Twisted Kähler differential forms

Max Karoubi* Mariano Suarez Alvarez†

In [4], the first author has shown the interest of "quantum" differential forms in Algebraic Topology. They are obtained from the usual ones by a slight change of the rules of calculus on polynomials and series. In this paper, we make a more systematic study of these new quantum differential forms. Our starting point is a commutative algebra A with an endomorphism α; the differential graded algebra of "twisted" differential forms $\Omega^\bullet_\alpha A$ is then obtained as the quotient of the universal non-commutative differential graded algebra $\Omega^\bullet A$, defined by A. Connes and the first author, by the ideal generated by the relations (d being the differential)

$$da\, b - \alpha(b)\, da.$$

If α is the identity, we recover the classical commutative differential graded algebra of Kähler differential forms. If $A = k[x]$ and the endomorphism α is given by $\alpha(x^n) = q^n x^n$, where $q \in k$ is a "quantum" parameter, we find the differential graded algebra introduced in [4] for topological purposes.

The interest of this general definition lies essentially in the existence of a remarkable braided structure R on $\Omega^\bullet_\alpha A$, which reduces to the ordinary flip if α is the identity, in the way defined in [4], p. 2—see the precise definition below. As a matter of fact, we show at the same time its *uniqueness* under the condition that $R(a \otimes b) = b \otimes a$ when both a and b belong to A, identified to the degree zero part of $\Omega^\bullet_\alpha A$. If α is an automorphism, we produce in this way a lot of examples of representations of the braid group $\mathscr{B}_n$ in a vector space or a module, by considering $(\Omega^\bullet_\alpha A)^{\otimes n}$ or, more generally, $J^{\otimes n}$, where J is any sub-quotient of $\Omega^\bullet_\alpha A$ stable by the braiding. For instance, if A is the algebra of polynomials in several variables and if α is induced by a linear transformation of these variables, filtrations by various degrees in the variables produce such sub-quotients.

1. Generalities and statement of the theorem

1.1. Let A be an associative algebra. A *universal derivation* for A is a derivation $d : A \to \Omega^1 A$ of A such that for each derivation $\delta : A \to M$ of A with values in an A-bimodule M there exists exactly one morphism of bimodules $f : \Omega^1 A \to M$ such that $\delta = f \circ d$. Such an object always exists, and is unique up to an obvious notion of isomorphism; it can be concretely realized by taking $\Omega^1 A = \mathrm{Ker}(A \otimes A \to A)$, the kernel of the multiplication map, and defining $da = 1 \otimes a - a \otimes 1$ if $a \in A$.

1.2. The *algebra of universal differential forms* on A, which we shall write $\Omega^\bullet A$, is the tensor algebra $T_A \Omega^1 A$ of the A-bimodule $\Omega^1 A$; it has a natural grading, and the map $d : A \to \Omega^1 A$ induces in a unique way a derivation $d : \Omega^\bullet A \to \Omega^\bullet A$ with respect to which it becomes a cohomologically graded differential algebra; *cf.* [1, 2].

1.3. Let now A be a *commutative* algebra, and let $\alpha : A \to A$ be an algebra endomorphism; we write $\bar{a} = \alpha(a)$. Let $I_\alpha A$ be the differential ideal generated in $\Omega^\bullet A$ by the elements $da\, b - \bar{b}\, da$ for a, $b \in A$, and $\Omega^\bullet_\alpha A = \Omega^\bullet A / I_\alpha A$. This is again by construction a differential algebra, which is graded since the ideal I_α is homogeneous, and which is clearly natural with respect to maps in the category of pairs (A, α) as above, and where the morphisms are morphisms of the underlying algebras commuting with the given endomorphisms. We call $\Omega^\bullet_\alpha A$ the *differential graded algebra of twisted Kähler differential forms*.

*Université Paris 7 - Mathématiques - 2, place Jussieu 75251 Paris Cedex 05.
†Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires. Ciudad Universitaria. Pab I, Buenos Aires (1428) Argentina. e-mail: mariano@dm.uba.ar.
This work was supported by a grant from UBACYT TW69, the international cooperation project SECyT-ECOS A98E05, and a CONICET scholarship.

We note that since I_α is a differential ideal we have the relation $dudv = -d\bar{v}du$ in $\Omega^\bullet_\alpha$ for each pair of elements $u, v \in A$, as a simple computation shows.

1.4. Let A be an algebra. A *braiding* on A is a morphism $R : A \otimes A \to A \otimes A$ such that

$$R \circ (\eta \otimes 1) = 1 \otimes \eta, \qquad R \circ (1 \otimes \eta) = \eta \otimes 1; \tag{1}$$
$$(R \otimes 1) \circ (1 \otimes R) \circ (R \otimes 1) = (1 \otimes R) \circ (R \otimes 1) \circ (1 \otimes R); \tag{2}$$
$$(\mu \otimes 1) \circ (1 \otimes R) \circ (R \otimes 1) = R \circ (1 \otimes \mu); \tag{3}$$
$$(1 \otimes \mu) \circ (R \otimes 1) \circ (1 \otimes R) = R \circ (\mu \otimes 1); \tag{4}$$
$$\mu \circ R = \mu. \tag{5}$$

Here $\mu : A \otimes A \to A$ is the multiplication map, and $\eta : k \to A$ gives the identity element of A.

The operator R is regarded as an interchange operator. From this point of view, the condition (2), the Yang-Baxter equation, is a natural one to impose; in particular, it implies that there is an action of the braid group $\mathscr{B}_n$ on the tensor power $A^{\otimes n}$ whenever α is an automorphism. Relations (3) and (4) express compatibility of the braiding with the product. Finally, equation (5) is read as imposing a commutativity.

1.5. If A is a differential graded algebra, we will say that a morphism $R : A \otimes A \to A \otimes A$ is a *braiding of differential graded algebras* if it is simultaneously a braiding and a map of differential graded modules with respect to the usual structure on $A \otimes A$.

1.6. If A is a commutative algebra, we consider the morphism $\tau : A \otimes A \to A \otimes A$ given by $\tau(a \otimes b) = b \otimes a$; it is a braiding of A, the *trivial braiding* or *ordinary flip*.

1.7. With this vocabulary, we can now state our theorem:

1.8. Theorem. *There exists a unique functorial way of assigning to each endomorphism $\alpha : A \to A$ of a commutative algebra a braiding $R : \Omega^\bullet_\alpha A \otimes \Omega^\bullet_\alpha A \to \Omega^\bullet_\alpha A \otimes \Omega^\bullet_\alpha A$ of the differential graded algebra of twisted Kähler differential forms on (A, α) in such a way that its restriction to the degree zero submodule $\Omega^0_\alpha A \otimes \Omega^0_\alpha A = A \otimes A$ is the trivial braiding τ.*

2. Uniqueness

2.9. We write $R_{i,j}$ for the restriction of R to $\Omega^i_\alpha A \otimes \Omega^j_\alpha A$. Our strategy to show uniqueness is to relate the various restrictions $R_{i,j}$ to $R_{0,0}$ and $R_{1,0}$ and then to prove that these two morphisms are determined by the conditions stated in the theorem.

We start with a straightforward computation.

2.10. Let $i \geq 0$, $j \geq 1$. One has

$$\begin{aligned}
&R_{i,j}(u_0 du_1 \cdots du_i \otimes v_0 dv_1 \cdots dv_j)\\
&\quad = -R(u_0 du_1 \cdots du_i \otimes d(v_0 v_1) \cdots dv_j)\\
&\qquad - \sum_{k=1}^{j-1} (-1)^k R(u_0 du_1 \cdots du_i \otimes dv_0 dv_1 \cdots d(v_k v_{k+1}) \cdots dv_j)\\
&\qquad - (-1)^j R(u_0 du_1 \cdots du_i \otimes dv_0 dv_1 \cdots dv_{j-1} v_j)\\
&\quad = -(-1)^i Rd(u_0 du_1 \cdots du_i \otimes v_0 v_1 dv_2 \cdots dv_j) + (-1)^i R(du_0 du_1 \cdots du_i \otimes v_0 v_1 dv_2 \cdots dv_j)\\
&\qquad - \sum_{k=1}^{j-1} (-1)^{k+i} Rd(u_0 du_1 \cdots du_i \otimes v_0 dv_1 \cdots d(v_k v_{k+1}) \cdots dv_j)\\
&\qquad + \sum_{k=1}^{j-1} (-1)^{k+i} R(du_0 du_1 \cdots du_i \otimes v_0 dv_1 \cdots d(v_k v_{k+1}) \cdots dv_j)\\
&\qquad - (-1)^j R(u_0 du_1 \cdots du_i \otimes dv_0 dv_1 \cdots dv_{j-1} v_j)
\end{aligned}$$

$$
\begin{aligned}
&= -(-1)^i dR(u_0du_1\cdots du_i\otimes v_0[v_1dv_2\cdots dv_j+\sum_{k=1}^{j-1}(-1)^k)dv_1\cdots d(v_kv_{k+1})\cdots dv_j])\\
&\quad +(-1)^i R(du_0du_1\cdots du_i\otimes v_0[v_1dv_2\cdots dv_j+\sum_{k=1}^{j-1}(-1)^k)dv_1\cdots d(v_kv_{k+1})\cdots dv_j])\\
&\quad -(-1)^j R(u_0du_1\cdots du_i\otimes dv_0dv_1\cdots dv_{j-1}v_j)\\
&=(-1)^{i+j}dR_{i,j-1}(u_0du_1\cdots du_i\otimes v_0dv_1\cdots dv_{j-1}v_j)\\
&\quad -(-1)^{i+1}R_{i+1,j-1}(du_0du_1\cdots du_i\otimes v_0dv_1\cdots dv_{j-1}v_j)\\
&\quad -(-1)^j R(u_0du_1\cdots du_i\otimes dv_0dv_1\cdots dv_{j-1}v_j).
\end{aligned}
$$

Since

$$
\begin{aligned}
&R(u_0du_1\cdots du_i\otimes dv_0\cdots dv_{j-1}v_j)\\
&\quad=(R\circ 1\otimes\mu)(u_0du_1\cdots du_i\otimes dv_0\cdots dv_{j-1}\otimes v_j)\\
&\quad=(\mu\otimes 1\circ 1\otimes R\circ R\otimes 1)(u_0du_1\cdots du_i\otimes dv_0\cdots dv_{j-1}\otimes v_j)\\
&\quad=(\mu\otimes 1\circ 1\otimes R_{*,0})(R(u_0du_1\cdots du_i\otimes dv_0\cdots dv_{j-1})\otimes v_j)
\end{aligned}
$$

and

$$
\begin{aligned}
&R(u_0du_1\cdots du_i\otimes dv_0\cdots dv_{j-1})\\
&\quad=(-1)^i Rd(u_0du_1\cdots du_i\otimes dv_0\cdots dv_{j-1})-(-1)^i R(du_0du_1\cdots du_i\otimes v_0dv_1\cdots dv_{j-1})\\
&\quad=(-1)^i dR_{i,j-1}(u_0du_1\cdots du_i\otimes dv_0\cdots dv_{j-1})-(-1)^i R_{i+1,j-1}(du_0du_1\cdots du_i\otimes v_0dv_1\cdots dv_{j-1})
\end{aligned}
$$

we see that we can compute $R_{i,j}$ in terms of $R_{i,j-1}$, $R_{i+1,j-1}$ and $R_{\bullet,0}$; a simple inductive argument then shows that R is determined by $R_{\bullet,0}$.

2.11. Now, if $i\geq 0$,

$$
\begin{aligned}
&R_{i+1,0}(u_0du_1\cdots du_{i+1}\otimes v_0)=(R\circ\mu\otimes 1)(u_0du_1\cdots du_i\otimes du_{i+1}\otimes v_0)\\
&\quad=(1\otimes\mu\circ R\otimes 1\circ 1\otimes R)(u_0du_1\cdots du_i\otimes du_{i+1}\otimes v_0)\\
&\quad=(1\otimes\mu\circ R\otimes 1)(u_0du_1\cdots du_i\otimes R_{1,0}(du_{i+1}\otimes v_0)),
\end{aligned}
$$

so that, if we assume

$$\operatorname{Im} R_{1,0}\subset\Omega^0_\alpha A\otimes\Omega^1_\alpha A, \tag{6}$$

we see that the maps $R_{\bullet,0}$ are determined by $R_{0,0}$ and $R_{1,0}$.

2.12. Part of our hypothesis is that $R_{0,0}=\tau$; the required uniqueness will follow then if we can show that the hypothesis also determines $R_{1,0}$ in a such a way that (6) is verified.

2.13. Let us consider the polynomial algebra $L_2=k[\{x_i,y_i\}_{i\geq 0}]$ on variables x_i and y_i, for $i\geq 0$, equipped with the endomorphism $\lambda:L_2\to L_2$ such that $\lambda(x_i)=x_{i+1}$ and $\lambda(y_i)=y_{i+1}$.

Since R is a braiding, we have that $\mu R(dx_0\otimes y_0)=\mu(dx_0\otimes y_0)=dx_0y_0=y_1dx_0=\mu(y_1\otimes dx_0)$, so there is an element $\omega\in\operatorname{Ker}(\mu:\Omega^\bullet_\lambda L_2\otimes\Omega^\bullet_\lambda L_2\to\Omega^\bullet_\lambda L_2)$ such that

$$R(dx_0\otimes y_0)=y_1\otimes dx_0+\omega.$$

2.14. Let A be a commutative algebra and $\alpha : A \to A$ be an endomorphism of A; if $a, b \in A$, there is exactly one morphism in the category of endomorphisms of algebras $\phi_{a,b} : (L_2, \lambda) \to (A, \alpha)$ such that $\phi_{a,b}(x_0) = a$ and $\phi_{a,b}(y_0) = b$, and it induces in turn a morphism of differential graded algebras, which we will write $\phi_{a,b}$ as well, $\phi_{a,b} : \Omega^\bullet_\lambda L_2 \to \Omega^\bullet_\alpha A$. Naturality of R implies that

$$\begin{aligned} R(da \otimes b) &= R\phi_{a,b}(dx_0 \otimes y_0) = \phi_{a,b}R(dx_0 \otimes y_0) \\ &= \phi_{a,b}(y_1 \otimes dx_0) + \phi_{a,b}(\omega) \\ &= \bar{b} \otimes da + \phi_{a,b}(\omega); \end{aligned}$$

so that ω determines $R_{1,0}$ on the elements of the form $da \otimes b$ in $\Omega^1_\alpha A \otimes \Omega^0_\alpha A$. In general, if $adb \otimes c \in \Omega^1_\alpha A \otimes \Omega^0_\alpha A$, we have

$$\begin{aligned} R(adb \otimes c) &= R(d(ab) \otimes c) - R(dab \otimes c) \\ &= R(d(ab) \otimes c) - (R \circ \mu \otimes 1)(da \otimes b \otimes c) \\ &= R(d(ab) \otimes c) - (1 \otimes \mu \circ R \otimes 1 \circ 1 \otimes R)(da \otimes b \otimes c) \\ &= R(d(ab) \otimes c) - (1 \otimes \mu \circ R \otimes 1)(da \otimes c \otimes b) \\ &= R(d(ab) \otimes c) - (1 \otimes \mu)(R(da \otimes c) \otimes b); \end{aligned}$$

observe that we have used the hypothesis that $R_{0,0} = \tau$. We conclude that ω actually determines $R_{1,0}$.

Let us write $\omega(a, b) = \phi_{a,b}(\omega)$.

2.15. Now let $L_3 = k[\{x_i, y_i, z_i\}_{i \geq 0}]$ be endowed with the endomorphism $\lambda : L_3 \to L_3$ such that $\lambda(x_i) = x_{i+1}$, $\lambda(y_i) = y_{i+1}$ and $\lambda(z_i) = z_{i+1}$. We compute in $\Omega^\bullet_\lambda L_3$:

$$\begin{aligned} &(1 \otimes R \circ R \otimes 1 \circ 1 \otimes R)(dx_0 \otimes y_0 \otimes z_0) \\ &\quad = (1 \otimes R \circ R \otimes 1)(dx_0 \otimes z_0 \otimes y_0) \\ &\quad = (1 \otimes R)(z_1 \otimes dx_0 \otimes y_0 + \omega(x_0, z_0) \otimes y_0) \\ &\quad = z_1 \otimes y_1 \otimes dx_0 + z_1 \otimes \omega(x_0, y_0) + (1 \otimes R)(\omega(x_0, z_0) \otimes y_0) \\ &(R \otimes 1 \circ 1 \otimes R \circ R \otimes 1)(dx_0 \otimes y_0 \otimes z_0) \\ &\quad = (R \otimes 1 \circ 1 \otimes R)(y_1 \otimes dx_0 \otimes z_0 + \omega(x_0, y_0) \otimes z_0) \\ &\quad = (R \otimes 1)(y_1 \otimes z_1 \otimes dx_0 + y_1 \otimes \omega(x_0, z_0) + (1 \otimes R)(\omega(x_0, y_0) \otimes z_0) \\ &\quad = z_1 \otimes y_1 \otimes dx_0 + (R \otimes 1)(y_1 \otimes \omega(x_0, z_0)) + (R \otimes 1 \circ 1 \otimes R)(\omega(x_0, y_0) \otimes z_0) \end{aligned}$$

Since R satisfies the braid equation (2), we have then that

$$z_1 \otimes \omega(x_0, y_0) + (1 \otimes R)(\omega(x_0, z_0) \otimes y_0) = (R \otimes 1)(y_1 \otimes \omega(x_0, z_0)) + (R \otimes 1 \circ 1 \otimes R)(\omega(x_0, y_0) \otimes z_0)$$

Apply $1 \otimes \mu$ to both sides of this equality; on the left, we obtain

$$\begin{aligned} &(1 \otimes \mu)(z_0 \otimes \omega(x_0, y_0)) + (1 \otimes \mu R)(\omega(x_0, z_0) \otimes y_0) \\ &\quad = (1 \otimes \mu)(\omega(x_0, z_0) \otimes y_0) \\ &\quad = \omega(x_0, z_0)y_0 \end{aligned}$$

and, on the right,

$$\begin{aligned} &(1 \otimes \mu \circ R \otimes 1)(y_1 \otimes \omega(x_0, z_0)) + (1 \otimes \mu \circ R \otimes 1 \circ 1 \otimes R)(\omega(x_0, y_0) \otimes z_0) \\ &\quad = (1 \otimes \mu \circ R \otimes 1)(y_1 \otimes \omega(x_0, z_0)) + (\mu \otimes 1)(\omega(x_0, y_0) \otimes z_0) \\ &\quad = (1 \otimes \mu \circ R \otimes 1)(y_1 \otimes \omega(x_0, z_0)). \end{aligned}$$

so that

$$\omega(x_0, z_0)y_0 = (1 \otimes \mu \circ R \otimes 1)(y_1 \otimes \omega(x_0, z_0)).$$

Observe that the variable y_0 cannot appear on the right hand side of this equality because of naturality; in view of the left hand side, we must have $\omega = 0$.

This shows that, if $\alpha : A \to A$ is an endomorphism of a commutative algebra, we have in $\Omega^\bullet_\alpha A$ that

$$R(da \otimes b) = \bar{b} \otimes da.$$

In view of what has been said above, the uniqueness statement in the theorem follows from this.

3. Existence

3.16. Let us show now that there exists a braiding satisfying the conditions in the statement. We do this by explicitly constructing it.

3.17. Let A be a commutative and let $\alpha : A \to A$ be a endomorphism of A. We define a morphism of graded modules $I : \Omega^\bullet_\alpha A \to \Omega^\bullet_\alpha A$ of degree -1 by putting, on $\Omega^n_\alpha A$,

$$I(u_0 du_1 \cdots du_n) = \sum_{i=1}^{n} (-1)^{i+1} u_0 du_1 \cdots du_{i-1}(u_i - \bar{u}_i) d\bar{u}_{i+1} \cdots d\bar{u}_n.$$

It is easy to check that this is well defined. This operator is obviously left A-linear, and does not commute in general with the differential on $\Omega^\bullet_\alpha A$; in fact,

$$Id(u_0 du_1 \cdots du_n) = \sum_{i=0}^{n} (-1)^{i+2} du_0 \cdots du_{i-1}(u_i - \bar{u}_i) d\bar{u}_{i+1} \cdots d\bar{u}_n$$

and

$$\begin{aligned} dI(u_0 du_1 \cdots du_n) &= \sum_{i=1}^{n} (-1)^{i+1} d(u_0 \cdots du_{i-1}(u_i - \bar{u}_i) d\bar{u}_{i+1} \cdots d\bar{u}_n) \\ &= \sum_{i=1}^{n} (-1)^{i+1} du_0 \cdots du_{i-1}(u_i - \bar{u}_i) d\bar{u}_{i+1} \cdots d\bar{u}_n \\ &\quad + \sum_{i=1}^{n} (-1)^{i+1} u_0 \cdots du_{i-1}(du_i - d\bar{u}_i) d\bar{u}_{i+1} \cdots d\bar{u}_n \end{aligned}$$

so that

$$\begin{aligned} (Id + dI)(u_0 du_1 \cdots du_n) &= (u_0 - \bar{u}_0) d\bar{u}_1 \cdots d\bar{u}_n + \sum_{i=1}^{n} u_0 du_1 \cdots du_{i-1}(du_i - d\bar{u}_i) d\bar{u}_{i+1} \cdots d\bar{u}_n \\ &= -\bar{u}_0 d\bar{u}_1 \cdots d\bar{u}_n + u_0 du_1 \cdots du_n \\ &= (1 - \alpha)(u_0 du_1 \cdots du_n) \end{aligned}$$

Thus, I is a homotopy $1_{\Omega^\bullet_\alpha A} \simeq \alpha$.

3.18. This computation proves the first part of the following lemma. To state it and in order to simplify future formulas, we introduce some notation. In what follows we shall write $[n]$, for $n \in \mathbb{Z}$, instead of $(-1)^n$, and, in a context where there is an endomorphism of an algebra—α, say—we will write $\bar{a}^n$ instead of $\alpha^n(a)$. Also, we will agree that a homogeneous differential form stands for its degree when inside square brackets or in an exponent. For example, we will write $[\omega(\psi + 1)]\bar{\phi}^\omega$, when ω, ψ and ϕ are homogeneous differential forms, instead of $(-1)^{\deg \omega (\deg \psi + 1)} \alpha^{\deg \omega}(\phi)$.

3.19. Lemma. *Let A be a commutative algebra and let $\alpha : A \to A$ be an endomorphism of A. For $\omega, \psi \in \Omega^\bullet_\alpha A$ and $v \in A$ the following relations hold:*

$$I : 1_{\Omega^\bullet_\alpha A} \simeq \alpha \tag{7}$$

$$I(\omega\psi) = I\omega\bar{\psi} + [\omega]\omega I\psi \tag{8}$$

$$I(\omega dv) = I\omega d\bar{v} + [\omega]\omega(v - \bar{v}) \tag{9}$$

$$\omega v - v\omega = [\omega]I\omega dv \tag{10}$$

Moreover, we have $I^2 = 0$.

Proof. We have just shown (7); (8) and (9) follow immediately from the definition of I. Let us check (10) inductively on $\deg\omega$. If $\deg\omega = 0$, it is true because A is commutative and I is zero on 0-forms. Assume then the truth of (10) for an homogeneous form ω; then, if u, $v \in A$,

$$\begin{aligned}
\omega duv &= \omega d(uv) - \omega udv \\
&= \omega vdu + \omega dvu - \omega udv \\
&= \omega vdu - \omega(u - \bar{u})dv \\
&= v\omega du + [\omega]I\omega dv - \omega(u - \bar{u})dv \\
&= v\omega du + [\omega](I\omega - [\omega]\omega(u - \bar{u}))dv \\
&= v\omega du + [\omega]I(\omega du)dv
\end{aligned}$$

This shows (10) for all ω.

Finally, to show that $I^2 = 0$ inductively, we observe that it is trivially true on 0-forms, and if $I^2\omega = 0$ for an homogeneous form ω, we have

$$\begin{aligned}
I^2(\omega dv) &= I(I\omega d\bar{v} + [\omega]\omega(v - \bar{v})) \\
&= I^2\omega d\bar{v}^2 + [I\omega]I\omega I(d\bar{v}) + [\omega]I\omega(\bar{v} - \bar{v}^2) + [2\omega]\omega I(v - \bar{v}) \\
&= [\omega - 1]I\omega(\bar{v} - \bar{v}^2) + [w]I\omega(\bar{v} - \bar{v}^2) \\
&= 0
\end{aligned}$$

so that I^2 vanishes identically on $\Omega^\bullet_\alpha A$. ☐

3.20. Let us fix a commutative algebra A and an endomorphism $\alpha : A \to A$. Let $R : \Omega^\bullet_\alpha A \otimes \Omega^\bullet_\alpha A \to \Omega^\bullet_\alpha A \otimes \Omega^\bullet_\alpha A$ be given by

$$R(\omega \otimes \phi) = [\omega\phi]\bar{\phi}^\omega \otimes \omega - [(\omega + 1)\phi]I\bar{\phi}^\omega \otimes d\omega$$

We will verify that this operator satisfies the conditions in the theorem. From the definition, it is clear that R verifies (1).

3.21. Next, we have

$$\begin{aligned}
&\mu R(u_0du_1 \cdots du_n \otimes v_0dv_1 \cdots dv_m) \\
&\quad = [nm]\bar{v}_0^n d\bar{v}_1^n \cdots d\bar{v}_m^n u_0du_1 \cdots du_n - [(n + 1)m]I(\bar{v}_0^n d\bar{v}_1^n \cdots d\bar{v}_m^n)du_0du_1 \cdots du_n \\
&\quad = [nm]\bar{v}_0^n u_0 d\bar{v}_1^n \cdots d\bar{v}_m^n du_1 \cdots du_n + [(n + 1)m]\bar{v}_0^n I(d\bar{v}_1^n \cdots d\bar{v}_m^n)du_0du_1 \cdots du_n \\
&\qquad - [(n + 1)m]I(\bar{v}_0^n d\bar{v}_1^n \cdots d\bar{v}_m^n)du_0du_1 \cdots du_n \\
&\quad = u_0du_1 \cdots du_n v_0dv_1 \cdots dv_m
\end{aligned}$$

so that $\mu R = \mu$; this is (5).

3.22. We want to check that R satisfies the braid equation (2); evaluating both sides on $\omega \otimes \phi \otimes \psi$ for homogeneous forms ω, ϕ, $\psi \in \Omega^\bullet_\alpha A$, we find

$$
\begin{aligned}
&\omega \otimes \phi \otimes \psi \\
&\quad \xmapsto{R\otimes 1} [\phi\omega]\bar{\phi}^\omega \otimes \omega \otimes \psi - [\phi(\omega+1)]I\bar{\phi}^\omega \otimes d\omega \otimes \psi \\
&\quad \xmapsto{1\otimes R} [\phi\omega + \omega\psi]\bar{\phi}^\omega \otimes \bar{\psi}^\omega \otimes \omega \\
&\qquad - [\phi\omega + \omega\psi + \psi]\bar{\phi}^\omega \otimes I\bar{\psi}^\omega \otimes d\omega \\
&\qquad - [\phi\omega + \phi + (\omega+1)\psi]I\bar{\phi}^\omega \otimes \bar{\psi}^{\omega+1} \otimes d\omega \\
&\quad \xmapsto{R\otimes 1} [\phi\omega + \omega\psi + \phi\psi]\bar{\psi}^{\omega+\phi} \otimes \bar{\phi}^\omega \otimes \omega \\
&\qquad - [\phi\omega + \omega\psi + \phi\psi + \psi]I\bar{\psi}^{\omega+\phi} \otimes d\bar{\phi}^\omega \otimes \omega \\
&\qquad - [\phi\omega + \omega\psi + \psi + \phi(\psi-1)]I\bar{\psi}^{\omega+\phi} \otimes \bar{\phi}^\omega \otimes d\omega \\
&\qquad - [\phi\omega + \phi + (\omega+1)\psi + (\phi-1)\psi]\bar{\psi}^{\omega+\phi} \otimes I\bar{\phi}^\omega \otimes d\omega \\
&\qquad + [\phi\omega + \phi + (\omega+1)\psi + (\phi-1)\psi + \psi]I\bar{\psi}^{\omega+\phi} \otimes dI\bar{\psi}^\omega \otimes d\omega
\end{aligned}
$$

and

$$
\begin{aligned}
&\omega \otimes \phi \otimes \psi \\
&\quad \xmapsto{1\otimes R} [\phi\psi]\omega \otimes \bar{\psi}^\phi \otimes \phi \\
&\qquad - [\phi\psi + \psi]\omega \otimes I\bar{\psi}^\phi \otimes d\phi \\
&\quad \xmapsto{R\otimes 1} [\phi\psi + \omega\psi]\bar{\psi}^{\phi+\omega} \otimes \omega \otimes \phi \\
&\qquad - [\phi\psi + \psi + \omega(\psi-1)]I\bar{\psi}^{\phi+\omega} \otimes \omega \otimes d\phi \\
&\quad \xmapsto{1\otimes R} [\phi\psi + \omega\psi + \omega\phi]\bar{\psi}^{\phi+\omega} \otimes \bar{\phi}^\omega \otimes \omega \\
&\qquad - [\phi\psi + \omega\psi + \omega\phi + \phi]\bar{\psi}^{\phi+\omega} \otimes I\bar{\phi}^\omega \otimes d\omega \\
&\qquad - [\phi\psi + \omega\psi + \omega\phi + \phi]I\bar{\psi}^{\phi+\omega} \otimes \bar{\phi}^{\omega+1} \otimes d\omega \\
&\qquad - [\phi\psi + \psi + (\omega(\psi-1) + \omega(\phi+1)]I\bar{\psi}^{\phi+\omega} \otimes d\bar{\phi}^\omega \otimes \omega \\
&\qquad - [\phi\psi + \psi + \omega(\psi-1) + \omega(\phi+1) + \phi + 1]I\bar{\psi}^{\phi+\omega} \otimes Id\bar{\phi}^\omega \otimes d\omega
\end{aligned}
$$

These are equal, because

$$
\begin{aligned}
&- [\phi\omega + \omega\psi + \psi + \phi\psi - \phi]I\bar{\psi}^{\omega+\phi} \otimes \bar{\phi}^\omega \otimes d\omega + [\phi\omega + \phi + \omega\psi + \psi + \phi\psi]I\bar{\psi}^{\omega+\phi} \otimes dI\bar{\phi}^\omega \otimes d\omega = \\
&\qquad - [\phi\psi + \omega\psi + \psi + \omega\phi + \phi]I\bar{\psi}^{\phi\omega} \otimes \bar{\phi}^{\omega+1} \otimes d\omega + [\phi\psi + \psi + \omega\psi + \omega\phi + \phi + 1]I\bar{\psi}^{\phi+\omega} \otimes Id\bar{\phi}^\omega \otimes d\omega
\end{aligned}
$$

which in turn follows from

$$-[\psi - \phi]\bar{\phi}^\omega + [\phi + \psi]dI\bar{\phi}^\omega = -[\psi + \phi]\bar{\phi}^{\omega+1} - [\psi + \phi]Id\bar{\phi}^\omega$$

which is true, because lemma **3.19** implies that

$$-\bar{\phi}^\omega + dI\bar{\phi}^\omega = -\bar{\phi}^{\omega+1} - Id\bar{\phi}^\omega$$

3.23. Finally, our map R is compatible with multiplication in $\Omega_\alpha^\bullet A$, since, for example,

$$
\begin{aligned}
&\omega\otimes\phi\otimes\psi\\
&\quad\stackrel{1\otimes\mu}{\longmapsto}\omega\otimes\phi\psi\\
&\quad\stackrel{R}{\longmapsto}[\omega(\phi+\psi)]\bar\phi^\omega\bar\psi^\omega\otimes\omega-[\omega(\phi+\psi)+\phi+\psi]I(\bar\phi^\omega\bar\psi^\omega)\otimes d\omega\\
&\omega\otimes\phi\otimes\psi\\
&\quad\stackrel{R\otimes 1}{\longmapsto}[\omega\phi]\bar\phi^\omega\otimes\omega\otimes\psi-[\omega\phi+\phi]I\bar\psi^\omega\otimes d\omega\otimes\psi\\
&\quad\stackrel{1\otimes R}{\longmapsto}[\omega\phi+\omega\psi]\bar\phi^\omega\otimes\bar\psi^\omega\otimes\omega\\
&\qquad\quad-[\omega\phi+\omega\psi+\psi]\bar\phi^\omega\otimes I\bar\psi^\omega\otimes d\omega\\
&\qquad\quad-[\omega\phi+\phi+(\omega+1)\psi]I\bar\phi^\omega\otimes\bar\psi^{\omega+1}\otimes d\omega\\
&\quad\stackrel{\mu\otimes 1}{\longmapsto}[\omega\phi+\omega\psi]\bar\phi^\omega\bar\psi^\omega\otimes\omega\\
&\qquad\quad-[\omega\phi+\omega\psi+\psi+\psi]([\phi]\bar\phi^\omega I\bar\psi^\omega+I\bar\phi^\omega\bar\psi^{\omega+1})\otimes d\omega
\end{aligned}
$$

and these are equal by the lemma; this shows (3), and the other equation (4) is checked in the same way.

3.24. It is obvious that R depends naturally on α, and reduces to the trivial twist τ in degree 0. Since it satisfies the required conditions, theorem **1.8** is proved.

3.25. Two simple examples. Consider $A = k[x]$ and $q \in k$, and let $\alpha : A \to A$ be the endomorphism such that $\alpha(x) = qx$. Then we have $\Omega_\alpha^0 A = k[x]$, $\Omega_\alpha^1 A = k[x]dx$ and the twisted exterior differential is given by $dx^n = n_q x^{n-1}dx$ for each $n \geq 1$, where, for each n, we define the q-integer $n_q = (1-q^n)/(1-q)$ when $q \neq 1$, and $n_1 = n$. The operator $I : \Omega_\alpha^1 A \to \Omega_\alpha^0 A$ is given by q-integration of forms: $I(x^n dx) = (1-q)x^{n+1}$. Using this, we easily obtain the following formulas for the braiding constructed above on $\Omega_\alpha^\$ A$:

$$
\begin{aligned}
&R(x^n\otimes x^m)=x^m\otimes x^n\\
&R(x^n dx\otimes x^m)=q^m x^m\otimes x^n dx\\
&R(x^n\otimes x^m dx)=x^m dx\otimes x^n+(1-q)x^{m+1}\otimes x^{n-1}dx\\
&R(x^n dx\otimes x^m dx)=-q^{m+1}x^m dx\otimes x^n dx
\end{aligned}
$$

We thus recover the main example considered in [4]. More generally, one can replace A with a ring of convergent power series f with the endomorphism given by $\alpha(f)(x) = f(qx)$ like in [3].

3.26. Another familiar example is the following. Let $A = k[x]/(x^2 - x)$ and let $\alpha : A \to A$ be such that $\alpha(x) = 1 - x$. Then $\Omega_\alpha^\bullet A = \Omega^\bullet A$ can be identified with the differential graded algebra of normalized cochains on the simplicial set $\{0, 1\}$. Since $\alpha^2 = 1$, the action of the braid group $\mathscr{B}_n$ reduces to the action of the symmetric group $\mathscr{S}_n$.

4. References

[1] A. Connes, *Non-commutative differential geometry*. Publ. Math., Inst. Hautes Étud. Sci. **62**, 41-144 (1985).

[2] M. Karoubi, *Homologie cyclique et K-théorie*. Astérisque **149**. Paris: Société Mathématique de France, 1987.

[3] M. Karoubi, *Braiding of differential forms and homotopy types.* Comptes Rendus Acad. Sci. Paris **331**, série 1, 757–762 (2000).

[4] M. Karoubi, *Quantum methods in Algebraic Topology*. To appear in the Proceedings of the A.M.S. (2001).