xrightarrow{f^*} & KR_n(V\times Q;\mathbb{Z}/m) & \xrightarrow{f_*} & KR_n(V;\mathbb{Z}/m)
\end{array}$$

We just saw that the top horizontal composite is the identity for all n. Atiyah established in [1, Section 5] that the bottom composite is the identity for all n. If $n \geqslant \dim(V) = \dim(V \times Q) - 1$, the middle vertical is an isomorphism by Theorem 4.7; it follows that the left vertical is also an isomorphism in this range, as required. □

As an immediate consequence of this, we can show that the 2-primary Bott localization of algebraic K-theory is KR-theory. Recall from [21] that for each $m = 2^\nu$ (except $m \neq 2, 4$) there is a graded-commutative ring structure on $K_*(V;\mathbb{Z}/m)$ and $KR^*(X;\mathbb{Z}/m)$. For suitably large $N = 2^k$ there an isomorphism $K_{2N}(\mathbb{Z};\mathbb{Z}/m) \cong K_{2N}(\mathbb{C};\mathbb{Z}/m) \cong \mathbb{Z}/m$; see [38]. By abuse of notation, we write β^N for the element in $K_{2N}(\mathbb{Z};\mathbb{Z}/m)$ whose image under this isomorphism is the Nth power of the Bott element $\beta \in K_2(\mathbb{C};\mathbb{Z}/m)$. We define $K_*(V;\mathbb{Z}/m)[\beta^{-1}]$ to be the localization of $K_*(V;\mathbb{Z}/m)$ at the powers of β^N. Since the image of β^N in $KR^{-2N}(\text{point};\mathbb{Z}/m) \cong KU^{-2N}(\text{point};\mathbb{Z}/m) \cong \mathbb{Z}/m$ is a power of the periodicity element, Theorem 4.8 implies the following result.

Corollary 4.9. *If V is a smooth real variety, the ring map $K_*(V;\mathbb{Z}/m) \to KR_*(V;\mathbb{Z}/m)$ induces an isomorphism of graded rings*:

$$\alpha_*\colon K_*(V;\mathbb{Z}/m)[\beta^{-1}] \to KR_*(V;\mathbb{Z}/m), \quad m = 2^\nu,\ m \neq 2,4.$$

Remark 4.9.1. Although we shall avoid using étale K-theory directly, we should point out that it is lurking in the shadows. Indeed, we know by [9, 7.1] that $K^{\text{et}}(V) \simeq K^{\text{et}}(V_{\mathbb{C}})^{hG}$ after completing at the prime 2. Since $K^{\text{et}}(V_{\mathbb{C}})$ is weakly equivalent to the 2-completion of $KU(X)$, and 2-completion commutes with homotopy limits, it follows from Theorem 1.1 that the 2-completion of $KR(X)$ is weakly equivalent to $K^{\text{et}}(V)$. This observation was also made by Friedlander and Walker in [12, 4.7].

Theorem 4.8 also gives $2N$-periodicity information about the G-theory of V, the K-theory of coherent modules on V. A typical example of this occurs when $m = 8, 16$; in this case we know (see [38]) that β^4 lifts to $K_8(\mathbb{Z};\mathbb{Z}/m)$.

Corollary 4.10. *If V is a real variety with $V(\mathbb{R})=\emptyset$ and m is 8 or 16 then multiplication by β^4,*

$$G_n(V;\mathbb{Z}/m)\to G_{n+8}(V;\mathbb{Z}/m),$$

is an isomorphism for all $n\geqslant \dim(V)$, and an injection for $n=\dim(V)-1$.

Proof (Suslin [29, p. 350]). Using the localization sequence in G-theory, we see by induction on $\dim(V)$ that it suffices to prove the result for the function field $k=\mathbb{R}(V)$ of every such variety. But this case follows from 4.8, because $K_*(k;\mathbb{Z}/m)\cong G_*(k;\mathbb{Z}/m)$. □

We conclude with a small improvement for curves, using [23]. The following result was implicit in [23] but inexplicably does not appear in loc. cit. Note that for a smooth curve we have $K_0(V;\mathbb{Z}/m)\cong K_0(V)/m=\mathbb{Z}/m\oplus \mathrm{Pic}(V)/m$.

Theorem 4.11 (Pedrini–Weibel). *Let V be a smooth curve defined over $\mathbb{R}$. Then multiplication by the Bott element $\beta\in K_8(V;\mathbb{Z}/2)$ induces an isomorphism between $K_0(V)/2\cong\mathbb{Z}/2\oplus\mathrm{Pic}(V)/2$ and $K_8(V;\mathbb{Z}/2)$.*

Proof. We may regard the Bott element as a generator of the $E_2^{-4,-4}$ term $H^0(V,\mathbb{Z}/2(4))$ in the Friedlander–Suslin spectral sequence (5.2.2) below (see [38, Proposition 4]). There is a multiplicative pairing with the integral spectral sequence by [11, 16.2]. Thus multiplication by β gives an injection from $\mathrm{Pic}(V)/2=H^2(V,\mathbb{Z}(1))/2$ into the subgroup of $E_2^{-3,-5}=H^2(V,\mathbb{Z}/2(5))$ consisting of cycles, i.e., an injection of $\mathrm{Pic}(V)/2$ into $E_3^{-3,-5}$. It also gives the canonical isomorphism from $\mathbb{Z}/2=H^0(V,\mathbb{Z}(0))/2$ to $H^0(V,\mathbb{Z}(4))$ sending 1 to β, which survives to E_∞. We now proceed on a case-by-case basis.

If $V(\mathbb{R})$ is empty, there are no differentials and the spectral sequence collapses to yield the isomorphism. If $V(\mathbb{R})$ is non-empty but has no loops, we see from (6.7) of [23] that $\mathrm{Pic}(V)/2$ is $E_3^{-3,-5}=E_\infty^{-3,-5}$, and that all other terms in the associated graded group for $K_8(V;\mathbb{Z}/8)$ vanish except for $E_\infty^{-4,-4}=\mathbb{Z}/m$ (on the Bott element). If V is affine and $V(\mathbb{R})$ has a loop, the same argument applies using (7.5) of [23]. Finally, the projective case follows by piecing this together with [23, 7.1]. □

Remark 4.11.1. If $V(\mathbb{R})\neq\emptyset$ then $\mathrm{Pic}(V)/2$ is $(\mathbb{Z}/2)^\lambda$, where λ is the number of compact components (circles) of $V(\mathbb{R})$. If $V(\mathbb{R})=\emptyset$ then $\mathrm{Pic}(V)/2$ is $\mathbb{Z}/2$ if V is projective, and is zero if V is affine. See [23].

Corollary 4.12. *If V is a smooth curve over $\mathbb{R}$ then α_n: $K_n(V;\mathbb{Z}/m)\to KR_n(V;\mathbb{Z}/m)$ is an isomorphism for all $n\geqslant 0$.*

Proof. If m is odd, this is 4.4, so we may assume that $m=2^\nu$. Using the coefficient sequences for $\mathbb{Z}/2^\nu\to\mathbb{Z}/2$, we see that it suffices to prove the result for $m=2$. By our Main Theorem 4.8, the result is true for $n>0$. But α is a map of ring spectra, so the result follows from 4.11 and the following square:

$$\begin{array}{ccc} K_0(V)/2 & \xrightarrow{\alpha_0} & KR_0(V;\mathbb{Z}/2) \\ \downarrow \cong & & \downarrow \cong \\ K_8(V;\mathbb{Z}/m) & \xrightarrow{\cong \alpha_8} & KR_8(V;\mathbb{Z}/2). \end{array}$$

□

Remark 4.12.1. In spite of this periodicity, there is a little difference between the integral group $K_0(V)$ and $K_8(V)$. Suppose that m is a power of 2, and X is smooth projective curve with a real point. Then $K_0(X)/m = (\mathbb{Z}/m)^2 \oplus (Z/2)^{\lambda-1}$, but $K_8(X)/m = (\mathbb{Z}/2)^{\lambda-1}$ and ${}_mK_7(X) = (\mathbb{Z}/m)^2$ (see the main theorem of [23]). Thus there is a migration in the universal coefficient filtration of $K_*(V)$.

5. Varieties with no Real points

In this section we prove the following result, which was used in the proof of Theorem 4.7. For each V, the canonical map $\mathbf{K}(V) \to \mathbf{K}(V_{\mathbb{C}})^{hG}$ induces homomorphisms $K_n(V;\mathbb{Z}/m) \to \pi_n(\mathbf{K}(V_{\mathbb{C}})^{hG};\mathbb{Z}/m)$.

Theorem 5.1. *If V is a smooth real variety with $V(\mathbb{R}) = \emptyset$, and $m = 2^v$, then the canonical maps*

$$K_n(V;\mathbb{Z}/m) \to \pi_n(\mathbf{K}(V_{\mathbb{C}})^{hG};\mathbb{Z}/m)$$

are isomorphisms for all $n \geqslant \dim(V) - 1$.

Remark 5.1.1. If E is the homotopy fiber of $\mathbf{K}(V) \to \mathbf{K}(V_{\mathbb{C}})^{hG}$, Theorem 5.1 implies that $\pi_n(E;\mathbb{Z}/m) = 0$ for all $n \geqslant \dim(V) - 1$.

Our proof will use the connection between motivic cohomology and algebraic K-theory. For each non-negative q there is a cochain complex $\mathbb{Z}(q)$ of Zariski sheaves on V, constructed in [13], and its hypercohomology $H^n(V,\mathbb{Z}(q))$ is called the *motivic cohomology* of V. Recall that the hypercohomology of any complex C of sheaves on V is defined as the cohomology of a chain complex of abelian groups $\mathbb{R}\Gamma_V C$, quasi-isomorphic to the complex of global sections of the canonical flasque resolution of C.

5.2 There is a motivic-to-K-theory spectral sequence of the form

$$E_2^{p,q} = H^{p-q}(V,\mathbb{Z}(-q)) \Rightarrow K_{-p-q}(V). \tag{5.2.1}$$

If V is a field, it was constructed by Bloch and Lichtenbaum [33]. Friedlander and Suslin give a general construction in [11, 13.6], using the tower of spaces:

$$T_{q+1} \to T_q \to \cdots \to T_0, \qquad T_q = \Omega\mathbb{R}\Gamma_V(\Omega^{-1}\mathscr{K}^q),$$

ending in $T_0 \simeq \mathbf{K}(V)$. The homotopy cofibers F_q in this tower are the generalized Eilenberg–MacLane spectra associated to the global sections $\mathbb{R}\Gamma_V\mathbb{Z}(q)$ of the canonical flasque resolution of $\mathbb{Z}(q)$, suspended $2q$ times; see 13.7 and 13.11.1 of [11]. That is, $\pi_n F_q = H^{2q-n}(V,\mathbb{Z}/m(q))$. In fact, (5.2.1) is just the usual spectral sequence of a tower, $E_{pq}^2 = \pi_{p+q}(F_q) \Rightarrow \pi_{p+q}T_0$, reindexed as a cohomology spectral sequence.

Smashing the tower with a Moore spectrum M, this gives rise to the motivic-to-K-theory spectral sequence of [11, 16.2]:

$$E_2^{p,q} = H^{p-q}(V, \mathbb{Z}/m(-q)) \Rightarrow K_{-p-q}(V; \mathbb{Z}/m). \tag{5.2.2}$$

The Galois group G acts upon the tower T_* for $V_{\mathbb{C}}$. Taking homotopy fixed points yields a new tower $(T_*)^{hG}$ ending in $\mathbf{K}(V_{\mathbb{C}})^{hG}$, and two spectral sequences converging to its homotopy groups, $\pi_*(\mathbf{K}(V_{\mathbb{C}})^{hG})$ and $\pi_*(\mathbf{K}(V_{\mathbb{C}})^{hG}; \mathbb{Z}/m)$. In order to identify the E_2-terms $\pi_*(F_q^{hG})$, we need a remark about group hypercohomology.

Suppose that G acts on a cochain complex C^*. We write C^{hG} for the right derived cochain complex $\mathbb{R}F(C^*)$, where $F(C) = C^G$. This notation is justified by the well known fact (see [32, p. 533]) that if E is the generalized Eilenberg–MacLane spectrum of a complex C^* of abelian groups then the (topologist's) homotopy fixed point spectrum E^{hG} is the generalized Eilenberg–MacLane spectrum for C^{hG}. Thus the homotopy groups of E^{hG} give the group hypercohomology $H^*(G; C^*)$.

In our case, G acts on the complexes $\mathbb{R}\Gamma_{V_{\mathbb{C}}}\mathbb{Z}(i)$ of abelian groups associated to the complexes of sheaves $\mathbb{Z}(i)$ on $V_{\mathbb{C}}$. Hence $\pi_n(F_q^{hG}) = H^{2q-n}(G; \mathbb{R}\Gamma_{V_{\mathbb{C}}}\mathbb{Z}(-q))$ and $\pi_n(F_q^{hG}; \mathbb{Z}/m) = H^{2q-n}(G; \mathbb{R}\Gamma_{V_{\mathbb{C}}}\mathbb{Z}/m(-q))$. Therefore the new spectral sequence may be written as

$$^hE_2^{p,q} = H^{p-q}(G, \mathbb{R}\Gamma_{V_{\mathbb{C}}}\mathbb{Z}/m(-q)) \Rightarrow \pi_{-p-q}(\mathbf{K}(V_{\mathbb{C}})^{hG}; \mathbb{Z}/m). \tag{5.3}$$

We will need the following elementary result. Recall [39, 1.2.7] that the good truncation $\tau_{\leqslant i}C$ of a cochain complex C is the subcomplex with $(\tau C)^n$ equal to: C^n for $n < i$; Z^i for $n = i$; 0 for $n > i$. It satisfies: $H^n\tau_{\leqslant i}C$ is $H^n(C)$ if $n \leqslant i$, and zero otherwise.

Lemma 5.4. *Let $\mathscr{A}$ be an abelian category with enough injectives, and $F: \mathscr{A} \to \mathscr{B}$ an additive functor. Then for every good truncation $\tau = \tau_{\leqslant i}$ the total right derived functor $\mathbb{R}F$ satisfies $\tau\mathbb{R}F \cong \tau\mathbb{R}F\tau$. That is, for every cochain complex C in $\mathscr{A}$ the canonical map $\tau C \to C$ induces an isomorphism*:

$$\tau\mathbb{R}F(\tau C) \xrightarrow{\cong} \tau(\mathbb{R}FC).$$

Proof. Choose a Cartan–Eilenberg resolution $C^* \to I^{*,*}$ and recall that $d(C^i) \to d^h(I^{i,*})$ is an injective resolution. Hence if $\tau^h I$ is the double complex obtained by applying τ to each row $I^{*,q}$ then $\tau C \to \tau^h I$ is also a Cartan–Eilenberg resolution. Applying F to $\tau^h I \to I$ yields a morphism of double complexes which on total complexes gives $\mathbb{R}F(\tau C) \to \mathbb{R}F(C)$. This is an injection, and the cokernel complex K is concentrated in degrees $> i$.

$$0 \to I^{i+1,0}/dI^{i,0} \to I^{i+2,0} \oplus I^{i+1,1}/dI^{i,1} \to \cdots$$

Since $\tau(K) = 0$ and τ is a triangulated functor, we have $\tau\mathbb{R}F(\tau C) \cong \tau\mathbb{R}F(C)$. □

Example 5.5. (a) For every morphism $f: X \to Y$, the higher direct image $\mathbb{R}f_*$ satisfies $\tau\mathbb{R}f_* = \tau\mathbb{R}f_*\tau$. Indeed, for every complex C of sheaves on X, $\tau\mathbb{R}f_*(\tau C) \cong \tau(\mathbb{R}f_*C)$.

(b) Applying Lemma 5.4 to $F: \mathbb{Z}[G]$-**mod** $\to$ **Ab**, $F(C) = C^G$, and writing C^{hG} for $\mathbb{R}F(C)$ as above, we obtain the formula:

$$\tau((\tau C)^{hG}) \xrightarrow{\cong} \tau(C^{hG}).$$

(c) For the same reasons, if $\mathscr{B}$ is the category of sheaves of abelian groups on some space, and $\mathscr{A}$ is the category of sheaves of G-modules then the formula of (b) holds.

We are now ready to prove Theorem 5.1. For reasons of exposition, we shall treat the function field first (in 5.8). The reason is that if k is a field then a sheaf of abelian groups on Spec(k) is just an abelian group. To emphasize the dependence on k, we will write $\mathbb{Z}(i)_k$ for the cochain complex of abelian groups whose cohomology is the motivic cohomology of Spec(k).

If $k \subset \ell$ is a Galois field extension with Galois group G, then G acts on the cochain complexes $\mathbb{Z}(i)_\ell$. In fact, since $\mathbb{Z}(i)$ is a complex of étale sheaves [30, 3.1] we even have $\mathbb{Z}(i)_k = \mathbb{Z}(i)_\ell^G$. Similarly, $\mathbb{Z}/m(i)_k = \mathbb{Z}/m(i)_\ell^G$ for each m.

Theorem 5.6. *Let $k \subset \ell$ be a Galois field extension with Galois group G, and $m = 2^\nu$. Then for all i:*

$$\mathbb{Z}/m(i)_k \simeq \tau_{\leqslant i}(\mathbb{Z}/m(i)_\ell^{hG}).$$

When $i \geqslant cd_m(\ell)$ and $i \geqslant cd_m(k)$ this simplifies to: $\mathbb{Z}/m(i)_k \simeq \mathbb{Z}/m(i)_\ell^{hG}$.

Proof. Write Γ and Γ' for the absolute Galois groups of k_{sep} over k and ℓ, respectively, so that $G = \Gamma/\Gamma'$. If μ is a Galois module for Γ then $\mu^\Gamma = \pi_*(\mu)$, where π: Spec$(k)_{\text{et}} \to$ Spec$(k)_{\text{zar}}$, and similarly $\mu^{\Gamma'} = \pi'_*(\mu)$, where π': Spec$(\ell)_{\text{et}} \to$ Spec$(\ell)_{\text{zar}}$, Hence the total derived functor for $\pi_*(\mu) = \mu^\Gamma$ is a cochain complex $\mathbb{R}\pi_*\mu$ whose cohomology gives $H^*_{\text{et}}(\text{Spec}(k), \mu) = H^*(\Gamma, \mu)$. As in [39, 10.8.3], [20, p. 105], we can replace the Hochschild–Serre spectral sequence $H^p(G, H^q(\Gamma', \mu)) \Rightarrow H^{p+q}(\Gamma, \mu)$ by the equation $\mathbb{R}\pi_*\mu \cong (\mathbb{R}\pi'_*\mu)^{hG}$. Example 5.5(b) applied to $C = \mathbb{R}\pi'_*\mu$ shows that we have

$$\tau(\mathbb{R}\pi_*\mu) \cong \tau((\tau\mathbb{R}\pi'_*\mu)^{hG}).$$

We apply this with $\mu = \mu_m^{\otimes i}$ and $\tau = \tau_{\leqslant i}$. The Beilinson–Lichtenbaum conjecture, which follows for $m = 2^\nu$ by Voevodsky's theorem [36] and [30, 7.4], $\mathbb{Z}/m(i)_k \cong \tau\mathbb{R}\pi_*\mu$ and similarly $\mathbb{Z}/m(i)_\ell \cong \tau\mathbb{R}\pi'_*\mu$. Thus we have $\mathbb{Z}/m(i)_k \cong \tau(\mathbb{Z}/m(i)_\ell)^{hG}$, as desired.

When i is at least the étale cohomological dimension of k and ℓ, this simplifies. Indeed, $\mathbb{Z}/m(i)_k \cong \tau(\mathbb{R}\pi_*\mu) \cong \mathbb{R}\pi_*\mu$ and $\mathbb{Z}/m(i)_\ell \cong \tau(\mathbb{R}\pi'_*\mu) \cong \mathbb{R}\pi'_*\mu$. Translating $\mathbb{R}\pi_*\mu \cong (\mathbb{R}\pi'_*\mu)^{hG}$ into motivic language yields the second assertion. □

Example 5.7. Applying cohomology, we see that $H^n(\mathbb{Z}/m(i)_k) \xrightarrow{\cong} H^n(\mathbb{Z}/m(i)_\ell^{hG})$ if either $n \leqslant i$ or $i \geqslant d = \max\{cd_m(k), cd_m(\ell)\}$. Moreover, both vanish if $n > d$ and $i \geqslant d$.

Now the class of function fields $\mathbb{R}(V)$ of varieties with $V(\mathbb{R}) = \emptyset$ is closed under taking finite extensions. Moreover, the function field $k = \mathbb{R}(V)$ has étale cohomological dimension $cd_2\mathbb{R}(V) = \dim(V)$; see [8, 1.2.1] for a proof.

Theorem 5.8. *Let $k = \mathbb{R}(V)$ be the function field of a real variety V with $V(\mathbb{R}) = \emptyset$. If $m = 2^\nu$, $d = \dim(V)$ and $\ell = k \otimes_{\mathbb{R}} \mathbb{C}$, then $K_n(k; \mathbb{Z}/m) \to \pi_n(\mathbf{K}(\ell)^{hG}; \mathbb{Z}/m)$ is an injection for all n, and an isomorphism for all $n \geqslant d - 1$.*

Proof. There is a morphism from the spectral sequence (5.2.2) for $K_*(k; \mathbb{Z}/m)$ to the spectral sequence (5.3) for $\pi_*(\mathbf{K}(\ell)^{hG}; \mathbb{Z}/m)$. On the E_2-terms it is the cohomology of the maps $\mathbb{Z}/m(i)_k \to$

$$
\begin{array}{ccccccccccc}
 & & & & & H^{0,0} & * & * & * & & (q=0) \\
 & & & \cdots & \cdots & \cdots & * & * & 0 & & \\
 & & H^{0,d-1} & \cdots & H^{d-2,d-1} & H^{d-1,d-1} & * & 0 & & & \\
 & H^{0,d} & H^{1,d} & \cdots & H^{d-1,d} & H^{d,d} & 0 & 0 & & & (q=-d) \\
H^{0,d+1} & H^{1,d+1} & H^{2,d+1} & \cdots & H^{d,d+1} & 0 & 0 & 0 & & &
\end{array}
$$

Fig. 1. The motivic spectral sequence (5.3) for $\pi_*(\mathbf{K}(\ell)^{hG};\mathbb{Z}/m)$. For $q \leqslant -d$ it agrees with the spectral sequence (5.2.2) for $K_*(k;\mathbb{Z}/m)$.

$\mathbb{Z}/m(i)_\ell^{hG}$. By Example 5.7, the groups $E_2^{p,q}$ and ${}^hE_2^{p,q}$ are isomorphic when $p \leqslant 0$ or $q \leqslant -d$; both vanish if $p > 0$ and $q \leqslant -d$. In particular, $E_2^{p,q} \to {}^hE_2^{p,q}$ is an isomorphism if $p+q \leqslant 1-d$ and an injection if $p+q = 2-d$. It follows from Lemma 5.8.2 below that $K_n(k;\mathbb{Z}/m) \to \pi_n(\mathbf{K}(\ell)^{hG};\mathbb{Z}/m)$ is an isomorphism for $n \geqslant d-1$, and an injection for $n = d-2$. (See Fig. 1.)

To conclude, it suffices to show that if $n < d-1$ then $K_n(k;\mathbb{Z}/m) \to \pi_n(\mathbf{K}(\ell)^{hG};\mathbb{Z}/m)$ is an injection. For this it suffices to show that for every $r \geqslant 2$ and every p with $-r < p \leqslant 0$ the differential ${}^hd_r$: ${}^hE_r^{p,q} \to {}^hE_r^{p+r,q-r+1}$ vanishes in (5.3). But because $E_r^{p,q} \cong {}^hE_r^{p,q}$, this factors through the corresponding differential d_r: $E_r^{p,q} \to 0 = E_r^{p+r,q-r+1}$ in the spectral sequence (5.2.2). □

Example 5.8.1. The bound in Theorem 5.8 is best possible. For example, if $k = \mathbb{R}(V)$ with $\dim(V) = d = 2$ then $K_0(k;\mathbb{Z}/m) = \mathbb{Z}/m$ but $\pi_0(\mathbf{K}(\ell)^{hG};\mathbb{Z}/m) \cong \mathbb{Z}/m \oplus \mathbb{Z}/2$. This follows from (5.3) given the calculation: $H^2(\mathbb{Z}/m(1)_\ell^{hG}) = H^2(\mu_m(\ell)^{hG}) = H^2(G,\mu_m(\ell)) \cong \mu_2$.

Lemma 5.8.2. *Let $f_r^{p,q}: E_r^{p,q} \to {}'E_r^{p,q}$ be a morphism of bounded spectral sequences. Suppose, for some N and r_0, that $f_{r_0}^{p,q}$ is an isomorphism for all (p,q) with $p+q \leqslant N$ and an injection for $p+q = N+1$. Then $E_\infty^{p,q} \cong {}'E_\infty^{p,q}$ for all (p,q) with $p+q \leqslant N$.*

Proof. It suffices to show that, for all $r \geqslant r_0$, the map $E_r^{pq} \to {}'E_r^{pq}$ is an isomorphism for all (p,q) with $p+q \leqslant N$ and an injection for $p+q = N+1$. We proceed by induction on r. Fix p and q with $p+q \leqslant N$ and set $p' = p+r$, $q' = q+1-r$. A diagram chase upon the homology of

$$
\begin{array}{ccccccc}
* & \longrightarrow & E_r^{p,q} & \xrightarrow{d_r} & E_r^{p',q'} & \longrightarrow & * \\
\cong\downarrow & & \cong\downarrow & & \text{into}\downarrow & & \downarrow \\
* & \longrightarrow & {}'E_r^{p,q} & \xrightarrow{'d_r} & {}'E_r^{p',q'} & \longrightarrow & *
\end{array}
$$

shows that $E_{r+1}^{p,q} \to {}'E_{r+1}^{p,q}$ is an isomorphism and that $E_{r+1}^{p',q'} \to {}'E_{r+1}^{p',q'}$ is an injection. □

The proof of 5.1 for varieties is similar but uses more technical machinery. Suppose that μ is an étale sheaf on V, and write μ' for its restriction to $V_{\mathbb{C}}$. We have a diagram of sites:

$$
\begin{array}{ccccc}
(V_{\mathbb{C}})_{et} & \xrightarrow{\pi'} & (V_{\mathbb{C}})_{zar} & & \\
\gamma'\downarrow & & \downarrow\gamma & & \\
V_{et} & \xrightarrow{\pi} & V_{zar} & \xrightarrow{\Gamma_V} & \text{point.}
\end{array}
$$

The direct image $\gamma_*\pi'_*\mu' = \pi_*\gamma'_*\mu'$ of μ' is the Zariski sheaf $U \mapsto \mu(U_{\mathbb{C}})$ on V. The Galois group G acts on $\gamma'_*\mu'$, and since $\mu(U_{\mathbb{C}})^G = \mu(U)$ we have $\pi_*(\gamma'_*\mu')^G = (\pi_*\gamma'_*\mu')^G = \pi_*\mu$.

Lemma 5.9. *For every étale sheaf* μ on V, $\mathbb{R}\pi_*\mu \cong (\mathbb{R}\gamma_*\mathbb{R}\pi'_*\mu')^{hG}$ *in the derived category* $\mathbf{D}(V_{\text{zar}})$. *For each truncation* $\tau = \tau_{\leqslant i}$ *we have a canonical isomorphism*

$$\tau[(\tau\mathbb{R}\gamma_*(\tau\mathbb{R}\pi'_*\mu'))^{hG}] \xrightarrow{\cong} \tau\mathbb{R}\pi_*\mu.$$

Proof. We may regard the functor $\mu \mapsto \gamma_*\pi'_*\mu'$ as landing in the abelian category $\mathscr{A}$ of Zariski sheaves of G-modules on V. The composition with $\mathscr{F} \mapsto \mathscr{F}^G$ is the functor π_*. As observed in [20, III.2.20], if $\mathscr{I}$ is an injective étale sheaf on V then $\gamma_*\pi'_*\mathscr{I}'$ is G-acyclic. By [39, 10.8.3], this yields an isomorphism $\mathbb{R}\pi_*\mu \cong (\mathbb{R}\gamma_*\mathbb{R}\pi'_*\mu')^{hG}$ for any étale sheaf μ. This is our first assertion.

We now apply the truncation τ, and use examples 5.5(a, c) to see that $\tau(\mathbb{R}\gamma_*C)^{hG} \cong \tau(\tau\mathbb{R}\gamma_*\tau C)^{hG}$ for $C = \mathbb{R}\pi'_*\mu'$. This is our second assertion. □

Remark 5.9.1. Let $\Gamma_V(\mathscr{F}) = \mathscr{F}(V)$ be the global sections functor. If $\mathscr{F}$ is a Zariski sheaf of G-modules then $\Gamma_V(\mathscr{F}^G) = (\Gamma_V\mathscr{F})^G$. It follows that the two derived functors commute: $\mathbb{R}\Gamma_V(\mathscr{F}^{hG}) = (\mathbb{R}\Gamma_V\mathscr{F})^{hG}$.

Remark 5.9.2. The Hochschild–Serre spectral sequence

$$H^p(G, H^q_{\text{et}}(V_{\mathbb{C}}, \mu')) \Rightarrow H^{p+q}_{\text{et}}(V, \mu)$$

(see [20, III.2.20]) follows from Lemma 5.9 and Remark 5.9.1. Indeed, applying $\mathbb{R}\Gamma_V$ to Lemma 5.9 yields the isomorphism $\mathbb{R}\Gamma_V\mathbb{R}\pi_*\mu \cong (\mathbb{R}\Gamma_V\mathbb{R}\gamma_*\mathbb{R}\pi'_*\mu')^{hG}$ in the derived category $\mathbf{D}(\mathbf{Ab})$. As observed in loc. cit., the Hochschild–Serre spectral sequence is just the group hypercohomology spectral sequence of the right-hand side.

Write $\mathbb{Z}/m(i)_V$ and $\mathbb{Z}/m(i)_{V_{\mathbb{C}}}$ for the restrictions of $\mathbb{Z}/m(i)$ to V and $V_{\mathbb{C}}$, respectively. Here is the analogue of Theorem 5.6.

Proposition 5.10. *When* $m = 2^\nu$, *we have*: $\mathbb{Z}/m(i)_V \cong \tau_{\leqslant i}[(\mathbb{R}\gamma_*\mathbb{Z}/m(i)_{V_{\mathbb{C}}})^{hG}]$. *When* $V(\mathbb{R}) = \emptyset$ *and* $i \geqslant \dim(V)$, *this simplifies to*

$$\mathbb{Z}/m(i)_V \cong \mathbb{R}\pi_*\mu_m^{\otimes i} \cong (\mathbb{R}\gamma_*\mathbb{Z}/m(i)_{V_{\mathbb{C}}})^{hG}.$$

Proof. Set $\mu = \mu_m^{\otimes i}$. By Voevodsky's theorem [36] and [30, 7.4], we have $\mathbb{Z}/m(i)_V \cong \tau\mathbb{R}\pi_*\mu$ and $\mathbb{Z}/m(i)_{V_{\mathbb{C}}} \cong \tau\mathbb{R}\pi'_*\mu'$. Applying 5.5(c) to $C = \mathbb{R}\gamma_*\mathbb{Z}/m(i)_{V_{\mathbb{C}}}$ in Lemma 5.9, we get the first assertion:

$$\mathbb{Z}/m(i)_V \cong \tau[(\tau\mathbb{R}\gamma_*\mathbb{Z}/m(i)_{V_{\mathbb{C}}})^{hG}] \cong \tau[(\mathbb{R}\gamma_*\mathbb{Z}/m(i)_{V_{\mathbb{C}}})^{hG}].$$

When $V(\mathbb{R}) = \emptyset$, the local rings of V and $V_{\mathbb{C}}$ have étale cohomological dimension at most d. Thus if $i \geqslant \dim(V)$ then $\mathbb{Z}/m(i)_V \cong \mathbb{R}\pi_*\mu$ and $\mathbb{Z}/m(i)_{V_{\mathbb{C}}} \cong \mathbb{R}\pi'_*\mu'$, and the second assertion is just the isomorphism $\mathbb{R}\pi_*\mu \cong (\mathbb{R}\gamma_*\mathbb{R}\pi'_*\mu')^{hG}$ of Lemma 5.9. □

Now $(\Gamma_V)\gamma_* = \Gamma_{V_{\mathbb{C}}}$ is the global sections functor on $V_{\mathbb{C}}$. Applying $\mathbb{R}\Gamma_V$ to the universal map from $\mathbb{Z}/m(i)$ to $\mathbb{R}\gamma_*\mathbb{Z}/m(i)^{hG}_{V_{\mathbb{C}}}$ and using 5.9.1, we get a canonical map from $\mathbb{R}\Gamma_V\mathbb{Z}/m(i)$ to $\mathbb{R}\Gamma_V(\mathbb{R}\gamma_*\mathbb{Z}/m(i)^{hG}_{V_{\mathbb{C}}}) = (\mathbb{R}\Gamma_{V_{\mathbb{C}}}\mathbb{Z}/m(i))^{hG}$ in the derived category $\mathbf{D}(\mathbf{Ab})$. Applying cohomology gives a canonical map

$$H^n(V, \mathbb{Z}/m(i)) \to H^n(V, (\mathbb{R}\gamma_*\mathbb{Z}/m(i)_{V_{\mathbb{C}}})^{hG}) \cong H^n(G, \mathbb{R}\Gamma_{V_{\mathbb{C}}}\mathbb{Z}/m(i)_{V_{\mathbb{C}}}).$$

Corollary 5.11. *If either $n \leqslant i$, or $V(\mathbb{R}) = \emptyset$ and $i \geqslant \dim(V)$, then the canonical map is an isomorphism*:

$$H^n(V, \mathbb{Z}/m(i)) \cong H^n_{\text{et}}(V, \mu_m^{\otimes i}) \xrightarrow{\cong} H^n(G, \mathbb{R}\Gamma_{V_{\mathbb{C}}}\mathbb{Z}/m(i)_{V_{\mathbb{C}}}).$$

These groups vanish if $i \geqslant \dim(V)$ and $n > cd_m(V)$.

If $n = i+1$, the maps $H^{i+1}(V, \mathbb{Z}/m(i)) \to H^{i+1}(G, \mathbb{R}\Gamma_{V_{\mathbb{C}}}\mathbb{Z}/m(i)_{V_{\mathbb{C}}})$ are injections.

Proof. Suppose first that $i \geqslant d = \dim(V)$. Applying $\mathbb{R}\Gamma_V$ to 5.10, we see from 5.9.1 that

$$\mathbb{R}\Gamma_V \mathbb{Z}/m(i) \cong \mathbb{R}(\Gamma_V \pi_*)\mu_m^{\otimes i} \cong (\mathbb{R}\Gamma_V \mathbb{R}\gamma_* \mathbb{Z}/m(i)_{V_{\mathbb{C}}})^{hG} \cong (\mathbb{R}\Gamma_{V_{\mathbb{C}}}\mathbb{Z}/m(i)_{V_{\mathbb{C}}})^{hG}$$

in $\mathbf{D}(\mathbf{Ab})$. Applying H^n yields the result for all n, since $H^n_{\text{et}}(V, \mu) = 0$ for $n > cd_m(V)$.

If $i < d$, we apply $\tau\mathbb{R}\Gamma_V$ to 5.10. Using the formula $\tau(\mathbb{R}\Gamma_V)\tau = \tau\mathbb{R}\Gamma_V$ of 5.5(a) and 5.9.1, we get $\tau\mathbb{R}\Gamma_V\mathbb{Z}/m(i) \cong \tau(\mathbb{R}\Gamma_{V_{\mathbb{C}}}\mathbb{Z}/m(i))^{hG}$. Applying H^n for $n \leqslant i$ yields the first assertion, since in this case $H^n\tau\mathbb{R}\Gamma_V C = H^n\mathbb{R}\Gamma_V C = H^n(V, C)$ for all C.

When $n = i+1$, set $C = (\mathbb{R}\gamma_*\mathbb{Z}/m(i)_{V_{\mathbb{C}}})^{hG}$. Since the complex $C/\tau C$ is zero in degrees at most i, we always have $H^i(V, C/\tau C) = 0$. Hence $H^{i+1}(V, \tau C)$ always injects into $H^{i+1}(V, C)$. □

We may now modify the proof of Theorem 5.8 to prove Theorem 5.1 in the general case.

Proof of 5.1. There is a morphism from the spectral sequence (5.2.2) for $K_*(V; \mathbb{Z}/m)$ to the spectral sequence (5.3) for $\pi_*(\mathbf{K}(V_{\mathbb{C}})^{hG}; \mathbb{Z}/m)$. On the $E_2^{p,q}$-terms it is the canonical map from $H^{p-q}(V, \mathbb{Z}/m(-q)) = H^{p-q}\mathbb{R}\Gamma_V\mathbb{Z}/m(-q)$ to $H^{p-q}(\mathbb{R}\Gamma_{V_{\mathbb{C}}}\mathbb{Z}/m(-q)_{V_{\mathbb{C}}})^{hG}$.

Now assume that $V(\mathbb{R}) = \emptyset$, so that $cd_m(V) \leqslant 2\dim(V)$. By 5.11, the groups $E_2^{p,q}$ and ${}^hE_2^{p,q}$ are isomorphic when $p \leqslant 0$ or $q \leqslant -d$, and vanish in the region $q \leqslant -d$, $p > q + cd_m(V)$. It follows from Lemma 5.8.2 that $K_n(V; \mathbb{Z}/m) \to \pi_n(\mathbf{K}(V_{\mathbb{C}})^{hG}; \mathbb{Z}/m)$ is an isomorphism for $n \geqslant d$, and that both spectral sequences are bounded and convergent.

If $p+q = 1-d$, we also see immediately that $E_r^{p,q} \cong {}^hE_r^{p,q}$ if either $p \geqslant 0$ or $p \leqslant -r$. We must analyze d_r: $E_r^{p,q} \to E_r^{1,1-d}$ for $p = -r+1$. Because $E_2^{1,1-d} = H^d(V, \mathbb{Z}/m(d-1))$, $E_r^{1,1-d} \to {}^hE_r^{1,1-d}$ is an injection for $r = 2$ by 5.11. It follows by induction that it is an injection for every $r > 2$, and that $E_r^{p,q} \to {}^hE_r^{p,q}$ is an isomorphism for all r when $p+q = 1-d$. This implies that $K_{d-1}(V; \mathbb{Z}/m) \cong \pi_{d-1}(\mathbf{K}(V_{\mathbb{C}})^{hG}; \mathbb{Z}/m)$ as well. □

6. Integral descent with no Real Points

In this section we fix a real variety V with no real points, and study the canonical map $\mathbf{K}(V) \to \mathbf{K}(V_{\mathbb{C}})^{hG}$, where $G = \text{Gal}(\mathbb{C}/R)$. Here is our main result.

Theorem 6.1. *Let V be a real variety with no real points, and set $d = \dim(V)$. Then the map $K_n(V) \to \pi_n\mathbf{K}(V_{\mathbb{C}})^{hG}$ is an isomorphism for all $n \geqslant d-1$ and an injection for $n = d-2$. For all n, the kernel and cokernel of this map are 2-primary torsion groups of bounded exponent.*

As in the previous section, we begin by establishing the result for the function field $k = \mathbb{R}(V)$ of V (in 6.5), in order to present the thrust of the argument without the distraction of certain technical group hypercohomology issues for G-sheaves.

For the next few lemmas we will fix i, $d=\dim(V)$, $k=\mathbb{R}(V)$ and $\ell = k\otimes_{\mathbb{R}}\mathbb{C}$, as in the last section. It is convenient to introduce the Serre subcategory $\mathscr{T}$ of 2-primary torsion abelian groups of bounded exponent. A homomorphism of abelian groups is an isomorphism modulo $\mathscr{T}$ just in case its cokernel and cokernel lie in $\mathscr{T}$.

Lemma 6.2. *If $q>i$ then $H^q(k,\mathbb{Z}(i)) = H^q(\ell,\mathbb{Z}(i)) = 0$. If either $q<0$ or $d+2\leqslant q\leqslant i$, both $H^q(k,\mathbb{Z}(i))$ and $H^q(\ell,\mathbb{Z}(i))$ are uniquely 2-divisible groups.*

Proof. It suffices to prove the result for k, since $\ell=\mathbb{R}(V_{\mathbb{C}})$. Set $\mathbb{Z}(i)=\mathbb{Z}(i)_k$, so that $H^q(k,\mathbb{Z}(i))= H^q\mathbb{Z}(i)$. Because $\mathbb{Z}(i)$ is defined to be zero in degrees $>i$, $H^q\mathbb{Z}(i)=0$ for $q>i$. Now recall that $\mathbb{Z}(i)\otimes^{\boldsymbol{L}}\mathbb{Z}/2=\mathbb{Z}/2(i)$, so that there is an exact sequence

$$H^{q-1}\mathbb{Z}/2(i)\to H^q\mathbb{Z}(i)\xrightarrow{2} H^q\mathbb{Z}(i)\to H^q\mathbb{Z}/2(i). \tag{6.2.1}$$

For $q\leqslant i$ we have $H^q\mathbb{Z}/2(i)\cong H^q_{\text{et}}(k,\mathbb{Z}/2)$, and this group vanishes unless $0\leqslant q\leqslant d = cd_2(k)$. From (6.2.1) we see that $H^q\mathbb{Z}(i)$ is uniquely 2-divisible unless $0\leqslant q\leqslant d+1$. □

As noted in Section 5, G acts on the chain complex of abelian groups $\mathbb{Z}(i)_\ell$, and hence on the cohomology $H^n(\ell,\mathbb{Z}(i)) = H^n\mathbb{Z}(i)_\ell$.

Lemma 6.3. *For each i, the edge map η: $H^q(G,\mathbb{Z}(i)_\ell)\to H^q(\ell,\mathbb{Z}(i))^G$ is an isomorphism for $q\leqslant 0$. For $q\geqslant 0$, it is an isomorphism modulo $\mathscr{T}$. If $q>i\geqslant d$, both groups are zero.*

Proof. By 6.2 and [39, 6.1.10], we have $H^p(G,H^q(\ell,\mathbb{Z}(i)))=0$ for $p\neq 0$, provided that either $q<0$ or $q>\min\{i,d+1\}$. Hence the group hypercohomology spectral sequence [39, 6.1.15] is bounded and converges:

$$E_2^{p,q} = H^p(G,H^q(\ell,\mathbb{Z}(i)))\Rightarrow H^{p+q}(G,\mathbb{Z}(i)_\ell).$$

Since the terms off the q-axis are all groups of exponent 2, it follows immediately from this spectral sequence that the edge map $H^q(G,\mathbb{Z}(i)_\ell)\to H^q(\ell,\mathbb{Z}(i))^G$ is an isomorphism for $q\leqslant 0$, and that for $q>0$ the kernel and cokernel have exponent at most $\min\{2^q,2^{i+1},2^{d+2}\}$.

For $q>i$ this implies that $H^q(G,\mathbb{Z}(i)_\ell)$ is in $\mathscr{T}$. Since $H^q(G,\mathbb{Z}/2(i)_\ell)=0$ for $i\geqslant d$ by 5.7, the universal coefficient sequence shows that $H^q(G,\mathbb{Z}(i)_\ell)$ must be 2-divisible, and hence zero. Since $H^q(\ell,\mathbb{Z}(i))=0$ for $q>i$ as well, we are done. □

Corollary 6.4. *$H^q(k,\mathbb{Z}(i))\to H^q(G,\mathbb{Z}(i)_\ell)$ is an isomorphism for $q>i\geqslant d$ and $q<0$. It is an isomorphism modulo $\mathscr{T}$ for all q.*

Proof. If $q>i\geqslant d$, both groups are zero by 6.2 and 6.3. For other q it suffices by 6.3 to consider the natural map $H^q(k,\mathbb{Z}(i))\to H^q(\ell,\mathbb{Z}(i))^G$. Its kernel and cokernel have exponent 2 because both of its compositions with the transfer map $H^q(\ell,\mathbb{Z}(i))^G\to H^q(k,\mathbb{Z}(i))$ are multiplication by 2. If $q<0$ then both groups are uniquely 2-divisible by 6.2, so these maps are isomorphisms inverse to each other. □

Proposition 6.5. *Fix a function field $k=\mathbb{R}(V)$ of a real variety V with no real points, and set $d=\dim(V)$, $\ell=k\otimes_{\mathbb{R}}\mathbb{C}$. Then the map $K_n(k)\to\pi_n\mathbf{K}(\ell)^{hG}$ is an isomorphism for all $n\geqslant d-1$ and* *an injection for $n = d - 2$. For all n, the kernel and cokernel of this map are 2-primary torsion groups of bounded exponent.*

Proof. Now $H^n(G, \mathbb{Z}(i)) = 0$ for $i < 0$ (as $\mathbb{Z}(i) = 0$ by definition), and for $n > i \geqslant d$ by 6.3. This shows that the integral version of (5.3) is a bounded spectral sequence:

$$^hE_2^{p,q} = H^{p-q}(G, \mathbb{Z}(-q)_\ell) \Rightarrow \pi_{-p-q}\mathbf{K}(\ell)^{hG}. \tag{6.5.1}$$

In the morphism of spectral sequences, from (5.2.1) to (6.5.1), the E_2 terms are isomorphisms modulo $\mathscr{T}$ by 6.4. By the Comparison Theorem [39, 5.2.12], the maps $K_n(k) \to \pi_n\mathbf{K}(\ell)^{hG}$ are all isomorphisms modulo $\mathscr{T}$. It follows that the homotopy groups $\pi_*(E)$ of the homotopy fiber E are all in $\mathscr{T}$. By 5.1.1, $\pi_n(E; \mathbb{Z}/2) = 0$ for all $n \geqslant d - 1$. It follows that $\pi_n(E) = 0$ for all $n \geqslant d - 2$. Hence $K_n(k) \to \pi_n\mathbf{K}(\ell)^{hG}$ is an isomorphism for all $n \geqslant d - 1$, and an injection for $n = d - 2$. □

The above argument does not go through smoothly for sheaves on V, because of the following technical problem. Suppose we are given an unbounded chain complex C of sheaves of G-modules, such as $\mathbb{R}\gamma_*\mathbb{Z}(i)$. Although we know that the group hypercohomology complex C^{hG} exists by [27], we do not know if we can use the usual Cartan–Eilenberg resolution to construct it, because the category $\mathscr{S}$ of sheaves of G-modules does not satisfy (AB4*); products are not exact. It is difficult to compute the hypercohomology sheaves $H^*(G; C)$ of a general complex C for the same reason.

Here is how to construct group hypercohomology of a complex C in $\mathscr{S}$. We say that a complex L in $\mathscr{S}$ is *fibrant* (in $\mathscr{S}$) if for every acyclic complex A in $\mathscr{S}$ the complex $\mathrm{Hom}_{\mathscr{S}}(A, L)$ is acyclic. If $C \to L$ is a quasi-isomorphism with L fibrant then C^{hG} is defined to be L^G, and $H^*(G, C)$ is defined to be $H^*(C^{hG})$.

Lemma 6.6. *If C is a complex of sheaves of $\mathbb{Z}[\frac{1}{2}]G$-modules on V_{zar} then $C^{hG} \xrightarrow{\simeq} C^G$ and $H^*(G, C) \cong H^*(C^G) = [H^*(C)]^G$.*

Proof. Write $\mathscr{S}[\frac{1}{2}]$ for the category of sheaves of R-modules, where $R = \mathbb{Z}[\frac{1}{2}]G$. Choose a quasi-isomorphism $C \to L$ in $\mathscr{S}[\frac{1}{2}]$ with L fibrant. Because $C \mapsto C[\frac{1}{2}]$ is exact, every fibrant L in $\mathscr{S}[\frac{1}{2}]$ is also fibrant in $\mathscr{S}$. Therefore $C^{hG} = L^G$.

As a ring, $\mathbb{Z}[\frac{1}{2}]G \cong \mathbb{Z}[\frac{1}{2}] \times \mathbb{Z}[\frac{1}{2}]$. Therefore every uniquely 2-divisible G-complex C is naturally the direct sum of the trivial G-complex C^G and a complex C_- on which G acts via the sign representation $\mathbb{Z}[G] \to \mathbb{Z}$. By construction, the quasi-isomorphism $C \simeq L$ induces quasi-isomorphisms $C^G \simeq L^G$ and $C_- \simeq L_-$. □

We are now ready to give the analogues of Lemmas 6.2 and 6.3.

Lemma 6.7. *The Zariski sheaves $H^q\mathbb{Z}(i)_V$ and $R^q\gamma_*\mathbb{Z}(i)_{V_{\mathbb{C}}}$ vanish if $q > i$, and are uniquely 2-divisible sheaves if $q < 0$. If $V(\mathbb{R}) = \emptyset$ and $d = \dim(V)$ they are also uniquely 2-divisible for $d + 2 \leqslant q \leqslant i$.*

Proof. Let S be a local ring of a point $v \in V$. Then the stalk of $H^q\mathbb{Z}(i)$ at v is $H^q(S, \mathbb{Z}(i)) = H^q\mathbb{Z}(i)(S)$. Similarly, the inverse image $S' = \gamma^{-1}S$ is a semilocal scheme, and its stalk at v is

$H^q(S', \mathbb{Z}(i))$, which equals $H^q\mathbb{Z}(i)(S')$ because its terms are Γ-acyclic by [37, 4.27]. Both vanish if $q > i$.

For $q \leqslant i$, we argue as in Lemma 6.2. The stalks are $H^q(S, \mathbb{Z}/2(i)) \cong H^n_{\text{et}}(S, \mu_m^{\otimes i})$ and $H^q(S', \mathbb{Z}/2(i)) \cong H^n_{\text{et}}(S', \mu_m^{\otimes i})$ by 5.11. Since S and S' have étale cohomological dimension $\dim(S)$, the stalks vanish unless $0 \leqslant q \leqslant \dim(S) \leqslant \dim(V)$. □

Corollary 6.8. *If $q < 0$ then the sheaf $H^q(G, \mathbb{R}\gamma_*\mathbb{Z}(i))$ is naturally isomorphic to the sheaf $H^q\mathbb{Z}(i)_V \cong [R^q\gamma_*\mathbb{Z}(i)_{V_{\mathbb{C}}}]^G$.*

Proof. Let $C = \tau_{<0}\mathbb{R}\gamma_*\mathbb{Z}(i)_{V_{\mathbb{C}}}$. By 6.7, C is quasi-isomorphic to $C[\frac{1}{2}]$. But then 6.6 yields

$$H^q(G, \mathbb{R}\gamma_*\mathbb{Z}(i)) = H^q(G, C) \cong H^q(C^G) = [H^q(C)]^G.$$

Again by 6.6, the G-sheaf $H^q(C) = R^q\gamma_*\mathbb{Z}(i)_{V_{\mathbb{C}}}$ is uniquely 2-divisible. The usual transfer argument now shows that $H^q(C)^G$ is isomorphic to the sheaf $H^q\mathbb{Z}(i)_V$. □

Lemma 6.9. *For each i, the edge map $H^q(G, \mathbb{R}\gamma_*\mathbb{Z}(i)_{V_{\mathbb{C}}}) \xrightarrow{\eta^q} [R^q\gamma_*\mathbb{Z}(i)_{V_{\mathbb{C}}}]^G$ is an isomorphism for $q \leqslant 0$. For $q \geqslant 0$, it is an isomorphism modulo $\mathscr{T}$. If $q > i \geqslant d$ and $V(\mathbb{R}) = \emptyset$, both sheaves are zero.*

Proof. If $q < 0$, this follows from 6.8. If $q > 0$ then $H^q(G, \mathbb{R}\gamma_*\mathbb{Z}(i)) = H^q(G, \tau_{\geqslant 0}\mathbb{R}\gamma_*\mathbb{Z}(i))$ because $H^q(G, \tau_{<0}\mathbb{R}\gamma_*\mathbb{Z}(i)) = 0$ by 6.6. Therefore the group hypercohomology spectral sequence (see [39, 6.1.15]) is bounded and converges:

$$H^p(G, R^q\gamma_*\mathbb{Z}(i)) \Rightarrow H^{p+q}(G, \mathbb{R}\gamma_*\mathbb{Z}(i)).$$

It follows that η^0 is also an isomorphism, and that η^q is an isomorphism modulo $\mathscr{T}$ in general. Since $\mathbb{R}\gamma_*\mathbb{Z}/2(i)^{hG} \cong \mathbb{Z}/2(i)$ for $i \geqslant d$ by 5.10, the universal coefficient sequence shows that if $q > i \geqslant d$ then $H^q(G, \mathbb{R}\gamma_*\mathbb{Z}(i))$ is uniquely 2-divisible and hence zero. □

Corollary 6.10. *The map of sheaves $H^q\mathbb{Z}(i) \to H^q(G, \mathbb{R}\gamma_*\mathbb{Z}(i))$ is an isomorphism modulo $\mathscr{T}$ for all q and i. It is an isomorphism if $q < 0$ or $q > i \geqslant d$.*

Proof. The proof of 6.4 goes through, substituting 6.7 for 6.2 and 6.9 for 6.3. □

Proof of 6.1. Let C denote the mapping cone of $\mathbb{Z}(i)_V \to [\mathbb{R}\gamma_*\mathbb{Z}(i)_{V_{\mathbb{C}}}]^{hG}$. By 6.9, the cohomology of C is bounded and lies in $\mathscr{T}$. Hence the cohomology of $\mathbb{R}\Gamma_V C$ is bounded and lies in $\mathscr{T}$, by the hypercohomology spectral sequence. Applying $\mathbb{R}\Gamma_V$ to the map $\mathbb{Z}(i)_V \to [\mathbb{R}\gamma_*\mathbb{Z}(i)_{V_{\mathbb{C}}}]^{hG}$ and using 5.9.1 yields a map from $\mathbb{R}\Gamma_V\mathbb{Z}(i)_V$ to $[\mathbb{R}\Gamma_{V_{\mathbb{C}}}\mathbb{Z}(i)_{V_{\mathbb{C}}}]^{hG}$ whose cone is quasi-isomorphic to $\mathbb{R}\Gamma_V C$. Applying H^n yields a map $H^n(V, \mathbb{Z}(i)) \to H^n(G, \mathbb{R}\Gamma_{V_{\mathbb{C}}}\mathbb{Z}(i)_{V_{\mathbb{C}}})$ which is an isomorphism modulo $\mathscr{T}$. For $i = -q$ and $n = p - q$ this is the map of E_2-terms in the morphism of spectral sequences from (5.2.1) to the integral analogue of (5.3):

$${}^hE_2^{p,q} = H^{p-q}(G, \mathbb{R}\Gamma_{V_{\mathbb{C}}}\mathbb{Z}(-q)) \Rightarrow \pi_{-p-q}(\mathbf{K}(V_{\mathbb{C}})^{hG}).$$

Since these spectral sequences are bounded when $V(\mathbb{R}) = \emptyset$, the morphism of spectral sequences converges to a map $K_*(V) \to \pi_*(\mathbf{K}(V_{\mathbb{C}})^{hG})$ which is an isomorphism modulo $\mathscr{T}$.

Let us write E for the homotopy fiber of the map $\mathbf{K}(V) \to \mathbf{K}(V_{\mathbb{C}})^{hG}$. It follows that the homotopy groups $\pi_*(E)$ of E are all in $\mathscr{T}$. By 5.1.1, $\pi_n(E;\mathbb{Z}/2)=0$ for all $n \geqslant d-1$. It follows that $\pi_n(E)=0$ for all $n \geqslant d-2$. Hence $K_n(V) \to \pi_n\mathbf{K}(V_{\mathbb{C}})^{hG}$ is an isomorphism for all $n \geqslant d-1$, and an injection for $n=d-2$. □

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Appendix A. Calculations

The point of this short section is to show how $KR^*(X)$ may be computed in some cases of interest.

Theorem A.1. *Let R_d denote the coordinate ring $\mathbb{R}[x_0,\dots,x_d]/(\sum x_j^2 = 1)$ of the d-sphere, and $V = \mathrm{Spec}(R_d)$ the corresponding affine variety. Then (for $m = 2^\nu$):*

$$K_n(R_d;\mathbb{Z}/m) \cong KO^{-n}(S^d;\mathbb{Z}/m), \quad n \geqslant 0.$$

Proof. We claim that there is an equivariant deformation retraction from $V(\mathbb{C})$ onto $S^d = V(\mathbb{C})^G$. Since G acts trivially on S^d, this yields $KR^*(V(\mathbb{C})) \cong KR^*(S^d) = KO^*(S^d)$, and similarly with finite coefficients. Therefore our main theorem yields the result for all $n \geqslant d-1$. To extend the result to all $n \geqslant 0$, we need the fact that the even part C_0 of every Clifford algebra over $\mathbb{R}$ is a matrix ring over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Therefore the groups $K_*(C_0;\mathbb{Z}/m)$ are 8-periodic by [28, 2.9 and 3.5]. It follows from [31, Theorem 2] that the groups $K_n(R_d;\mathbb{Z}/m)$ are 8-periodic for all $n \geqslant 0$, whence the result.

It remains to establish the claim. Write $z_j = a_j + ib_j$ with the a_j, b_j real. Then $z = (z_0,\dots,z_d)$ is a point on $V(\mathbb{C})$ iff $\sum a_j^2 = 1 + \sum b_j^2$ and $\sum a_j b_j = 0$. For $t \in [0,1]$, set $R(t) = \sqrt{\sum a_j^2 - t^2 \sum b_j^2}$ and

$$h_t(z) = \left(\frac{a_0 + itb_0}{R(t)},\dots,\frac{a_j + itb_j}{R(t)},\dots,\frac{a_d + itb_d}{R(t)}\right).$$

Then $h_t(z) \in V(\mathbb{C})$, and $h_t\colon V(\mathbb{C}) \to V(\mathbb{C})$ is an equivariant deformation retraction from $V(\mathbb{C})$ onto $S^d = V(\mathbb{C})^G$. □

Sometimes it is appropriate to use the following technique. If X has the G-homotopy type of a finite G-CW complex, we may compute its KR-theory using the equivariant Atiyah–Hirzebruch spectral sequence (Bredon's main spectral sequence [7, IV.4]):

$$E_2^{p,q} = H_G^p(X; KR^q) \Rightarrow KR^{p+q}(X), \tag{A.2}$$

where H_G^* denotes equivariant cohomology and KR^* denotes the coefficient system for G associated to KR: in the notation of [7, I.4], $KR^*(G) = KU^*$ and $KR^*(\text{point}) = KO^*$. By definition [7, I.6.5], $H_G^p(X, KR^q)$ is the cohomology of the equivariant cochain complex $C_G^*(X, KR^q)$.

Example A.3. Suppose that $X^G = \emptyset$. For q odd, the equivariant cochain complex and hence $H_G^p(X, KR^q)$ is zero. For $q = 2i$ even, the cochain complex is the one used to compute the cohomology of X/G with coefficients in the twisted (local) coefficient system $\mathbb{Z}(i)$, which is constant ($\mathbb{Z}$) for even i and the sign representation for odd i. Hence $H_G^p(X, KR^{2i}) \cong H^p(X/G, \mathbb{Z}(i))$.

In particular, if $X = V(\mathbb{C})$ for a smooth real curve V with no real points, then the spectral sequence (A.2) collapses to yield the result:

$$KR^{2i}(X) \cong H^0(X/G, \mathbb{Z}(i)) \oplus H^2(X/G, \mathbb{Z}(i+1)), \quad KR^{2i+1}(X) \cong H^1(X/G, \mathbb{Z}(i)).$$

If V is a smooth projective curve of genus g with no real points, we claim that

$$KR^0(X) \cong \mathbb{Z}^2; \quad KR^{-1}(X) \cong \mathbb{Z}^g \oplus (\mathbb{Z}/2);$$

$$KR^{-2}(X) \cong (\mathbb{Z}/2); \quad KR^{-3}(X) \cong \mathbb{Z}^g.$$

(The other groups are determined by 4-periodicity; see 1.8.)

In fact, it is easy to compute that $H^0(X/G, \mathbb{Z}) = \mathbb{Z}$, $H^2(X/G, \mathbb{Z}) = \mathbb{Z}/2$, and $H^0(X/G, \mathbb{Z}(1)) = H^2(X/G, \mathbb{Z}(1)) = 0$. It remains to show that $H^1(X/G, \mathbb{Z}) \cong \mathbb{Z}^g$ but $H^1(X/G, \mathbb{Z}(-1)) \cong \mathbb{Z}^g \oplus \mathbb{Z}/2$. This follows from Commessatti's theorem that $H^1(X, \mathbb{Z}) \cong \mathbb{Z}[G]^g$ (see Section 2 of [23]), and a standard argument using the spectral sequence $E_2^{p,q} = H^p(G, H^q(X, \mathbb{Z}(i))) \Rightarrow H^{p+q}(X/G, \mathbb{Z}(i))$.

Example A.4. Suppose that V is an irreducible affine real curve. Then $X = V(\mathbb{C})$ is equivariantly homotopic to a 1-dimensional G-CW complex, so again the spectral sequence (A.2) collapses. In addition, the qth row is zero when $-q \equiv 3, 5, 7 \pmod 8$ and $H_G^*(X; KR^{-1}) \cong H^*(V(\mathbb{R}); \mathbb{Z}/2)$. From this we immediately deduce that if $V(\mathbb{R})$ has λ compact components then $KR^0(X) \cong \mathbb{Z} \oplus (\mathbb{Z}/2)^\lambda$, while $KR^{-6}(X) = 0$. We leave the rest of the calculations as an exercise for the interested reader, noting that the eventual result may be read off from [23, 7.2].

Example A.5. If V is a smooth projective curve over $\mathbb{R}$ with $V(\mathbb{R}) \neq \emptyset$, the same reasoning shows that the groups $KR^*(V(\mathbb{C}))$ are determined by the genus g and the number λ of components of $V(\mathbb{R})$ (each of which is a circle). Again, the actual calculation may be read off from [23]. In particular, $KR^0(V(\mathbb{C})) \cong \mathbb{Z}^2 \oplus (\mathbb{Z}/2)^{\lambda-1}$ and $KR^{-1}(V(\mathbb{C})) \cong \mathbb{Z}^g \oplus (\mathbb{Z}/2)^{\lambda+1}$.

Note that $\lambda \leqslant g+1$ by Harnack's theorem. For example, if V is a projective curve of genus $g = 1$, then $V(\mathbb{R})$ has either $0, 1$ or 2 components. The corresponding Real spaces $V(\mathbb{C})$ are such that $V(\mathbb{C})/G$ is a Klein bottle, Möbius strip or annulus, respectively. As observed in [3, Section 9], there are 13 isomorphism classes of such V.

Sometimes $KR^*(X)$ can be computed using Mayer–Vietoris sequences.

Example A.6. Suppose that X is a Riemann surface of genus G with the top-bottom involution. Then X/G is a disk with g interior holes, and X^G consists of $g+1$ circles. For example, $X = V(\mathbb{C})$ has this property whenever V is a smooth projective curve of genus g and $V(\mathbb{R})$ has $g+1$ components.

To compute $KR^*(X)$, we disect the space as follows. Let V denote a small tubular neighborhood of X^G in X, and write $U = X - X^G$ as the disjoint union of two conjugate pieces U_1 and U_2, both homotopy equivalent to X/G. By inspection, $KR^*(U) = KU^*(X/G)$ and $KR^*(U \cap V) = KU^*(V)$, while $KR^*(V) \cong KO^*(V) \cong KO^*(\coprod_0^g S^1)$. Using Bott's exact Gysin sequence

$$\cdots \to KO^{p+1}(V) \to KO^p(V) \to KU^p(V) \to KO^{p+2}(V) \to \cdots$$

and a diagram chase, we obtain the exact sequence

$$\cdots \to KO^{p+1}\left(\coprod_0^g S^1\right) \to KR^p(X) \to KU^p\left(\bigvee_g S^1\right) \xrightarrow{\gamma} KO^{p+2}\left(\coprod_0^g S^1\right) \to \cdots.$$

The map γ is explicit: on the first g factors it is the projection onto the appropriate component of $\bigvee_g S^1$, while the last circle is mapped by the "sum" map. The groups $KR^p(X)$ may be now be determined, and the answer will agree with the answer in A.5 above.

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