HERMITIAN K-THEORY OF THE INTEGERS

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Abstract. Rognes and Weibel used Voevodsky's work on the Milnor conjecture to deduce the strong Dwyer-Friedlander form of the Lichtenbaum-Quillen conjecture at the prime 2. In consequence (the 2-completion of) the classifying space for algebraic K-theory of the integers $\mathbb{Z}[1/2]$ can be expressed as a fiber product of well-understood spaces BO and $BGL(\mathbb{F}_3)^+$ over BU. Similar results are now obtained for Hermitian K-theory and the classifying spaces of the integral symplectic and orthogonal groups. For the integers $\mathbb{Z}[1/2]$, this leads to computations of the 2-primary Hermitian K-groups and affirmation of the Lichtenbaum-Quillen conjecture in the framework of Hermitian K-theory.

0. Introduction. In [9], Bökstedt introduced the study of the commuting square

$$(0\text{-}1) \qquad \begin{array}{ccc} BGL(\mathbb{Z}')_{\#}^+ & \longrightarrow & BGL(\mathbb{R})_{\#} \\ \downarrow & & \downarrow \\ BGL(\mathbb{F}_3)_{\#}^+ & \xrightarrow{b} & BGL(\mathbb{C})_{\#}. \end{array}$$

Here $\mathbb{Z}'$ denotes the ring $\mathbb{Z}[1/2]$, $\mathbb{F}_3$ the finite field with three elements, and, for $F = \mathbb{R}, \mathbb{C}$, $BGL(F)$ is the classifying space of the infinite general linear group $GL(F)$ with the usual topology. The symbol # indicates the 2-adic completion, and the map b the Brauer lift, corresponding to the fibring of Adams' map $\psi^3 - 1$ on $BU = BGL(\mathbb{C})$. The remaining maps are induced from the obvious ring homomorphisms.

The Dwyer-Friedlander formulation of the Lichtenbaum-Quillen conjecture for $\mathbb{Z}$ at the prime 2 is that the above square is homotopy cartesian [15] Conjecture 1.3, Proposition 4.2. This has been affirmed in work of Rognes and Weibel [48]— see [57] Corollary 8.

Since the homotopy fiber of the map $BGL(\mathbb{R}) \to BGL(\mathbb{C})$ is the homogeneous space $GL(\mathbb{C})/GL(\mathbb{R})$, which has the homotopy type of $\Omega^7(BGL(\mathbb{R}))$ by Bott periodicity [11], we may also write the homotopy fibration

$$\Omega^7(BGL(\mathbb{R}))_{\#} \longrightarrow BGL(\mathbb{Z}')_{\#}^+ \longrightarrow BGL(\mathbb{F}_3)_{\#}^+.$$

The purpose of this paper is to prove analogous results (see Section 2 below) with the orthogonal and symplectic groups over $\mathbb{Z}'$ substituted for the general linear

group. Our motivation comes from previous work by the first author involving the mapping class group [8], and by the second author on the analogue of Bott periodicity in Hermitian K-theory [26]. This subject is of course related to the computations by A. Borel [10] of the rational cohomology of arithmetic groups.

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1. Motivational background.   The commutative diagram below appears in [8]. In it, Br_g denotes the g-strand braid group and the map Artin is Artin's representation of Br_g as automorphisms of the free group Fr_g on g generators. The group $\Gamma_{g,1}$ is the mapping class group of a surface of genus g with one boundary component. The map ψ_g is constructed by Vershinin [54] p. 1000. H is the hyperbolic map sending a matrix A to the matrix $\begin{pmatrix} A & O \\ O & {}^tA^{-1} \end{pmatrix}$.

$$
\begin{array}{ccccccc}
 & & \Gamma_{g,1} & \xrightarrow{\ \mathrm{id}\ } & \Gamma_{g,1} & & \\
 & \nearrow\ \psi_g & & & & \searrow & \\
\mathrm{Br}_g & \twoheadrightarrow & \Sigma_g & \rightarrowtail & O_g(\mathbb{Z}) & \xrightarrow{\ H\ } & \mathrm{Sp}_{2g}(\mathbb{Z}) \\
 & {}_{\mathrm{Artin}}\searrow & & & \downarrow & & \downarrow \\
 & & \mathrm{Aut}(\mathrm{Fr}_g) & \twoheadrightarrow & \mathrm{GL}_g(\mathbb{Z}) & \xrightarrow{\ H\ } & \mathrm{GL}_{2g}(\mathbb{Z}).
\end{array}
$$

Here all maps are injective except for the surjections $\mathrm{Aut}(\mathrm{Fr}_g) \twoheadrightarrow \mathrm{GL}_g(\mathbb{Z})$, $\mathrm{Br}_g \twoheadrightarrow \Sigma_g$ and $\Gamma_{g,1} \twoheadrightarrow \mathrm{Sp}_{2g}(\mathbb{Z})$.

Now, as in [8], combine with the inclusions in the real symplectic and general linear groups, stabilize, and take $B(\,\cdot\,)^+$, to get

$$
\begin{array}{ccccccccc}
 & & B\Gamma_\infty^+ & \xrightarrow{\ \mathrm{id}\ } & B\Gamma_\infty^+ & & & & BU \\
 & \nearrow & & & & \searrow & & & \downarrow{\cong} \\
B\mathrm{Br}_\infty^+ & \rightarrow & B\Sigma_\infty^+ & \rightarrow & BO(\mathbb{Z})^+ & \rightarrow & B\mathrm{Sp}(\mathbb{Z})^+ & \rightarrow & B\mathrm{Sp}\mathbb{R} \\
 & \searrow & & & \downarrow & & \downarrow & & \downarrow \\
 & & B\mathrm{Aut}(\mathrm{Fr}_\infty)^+ & \rightarrow & B\mathrm{GL}(\mathbb{Z})^+ & \rightarrow & B\mathrm{GL}(\mathbb{Z})^+ & \rightarrow & B\mathrm{GL}\mathbb{R}.
\end{array}
$$

The challenge is to describe these maps in homotopy theory. For instance, [14] discusses $B\Gamma_\infty^+ \to B\mathrm{GL}(\mathbb{Z})^+$, while [36] looks at $B\Gamma_\infty^+ \to BU$.

In this work we focus on $BO(\mathbb{Z}')^+$ and $B\mathrm{Sp}(\mathbb{Z}')^+$ as accessible approximations to $BO(\mathbb{Z})^+$ and $B\mathrm{Sp}(\mathbb{Z})^+$. Indeed, if we extrapolate the results known in the higher Hermitian K-theory of rings A (where 2 is invertible) and the results known in lower K-theory, then it is reasonable to conjecture that, up to odd finite torsion,

for $i > 1$ we have the isomorphisms

$$\pi_i(BO(\mathbb{Z})^+) \cong \pi_i(BO(\mathbb{Z}')^+) \quad \text{and} \quad \pi_i(BSp(\mathbb{Z})^+) \cong \pi_i(BSp(\mathbb{Z}')^+).$$

2. Main results. We recall first some notations from [26]. Let A be a ring provided with an anti-involution $x \mapsto \check{x}$, and let ε be an element of the center of A such that $\varepsilon\check{\varepsilon} = 1$. We assume also the existence of an element λ of the center such that $\lambda + \check{\lambda} = 1$. In most cases, $\lambda = 1/2$ and $\varepsilon = \pm 1$. In this setting, a central role is played by the ε-*orthogonal group*, which is the group of automorphisms of the ε-hyperbolic module $_\varepsilon H(A^n)$, denoted by $_\varepsilon O_{n,n}(A)$: its elements can be described as 2×2 matrices written in n-blocks

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

such that $M^*M = MM^* = I$, where the "ε-hyperbolic adjoint" M^* is defined as

$$M^* = \begin{pmatrix} {}^t\check{d} & \varepsilon\, {}^t\check{b} \\ \check{\varepsilon}\, {}^t\check{c} & {}^t\check{a} \end{pmatrix}$$

given by conjugating the involute transpose of M by the matrix

$$_\varepsilon J_n = \begin{pmatrix} 0 & \varepsilon I_n \\ I_n & 0 \end{pmatrix}.$$

For instance, when A is commutative (corresponding to $\check{} = $ id also being an involution), the usual symplectic group $\mathrm{Sp}_{2n}(A)$ is just $_{-1}O_{n,n}(A)$ with our notations. As other examples, the classical orthogonal groups $_1O_{n,n}(\mathbb{R})$ and $_1O_{n,n}(\mathbb{C})$ have the homotopy type of $O(n) \times O(n)$ and $O(2n)$ respectively, whereas $\mathrm{Sp}_{2n}(\mathbb{R})$ has the homotopy type of the unitary group $U(n)$ (see, for example [19]). We denote by $_\varepsilon O(A)$ the direct limit of the $_\varepsilon O_{n,n}(A)$ with respect to the obvious inclusions within each of the four component blocks. If $A = \mathbb{R}$ or $\mathbb{C}$, we provide $_\varepsilon O(A)$ with its usual topology unless otherwise stated.

The first aim of this paper is to prove the following squares homotopy cartesian:

$$(2\text{-}2) \qquad \begin{array}{ccc} B_1O(\mathbb{Z}')^+_\# & \longrightarrow & B_1O(\mathbb{R})_\# \\ \downarrow & & \downarrow \\ B_1O(\mathbb{F}_3)^+_\# & \longrightarrow & B_1O(\mathbb{C})_\# \end{array}$$

and

$$(2\text{-}3) \qquad \begin{array}{ccc} B_{-1}O(\mathbb{Z}')^+_\# & \longrightarrow & B_{-1}O(\mathbb{R})_\# \\ \downarrow & & \downarrow \\ B_{-1}O(\mathbb{F}_3)^+_\# & \longrightarrow & B_{-1}O(\mathbb{C})_\#. \end{array}$$

Since we have the well-known homotopy equivalences

$$_1O(\mathbb{R}) \simeq O \times O, \qquad _1O(\mathbb{C}) \simeq O, \qquad _{-1}O(\mathbb{R}) \simeq U, \qquad _{-1}O(\mathbb{C}) \simeq \mathrm{Sp},$$

on the one hand, and (cf. [22] p. 311)

$$\mathrm{Sp}/U \simeq \Omega^6(BO)$$

on the other, we obtain homotopy fibrations

$$BO_\# \longrightarrow B_1O(\mathbb{Z}')^+_\# \longrightarrow B_1O(\mathbb{F}_3)^+_\#.$$

$$\Omega^6(BO)_\# \longrightarrow B\mathrm{Sp}(\mathbb{Z}')^+_\# \longrightarrow B\mathrm{Sp}(\mathbb{F}_3)^+_\#.$$

Before starting to prove these statements, we should put them in a slightly more general framework. We have to consider the full classifying spaces of algebraic K-theory and Hermitian K-theory which we denote by $\mathcal{K}(A)$ and $_\varepsilon\mathcal{L}(A)$ respectively, following the notations of [26] (note that these notations are different from the ones generally used in surgery theory). Specifically, we have homotopy equivalences

$$\mathcal{K}(A) \simeq K_0(A) \times BGL(A)^+ \qquad \text{and} \qquad _\varepsilon\mathcal{L}(A) \simeq {}_\varepsilon L_0(A) \times B_\varepsilon O(A)^+.$$

Here $K_0(A)$ denotes the usual Grothendieck group of the category of finitely generated projective A-modules and $_\varepsilon L_0(A)$ the 'Witt-Grothendieck group' of the category constructed from the same objects with the extra structure of a nondegenerate ε-Hermitian form. There are obvious modifications when $A = \mathbb{R}$ or $\mathbb{C}$. A more functorial encoding of the π_0 information, which we need for consideration of fixed points under the involution, is given by the formulations $\mathcal{K}(A) = \Omega(BGL(SA)^+)$ and $_\varepsilon\mathcal{L}(A) = \Omega(B_\varepsilon O(SA)^+)$—see Appendix A. For our notation involving the 2-adic completion $X_\#$ of a nonconnected space X, we use the convention that when, as here, X is the loop space $X = \Omega Y$ of a connected space Y, then $X_\#$ is the loop space $\Omega(Y_\#)$.

For instance, the first diagram (0-1) may be written in an equivalent way as

$$(2\text{-}4) \qquad \begin{array}{ccc} \mathcal{K}(\mathbb{Z}')_\# & \longrightarrow & \mathcal{K}(\mathbb{R})_\# \\ \downarrow & & \downarrow \\ \mathcal{K}(\mathbb{F}_3)_\# & \longrightarrow & \mathcal{K}(\mathbb{C})_\# \end{array}$$

(we just cross the spaces by $K_0(A)_\# = \mathbb{Z}_\#$). The second and third diagrams may likewise be written in the following form (with $\varepsilon = \pm 1$):

$$(2\text{-}5) \qquad \begin{array}{ccc} _\varepsilon\mathcal{L}(\mathbb{Z}')_\# & \longrightarrow & {}_\varepsilon\mathcal{L}(\mathbb{R})_\# \\ \downarrow & & \downarrow \\ _\varepsilon\mathcal{L}(\mathbb{F}_3)_\# & \longrightarrow & {}_\varepsilon\mathcal{L}(\mathbb{C})_\#. \end{array}$$

This is clear for $\varepsilon = -1$ since $_{-1}L_0(A) \cong \mathbb{Z}$ for $A = \mathbb{Z}'$, $\mathbb{F}_3$, $\mathbb{R}$ or $\mathbb{C}$, as detailed in (4.1) below. For $\varepsilon = +1$ it requires a little more care, and uses the fact, proven in (4.3) below, that the following square of Witt-Grothendieck groups is cartesian:

$$
\begin{array}{ccc}
_1L_0(\mathbb{Z}') \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2 & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} \cong {}_1L_0(\mathbb{R}) \\
\downarrow & & \downarrow \\
_1L_0(\mathbb{F}_3) \cong \mathbb{Z} \oplus \mathbb{Z}/2 & \longrightarrow & \mathbb{Z} \cong {}_1L_0(\mathbb{C}).
\end{array}
$$

In this setting, we can now state our first main theorem.

THEOREM A.   *Diagram 2-5 above (for $\varepsilon = \pm 1$)*

$$
\begin{array}{ccc}
_\varepsilon\mathcal{L}(\mathbb{Z}')_\# & \longrightarrow & _\varepsilon\mathcal{L}(\mathbb{R})_\# \\
\downarrow & & \downarrow \\
_\varepsilon\mathcal{L}(\mathbb{F}_3)_\# & \longrightarrow & _\varepsilon\mathcal{L}(\mathbb{C})_\#
\end{array}
$$

is homotopy cartesian.

In consequence, we calculate the groups $_\varepsilon L_i(\mathbb{Z}')$, up to finite odd order subgroups. For comparison we also include $K_i(\mathbb{Z}')$ information, which follows immediately from what is known about $K_i(\mathbb{Z})$ [57], and the localization exact sequence

$$
K_{i+1}(\mathbb{Z}) \to K_{i+1}(\mathbb{Z}') \to K_i(\mathbb{F}_2) \to K_i(\mathbb{Z}) \to K_i(\mathbb{Z}'),
$$

given that $K_i(\mathbb{F}_2)$ is a finite group of odd order for $i > 0$.

THEOREM B.   *Modulo a finite group of odd order, the groups $K_i(\mathbb{Z}')$ and $_\varepsilon L_i(\mathbb{Z}')$ for $i \geq 0$ are as follows, where δ_{i0} denotes the Kronecker delta, and 2^t is the 2-primary part of $i + 1$.*

$i \pmod 8$	$K_i(\mathbb{Z}')$	$_{-1}L_i(\mathbb{Z}')$	$_1L_i(\mathbb{Z}')$
0	$\delta_{i0}\mathbb{Z}$	$\delta_{i0}\mathbb{Z}$	$\delta_{i0}\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2$
1	$\mathbb{Z} \oplus \mathbb{Z}/2$	0	$\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$
2	$\mathbb{Z}/2$	$\mathbb{Z}$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$
3	$\mathbb{Z}/16$	$\mathbb{Z}/16$	$\mathbb{Z}/8$
4	0	$\mathbb{Z}/2$	$\mathbb{Z}$
5	$\mathbb{Z}$	$\mathbb{Z}/2$	0
6	0	$\mathbb{Z}$	0
7	$\mathbb{Z}/2^{t+1}$	$\mathbb{Z}/2^{t+1}$	$\mathbb{Z}/2^{t+1}$

A further consequence is affirmation of the "Lichtenbaum-Quillen conjecture" for Hermitian K-theory, as follows. (Here the notation $_\varepsilon\mathbb{Z}/2$ keeps record of the particular $\mathbb{Z}/2$ action. Thus the homotopy fixed point set

$$
\mathcal{K}(\mathbb{Z}')^{h(_\varepsilon\mathbb{Z}/2)} := \mathrm{map}_{_\varepsilon\mathbb{Z}/2}(E\mathbb{Z}/2, \mathcal{K}(\mathbb{Z}'))
$$

is the space of maps equivariant under the $_\varepsilon\mathbb{Z}/2$ action, where $E\mathbb{Z}/2$ is a contractible free $\mathbb{Z}/2$-space, such as S^∞ with antipodal action, and $\mathbb{Z}/2$ acts on $\mathcal{K}(\mathbb{Z}')$ via conjugation by the matrices $_\varepsilon J_n$ as described above.)

THEOREM C. *For $\varepsilon = \pm 1$, the natural map*

$$_\varepsilon\mathcal{L}(\mathbb{Z}') = \mathcal{K}(\mathbb{Z}')^{\varepsilon\mathbb{Z}/2} \longrightarrow \mathcal{K}(\mathbb{Z}')^{h(_\varepsilon\mathbb{Z}/2)},$$

from the fixed point set to the homotopy fixed point set of the $_\varepsilon\mathbb{Z}/2$ action on $\mathcal{K}(\mathbb{Z}')$, becomes a homotopy equivalence after 2-adic completion.

For comparison, we remark that in [47] it is shown that, for the spectrum $T(\mathbb{Z})$ for topological Hochschild homology of the integers, the natural map from the fixed point set to the homotopy fixed point set, $T(\mathbb{Z})^{\mathbb{Z}/2} \to T(\mathbb{Z})^{h\mathbb{Z}/2}$, also induces a homotopy equivalence of 2-completed connective covers.

Our results also hold at the level of spectra. For, as in [46], Diagram (0-1) above may be recast as a diagram of 2-completed connective spectra. Then, because in order to achieve the passage from algebraic to Hermitian K-theory we arrange that maps of infinite loop spaces are induced by maps of rings and so are infinitely deloopable, our results similarly can be expressed in terms of 2-completed connective spectra. (We note that it is essential to consider connective spectra, since the negative K-groups of any discrete regular Noetherian ring like $\mathbb{Z}'$ and $\mathbb{F}_3$ are all zero, according to a well-known theorem of Bass; in contrast, the negative topological K-groups of $\mathbb{R}$ and $\mathbb{C}$, by periodicity, do not always coincide. By our induction results below, similar facts hold for Hermitian K-groups in negative dimensions.)

Strategy of the proofs, and organization of the paper. Let us denote by $_\varepsilon\overline{\mathcal{L}}(\mathbb{Z}')$ the homotopy cartesian product of $_\varepsilon\mathcal{L}(\mathbb{F}_3)$ and $_\varepsilon\mathcal{L}(\mathbb{R})$ over $_\varepsilon\mathcal{L}(\mathbb{C})$. Tautologically, for Theorem A we want to prove that the map

$$_\varepsilon\phi : \ _\varepsilon\mathcal{L}(\mathbb{Z}')_\# \longrightarrow \ _\varepsilon\overline{\mathcal{L}}(\mathbb{Z}')_\#$$

is a homotopy equivalence. Our strategy of proof is as follows:

The first step is to check that $_\varepsilon\phi$ induces an isomorphism on π_0 and π_1 for $\varepsilon = \pm 1$. This requires verifications that are detailed in Section 4 below.

As discussed at the outset, we have the homotopy cartesian diagram (2-4):

$$
\begin{array}{ccc}
\mathcal{K}(\mathbb{Z}')_\# & \longrightarrow & \mathcal{K}(\mathbb{R})_\# \\
\downarrow & & \downarrow \\
\mathcal{K}(\mathbb{F}_3)_\# & \longrightarrow & \mathcal{K}(\mathbb{C})_\#.
\end{array}
$$

In other words, if we denote by $\overline{\mathcal{K}}(\mathbb{Z}')$ the homotopy cartesian product of $\mathcal{K}(\mathbb{F}_3)$

and $\mathcal{K}(\mathbb{R})$ over $\mathcal{K}(\mathbb{C})$, the obvious map

$$\psi : \mathcal{K}(\mathbb{Z}')_{\#} \longrightarrow \overline{\mathcal{K}}(\mathbb{Z}')_{\#}$$

is a homotopy equivalence.

The last step of the proof, in Section 5, is to show that the verifications above and the fact that ψ is a homotopy equivalence imply that $_\varepsilon\phi$ is also a homotopy equivalence. This step is similar in spirit to the argument used already in [28], and is discussed in Section 3.1, together with several other "Karoubi induction" methods that we construct for later use. Here however it requires a careful study of the map $\mathcal{K}(\mathbb{F}_3) \longrightarrow \mathcal{K}(\mathbb{C})$ (the Brauer lifting) and its analogue in Hermitian *K*-theory, in order that required equivalences be induced from ring homomorphisms. A further ingredient is that rational and mod 2 arguments need to be handled separately.

Theorem B then follows readily from Theorem A by consideration in Section 6.1 of the various exact sequences of homotopy groups that arise from homotopy cartesian squares.

In order to prove Theorem C, we need to compare the homotopy cartesian square (2-5) of Theorem A with the homotopy fixed point sets that occur in the homotopy cartesian square of Diagram 2-4. Again, the behavior of $\mathcal{K}(\mathbb{F}_3)$ requires some care. However, the most treacherous part of the investigation concerns the equivalence $_\varepsilon\mathcal{L}(\mathbb{R}) \longrightarrow \mathcal{K}(\mathbb{R})^{h(_\varepsilon\mathbb{Z}/2)}$. Our treatment draws on the representation of topological *K*-theory by means of Fredholm operators on a Hilbert space, and leads to a result of independent interest, on the algebraic *K*-theory of simple algebras with involution.

3. Induction methods.

THEOREM 3.1. *Let* $n_0 \in \mathbb{Z}$ *be arbitrary. Suppose that* $\gamma : A \longrightarrow B$ *is a Hermitian ring map of* $\mathbb{Z}'$*-algebras inducing an isomorphism*

$$_\varepsilon L_n(\gamma) : {}_\varepsilon L_n(A) \overset{\cong}{\longrightarrow} {}_\varepsilon L_n(B)$$

for $\varepsilon = \pm 1$ *and* $n = n_0$, $n_0 + 1$.

(a) Upward induction. *Let* $r \geq 1$*, and suppose that* γ *induces an isomorphism*

$$K_n(\gamma) : K_n(A) \overset{\cong}{\longrightarrow} K_n(B)$$

whenever $n_0 \leq n \leq n_0 + r$*. Then* γ *also induces an isomorphism*

$$_\varepsilon L_n(\gamma) : {}_\varepsilon L_n(A) \overset{\cong}{\longrightarrow} {}_\varepsilon L_n(B)$$

whenever $\varepsilon = \pm 1$ *and* $n_0 \leq n \leq n_0 + r$.

(b) Downward induction. *Let $s \geq 1$, and suppose that γ induces an isomorphism*

$$K_n(\gamma) : K_n(A) \xrightarrow{\cong} K_n(B)$$

whenever $n_0 - s \leq n \leq n_0$ and an epimorphism when $n = n_0 + 1$. Then γ also induces an isomorphism

$$\varepsilon L_n(\gamma) : \varepsilon L_n(A) \xrightarrow{\cong} \varepsilon L_n(B)$$

whenever $\varepsilon = \pm 1$ and $n_0 - s \leq n \leq n_0$.

Remark. It will be clear from the proof that the assertion is also valid for homotopy groups with finite coefficients, and modulo a Serre class of abelian groups.

Proof. (a) We prove the theorem by induction on r, the case $r = 1$ being given. Suppose therefore that the theorem holds for a given $r \geq 1$. Following the notations of [26], we have for all n a commutative diagram of exact sequences

$$\varepsilon L_{n+1}(A) \longrightarrow K_{n+1}(A) \longrightarrow \varepsilon V_n(A) \longrightarrow \varepsilon L_n(A) \longrightarrow K_n(A)$$
$$\downarrow {\varepsilon L_{n+1}(\gamma)} \qquad \downarrow {K_{n+1}(\gamma)} \qquad \downarrow {\varphi_n} \qquad \downarrow {\varepsilon L_n(\gamma)} \qquad \downarrow {K_n(\gamma)}$$
$$\varepsilon L_{n+1}(B) \longrightarrow K_{n+1}(B) \longrightarrow \varepsilon V_n(B) \longrightarrow \varepsilon L_n(B) \longrightarrow K_n(B).$$

Now write $m = n_0 + r$. The five lemma and the induction hypothesis imply that φ_{m-1} is an isomorphism and φ_m is an epimorphism. By the fundamental theorem in Hermitian K-theory [26], the groups εV_n and $-\varepsilon U_{n+1}$ are naturally isomorphic. Therefore, we can write a second commutative diagram of exact sequences:

$$-\varepsilon U_{m+1}(A) \to K_{m+1}(A) \to -\varepsilon L_{m+1}(A) \to -\varepsilon U_m(A) \to K_m(A)$$
$$\downarrow {\varphi_m} \qquad \downarrow {K_{m+1}(\gamma)} \qquad \downarrow {-\varepsilon L_{m+1}(\gamma)} \qquad \downarrow {\varphi_{m-1}} \qquad \downarrow {K_m(\gamma)}$$
$$-\varepsilon U_{m+1}(B) \to K_{m+1}(B) \to -\varepsilon L_{m+1}(B) \to -\varepsilon U_m(B) \to K_m(B).$$

According to the above, $K_{m+1}(\gamma)$, φ_{m-1}, $K_m(\gamma)$ are isomorphisms and φ_m is an epimorphism. Therefore, $-\varepsilon L_{m+1}(\gamma)$ is an isomorphism by the five lemma.

(b) Given that $\pm 1 L_n(\gamma), \pm 1 L_{n+1}(\gamma), K_{n-1}(\gamma)$ and $K_n(\gamma)$ are isomorphisms, and that $K_{n+1}(\gamma)$ is an epimorphism, applying the five lemma to the map of exact sequences (with vertical arrows and γ omitted for brevity)

$$K_{n+1} \qquad -\varepsilon L_{n+1} \qquad \varphi_{n-1} \qquad K_n \qquad -\varepsilon L_n \qquad \varphi_{n-2} \qquad K_{n-1}$$

reveals that φ_{n-1} is an isomorphism and φ_{n-2} a monomorphism. Then the five

lemma for the map of exact sequences

$$K_n \qquad \varphi_{n-1} \qquad {}_\varepsilon L_{n-1} \qquad K_{n-1} \qquad \varphi_{n-2}$$

gives ${}_\varepsilon L_{n-1}(\gamma)$ as an isomorphism, and so the induction proceeds. $\qquad \square$

We now present several consequences of this result that we shall need later, and which are of independent interest. The first applications are to mod ℓ L-theory, where $\ell \geq 2$. To simplify notation, for $n \in \mathbb{Z}$ we write

$$_\varepsilon \tilde{L}_n(A) = {}_\varepsilon L_n(A; \mathbb{Z}/\ell),$$

being the n th $\mathbb{Z}/\ell$-homotopy group of the spectrum $_\varepsilon \mathcal{L}(A)$ continued into negative dimensions by iterated suspension; likewise for mod ℓ K-groups. The first result is a technical device for moving from integral to mod ℓ L-theory.

COROLLARY 3.2. *In the notation of the theorem above, suppose that, for both $\varepsilon = 1$ and $\varepsilon = -1$, γ induces isomorphisms on $_\varepsilon L_{n_0}, {}_\varepsilon L_{n_0+1}, K_{n_0-1}$ and K_{n_0}, and an epimorphism on K_{n_0+1}. If γ induces isomorphisms on the mod ℓ K-groups $\tilde{K}_n$ for $n_0 - s \leq n \leq n_0 + r$, then it also induces isomorphisms on all mod ℓ L-groups $_\varepsilon \tilde{L}_n$ in this range.*

Proof. From (b) of the theorem, we obtain an isomorphism on $_\varepsilon L_{n_0-1}$. Then the universal coefficients formula

$$_\varepsilon L_n(A) \otimes \mathbb{Z}/\ell \rightarrowtail {}_\varepsilon \tilde{L}_n(A) \twoheadrightarrow {}_\ell({}_\varepsilon L_{n-1}(A))$$

with $n = n_0, n_0+1$ gives the isomorphism on $_\varepsilon \tilde{L}_{n_0}, {}_\varepsilon \tilde{L}_{n_0+1}$ that enables both upward and downward induction to commence. $\qquad \square$

Next, we have a Mayer-Vietoris sequence.

THEOREM 3.3. *For a prime $\ell \geq 2$, let*

$$\begin{array}{ccc} A_1 & \overset{\alpha}{\twoheadrightarrow} & A_2 \\ \downarrow & \lrcorner & \downarrow \\ A_3 & \underset{\beta}{\twoheadrightarrow} & A_4 \end{array}$$

be a cartesian square of Hermitian rings and homomorphisms, with β (and therefore α) surjective, where each ring is a $\mathbb{Z}[1/2\ell]$-algebra. Then for $\varepsilon \in \{\pm 1\}$ and all integers n there is a natural exact sequence

$$\cdots \rightarrow {}_\varepsilon \tilde{L}_{n+1}(A_4) \rightarrow {}_\varepsilon \tilde{L}_n(A_1) \rightarrow {}_\varepsilon \tilde{L}_n(A_2) \oplus {}_\varepsilon \tilde{L}_n(A_3) \rightarrow {}_\varepsilon \tilde{L}_n(A_4) \rightarrow \cdots.$$

Proof. As usual, the Mayer-Vietoris sequence follows from an excision isomorphism between the relative terms obtained from the surjections α and β. To obtain this isomorphism, first recall from [26] p. 262 that associated to a ring map $\varphi : B \to D$ there is the fiber product $\Gamma(\varphi)$

$$\begin{array}{ccc} \Gamma(\varphi) & \twoheadrightarrow & CD \\ \downarrow & \lrcorner & \downarrow \\ SB & \xrightarrow{\varphi} & SD \end{array}$$

where C and S stand for cone and suspension respectively. This ring has the property that $\Omega\mathcal{K}(\Gamma(\varphi))$ and $\Omega_\varepsilon\mathcal{L}(\Gamma(\varphi))$ have the homotopy type of the homotopy fibers of $\mathcal{K}(\varphi)$ and $_\varepsilon\mathcal{L}(\varphi)$. So here it suffices to check that the map $\Gamma(\alpha) \to \Gamma(\beta)$ induces an isomorphism of $_\varepsilon\tilde{L}_*$-groups. This is an application of the preceding theorem to the data that the excision isomorphism for $\mathbb{Z}[1/\ell]$-algebras holds: first, for mod ℓ K-theory (using surjectivity) by [56]; and, second, for negatively indexed $_\varepsilon\tilde{L}_*$-groups by application of universal coefficients to the results of [32]. $\square$

Homotopy invariance also follows.

THEOREM 3.4. *For a prime $\ell \geq 2$, a $\mathbb{Z}[1/2\ell]$-algebra A and $\varepsilon \in \{\pm 1\}$, there is a natural isomorphism*

$$_\varepsilon\tilde{L}_*(A[x]) \cong {}_\varepsilon\tilde{L}_*(A).$$

Proof. The corresponding result for mod ℓ K-theory is proved in [56]. Again, we check the result for negatively indexed $_\varepsilon\tilde{L}_*$-groups. Since for suspension rings we have from Appendix A that $S(A[x]) \cong (SA)[x]$, the result is true in all negative dimensions provided it holds for $_\varepsilon\tilde{L}_0$. As the injection $A \rightarrowtail A[x]$ is split, the universal coefficients formula

$$K_0(A) \otimes \mathbb{Z}/\ell \rightarrowtail \tilde{K}_0(A) \twoheadrightarrow {}_\ell K_{-1}(A)$$

shows that homotopy invariance holds for $\tilde{K}_0$ precisely when it holds for both $\mathbb{Z}/\ell \otimes K_0$ and $_\ell K_{-1}$; likewise for $_\varepsilon\tilde{L}_0$. From homotopy invariance of $_\varepsilon W_0 = \mathrm{Coker}(K_0 \to {}_\varepsilon L_0)$ established in [23] (and so, by suspension, of $_\varepsilon W_{-1}$), we use the invariance of $\mathbb{Z}/\ell \otimes K_0$ (respectively $_\ell K_{-1}$) to deduce the invariance of $\mathbb{Z}/\ell \otimes {}_\varepsilon L_0$ (respectively $_\ell({}_\varepsilon L_{-1})$). Hence we have homotopy invariance for $_\varepsilon\tilde{L}_0$ too. $\square$

Here is another useful application of the methodology.

PROPOSITION 3.5. *Let A be a ring with 2 invertible in A, and let $r \geq 1$, $s \geq 0$. Suppose that the groups $K_n(A)$ (whenever $n_0 - s \leq n \leq n_0 + r$) and $_\varepsilon L_i(A)$ (for*

$\varepsilon \in \{\pm 1\}$, $n \in \{n_0, n_0 + 1\}$) *are finitely generated. Then the groups* $_\varepsilon L_n(A)$ *are finitely generated whenever* $n_0 - s \leq n \leq n_0 + r$.

Proof. The result follows from Theorem 3.1 above by considering the embedding of A in its cone ring CA, which has trivial K- and L-theory. (Clearly the argument works more generally with finite generation replaced by any Serre class of abelian groups.) □

This of course notably applies to Quillen's result [43] on finite generation of K-groups of rings of S-integers.

COROLLARY 3.6. *Let A be a ring of S-integers in a number field such that* $\mathbb{Z}' \subseteq A$. *Then for all integers n the groups* $_\varepsilon L_n(A)$ *are finitely generated.*

4. Low-dimensional checks. We write $_\varepsilon L_n(A)$ (resp. $_\varepsilon L_n^{\mathrm{top}}(A)$) for the homotopy groups $\pi_n(_\varepsilon \mathcal{L}(A))$ in the discrete case (resp. in the continuous case). Analogously, we write $K_n(A)$ (resp. $K_n^{\mathrm{top}}(A)$) for the homotopy groups $\pi_n(\mathcal{K}(A))$ in the same situations. Finally, $_\varepsilon \overline{L}_n(\mathbb{Z}')$ denotes the homotopy groups $\pi_n(_\varepsilon \overline{\mathcal{L}}(\mathbb{Z}'))$.

4.1. The map $_{-1}\phi : {}_{-1}\mathcal{L}(\mathbb{Z}')_\# \longrightarrow {}_{-1}\overline{\mathcal{L}}(\mathbb{Z}')_\#$ **induces an isomorphism on** π_0. In fact, this result holds even before 2-completion. For, because all four rings $A = \mathbb{Z}'$, $\mathbb{F}_3$, $\mathbb{R}$, $\mathbb{C}$ are Dedekind, each $\pi_0(_{-1}\mathcal{L}(A))$ equals $\mathbb{Z}$, detected by the (even) rank of the free symplectic A-inner product space [37] p. 7. Since each map of rings preserves rank, there is a cartesian square

$$_{-1}L_0(\mathbb{Z}') \cong \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \cong {}_{-1}L_0(\mathbb{R})$$
$$\downarrow \cong \qquad\qquad\qquad \downarrow \cong$$
$$_{-1}L_0(\mathbb{F}_3) \cong \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \cong {}_{-1}L_0(\mathbb{C}).$$

On the other hand, the exact sequence that enables us to compute $\pi_0(_{-1}\overline{\mathcal{L}}(\mathbb{Z}'))$ $= {}_{-1}\overline{L}_0(\mathbb{Z}')$ is extracted from the homotopy cartesian diagram

$$_{-1}\overline{\mathcal{L}}(\mathbb{Z}') \longrightarrow {}_{-1}\mathcal{L}(\mathbb{R})$$
$$\downarrow \qquad\qquad \downarrow$$
$$_{-1}\check{\mathcal{L}}(\mathbb{F}_3) \longrightarrow {}_{-1}\check{\mathcal{L}}(\mathbb{C})$$

and reduces to the following:

$$_{-1}L_1^{\mathrm{top}}(\mathbb{C}) \longrightarrow {}_{-1}\overline{L}_0(\mathbb{Z}') \longrightarrow {}_{-1}L_0(\mathbb{F}_3) \oplus {}_{-1}L_0(\mathbb{R}) \twoheadrightarrow {}_{-1}L_0(\mathbb{C}).$$

Since

$$_{-1}L_1^{\mathrm{top}}(\mathbb{C}) = \pi_1(B\mathrm{Sp}) = \pi_5(BO) = 0,$$

this sequence shows that $_{-1}\overline{L}_0(\mathbb{Z}')$ is the same pull-back as $_{-1}L_0(\mathbb{Z}')$, making $\pi_0(_{-1}\phi)$ an isomorphism.

4.2. **The map** $_{-1}\phi : \,_{-1}\mathcal{L}(\mathbb{Z}')_{\#} \longrightarrow \,_{-1}\overline{\mathcal{L}}(\mathbb{Z}')_{\#}$ **induces an isomorphism on** π_1. After [22] p. 382, we have

$$_{-1}L_1(\mathbb{Z}') = \mathrm{Sp}(\mathbb{Z}')\,/[\mathrm{Sp}(\mathbb{Z}'),\,\mathrm{Sp}(\mathbb{Z}')] = 0.$$

On the other hand, the same homotopy cartesian square as above gives rise to the following exact sequence (after completion)

$$_{-1}L_2^{\mathrm{top}}(\mathbb{C}) \longrightarrow \,_{-1}\overline{L}_1(\mathbb{Z}')_{\#} \longrightarrow \,_{-1}L_1(\mathbb{F}_3)_{\#} \oplus \,_{-1}L_1^{\mathrm{top}}(\mathbb{R})_{\#} \longrightarrow \,_{-1}L_1^{\mathrm{top}}(\mathbb{C})_{\#}.$$

We have here, from $_{-1}O(\mathbb{C}) \simeq \mathrm{Sp}$ and $_{-1}O(\mathbb{R}) \simeq U$,

$$_{-1}L_2^{\mathrm{top}}(\mathbb{C}) = \pi_2(B\mathrm{Sp}) = \pi_6(BO) = 0;$$

$$_{-1}L_1^{\mathrm{top}}(\mathbb{R}) = \pi_1(BU) = 0;$$

$$_{-1}L_1(\mathbb{F}_3) = \mathrm{Sp}(\mathbb{F}_3)\,/\big[\mathrm{Sp}(\mathbb{F}_3),\,\mathrm{Sp}(\mathbb{F}_3)\big] = 0.$$

Therefore, $_{-1}\overline{L_1}(\mathbb{Z}')$ is also reduced to 0.

4.3. **The map** $_1\phi : \,_1\mathcal{L}(\mathbb{Z}')_{\#} \longrightarrow \,_1\overline{\mathcal{L}}(\mathbb{Z}')_{\#}$ **induces an isomorphism on** π_0. It is well known that the Witt-Grothendieck group of $\mathbb{Z}'$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2$ (signature, rank and discriminant; note that the discriminant is positive on real hyperbolic forms).

For the determination of $_1\overline{L_0}(\mathbb{Z}')$, we observe that the homotopy cartesian diagram

$$\begin{array}{ccc} _1\overline{\mathcal{L}}(\mathbb{Z}') & \longrightarrow & _1\mathcal{L}(\mathbb{R}) \\ \downarrow & & \downarrow \\ _1\mathcal{L}(\mathbb{F}_3) & \longrightarrow & _1\mathcal{L}(\mathbb{C}) \end{array}$$

gives rise to the following exact sequence:

$$_1L_1(\mathbb{F}_3) \oplus \,_1L_1^{\mathrm{top}}(\mathbb{R}) \to \,_1L_1^{\mathrm{top}}(\mathbb{C}) \to \,_1\overline{L_0}(\mathbb{Z}') \to \,_1L_0(\mathbb{F}_3) \oplus \,_1L_0(\mathbb{R}) \twoheadrightarrow \,_1L_0(\mathbb{C}).$$

Recalling that $_1O(\mathbb{C}) \simeq O$ and $_1O(\mathbb{R}) \simeq O \times O$, we have

$$_1L_1^{\mathrm{top}}(\mathbb{C}) \cong K_1^{\mathrm{top}}(\mathbb{R}) \cong \mathbb{Z}/2$$

and

$$_1L_1^{\mathrm{top}}(\mathbb{R}) \cong K_1^{\mathrm{top}}(\mathbb{R}) \oplus K_1^{\mathrm{top}}(\mathbb{R}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2,$$

and the first arrow of the above sequence is surjective. On the other hand, we have the well-known isomorphisms

$$_1L_0(\mathbb{C}) \cong \mathbb{Z} \qquad\qquad \text{[27] p. 248}$$
$$_1L_0(\mathbb{R}) \cong \mathbb{Z} \oplus \mathbb{Z} \qquad\quad \text{[22] p. 306}$$
$$_1L_0(\mathbb{F}_3) \cong \mathbb{Z} \oplus \mathbb{Z}/2 \qquad \text{[50] IV.1.7}$$

which show that $_1\overline{L}_0(\mathbb{Z}')$ is also isomorphic to $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2$.

It remains to check that the given map between these two isomorphic groups is itself an isomorphism. The generator of the summand $\mathbb{Z}/2$ in $_1L_0(\mathbb{Z}')$ is the difference of the two quadratic forms $\langle 1 \rangle - \langle 2 \rangle$ with the usual notations. This element maps into the generator of the summand $\mathbb{Z}/2$ in $_1L_0(\mathbb{F}_3)$ which is detected by the discriminant in $(\mathbb{F}_3^\bullet)/(\mathbb{F}_3^\bullet)^2$. This discriminant is just the Legendre symbol (for $p = 3$)

$$\left(\frac{2}{p}\right) = (-1)^{(p^2-1)/8}$$

which is equal to -1 (more simply, 2 is not a square mod 3). The torsion-free summands correspond to the rank and index/signature invariants [50] IV, V, which are preserved respectively by the ring maps and the inclusion $\mathbb{Z}' \hookrightarrow \mathbb{R}$. Thus we obtain the required isomorphism from $_1L_0(\mathbb{Z}') \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2$ to the kernel $_1\overline{L}_0(\mathbb{Z}')$ of $_1L_0(\mathbb{F}_3) \oplus {}_1L_0(\mathbb{R}) \twoheadrightarrow {}_1L_0(\mathbb{C})$.

4.4. **The map** $_1\phi : {}_1\mathcal{L}(\mathbb{Z}')_\# \longrightarrow {}_1\overline{\mathcal{L}}(\mathbb{Z}')_\#$ **induces an isomorphism on** π_1. From [6] (4.7.6), there is an exact sequence

$$\mathrm{SK}_1(\mathbb{Z}') \xrightarrow{H} {}_1L_1(\mathbb{Z}') \xrightarrow{\text{(det, SN)}} \mathrm{Ip}(\mathbb{Z}') \oplus \mathrm{Discr}(\mathbb{Z}') \to 0,$$

where $\mathrm{Ip}(\mathbb{Z}') = \{0_{\mathrm{Ip}\mathbb{Z}'}, 1_{\mathrm{Ip}\mathbb{Z}'}\}$ denotes the group of idempotents, and there is a short exact sequence [6] p. 156

$$\mathbb{Z}'^\bullet/(\mathbb{Z}'^\bullet)^2 \rightarrowtail \mathrm{Discr}(\mathbb{Z}') \twoheadrightarrow {}_2\mathrm{Pic}(\mathbb{Z}').$$

Now $\mathrm{Pic}(\mathbb{Z}') = 0$ since $\mathbb{Z}'$ is a principal ideal domain. Further, from the localization sequence

$$K_1(\mathbb{F}_2) \to K_1(\mathbb{Z}) \to K_1(\mathbb{Z}') \to K_0(\mathbb{F}_2) \to K_0(\mathbb{Z}) \to K_0(\mathbb{Z}'),$$

namely

$$0 \to \mathbb{Z}/2 \to K_1(\mathbb{Z}') \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z},$$

we have that $K_1(\mathbb{Z}') \cong \mathbb{Z}/2 \oplus \mathbb{Z} \cong \mathbb{Z}'^{\bullet}$. So $\mathrm{SK}_1(\mathbb{Z}') = 0$. Hence we have

$$_1L_1(\mathbb{Z}') \cong \mathbb{Z}/2 \oplus (\mathbb{Z}')^{\bullet}/(\mathbb{Z}'^{\bullet})^2 \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$$

generated respectively by $1_{\mathrm{Ip}\mathbb{Z}'}$, $2(\mathbb{Z}'^{\bullet})^2$ and $(-1)(\mathbb{Z}'^{\bullet})^2$.

On the other hand, from the homotopy cartesian diagram above we deduce the exact sequence

$$_1L_2(\mathbb{F}_3) \oplus {}_1L_2^{\mathrm{top}}(\mathbb{R}) \to {}_1L_2^{\mathrm{top}}(\mathbb{C}) \to {}_1\overline{L}_1(\mathbb{Z}') \to {}_1L_1(\mathbb{F}_3) \oplus {}_1L_1^{\mathrm{top}}(\mathbb{R}) \to {}_1L_1^{\mathrm{top}}(\mathbb{C}).$$

Again, from the homotopy equivalences $_1O(\mathbb{R}) \simeq O \times O$ and $_1O(\mathbb{C}) \simeq O$ it follows that the map

$$\mathbb{Z}/2 \oplus \mathbb{Z}/2 \cong {}_1L_2^{\mathrm{top}}(\mathbb{R}) \longrightarrow {}_1L_2^{\mathrm{top}}(\mathbb{C}) \cong \mathbb{Z}/2$$

is surjective. Therefore, $_1\overline{L}_1(\mathbb{Z}')$ is identified with the kernel of the map

$$\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \cong {}_1L_1(\mathbb{F}_3) \oplus {}_1L_1^{\mathrm{top}}(\mathbb{R}) \longrightarrow {}_1L_1^{\mathrm{top}}(\mathbb{C}) \cong \mathbb{Z}/2$$

which is also the sum of three copies of $\mathbb{Z}/2$.

It remains to check that $_1\phi$ yields this isomorphism. First recall from [32] (5.6) that for a discrete field F we have

$$_1L_1(F) \cong \mathrm{Ip}(F) \oplus \mathrm{Discr}(F) \cong \mathbb{Z}/2 \oplus F^{\bullet}/(F^{\bullet})^2.$$

In particular, this gives canonical generators for the cases of real and complex fields, where the algebraic and topological $_1L_1$ groups coincide. Then, by naturality of Ip and Discr, we have the three given generators of $_1L_1(\mathbb{Z}')$ mapping respectively under $_1\phi$ to the generators

$$1_{\mathrm{Ip}\mathbb{F}_3} + 1_{\mathrm{Ip}\mathbb{R}}, \quad 2(\mathbb{F}_3^{\bullet})^2 + (-1)(\mathbb{R}^{\bullet})^2, \quad (-1)(\mathbb{R}^{\bullet})^2$$

of

$$\mathrm{Ker}\left[\left\langle 1_{\mathrm{Ip}\mathbb{F}_3}\right\rangle \oplus \left\langle 2(\mathbb{F}_3^{\bullet})^2\right\rangle \oplus \left\langle 1_{\mathrm{Ip}\mathbb{R}}\right\rangle \oplus \left\langle (-1)(\mathbb{R}^{\bullet})^2\right\rangle \longrightarrow \left\langle 1_{\mathrm{Ip}\mathbb{C}}\right\rangle\right].$$

So we have the desired isomorphism after all.

4.5. **Summary: the** mod 2 **equivalence for** π_0, π_1. We now use the data collected above to display the low-dimensional information in the form we need.

LEMMA 4.6. *For* $\varepsilon \in \{\pm 1\}$, $i \in \{0, 1\}$, *the map* $_\varepsilon \phi : {_\varepsilon}\mathcal{L}(\mathbb{Z}')_\# \longrightarrow {_\varepsilon}\overline{\mathcal{L}}(\mathbb{Z}')_\#$ *induces an isomorphism on* π_i *with* $\mathbb{Z}/2$ *coefficients.*

Proof. Since we already have this result with integer coefficients, to obtain the mod 2 version via the universal coefficient sequence, it suffices to check that there is induced an integral isomorphism on π_{-1} of these spectra. Here the spaces $B_\varepsilon O(\mathbb{R})$, $B_\varepsilon O(\mathbb{C})$ are expressible in terms of *BO*, *BU* and *B*Sp, as noted above, and so contribute zero groups. Therefore we need only check that $_\varepsilon \phi$ induces isomorphisms from $_\varepsilon L_{-1}(\mathbb{Z}')$ to $_\varepsilon L_{-1}(\mathbb{F}_3)$. To do this, essentially in a reprise of downward induction in Theorem 3.1, we exploit the natural exact sequences (for $A = \mathbb{Z}', \mathbb{F}_3$)

$$K_1(A) \to {_{-\varepsilon}}L_1(A) \to {_{-\varepsilon}}U_0(A) \to K_0(A) \to {_{-\varepsilon}}L_0(A)$$

and (via the fundamental theorem [26])

$$K_0(A) \to {_{-\varepsilon}}U_0(A) \to {_\varepsilon}L_{-1}(A) \to K_{-1}(A).$$

We of course use the information we've already gathered about these groups.

$\varepsilon = 1.$ For both $A = \mathbb{Z}', \mathbb{F}_3$ the first sequence terminates with

$$0 \to {_{-1}}U_0(A) \to \mathbb{Z} \to \mathbb{Z}$$

where the final map is given by the rank (see (4.1)) and so is nonzero. So $_{-1}U_0(A)$ vanishes, and the second sequence finishes with

$$0 \to {_1}L_{-1}(A) \to 0,$$

leaving $_1L_{-1}(\mathbb{Z}') = {_1}L_{-1}(\mathbb{F}_3) = 0$.

$\varepsilon = -1.$ Here the first map of exact sequences becomes

$$\begin{array}{ccccccccc}
\mathbb{Z} \oplus \mathbb{Z}/2 & \to & (\mathbb{Z}/2)^3 & \to & {_1}U_0(\mathbb{Z}') & \to & \mathbb{Z} & \to & \mathbb{Z}^2 \oplus \mathbb{Z}/2 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{Z}/2 & \to & (\mathbb{Z}/2)^2 & \to & {_1}U_0(\mathbb{F}_3) & \to & \mathbb{Z} & \to & \mathbb{Z} \oplus \mathbb{Z}/2
\end{array}$$

where in each sequence the final map is injective and the first has cokernel $\mathbb{Z}/2$, by our previous considerations. We therefore obtain that $_\varepsilon \phi$ induces an isomorphism $_1U_0(\mathbb{Z}') \xrightarrow{\cong} {_1}U_0(\mathbb{F}_3) \cong \mathbb{Z}/2$. Since $K_i(\mathbb{Z}') \to K_i(\mathbb{F}_3)$ is an isomorphism for $i = 0, -1$, the desired isomorphism on $_{-1}L_{-1}(A)$ now follows by applying the five lemma to the second sequence. $\square$

5. The homotopy equivalence for Theorem A. In accordance with the argument of [52] p. 32 (exploiting Bockstein homomorphisms), our claimed homotopy equivalence

$$_\varepsilon \phi : {}_\varepsilon \mathcal{L}(\mathbb{Z}')_\# \longrightarrow {}_\varepsilon \overline{\mathcal{L}}(\mathbb{Z}')_\#$$

of 2-completed spaces may be deduced by proving that the map is both a rational and a 2-local equivalence. Thus our argument breaks into two parts.

5.1. **We work rationally**. Since by [18] the space $_\varepsilon \mathcal{L}(\mathbb{F}_3)_\#$ is rationally $\mathbb{Q}_\#$, the space $_\varepsilon \overline{\mathcal{L}}(\mathbb{Z}')_\#$ rationally reduces to the homotopy fiber of the rationalized map from $_\varepsilon \mathcal{L}(\mathbb{R})_\#$ to $_\varepsilon \mathcal{L}(\mathbb{C})_\#$, namely $\mathbb{Q}_\# \times \Omega^{7+\varepsilon}(BO)$. Now by [10] §12, the map from $_\varepsilon \mathcal{L}(\mathbb{Z}')_\#$ to $\mathbb{Q}_\# \times \Omega^{7+\varepsilon}(BO)$ is a rational homotopy equivalence. (Recall that finite generation of the homotopy groups of $_\varepsilon \mathcal{L}(\mathbb{Z}')$ was established in Corollary 3.6.) For the situation with $\mathbb{Z}'$ coefficients, see [25] p. 253 et seq.

5.2. **We work** $\bmod 2$ **(which is prime to 3)**. In fact, for the arguments we present, it is no more difficult to work in the more general setting mod ℓ, and consider the finite field $\mathbb{F}_q$ with q elements, q odd, and ℓ coprime to q. We require some analysis of Quillen's Brauer lifting $\mathcal{K}(\mathbb{F}_q) \longrightarrow \mathcal{K}^{\mathrm{top}}(\mathbb{C})$ [41] and its refinement by E. M. Friedlander [18] in the Hermitian case as a map $_\varepsilon \mathcal{L}(\mathbb{F}_q) \longrightarrow {}_\varepsilon \mathcal{L}^{\mathrm{top}}(\mathbb{C})$ (see Theorem 5.5 below). By the theorem of Quillen we have a homotopy fibration

$$(5\text{-}6) \qquad\qquad \mathcal{K}(\mathbb{F}_q) \longrightarrow \mathcal{K}^{\mathrm{top}}(\mathbb{C}) \longrightarrow \mathcal{K}^{\mathrm{top}}(\mathbb{C})$$

where the second arrow is defined by $\psi^q - 1$, ψ^q being the Adams operation. In other words, we have a homotopy equivalence between $\mathcal{K}(\mathbb{F}_q)$ and a certain homotopy fiber.

Our strategy is to replace the map $_\varepsilon \mathcal{L}(\mathbb{F}_q) \longrightarrow {}_\varepsilon \mathcal{L}^{\mathrm{top}}(\mathbb{C})$ by one induced from a map of rings, so that we obtain a ring B whose $_\varepsilon \mathcal{L}(B)$ serves as a candidate for $_\varepsilon \overline{\mathcal{L}}(\mathbb{Z}')$. To do this, we of course take advantage of the fact that we are interested only in homotopy equivalences mod ℓ. Thus, for example, by [53] it does not matter whether we use the usual or discrete topology on $\mathbb{R}$ and $\mathbb{C}$. In order to fix ideas (the usual topology leads to alternative arguments), we take the discrete topology. We proceed in five steps.

First, we exploit the fact [53] that in mod ℓ K-theory we may replace $\mathbb{F}_q$ by the ring $\mathbb{Z}_q$ of q-adic integers, the Witt ring of $\mathbb{F}_q$. Likewise, $\bar{\mathbb{Z}}_q$ is the ring of Witt vectors of the algebraic closure $\bar{\mathbb{F}}_q$ of $\mathbb{F}_q$, while $\bar{\mathbb{Q}}_q$ is the field of fractions of $\bar{\mathbb{Z}}_q$.

LEMMA 5.3. *The homomorphism $\mathbb{Z}_q \to \mathbb{F}_q$ induces an isomorphism of all* mod *ℓ L-groups.*

Proof. According to Corollary 3.2 above, we just have to check that $\mathbb{Z}_q \to \mathbb{F}_q$ has the following effects:

Isomorphism on $_\varepsilon L_0$. This follows from [37] p. 7 when $\varepsilon = 1$ (both groups $\mathbb{Z}$), and from [4] when $\varepsilon = -1$ (both groups $\mathbb{Z} \oplus \mathbb{Z}/2$).

Isomorphism on $_\varepsilon L_1$. For $\varepsilon = 1$ ($\mathbb{Z}/2 \oplus \mathbb{Z}/2$), see [6], and for $\varepsilon = -1$ (both groups vanish), see [7] (13.2).

Isomorphism on K_{-1}. Both groups vanish since the rings are regular Noetherian.

Isomorphism on K_0. Since each ring is local, the groups are $\mathbb{Z}$.

Epimorphism on K_1. We have $K_1(\mathbb{Z}_q) = \mathrm{GL}_1(\mathbb{Z}_q) \twoheadrightarrow \mathrm{GL}_1(\mathbb{F}_q) = K_1(\mathbb{F}_q)$.   □

Second, we observe that in *K*-theory with finite coefficients the Adams operation ψ^q on $\mathcal{K}(\mathbb{C})$ is induced by a certain (non-unique) discontinuous automorphism of $\mathbb{C}$ described as follows. First, we choose an embedding of the *q*-adic numbers $\mathbb{Q}_q$ in $\mathbb{C}$, and consider the union of all the cyclotomic fields (over $\mathbb{Q}_q$) generated by the *n*th roots of unity where $n = q^r - 1$ with *r* a power of *q*. This field admits an automorphism induced by the map sending a primitive root ζ to ζ^q. After Zorn's lemma, the automorphism extends to a (discontinuous) automorphism ψ of $\mathbb{C}$ that induces ψ^q [40]. The homotopy fiber of $\psi^q - 1 : \mathcal{K}(\mathbb{C}) \to \mathcal{K}(\mathbb{C})$ may now be realized as $\mathcal{K}(F)$ where the equalizer ring F of ψ and id is constructed in Appendix B as the pull-back

$$
\begin{array}{ccc}
F & \twoheadrightarrow & \mathbb{C} \\
\downarrow & \lrcorner & \downarrow {\scriptstyle (\psi,\mathrm{id})} \\
M(\Delta) & \xrightarrow{\tilde{\Delta}} & \mathbb{C} \times \mathbb{C}.
\end{array}
$$

Next, we use the fact that $\mathbb{Z}_q$ is fixed by ψ^q, in order to factorize its inclusion in $\mathbb{C}$ as

$$
\mathbb{Z}_q \longrightarrow F \xrightarrow{\eta} \mathbb{C}.
$$

The induced map $\mathcal{K}(\mathbb{Z}_q) \to \mathcal{K}(F)$ is then a mod ℓ homotopy equivalence, by the upper part of the folklore commuting diagram (*q* odd)

(5-7)
$$
\begin{array}{ccccc}
\mathcal{K}(\mathbb{F}_q) & \longrightarrow & \mathcal{K}^{\mathrm{top}}(\mathbb{C}) & \xrightarrow{\psi^q - 1} & \mathcal{K}^{\mathrm{top}}(\mathbb{C}) \\
\uparrow & & \uparrow & & \uparrow \\
\mathcal{K}(\mathbb{Z}_q) & \longrightarrow & \mathcal{K}(\bar{\mathbb{Q}}_q) & \xrightarrow{\psi - 1} & \mathcal{K}(\bar{\mathbb{Q}}_q) \\
& & \uparrow & & \uparrow \\
\downarrow & & \mathcal{K}(\bar{\mathbb{Z}}_q) & \xrightarrow{\psi - 1} & \mathcal{K}(\bar{\mathbb{Z}}_q) \\
& & \downarrow & & \downarrow \\
\mathcal{K}(\mathbb{F}_q) & \longrightarrow & \mathcal{K}(\bar{\mathbb{F}}_q) & \xrightarrow{\psi - 1} & \mathcal{K}(\bar{\mathbb{F}}_q)
\end{array}
$$

of Quillen [42], Friedlander and Suslin [53], in which all vertical arrows, induced by ring homomorphisms, are mod ℓ equivalences (ℓ prime to *q*), and ψ is the Frobenius map.

Fourth, we consider the commuting diagram of ring homomorphisms, in which the two right-hand squares are pull-backs, the bottom composite is just the diagonal map $\Delta : \mathbb{C} \to \mathbb{C} \times \mathbb{C}$, and the maps φ, φ' derive from the fact that ψ fixes $\mathbb{Z}_q$.

$$
\begin{array}{ccccc}
\mathbb{Z}' & \xrightarrow{\varphi'} & F' & \twoheadrightarrow & \mathbb{R} \\
\downarrow & & \downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{inc}} \\
\mathbb{Z}_q & \xrightarrow{\varphi} & F & \xrightarrow[\twoheadrightarrow]{\eta} & \mathbb{C} \\
\downarrow & & \downarrow & \lrcorner & \downarrow{\scriptstyle (\psi,\mathrm{id})} \\
\mathbb{C} & \longrightarrow & M(\Delta) & \xrightarrow[\twoheadrightarrow]{\tilde{\Delta}} & \mathbb{C} \times \mathbb{C}.
\end{array}
$$

Finally, from Theorem 3.3 above we have the following:

LEMMA 5.4. *There are* mod ℓ *homotopy equivalences between* $\mathcal{K}(F')$ *and* $\bar{\mathcal{K}}(\mathbb{Z}')$, *and between* $_\varepsilon\mathcal{L}(F')$ *and* $_\varepsilon\overline{\mathcal{L}}(\mathbb{Z}')$. $\square$

We remark that alternative proofs of the lemma above may be obtained from the fact, as in [38], that the inclusion $\mathbb{Z}_q \hookrightarrow \mathbb{C}$ induces Brauer lifting. One can proceed with the discrete topology on $\mathbb{C}$, as above, or, with the usual topology on $\mathbb{C}$ by replacing $\mathbb{C}[x]$ by the Banach ring $\mathbb{C}\langle x \rangle$ of complex convergent power series, as in [31].

To finish the proof of Theorem A, it now remains to compare $_\varepsilon\mathcal{L}(\mathbb{Z}')$ and $_\varepsilon\mathcal{L}(F')$. By combining the above lemma with the equivalence $\mathcal{K}(\mathbb{Z}')_\# \to \bar{\mathcal{K}}(\mathbb{Z}')_\#$ discussed at the beginning of the paper, we have that the ring homomorphism $\varphi' : \mathbb{Z}' \to F'$ induces a mod 2 homotopy equivalence $\mathcal{K}(\mathbb{Z}') \to \mathcal{K}(F')$. From Section 4, the induced map $_\varepsilon\mathcal{L}(\mathbb{Z}') \to {_\varepsilon}\mathcal{L}(F')$ also induces an isomorphism on the homotopy groups π_0 and π_1 mod 2. Hence Theorem A is now a consequence of Theorem 3.1. $\square$

With the methods developed here, we can also prove the *L*-theory counterpart of the fibration (5-6), a slightly more general formulation of Friedlander's result [18], [17], to be used in Lemma 7.4.

THEOREM 5.5. *For any odd prime power q, there are homotopy fibrations of spaces*

$$
_\varepsilon\mathcal{L}(\mathbb{F}_q) \longrightarrow {_\varepsilon}\mathcal{L}^{\mathrm{top}}(\mathbb{C}) \longrightarrow {_\varepsilon}\mathcal{L}^{\mathrm{top}}(\mathbb{C}),
$$

where the second arrow is defined by $\psi^q - 1$*, and*

$$
_\varepsilon\mathcal{L}(\mathbb{F}_q) \longrightarrow {_\varepsilon}\mathcal{L}(\bar{\mathbb{F}}_q) \longrightarrow {_\varepsilon}\mathcal{L}(\bar{\mathbb{F}}_q),
$$

with the second arrow defined by $\psi - 1$ *(and ψ the Frobenius automorphism).*

Proof. We are essentially using Friedlander's computations [18] of $_1L(\mathbb{F}_q)$ for $i = 5, 6$ and of $_{-1}L(\mathbb{F}_q)$ for $i = 1, 2$.

First note that the theorem holds both rationally and modulo the characteristic of $\mathbb{F}_q$, since the action of $\psi^q - 1$ on π_{2n} is as $q^n - 1$, and so invertible. Therefore we now work mod ℓ where ℓ is prime to q.

For the first homotopy fibration, we can repeat the argument used above, but now with respect to a map instead of $\varphi' : \mathbb{Z}' \to F'$. In other words, we write the homotopy fiber of $_\varepsilon\mathcal{L}^{\text{top}}(\mathbb{C}) \xrightarrow{\psi^q - 1} {}_\varepsilon\mathcal{L}^{\text{top}}(\mathbb{C})$ as $_\varepsilon\mathcal{L}(F)$, and we have a ring map $\varphi : \mathbb{Z}_q \to F$ that we want to show induces an isomorphism $_\varepsilon\tilde{L}_i(\mathbb{Z}_q) \to {}_\varepsilon\tilde{L}_i(F)$ of mod ℓ *L*-groups. By combining Lemma 5.3 and Friedlander's computation of $_\varepsilon\tilde{L}(\mathbb{F}_q)$, we know that

$$\begin{aligned}
{}_1\tilde{L}_i(\mathbb{Z}_q) &= {}_1\tilde{L}_i(F) = 0 &\qquad \text{for } i = 5, 6, \\
{}_{-1}\tilde{L}_i(\mathbb{Z}_q) &= {}_{-1}\tilde{L}_i(F) = 0 &\qquad \text{for } i = 1, 2.
\end{aligned}$$

Therefore, using Theorem 3.1 in both directions, we have

$$_\varepsilon\tilde{L}_i(\mathbb{Z}_q) = {}_\varepsilon\tilde{L}_i(F) \qquad \text{for } i \geq 0.$$

(Note however that $_\varepsilon\tilde{L}_{-1}(F) \neq 0 = {}_\varepsilon\tilde{L}_{-1}(\mathbb{F}_q)$.)

For the second fibration, we again use the folklore diagram (5-7). According to Appendix B, the homotopy fiber of the $_\varepsilon\mathcal{L}$ space map induced by $\psi - 1$ may be thought of as $_\varepsilon\mathcal{L}(B)$, where B is the pull-back

$$\begin{array}{ccc}
B & \twoheadrightarrow & \bar{\mathbb{F}}_q \\
\downarrow & \lrcorner & \downarrow {\scriptstyle(\psi,\text{id})} \\
M(\Delta) & \xrightarrow{\tilde{\Delta}} & \bar{\mathbb{F}}_q \times \bar{\mathbb{F}}_q.
\end{array}$$

Because ψ fixes $\mathbb{F}_q$, there is a canonical map $\mathbb{F}_q \longrightarrow B$ which we seek to show induces a mod ℓ homotopy equivalence on $_\varepsilon\mathcal{L}$ spaces. After Quillen [42], this is known for the $\mathcal{K}$ spaces. So the result follows from Corollary 3.2 as before, since the *L*-theories of $\bar{\mathbb{F}}_q$ and $\mathbb{C}$ with finite coefficients (coprime to q) coincide by [27]. $\qquad\square$

6. Computations modulo odd torsion.

THEOREM 6.1. *Modulo a finite group of odd order, the groups $_\varepsilon L_i(\mathbb{Z}')$ for $i \geq 0$ are as follows, where δ_{i0} denotes the Kronecker delta, and 2^t is the 2-primary part*

of $i + 1$.

$i \pmod 8$	$_{-1}L_i(\mathbb{Z}')$	$_1L_i(\mathbb{Z}')$
0	$\delta_{i0}\mathbb{Z}$	$\delta_{i0}\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2$
1	0	$\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$
2	$\mathbb{Z}$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$
3	$\mathbb{Z}/16$	$\mathbb{Z}/8$
4	$\mathbb{Z}/2$	$\mathbb{Z}$
5	$\mathbb{Z}/2$	0
6	$\mathbb{Z}$	0
7	$\mathbb{Z}/2^{t+1}$	$\mathbb{Z}/2^{t+1}$

Moreover, the odd torsion subgroup of $_\varepsilon L_i(\mathbb{Z}')$ is the invariant part of the odd torsion of $K_i(\mathbb{Z})$ induced by the involution $M \mapsto {}^{t}M^{-1}$ of $\mathrm{GL}(\mathbb{Z})$.

Proof. We first consider odd torsion. From the cases $i = 0, 1$ dealt with above, neither $K_i(\mathbb{Z}')$ nor $_\varepsilon L_i(\mathbb{Z}')$ contains (nontrivial) odd torsion, so that the Witt and co-Witt groups

$$\begin{aligned}
_\varepsilon W_i(\mathbb{Z}') &= \mathrm{Coker}\,[K_i(\mathbb{Z}') \longrightarrow {}_\varepsilon L_i(\mathbb{Z}')], \\
_\varepsilon W_i'(\mathbb{Z}') &= \mathrm{Ker}\,[{}_\varepsilon L_i(\mathbb{Z}') \longrightarrow K_i(\mathbb{Z}')]
\end{aligned}$$

are odd-torsion-free for $\varepsilon \in \{\pm 1\}$, $i \in \{0, 1\}$. Then by Proposition 1.35 and Théorème 3.7 of [25] these groups are odd-torsion-free for all i. In particular, the maps $K_i(\mathbb{Z}') \longrightarrow {}_\varepsilon L_i(\mathbb{Z}')$ are surjective on odd torsion. The assertion about the odd torsion subgroups then follows from the argument of [25] p. 253 (specifically, using the weak equivalence there denoted $\mathcal{C}(A) \sim {}_\varepsilon \mathcal{L}(A)$).

We can now work modulo odd torsion. The groups for $i = 0$ have been discussed above, so we may assume that $i \geq 1$.

$\varepsilon = 1$. Since the homotopy cartesian diagram

$$\begin{array}{ccc}
B_1O(\mathbb{Z}')_\#^+ & \longrightarrow & B_1O(\mathbb{R})_\# \\
\downarrow & & \downarrow \\
B_1O(\mathbb{F}_3)_\#^+ & \longrightarrow & B_1O(\mathbb{C})_\#
\end{array}$$

may also be written as

$$\begin{array}{ccc}
B_1O(\mathbb{Z}')_\#^+ & \longrightarrow & BO_\# \times BO_\# \\
\downarrow & & \downarrow \\
B_1O(\mathbb{F}_3)_\#^+ & \longrightarrow & BO_\#
\end{array}$$

we have a split short exact sequence:

$$0 \to {}_1L_i(\mathbb{Z}') \to \pi_i(BO) \oplus \pi_i(BO) \oplus \pi_i(BO(\mathbb{F}_3)^+) \to \pi_i(BO) \to 0$$

which can simply be written as an isomorphism

$$_1L_i(\mathbb{Z}') \cong \pi_i(BO) \oplus \pi_i(BO(\mathbb{F}_3)^+)$$

for $i > 0$. (In fact, more precisely one has a homotopy decomposition $B_1O(\mathbb{Z}')^+ \simeq BO \times B_1O(\mathbb{F}_3)^+$.) Then the results tabulated above follow immediately from the computations of Friedlander [18], given the following well-known lemma, easily proven by induction (or factorization). The relevance of $3^r - 1$ here is that it is the order of $K_{2r-1}(\mathbb{F}_3)$.

Numerical Claim. Write $(r)_2$ for the 2-primary part of r. Then, for even r,

$$\left(3^r - 1\right)_2 = 4(r)_2.$$

$\varepsilon = -1$.   Using the homotopy cartesian square

$$\begin{array}{ccc} B_{-1}O(\mathbb{Z}')_\#^+ & \longrightarrow & B_{-1}O(\mathbb{R})_\# \\ \downarrow & & \downarrow \\ B_{-1}O(\mathbb{F}_3)_\#^+ & \longrightarrow & B_{-1}O(\mathbb{C})_\# \end{array}$$

we argue as follows, with all spaces and homotopy groups 2-completed. The above reduces to the homotopy cartesian square

$$\begin{array}{ccc} BSp\mathbb{Z}'^+ & \longrightarrow & BU \\ \downarrow & & \downarrow^H \\ BSp\mathbb{F}_3^+ & \longrightarrow & BSp \end{array}$$

with vertical homotopy fiber $Sp/U \simeq \Omega^6BO$ induced by the hyperbolic map H, and horizontal homotopy fiber $Sp \simeq \Omega^5BO$. This gives rise to three exact homotopy sequences for each $i \geq 0$:

$$A_i : \qquad \pi_{i+1}BSp\mathbb{F}_3^+ \to \pi_{i+6}BO \to \pi_iBSp\mathbb{Z}'^+ \to \pi_iBSp\mathbb{F}_3^+ \to \pi_{i+5}BO$$

$$B_i : \qquad \pi_{i+1}BSp\mathbb{F}_3^+ \oplus \pi_{i+1}BU \to \pi_{i+5}BO \to \pi_iBSp\mathbb{Z}'^+ \to$$
$$\to \pi_iBSp\mathbb{F}_3^+ \oplus \pi_iBU \to \pi_{i+4}BO$$

$$C_i : \qquad \pi_{i+1}BU \to \pi_{i+5}BO \to \pi_iBSp\mathbb{Z}'^+ \to \pi_iBU \to \pi_{i+4}BO.$$

The groups $\pi_iBSp\mathbb{F}_3^+$, together with information about the map $H_* : \pi_iBU \to \pi_{i+4}BO$ appearing in B_i, are calculated in [18]. It follows immediately from A_i that for $i \equiv 0, 1, 2 \pmod 8$ the map $\pi_{i+6}BO \to \pi_iBSp\mathbb{Z}'^+$ is an isomorphism, while for $i \equiv 7 \pmod 8$ it is $\pi_iBSp\mathbb{Z}'^+ \to \pi_iBSp\mathbb{F}_3^+$ that is an isomorphism (in this case one again appeals to the above numerical claim). Likewise, when $i \equiv 6 \pmod 8$

the sequence C_i immediately becomes:

$$0 \to 0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2.$$

For $i \equiv 3 \pmod 8$, since H_* is multiplication by 2 on $\pi_{i+1}BU \cong \pi_{i+5}BO \cong \mathbb{Z}$, from B_i we have (using the numerical claim above) that $\pi_i B\mathrm{Sp}\mathbb{Z}'^+$ has order 16. On the other hand, from C_i this must be a cyclic group.

Then for $i \equiv 4 \pmod 8$ the fact that H_* is injective in B_i forces $\pi_i B\mathrm{Sp}\mathbb{Z}'^+$ to be finite. Hence C_i reduces to the sequence

$$0 \to \mathbb{Z}/2 \to \mathbb{Z}/2 \to \mathbb{Z} \rightarrowtail \mathbb{Z}.$$

Since the sequence A_{i-1} began with the zero map, so A_i ends with it, and becomes:

$$\mathbb{Z}/2 \oplus \mathbb{Z}/2 \twoheadrightarrow \mathbb{Z}/2 \to \mathbb{Z}/2 \to \mathbb{Z}/2 \xrightarrow{\ 0\ } \mathbb{Z}/2.$$

Finally, it follows that when $i \equiv 5 \pmod 8$ the final map of the sequence A_i (being the initial map of the sequence A_{i-1}) is an epimorphism with kernel $\pi_i B\mathrm{Sp}\mathbb{Z}'^+ = \mathbb{Z}/2$. $\qquad\square$

Remarks 6.2. (1) Of course the 2-primary torsion of $Z/2^{t+1}$ above is just the 2-primary torsion of Im J. Observe too that $_1L_3(\mathbb{Z}') = \mathbb{Z}/24$, a less exotic result than $K_3(\mathbb{Z}')$ which is $\mathbb{Z}/48$. On the other hand, $\mathbb{Z}/48$ is detected by $_{-1}L_3(\mathbb{Z}')$.

(2) Note that the groups $_{-1}L_i(\mathbb{Z}')$ for $i \equiv \pm 3 \pmod 8$ reveal that, in contrast to the orthogonal situation, the map $B\mathrm{Sp}\mathbb{Z}'^+_\# \to B\mathrm{Sp}\mathbb{F}_{3\#}^+$ above does not admit a section.

6.3. **Information on the integral orthogonal group conjecture.** The results tabulated above provide a certain amount of support for the conjectures of Section 1. To present this information, we consider the commuting diagram

$$
\begin{array}{ccccccccc}
\text{'Im}J\text{'} \to & B\Sigma_\infty^+ & \to & B_1O(\mathbb{Z})^+ & \to & B_1O(\mathbb{Z}')^+ & \to & BO \times BO & \xrightarrow{\ \mathrm{pr}_1\ } & BO \\
& & & \downarrow H & & \downarrow H & & \searrow{\scriptstyle\text{direct}\atop\scriptstyle\text{sum}} & & \\
& & & B\mathrm{GL}(\mathbb{Z})^+ & \to & B\mathrm{GL}(\mathbb{Z}')^+ & \to & B\mathrm{GL}\mathbb{R}^{\delta+} & \to & BO
\end{array}
$$

and apply (2-primary) input from the following sources. First, the homotopy groups of $B\Sigma_\infty^+$ correspond to stable homotopy groups of spheres by results of [5], [49]. In turn, the map from these groups to the K-theory of the integers (presented in [57]) is studied in [45]. The vanishing of this map on the image of the J-homomorphism in certain dimensions is shown in [55]. The KO-theory *degree map* (induced by group inclusion) from stable homotopy groups of spheres to the homotopy groups of BO (known after Bott) is given by the composite of the upper horizontal maps, and is analysed in [1]; it factors through the groups $_1L_i(\mathbb{Z}')$

computed in Theorem 6.1 above. For $i > 1$, the map from $BGL(\mathbb{Z})^+$ to $BGL(\mathbb{Z}')^+$ is known from the localization sequence of [51] to induce an isomorphism of homotopy groups. Finally, [53] compares $BGL\mathbb{R}^{\delta+}$ with $BGL\mathbb{R} \simeq BO$.

Applying π_i ($i > 1$) gives, modulo odd torsion:

$i \equiv 0 \pmod 8$.

$$
\begin{array}{ccccccccc}
\mathbb{Z}/2 \rightarrowtail\, ? \rightarrow\, ? & \rightarrow & \mathbb{Z} \oplus \mathbb{Z}/2 & \rightarrow & \mathbb{Z} \oplus \mathbb{Z} & \rightarrow & \mathbb{Z} \\
\downarrow \qquad\quad & & \downarrow & & & \searrow & \\
0 \longrightarrow \qquad & & 0 & \rightarrow & 0 & & \rightarrow \mathbb{Z}
\end{array}
$$

$i \equiv 1 \pmod 8$.

$$
\begin{array}{ccccccc}
\mathbb{Z}/2 \rightarrowtail\, ? \rightarrow & ? & \rightarrow & (\mathbb{Z}/2)^3 & \rightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \rightarrow \mathbb{Z}/2 \\
& \downarrow & & \downarrow & & \searrow \\
\mathbb{Z} \oplus \mathbb{Z}/2 & \xrightarrow{\text{id}} \mathbb{Z} \oplus \mathbb{Z}/2 \rightarrow & & \mathbb{Z}/2 & & \rightarrow \mathbb{Z}/2
\end{array}
$$

with the degree map surjective, and composite of top left two arrows followed by vertical arrow zero.

$i \equiv 2 \pmod 8$.

$$
\begin{array}{ccccccc}
0 \rightarrow\, ? \rightarrow & ? & \rightarrow & \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \rightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \rightarrow \mathbb{Z}/2 \\
\downarrow & \downarrow & & \downarrow & & \searrow \\
\mathbb{Z}/2 & \xrightarrow{\text{id}} & \mathbb{Z}/2 & & \rightarrow \quad \mathbb{Z}/2 & \rightarrow \mathbb{Z}/2
\end{array}
$$

with degree map surjective.

$i \equiv 3 \pmod 8$.

$$
\begin{array}{ccccccc}
\mathbb{Z}/8 \rightarrowtail\, ? \rightarrow & ? & \rightarrow & \mathbb{Z}/8 & \rightarrow & 0 & \rightarrow 0 \\
& \downarrow & & \downarrow & & \searrow \\
\mathbb{Z}/16 & \xrightarrow{\text{id}} & \mathbb{Z}/16 & \rightarrow \mathbb{Q}_2/\mathbb{Z}_2 & \rightarrow 0
\end{array}
$$

with composite of top three leftmost arrows followed by vertical arrow an injection.

$i \equiv 4 \pmod 8$.

$$
\begin{array}{ccccccc}
0 \rightarrow\, ? \rightarrow\, ? & \rightarrow & \mathbb{Z} & \rightarrow \mathbb{Z} \oplus \mathbb{Z} & \rightarrow \mathbb{Z} \\
& \downarrow & & \downarrow & & \searrow \\
0 & \longrightarrow 0 & \rightarrow & 0 & \rightarrow \mathbb{Z}
\end{array}
$$

$i \equiv 5 \pmod 8$.

$$
\begin{array}{ccccccc}
0 \rightarrow\, ? \rightarrow\, ? & \rightarrow & 0 & \rightarrow 0 & \rightarrow 0 \\
& \downarrow & & \downarrow & & \searrow \\
\mathbb{Z} & \xrightarrow{\text{id}} & \mathbb{Z} & \rightarrow 0 & \rightarrow 0
\end{array}
$$

$i \equiv 6 \pmod 8$.

$$
\begin{array}{ccccccccc}
0 & \to & ? & \to & ? & \to & 0 & \to & 0 & \to & 0 \\
& & & & \downarrow & & \downarrow & & \searrow & & \\
& & & & 0 & \longrightarrow & 0 & \to & 0 & \to & 0
\end{array}
$$

$i \equiv 7 \pmod 8$, $i + 1 = (\text{odd}) \cdot 2^t$.

$$
\begin{array}{ccccccccc}
\mathbb{Z}/2^{t+1} & \rightarrowtail & ? & \to & ? & \to & \mathbb{Z}/2^{t+1} & \to & 0 & \to 0 \\
& & & & \downarrow & & \downarrow & & \searrow & \\
& & & & \mathbb{Z}/2^{t+1} & \xrightarrow{\mathrm{id}} & \mathbb{Z}/2^{t+1} & \to & \mathbb{Q}_2/\mathbb{Z}_2 & \to 0
\end{array}
$$

with composite of top three leftmost arrows followed by vertical arrow an isomorphism.

7. Homotopy fixed points. In response to a query of B. Kahn, we now investigate the homotopy fixed point set of $\mathbb{Z}/2$ acting on $\mathcal{K}(\mathbb{Z}')$, at least after 2-adic completion. We recall that, for a ring Λ, $\mathcal{K}(\Lambda)$ is described *functorially* as the loop space $\Omega B\mathrm{GL}(S\Lambda)^+$. Therefore our main interest in the description below is in the case $A = S\Lambda$ (see Appendix A), and in particular $\Lambda = \mathbb{Z}'$.

In order to describe the $\mathbb{Z}/2$ action with respect to a ring A admitting an antiinvolution $x \mapsto \check{x}$, we recall that, for $\varepsilon = \pm 1$ fixed by the involution, $_\varepsilon O_{n,n}(A)$ as defined above is in effect the fixed subgroup $\mathrm{GL}_{2n}(A)^{\varepsilon \mathbb{Z}/2}$ of the action denoted $_\varepsilon \mathbb{Z}/2$ on $\mathrm{GL}_{2n}(A)$. (Our notation omits ε when it is safe to do so.) The nontrivial element of $_\varepsilon \mathbb{Z}/2$ sends an invertible matrix to the ε-hyperbolic adjoint of its inverse. Explicitly,

$$
M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_{2n}(A)
$$

is sent to the inverse of

$$
_\varepsilon J_n \cdot {}^{\mathrm{t}}\check{M} \cdot {}_\varepsilon J_n^{-1} = \begin{pmatrix} {}^{\mathrm{t}}\check{d} & \varepsilon\,{}^{\mathrm{t}}\check{b} \\ \varepsilon\,{}^{\mathrm{t}}\check{c} & {}^{\mathrm{t}}\check{a} \end{pmatrix}
$$

where

$$
_\varepsilon J_n = \begin{pmatrix} 0 & \varepsilon I_n \\ I_n & 0 \end{pmatrix}.
$$

We denote by $_\varepsilon O(A)$ the direct limit of the $_\varepsilon O_{n,n}(A)$ with respect to the obvious inclusions within each of the four component blocks. If $\Lambda = \mathbb{R}$ or $\mathbb{C}$, we take for $A = S\Lambda$ the topological suspension (as described in Appendix A).

As in [16], one observes that the $_\varepsilon \mathbb{Z}/2$ action is compatible with the process of stabilization, passage to classifying spaces and the plus-construction. Moreover, it behaves well with respect to suspension, and passing to loop spaces

commutes with taking fixed point sets. Thus the fixed point set of the space $\mathcal{K}(\Lambda) = \Omega(B\mathrm{GL}(S\Lambda)^+)$ is just

$$_\varepsilon\mathcal{L}(\Lambda) = \Omega(B_\varepsilon O(S\Lambda)^+),$$

a fact that we record.

LEMMA 7.1. *For Λ as above, and $\varepsilon = \pm1$, the fixed point set of the $_\varepsilon\mathbb{Z}/2$ action on $\mathcal{K}(\Lambda)$ is*

$$\mathcal{K}(\Lambda)^{\varepsilon\mathbb{Z}/2} \simeq {}_\varepsilon\mathcal{L}(\Lambda).$$

We now turn our attention to the homotopy fixed point set

$$\mathcal{K}(\Lambda)^{h(\varepsilon\mathbb{Z}/2)} := \mathrm{map}_{\varepsilon\mathbb{Z}/2}(E\mathbb{Z}/2, \mathcal{K}(\Lambda)),$$

the space of maps equivariant under the $_\varepsilon\mathbb{Z}/2$ action, where $E\mathbb{Z}/2$ is a contractible free $\mathbb{Z}/2$-space (usually taken to be S^∞ with antipodal action), and $\Lambda = \mathbb{Z}'$ and its related rings, such as $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{F}_3$.

We shall prove the following theorem.

THEOREM 7.2. *For $\varepsilon = \pm1$, the natural map*

$$_\varepsilon\mathcal{L}(\mathbb{Z}')_\# \longrightarrow [\mathcal{K}(\mathbb{Z}')^{h(\varepsilon\mathbb{Z}/2)}]_\#$$

is a homotopy equivalence.

Since the spaces $_\varepsilon\mathcal{L}(\mathbb{Z}')_\#$ and $[\mathcal{K}(\mathbb{Z}')^{h(\varepsilon\mathbb{Z}/2)}]_\#$ are both homotopy pull-backs, the proof largely reduces to three pairs of verifications (Lemmas 7.3, 7.4, 7.5 below) which may be found to some extent in the literature ([16], [20], [29], [33], [34]). Because the spaces of the theorem are not connected, the group of connected components needs special attention here.

LEMMA 7.3. *For $\varepsilon = \pm1$, the natural map*

$$_\varepsilon\mathcal{L}(\mathbb{C}) \longrightarrow \mathcal{K}(\mathbb{C})^{h(\varepsilon\mathbb{Z}/2)}$$

is a homotopy equivalence.

Proof. The lemma is true without the need of 2-adic completion. The case $\varepsilon = 1$ has a simple analysis, by means of the map of connected-component fibrations $X^0 \to X \to \pi_0(X)$:

$$\begin{array}{ccccc}
\Omega^0 B_1 O(S\mathbb{C}) = B_1 O(\mathbb{C}) & \longrightarrow & {}_1\mathcal{L}(\mathbb{C}) & \longrightarrow & {}_1 L_0(\mathbb{C}) \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^0 B\mathrm{GL}(S\mathbb{C}) = B\mathrm{GL}(\mathbb{C}) & \longrightarrow & \mathcal{K}(\mathbb{C}) & \longrightarrow & K_0(\mathbb{C})
\end{array}$$

that we claim represents the inclusion of the homotopy fixed point sets. Certainly this is true for the discrete base $K_0(\mathbb{C})$, since the action of $\mathbb{Z}/2$ is trivial and the map $\mathbb{Z} = {}_1L_0(\mathbb{C}) \to K_0(\mathbb{C}) = \mathbb{Z}$ is the identity. By polar decomposition of matrices, the required result for the fiber follows from a classical fact in algebraic topology: the map

$$BO \longrightarrow BU^{h({}_1\mathbb{Z}/2)}$$

is a homotopy equivalence [20], [16], [29].

For $\varepsilon = -1$, we cannot repeat the method above because the map ${}_{-1}L_0(\mathbb{C}) \to K_0(\mathbb{C})$ fails to be an isomorphism. Instead we use the general argument of [29] for Banach algebras. Specifically, we use the fact that, because $A = S\mathbb{C}$ is a C^*-algebra, the inclusion of $U_{2n}(A)$ in $\mathrm{GL}_{2n}(A)$ is a $\mathbb{Z}/2$-homotopy equivalence, where

$$U_{2n}(A) = \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid M^{-1} = \begin{pmatrix} {}^t\bar{a} & {}^t\bar{c} \\ {}^t\bar{b} & {}^t\bar{d} \end{pmatrix} \right\}$$

and the involution on $\mathrm{GL}_{2n}(A)$ is

$$M \longmapsto \begin{pmatrix} {}^td & -{}^tb \\ -{}^tc & {}^ta \end{pmatrix}^{-1}.$$

So, when $M \in U_{2n}(A)$, it is sent to

$$\begin{pmatrix} \bar{d} & -\bar{c} \\ -\bar{b} & \bar{a} \end{pmatrix} = {}_{-1}J_n\bar{M}_{-1}J_n^{-1}.$$

Therefore $U_{2n}(A)$ is in turn $\mathbb{Z}/2$-homotopy equivalent to $\mathrm{GL}_{2n}(A)$ equipped with the involution $M \mapsto {}_{-1}J_n\bar{M}_{-1}J_n^{-1}$. If we reinterpret $\mathrm{GL}_{2n}(S\mathbb{C})$ as $\mathrm{GL}_n(S\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C})$, with basis afforded by

$$I_2, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix},$$

then the involution now acts as complex conjugation. Therefore, by the main theorem of [29], there is a homotopy equivalence between $\mathcal{K}(S\mathbb{H})$ and $\mathcal{K}(S\mathbb{C})^{h(-{}_1\mathbb{Z}/2)}$, and hence, on passing to loop spaces, between $\mathcal{K}(\mathbb{H}) = {}_{-1}\mathcal{L}(\mathbb{C})$ and $\mathcal{K}(\mathbb{C})^{h(-{}_1\mathbb{Z}/2)}$. $\square$

LEMMA 7.4. *For $\varepsilon = \pm 1$ and q odd, the natural map*

$${}_\varepsilon\mathcal{L}(\mathbb{F}_q) \longrightarrow \mathcal{K}(\mathbb{F}_q)^{h(\varepsilon\mathbb{Z}/2)}$$

is a homotopy equivalence.

Proof.   This result also is true without the need of 2-adic completion.

From Theorem 5.5 and fibration (5-6) above, there is a map of homotopy fibrations

$$
\begin{array}{ccccc}
{}_\varepsilon\mathcal{L}(\mathbb{F}_q) & \longrightarrow & {}_\varepsilon\mathcal{L}(\mathbb{C}) & \xrightarrow{\psi^q-1} & {}_\varepsilon\mathcal{L}(\mathbb{C}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{K}(\mathbb{F}_q)^{h(\varepsilon\mathbb{Z}/2)} & \longrightarrow & \mathcal{K}(\mathbb{C})^{h(\varepsilon\mathbb{Z}/2)} & \xrightarrow{\psi^q-1} & \mathcal{K}(\mathbb{C})^{h(\varepsilon\mathbb{Z}/2)}.
\end{array}
$$

So the result follows from our previous discussion of the complex case. See also [33]. $\qquad\square$

To complete the proof of Theorem 7.2, we now turn to consideration of the homotopy fixed point set for $\mathbb{R}$. This verification is the most tricky one and really needs the 2-adic completion. Again, although this result may be found in the literature (at least, in positive dimensions), we present an alternative viewpoint using Fredholm operators that we feel is illuminating. The result that we require is the following. (Since we are considering $A = S\mathbb{R}$, $\mathcal{K}(\mathbb{R}) = \Omega B\mathrm{GL}(A)^+$ simplifies to $\mathrm{GL}(A)$; similarly for ${}_\varepsilon\mathcal{L}(\mathbb{R})$.)

LEMMA 7.5. *The natural map*

$$
{}_\varepsilon\mathcal{L}(\mathbb{R}) = {}_\varepsilon O(A) \longrightarrow \mathrm{GL}(A)^{h\mathbb{Z}/2} = \mathcal{K}(\mathbb{R})^{h\mathbb{Z}/2}
$$

becomes a homotopy equivalence after 2-*adic completion.*

We now prove this lemma for $\varepsilon = 1$, and then use elements of this argument when we later establish the case $\varepsilon = -1$.

7.6. The case $\varepsilon = 1$**.**   This case calls for analysis of the space ${}_1\mathcal{L}(\mathbb{R}) = {}_1O(A)$, where $A = S\mathbb{R}$. As above, we regard ${}_1O(A)$ as the direct limit of the ${}_1O_{n,n}(A)$. Using polar decomposition on matrices over the C^*-algebra A, we reinterpret ${}_1O_{n,n}(A)$ (up to homotopy) as the group of fixed points in $O_{2n}(A)$ under conjugation by the matrix $\begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}$. In other words, we have a commutative diagram of $\mathbb{Z}/2$-homotopy equivalence inclusions

$$
\begin{array}{ccc}
{}_1O_{n,n}(A) & \hookrightarrow & \mathrm{GL}_{2n}(A) \\
\uparrow & & \uparrow \\
O_n(A) \times O_n(A) & \hookrightarrow & O_{2n}(A) \\
\downarrow & & \downarrow \\
\mathrm{GL}_n(A) \times \mathrm{GL}_n(A) & \hookrightarrow & \mathrm{GL}_{2n}(A).
\end{array}
$$

Since by Kuiper's theorem the inclusion $\mathrm{GL}_n(A) \hookrightarrow \mathrm{GL}_{n+1}(A)$ is a homotopy equivalence, we just have to investigate the map

$$
\mathrm{GL}_1(A) \times \mathrm{GL}_1(A) \longrightarrow \mathrm{GL}_2(A)^{h\mathbb{Z}/2}
$$

or equivalently (as in Appendix A)

$$\chi : \mathcal{F}(H) \times \mathcal{F}(H) \longrightarrow \mathcal{F}(H \oplus H)^{h\mathbb{Z}/2}.$$

Here χ sends (D_1, D_2) to the constant map $E\mathbb{Z}/2 \to \mathcal{F}(H \oplus H)$ that takes value $D_1 \oplus D_2$. Recall from the Atiyah-Jänich theorem (see for example [2] or [21]) that the space $\mathcal{F}(H)$ of Fredholm operators in a (real or complex) Hilbert space H serves as a classifying space for the K-theory $K(X)$ of paracompact spaces X.

The statement that we want to prove for $\varepsilon = 1$ therefore amounts to the assertion that

$$\mathcal{K}(\mathbb{R}) \times \mathcal{K}(\mathbb{R}) \longrightarrow \mathcal{K}(M_2(\mathbb{R}))^{h\mathbb{Z}/2}$$

becomes a 2-adic equivalence, where the $\mathbb{Z}/2$-action on $M_2(S\mathbb{R})$ is given by conjugation with respect to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. For later use, we also need to prove that

$$\mathcal{K}(\mathbb{C}) \times \mathcal{K}(\mathbb{C}) \longrightarrow \mathcal{K}(M_2(\mathbb{C}))^{h\mathbb{Z}/2}$$

becomes a 2-adic equivalence. Therefore in what follows, H represents a real or complex Hilbert space. Also, we write map for the nonequivariant mapping space, and map$_*$ for the subspace of basepoint-preserving maps.

LEMMA 7.7. *The function space* map$_*(B\mathbb{Z}/2, \mathcal{F}(H))$ *is a version* $\mathcal{F}(H)_{\#}$ *of the 2-adic completion of* $\mathcal{F}(H)$ *(including the completion of* π_0*). With this version, the canonical map* $\mathcal{F}(H) \to \mathcal{F}(H)_{\#}$ *sends an element* x *of* $K(X) = [X, \mathcal{F}(H)]$ *to the map from* $X \times B\mathbb{Z}/2$ *to* $\mathcal{F}(H)$ *associated to the element* $x(L - 1)$ *of the relative K-group* $K(X \times B\mathbb{Z}/2, X)$*, where* L *is the canonical line bundle over* $B\mathbb{Z}/2$*.*

Proof. Let X be a sphere. We have to show that the map

$$\mathcal{F}(H) \to \text{map}_*(B\mathbb{Z}/2, \mathcal{F}(H))$$

described above induces an isomorphism

$$[X, \mathcal{F}(H)]_{\#} \to [X, \text{map}_*(B\mathbb{Z}/2, \mathcal{F}(H))]$$

where $[X, \mathcal{F}(H)]_{\#}$ is the 2-adic completion of the group $[X, \mathcal{F}(H)]$. The function space map$(X, \text{map}_*(B\mathbb{Z}/2, \mathcal{F}(H)))$ may be identified with the function space of maps $X \times B\mathbb{Z}/2 \to \mathcal{F}(H)$ that send $X \times \{*\}$ to the basepoint 1_H of $\mathcal{F}(H)$. In other words, we must calculate the relative group $K(X \times \mathbb{R}P^\infty, X)$, and investigate the map

$$K(X) \longrightarrow K(X \times \mathbb{R}P^\infty, X)$$

given by the correspondence $x \mapsto x(L - 1)$ as in the statement of the lemma.

Since the groups $K(X \times \mathbb{R}P^n, X)$ are finite for n even (by, for example, the Atiyah-Hirzebruch spectral sequence, or the exact sequences below), the projective limit $\varprojlim K(X \times \mathbb{R}P^n, X)$ has vanishing $\varprojlim^1$ term and is therefore isomorphic to $K(X \times \mathbb{R}P^\infty, X)$.

Now the group $K(X \times \mathbb{R}P^n, X)$ has been described concretely in [24] p. 249, Theorem 6.40 (see also, more recently, [30]). It is the middle term of an exact sequence

$$K(\mathcal{E}^{n,0}(X)) \to K(\mathcal{E}(X)) \xrightarrow{\alpha} K(X \times \mathbb{R}P^n, X) \to K^1(\mathcal{E}^{n,0}(X)) \xrightarrow{\Phi_n} K^1(\mathcal{E}(X)),$$

where $\mathcal{E}(X)$ denotes the category of (real or complex) vector bundles on X, and $\mathcal{E}^{n,0}(X)$ is the category of Clifford bundles (those with an action of the usual Clifford algebra $C^{n,0}$). Note that α is the cup-product by $(L - 1)$, where L is the canonical line bundle over $\mathbb{R}P^n$. Since we are taking the projective limit, we may assume that $n = 8k$. Then the Clifford algebra $C^{n,0}$ is isomorphic to the matrix algebra $M_{2^{4k}}(\mathbb{R})$, the category $\mathcal{E}^{n,0}(X)$ is equivalent to $\mathcal{E}(X)$, and the 'restriction of scalars' functor $\mathcal{E}^{n,0}(X) \longrightarrow \mathcal{E}(X)$ simply sends the class of a bundle E to that of its multiple $2^{4k}E$. On the other hand, since X is a sphere, $K^1(\mathcal{E}^{n,0}(X)) = K^1(X)$ is $\mathbb{Z}$ or 0 (in the complex case), else $\mathbb{Z}$, 0 or $\mathbb{Z}/2$ (in the real case).

From the diagram

$$\begin{array}{ccccccccc}
K(\mathcal{E}^{n+8,0}(X)) & \to & K(\mathcal{E}(X)) & \to & K(X \times \mathbb{R}P^{n+8}, X) & \to & K^1(\mathcal{E}^{n+8,0}(X)) & \xrightarrow{\Phi_{n+8}} & K^1(\mathcal{E}(X)) \\
\downarrow {\cdot 16} & & \downarrow {\mathrm{id}} & & \downarrow & & \downarrow {\cdot 16} & & \downarrow {\mathrm{id}} \\
K(\mathcal{E}^{n,0}(X)) & \to & K(\mathcal{E}(X)) & \to & K(X \times \mathbb{R}P^n, X) & \to & K^1(\mathcal{E}^{n,0}(X)) & \xrightarrow{\Phi_n} & K^1(\mathcal{E}(X))
\end{array}$$

we see that the induced map $\mathrm{Ker}\Phi_{n+8} \to \mathrm{Ker}\Phi_n$ is always zero. Therefore, $K(X \times \mathbb{R}P^\infty, X) = \varprojlim K(X \times \mathbb{R}P^n, X)$ is isomorphic to

$$\varprojlim K(X)/2^{4k}K(X) = K(X)_{\#}. \qquad \square$$

COROLLARY 7.8. *The above splitting*

$$K(X \times \mathbb{R}P^\infty) = K(X) \oplus K(X \times \mathbb{R}P^\infty, X)$$

corresponds to the splitting of the function space $\mathrm{map}(B\mathbb{Z}/2, \mathcal{F}(H))$ *as the product*

$$\theta : \mathrm{map}(B\mathbb{Z}/2, \mathcal{F}(H)) \xrightarrow{\simeq} \mathcal{F}(H) \times \mathrm{map}_*(B\mathbb{Z}/2, \mathcal{F}(H)) = \mathcal{F}(H) \times \mathcal{F}(H)_{\#}.$$

Proof. In view of Appendix A, the split fibration

$$\mathrm{map}_*(B\mathbb{Z}/2, \mathcal{F}(H)) \longrightarrow \mathrm{map}(B\mathbb{Z}/2, \mathcal{F}(H)) \xrightarrow{\mathrm{eval}} \mathcal{F}(H)$$

can be interpreted as the split fibration of topological groups (hence a product

fibration as spaces)

$$\mathrm{map}_*(B\mathbb{Z}/2, \mathrm{GL}_1(A)) \longrightarrow \mathrm{map}(B\mathbb{Z}/2, \mathrm{GL}_1(A)) \xrightarrow{e\mathit{v}\mathrm{al}} \mathrm{GL}_1(A)$$

where $\mathrm{GL}_1(A)$ is the group of invertible elements in the Calkin algebra $S\mathbb{R}$ or $S\mathbb{C}$, and the projection map is evaluation at the basepoint. One has to recall also that the group structure on $\mathrm{GL}_1(A)$ induces the group structure on $K(X) = [X, \mathrm{GL}_1(A)]$.   □

This result combines with the classical homotopy equivalence

$$\mathcal{F}(H) \simeq \mathcal{F}(H \oplus H)$$

of Kuiper [35] and Palais [39] to yield the following:

COROLLARY 7.9. *The above splitting induces a splitting*

$$\Psi : \mathcal{F}(H \oplus H)^{h\mathbb{Z}/2} \xrightarrow{\simeq} \mathcal{F}(H) \times \mathrm{map}_*(B\mathbb{Z}/2, \mathcal{F}(H)) = \mathcal{F}(H) \times \mathcal{F}(H)_{\#}.$$

We now have the composition

$$
\begin{aligned}
\mathcal{F}(H) \times \mathcal{F}(H) \;\; & \xrightarrow{\;\chi\;} \;\; \mathcal{F}(H \oplus H)^{h\mathbb{Z}/2} \\
& \xrightarrow{\;\simeq\;} \;\; \mathcal{F}(H) \times \mathrm{map}_*(B\mathbb{Z}/2, \mathcal{F}(H)) = \mathcal{F}(H) \times \mathcal{F}(H)_{\#}
\end{aligned}
$$

which we seek to show becomes an equivalence on 2-adic completion. This composite is analysed in terms of its effect on groups of homotopy classes of maps from a compact space X into the function spaces.

For this, we recall that another description of $K(X) = [X, \mathcal{F}(H)]$ consists of homotopy classes $d(E, F, D)$ of triples (E, F, D) where E and F are infinite-dimensional Hilbert bundles and $D : E \to F$ is a bundle morphism such that $D_x : E_x \to F_x$ is a Fredholm map for each $x \in X$ (see for instance [24] §II.2 for the general framework). As a matter of fact, Kuiper's theorem on the contractibility of $\mathrm{Aut}(H)$ implies that, without loss of generality, one can take E, F to be trivial Hilbert bundles. In this way D defines a map $X \to \mathcal{F}(H)$ which is unique up to homotopy. One advantage of this presentation is the following: for a finite-dimensional vector bundle G defining a class in $K(X)$, the cup-product of $x = d(E, F, D)$ by G is simply $d(E \otimes G, F \otimes G, D \otimes \mathrm{id}_G)$. Note also that the isomorphism $K(X) \to [X, \mathcal{F}(H)]$ is just induced by the correspondence associating to a vector bundle G over X the class $d(E, F, D)$ where

$$E = F = \ell^2(G) = G \oplus \cdots \oplus G \oplus \cdots$$

and

$$D(x_0, x_1, x_2, \ldots) = (x_1, x_2, \ldots).$$

Finally, for the group structure in $K(X)$, we have

$$d(E, F, D) + d(E', F', D') = d(E \oplus E', F \oplus F', D \oplus D').$$

That χ becomes an equivalence on 2-completion is immediate from the explicit description of the following lemma.

LEMMA 7.10. *The natural map*

$$\Psi \circ \chi : \mathcal{F}(H) \times \mathcal{F}(H) \longrightarrow \mathcal{F}(H \oplus H)^{h\mathbb{Z}/2} \xrightarrow{\simeq} \mathcal{F}(H) \times \mathrm{map}_*(B\mathbb{Z}/2, \mathcal{F}(H))$$

is induced by the correspondence (seen from the point of view of K-theory, as in Lemma 7.8) that associates to a pair of vector bundles (E_1, E_2) the virtual bundles $E_1 \oplus E_2$ and $E_2 \otimes (L - \mathbf{1})$, where L is the Hopf line bundle and $\mathbf{1}$ is the trivial line bundle on $B\mathbb{Z}/2 = \mathbb{R}P^\infty$. Thus, from Lemma 7.7, $\Psi \circ \chi$ is a 2-adic equivalence.

Proof. We check commutativity of the diagram

$$
\begin{array}{ccccc}
 & & \mathcal{F}(H) & & \\
 & {\scriptstyle \mathrm{in}_1} \nearrow & & \searrow {\scriptstyle \rho_1} & \\
\mathcal{F}(H) \times \mathcal{F}(H) & \xrightarrow{\ \chi\ } & \mathcal{F}(H \oplus H)^{h\mathbb{Z}/2} & \xleftarrow{\ \gamma\ } & \mathrm{map}(B\mathbb{Z}/2, \mathcal{F}(H)) \\
 & {\scriptstyle \mathrm{in}_2} \searrow & & \nearrow {\scriptstyle \rho_L} & \\
 & & \mathcal{F}(H) & & \\
\end{array}
$$

and then discuss the effect of composition of the diagram with the canonical map

$$\theta : \mathrm{map}(B\mathbb{Z}/2, \mathcal{F}(H)) \to \mathcal{F}(H) \times \mathrm{map}_*(B\mathbb{Z}/2, \mathcal{F}(H)).$$

Here $\rho_1(E) = E \otimes \mathbf{1}$ and $\rho_L(E) = E \otimes L$; we also write $\chi_j = \chi \circ \mathrm{in}_j$.

By Corollary 7.9, the obvious map

$$H \longrightarrow H \oplus 0 \hookrightarrow H \oplus H$$

induces a homotopy equivalence

$$\gamma : \mathrm{map}(B\mathbb{Z}/2, \mathcal{F}(H)) = \mathcal{F}(H \oplus 0)^{h\mathbb{Z}/2} \longrightarrow \mathcal{F}(H \oplus H)^{h\mathbb{Z}/2},$$

and so γ^{-1} is defined up to homotopy.

With this identification, the composition

$$\mathcal{F}(H) \xrightarrow{\ \chi_1\ } \mathcal{F}(H \oplus H)^{h\mathbb{Z}/2} \xrightarrow{\ \gamma^{-1}\ } \mathrm{map}(B\mathbb{Z}/2, \mathcal{F}(H))$$

associates to a vector bundle E_1 over a space X the vector bundle $\rho_1(E_1) = E_1 \otimes \mathbf{1}$ on the space $X \times B\mathbb{Z}/2$.

However, the crucial point is to make explicit the image of the 'second component' E_2, that is, to specify the composition

$$\mathcal{F}(H) \xrightarrow{\chi_2} \mathcal{F}(H \oplus H)^{h\mathbb{Z}/2} \xrightarrow{\gamma^{-1}} \mathrm{map}(B\mathbb{Z}/2, \mathcal{F}(H))$$

where $\chi_2(D) = \mathrm{id}_H \oplus D$.

In order to make $\gamma^{-1} \circ \chi_2$ explicit, we also translate this problem geometrically in terms of vector bundles: we start with the class $x = d(T, T, D)$ where T is the trivial Hilbert bundle $X \times H$, and D is represented by an endomorphism of T such that the induced map from X to $\mathrm{End}(H)$ is a family of Fredholm operators. By Kuiper's theorem that the topological group $\mathrm{Aut}(H)$ is contractible, we construct by induction on n a map $\sigma : S^n \longrightarrow \mathrm{Aut}(H)$ such that $\sigma(e) = \mathrm{id}_H$ and $\sigma(-x) = -\sigma(x)$, where e is the basepoint of the sphere. We denote again by $\sigma : S^\infty \longrightarrow \mathrm{Aut}(H)$ the map on the infinite sphere (recall that $B\mathbb{Z}/2 = \mathbb{R}P^\infty = S^\infty/(\mathbb{Z}/2)$). Now consider the equivariant homotopy $\lambda(x)\Delta\lambda(x)^{-1}$ in the space of Fredholm operators on $H \oplus H$, where $\lambda(x)$ and Δ are given by the following matrices:

$$\lambda(x) = \begin{pmatrix} \cos\theta & -\sigma(x)\sin\theta \\ \sigma(x)^{-1}\sin\theta & \cos\theta \end{pmatrix} \quad \text{and} \quad \Delta = \begin{pmatrix} I & 0 \\ 0 & D \end{pmatrix}$$

and $\theta \in [0, \pi/2]$. At the end of the homotopy we get the family of Fredholm operators

$$\sigma(x)D\sigma(x)^{-1} \oplus \mathrm{id}_H.$$

In our geometric interpretation the trick now is to remark that $\sigma(x)$ defines an isomorphism between the trivial Hilbert bundle $T = B\mathbb{Z}/2 \times H$ and its tensor product $T \otimes L$ with the Hopf bundle L on the infinite projective space (note that $T \otimes L$ is isomorphic to T, but not canonically). The result for $\gamma^{-1} \circ \chi_2(x)$ is then $d(T \otimes L, T \otimes L, D \otimes \mathrm{id}_L)$. In other words, a vector bundle E_2 on X goes to $\rho_L(E_2) = E_2 \otimes L$ on $X \times B\mathbb{Z}/2$.

Next recall from Corollary 7.8 that θ is geometrically defined by splitting the evaluation map $\mathrm{map}(B\mathbb{Z}/2, \mathcal{F}(H)) \to \mathrm{map}(*, \mathcal{F}(H))$. Thus, to a triple $d(E, F, D)$ as above (E and F being Hilbert bundles on $X \times B\mathbb{Z}/2$) it associates the pair

$$(d(E', F', D'), \ d(E, F, D) - d(E', F', D')),$$

where $D' : E' \to F'$ is the pull-back of $D : E \to F$ on $X \times B\mathbb{Z}/2$ via the composition map

$$X \times B\mathbb{Z}/2 \to X \times * \to X \times B\mathbb{Z}/2,$$

and $*$ is the basepoint of $B\mathbb{Z}/2$. Therefore, according to this geometric interpretation of θ, we have that $\theta(E_1 \otimes \mathbf{1}) = (E_1, 0)$ and $\theta(E_2 \otimes L) = (E_2, E_2 \otimes L - E_2 \otimes \mathbf{1})$. Consequently, on summing we find that the composition $\Psi \circ \chi = \theta \circ \gamma^{-1} \circ \chi$ sends (E_1, E_2) to $(E_1 \oplus E_2, E_2 \otimes (L - \mathbf{1}))$. $\qquad\qquad\square$

7.11. The case $\varepsilon = -1$. Here we show that $_{-1}\mathcal{L}(\mathbb{R})_{\#} \simeq (\mathcal{K}(\mathbb{R})^{h(-_1\mathbb{Z}/2)})_{\#}$. Using polar decomposition of matrices again, we are reduced to showing that

$$(S) \qquad\qquad \mathcal{K}(M_2(\mathbb{R}))^{\sigma}_{\#} \simeq \mathcal{K}(M_2(\mathbb{R}))^{h\sigma}_{\#}$$

where σ is the involution on $M_2(\mathbb{R})$ defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Note that $M_2(\mathbb{R})^{\sigma} = \mathbb{C}$, and so the statement is equivalent to:

$$\mathcal{K}(\mathbb{C})_{\#} \simeq \mathcal{K}(M_2(\mathbb{R}))^{h\sigma}_{\#},$$

an analogue of the well-known fact in Lemma 7.3 that

$$\mathcal{K}(\mathbb{R}) \simeq \mathcal{K}(\mathbb{C})^{\tau}$$

where τ denotes complex conjugation. The statement (7.11) is a particular case of the following theorem.

THEOREM 7.12. *Let A be a real or complex simple algebra, and let σ be a nontrivial involution. Then*

$$\omega : \mathcal{K}(A^{\sigma})_{\#} \longrightarrow (\mathcal{K}(A)^{h\sigma})_{\#}$$

is a homotopy equivalence.

Proof. Let us consider the complex case first, which means that $A = M_n(\mathbb{C})$. By the Skolem-Noether theorem, σ is the inner conjugation by a matrix whose square is a scalar matrix. So, after extracting square roots and effecting a change of basis of $\mathbb{C}^n$, we can assume that the matrix is of type $\begin{pmatrix} I_{\ell} & 0 \\ 0 & -I_q \end{pmatrix}$ where $0 < \ell = n - q < n$, since σ is nontrivial. Since we are stabilizing and suspending, we are reduced to proving the 2-adic equivalence

$$\mathcal{K}(\mathbb{C}) \times \mathcal{K}(\mathbb{C}) \longrightarrow \mathcal{K}(M_2(\mathbb{C}))^{h\mathbb{Z}/2}$$

shown in Lemma 7.10.

When A is a ring of matrices over $\mathbb{R}$ or $\mathbb{H}$, we consider

$$A' = A \otimes_{\mathbb{R}} \mathbb{C} \cong M_n(\mathbb{C}).$$

Let τ denote the complex conjugation on A', and write σ also to denote the extension of σ to A'. From the complex case, we already have the homotopy equivalence

$$\mathcal{K}(A'^{\sigma})_{\#} \simeq (\mathcal{K}(A')^{h\sigma})_{\#}.$$

So the result follows from the commutative diagram

$$
\begin{array}{ccc}
\mathcal{K}(A^{\sigma}) & \xrightarrow{\;\simeq\;} & \mathcal{K}((A^{\sigma})')^{h\tau} = \mathcal{K}((A')^{\sigma})^{h\tau} \\
{\scriptstyle \omega}\downarrow & & \downarrow \\
\mathcal{K}(A)^{h\sigma} & \xrightarrow{\;\simeq\;} & \mathcal{K}((A')^{h\tau})^{h\sigma} = \mathcal{K}((A')^{h\sigma})^{h\tau}
\end{array}
$$

in which the horizontal arrows are the descent homotopy equivalences of [29]. $\qquad\square$

Observe that in the above proof, the right-hand vertical arrow requires 2-adic completion to be an equivalence. Therefore this property is also true of the map ω.

7.13. Relation to the integral orthogonal group conjecture. Consider the diagram

$$
\begin{array}{ccc}
B_{\varepsilon}O(\mathbb{Z}')_{\#}^{+} & \xrightarrow{\;\alpha'\;} & B\mathrm{GL}(\mathbb{Z}')_{\#}^{+h(_{\varepsilon}\mathbb{Z}/2)} \\
{\scriptstyle \lambda_O}\uparrow & & \uparrow{\scriptstyle \lambda_{\mathrm{GL}}} \\
B_{\varepsilon}O(\mathbb{Z})_{\#}^{+} & \xrightarrow{\;\alpha\;} & B\mathrm{GL}(\mathbb{Z})_{\#}^{+h(_{\varepsilon}\mathbb{Z}/2)} \\
{\scriptstyle \rho_O}\downarrow & & \downarrow{\scriptstyle \rho_{\mathrm{GL}}} \\
B_{\varepsilon}O(\mathbb{F}_p)_{\#}^{+} & \xrightarrow{\;\alpha_p\;} & B\mathrm{GL}(\mathbb{F}_p)_{\#}^{+h(_{\varepsilon}\mathbb{Z}/2)}
\end{array}
$$

and its effect on homotopy groups of dimension at least 2. From Quillen's localization theorem, we know that λ_{GL} induces isomorphisms in this range. Likewise, isomorphisms for α' and α_p are proved in Theorem 7.2 and Lemma 7.4 respectively. Thus our conjecture that λ_O induces isomorphisms is equivalent to the "Lichtenbaum-Quillen" conjecture that α does also, again for the same range of dimension.

Appendix A. On suspensions. For a *discrete* ring Λ, we define the cone $C\Lambda$ of Λ to be the ring of infinite matrices (indexed by $\mathbb{N}$) over Λ for which there exists a natural number that bounds:
   (i) the number of nonzero entries in each row and column; and
   (ii) the number of distinct entries in the entire matrix.

If $\tilde{\Lambda}$ denotes the ideal of $C\Lambda$ comprising matrices with only finitely many nonzero entries, then the suspension $S\Lambda$ is defined to be the quotient ring $C\Lambda/\tilde{\Lambda}$. Note that we have canonical isomorphisms

$$C\Lambda = \Lambda \otimes_{\mathbb{Z}} C\mathbb{Z}, \qquad S\Lambda = \Lambda \otimes_{\mathbb{Z}} S\mathbb{Z};$$

from the second we readily obtain

$$S(\Lambda[x]) = (S\Lambda)[x].$$

On the other hand, if Λ is $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, with the usual topology, then we define $C\Lambda$ to be the ring of bounded operators in a real, complex or quaternionic Hilbert space H (with an ℓ^2-basis) and $\tilde{\Lambda}$ to be the ideal of compact operators in H. Then $S\Lambda$, defined to be $C\Lambda/\tilde{\Lambda}$, is called the Calkin algebra. It is related to the space of Fredholm operators $\mathcal{F}(H)$ by the surjective homotopy equivalence

$$\mathcal{F}(H) \longrightarrow \mathrm{GL}_1(S\Lambda).$$

By virtue of Kuiper's theorem on the contractibility of $\mathrm{Aut}(H)$, there are also chains of homotopy equivalences

$$\mathrm{GL}_1(S\Lambda) \longrightarrow \mathrm{GL}_2(S\Lambda) \longrightarrow \cdots \longrightarrow \mathrm{GL}(S\Lambda)$$

and

$$\mathcal{F}(H) \longrightarrow \mathcal{F}(H \oplus H) \longrightarrow \cdots \longrightarrow \mathcal{F}(H \oplus \cdots \oplus H \oplus \cdots)$$

as discussed in [21].

With these definitions, there are obvious maps

$$S\mathbb{Z}' \rightarrow S\mathbb{R} \rightarrow S\mathbb{C} \rightarrow S\mathbb{H}$$

inducing

$$\mathcal{K}(\mathbb{Z}') \rightarrow \mathcal{K}(\mathbb{R}) \rightarrow \mathcal{K}(\mathbb{C}) \rightarrow \mathcal{K}(\mathbb{H})$$

and (with the trivial antiinvolutions on $\mathbb{Z}'$, $\mathbb{R}$ and $\mathbb{C}$)

$$_{\varepsilon}\mathcal{L}(\mathbb{Z}') \rightarrow {_{\varepsilon}\mathcal{L}}(\mathbb{R}) \rightarrow {_{\varepsilon}\mathcal{L}}(\mathbb{C}).$$

See Section 2.

Appendix B. On equalizers. The purpose of this appendix is to review some background material that we need for analysis of the homotopy fiber on

$\mathcal{K}$ and $\mathcal{L}$ spaces of the difference map known as $\psi - 1$. Our requirement is that this map ψ be induced by a map of rings. So, as in Theorem 3.3, to obtain a Mayer-Vietoris sequence, first we need a method for replacing an arbitrary ring homomorphism by an epimorphism.

Mapping cylinder. Given any homomorphism $\theta : A \to B$ of $\mathbb{Z}[1/2\ell]$-algebras, construct the pull-back

$$M(\theta) = \{(a, \ell(x)) \in A \times B[x] \mid \theta(a) = \ell(0)\},$$

and so the diagram

$$
\begin{array}{ccc}
M(\theta) & \longrightarrow & B[x] \\
\downarrow{e_0'} \quad \searrow{\tilde{\theta}} & & \downarrow{e_0} \\
A & \xrightarrow{\theta} & B.
\end{array}
$$

Here $\tilde{\theta} : (a, \ell(x)) \mapsto \ell(1)$, $e_0 : \ell(x) \mapsto \ell(0)$, and e_0', are epimorphisms. (Note that $b \in B$ is the image under $\tilde{\theta}$ of $(1, 1 + (b - 1)x) \in M(\theta)$.) The triangles don't commute. However, because ℓ is invertible in B, homotopy invariance applies (for mod ℓ K-groups $\tilde{K}$ by [56], and for mod ℓ L-groups $\tilde{L}$ by Theorem 3.4 above); thus e_0 and e_1 induce the same isomorphism. Also, $f : A \to M(\theta)$ sending a to $(a, \theta(\hat{a}))$, where $\theta(\hat{a})$ denotes the constant polynomial at $\theta(a)$, has $\tilde{\theta} \circ f = \theta$ and $e_0' \circ f = \mathrm{id}$. Then the Mayer-Vietoris sequence (for mod ℓ K-groups $\tilde{K}$ by [56], and for mod ℓ L-groups $\tilde{L}$ by Theorem 3.3 above) makes the induced homomorphism f_* an isomorphism.

Equalizer. Suppose that we are given homomorphisms $f_1, f_2 : A \to B$ of $\mathbb{Z}[1/2\ell]$-algebras. Consider the ring $\bar{B} = B \times B$, with diagonal homomorphism $\Delta : B \to \bar{B}$. Use the mapping cylinder to replace Δ by $\tilde{\Delta} : M(\Delta) \to \bar{B}$. Now apply to

$$
\begin{array}{ccc}
F & \twoheadrightarrow & A \\
\downarrow{\gamma} & \lrcorner & \downarrow{(f_1, f_2)} \\
M(\Delta) & \xrightarrow{\tilde{\Delta}} & \bar{B}
\end{array}
$$

to obtain its exact Mayer-Vietoris sequence as the lower sequence of

$$
\begin{array}{ccccccccc}
\cdots \to & 0 & \to & \tilde{K}_n(M(\Delta)) & \to & \tilde{K}_n B & \to & 0 & \to \cdots \\
 & \downarrow & & \downarrow & & \downarrow{\Delta_*} & & \downarrow & \\
\cdots \to & \tilde{K}_n F \to \tilde{K}_n A \oplus \tilde{K}_n(M(\Delta)) & \to & \tilde{K}_n B \oplus \tilde{K}_n B & \to & \tilde{K}_{n-1} F \to \cdots.
\end{array}
$$

On passing to the cokernel of the vertical injection map, we retrieve the desired exact sequence

$$\cdots \to \tilde{K}_n F \to \tilde{K}_n A \overset{f_{1*}-f_{2*}}{\longrightarrow} \tilde{K}_n B \to \tilde{K}_{n-1} F \to \cdots$$

Similarly for $\tilde{L}$, making F the ring we seek with $\tilde{\mathcal{K}}(F)$ and $\tilde{\mathcal{L}}(F)$ the desired homotopy fiber.

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