ON THE BAUM–CONNES CONJECTURE IN THE REAL CASE

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Abstract

Let Γ be a countable discrete group. We prove that if the usual Baum–Connes conjecture is valid for Γ, then the real form of Baum–Connes is also valid for Γ. This is relevant to proving that Baum–Connes implies the stable Gromov–Lawson–Rosenberg conjecture about Riemannian metrics of positive scalar curvature.

0. Introduction

The classical Baum–Connes conjecture (for a given discrete countable group Γ) states that the index map [**1**]

$$\mu(\Gamma) : K_j^{\Gamma}(\underline{E}\Gamma) \longrightarrow K_j(C_r^*(\Gamma))$$

is an isomorphism (where $j = 0, 1 \mod 2$).

In this statement, $K_j(C_r^*(\Gamma))$ is the K-theory of the reduced C^*-algebra $C_r^*(\Gamma)$ (also denoted $C_r^*(\Gamma; \mathbb{C})$ in [**9**]) and $K_j^{\Gamma}(\underline{E}\Gamma)$ is the *complex* equivariant Kasparov K-homology (with Γ-compact supports) of the space $\underline{E}\Gamma$. This index map may also be defined in the *real* context, by using real Kasparov theory. In other words, there is an index map

$$\mu_{\mathbb{R}}(\Gamma) : KO_j^{\Gamma}(\underline{E}\Gamma) \longrightarrow K_j(C_r^*(\Gamma; \mathbb{R})),$$

where j takes its values in $\mathbb{Z} \mod 8$. We may now ask whether $\mu_{\mathbb{R}}(\Gamma)$ is also an isomorphism.

One source of interest in this question (for a given group Γ) is the result of S. Stolz (with contributions from J. Rosenberg, P. Gilkey and others): the injectivity of $\mu_{\mathbb{R}}(\Gamma)$ implies the stable Gromov–Lawson–Rosenberg conjecture [**2**] about the existence of a Riemannian metric of positive scalar curvature on compact connected spin manifolds with Γ as fundamental group [**10**].

The purpose of this paper is to show that the Baum–Connes conjecture in the real case follows from the usual (that is, complex) case. More precisely, our theorem is the following.

THEOREM *Let Γ be a discrete countable group. If $\mu(\Gamma)$ is an isomorphism then $\mu_{\mathbb{R}}(\Gamma)$ is also an isomorphism.*

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The proof relies on an interpretation of the index maps $\mu(\Gamma)$ and $\mu_{\mathbb{R}}(\Gamma)$ as K-theory connecting homomorphisms associated with exact sequences of (real or complex) C^*-algebras [**3**, **8**] and also on a general theorem for Banach algebras which follows directly from a 'descent theorem' in topological K-theory.

THEOREM [**4**] *Let A be a Banach algebra over the real numbers and let $A' = A \otimes_{\mathbb{R}} \mathbb{C}$ be its complexification. If $K_i(A') = 0$ for all $i \in \mathbb{Z} \mod 2$, then $K_j(A) = 0$ for all $j \in \mathbb{Z} \mod 8$.*

1. Definition of $\mu(\Gamma)$ and $\mu_{\mathbb{R}}(\Gamma)$

1.1 In this section, we recall the basic definitions of [**1**] and observe that these definitions extend immediately to the real case.

The universal proper Γ-space is denoted by $\underline{E}\Gamma$ and $K_j^{\Gamma}(\underline{E}\Gamma)$ denotes the following colimit:

$$\operatorname*{colim}_{\Delta} KK_{\Gamma}^{j}(C_0(\Delta), \mathbb{C}),$$

where Δ runs over all Γ-compact subspaces of $\underline{E}\Gamma$ (by definition, a Γ-subspace Δ is called Γ-compact if the quotient space Δ/Γ is compact). The composition of the two homomorphisms

$$KK_{\Gamma}^{j}(C_0(\Delta), \mathbb{C}) \longrightarrow KK_{\Gamma}^{j}(C_0(\Delta) \rtimes \Gamma, C_r^*(\Gamma)) \longrightarrow KK^{j}(\mathbb{C}, C_r^*(\Gamma))$$

induces (by taking the colimit) the map μ referred to in the Introduction. Here the first homomorphism is Kasparov's descent map [**5**] and the second one is induced by the Kasparov product with

$$1 \in KK^0(\mathbb{C}, C_0(\Delta) \rtimes \Gamma) = K_0(C_0(\Delta) \rtimes \Gamma).$$

REMARK 1.1 In this definition of μ, the specific space $\underline{E}\Gamma$ does not play a particular role. In other words, if X is *any* proper Γ-space, we could define in the same way an 'index map'

$$\mu(X, \Gamma) : K_j^{\Gamma}(X) \longrightarrow K_j(C_r^*(\Gamma)).$$

REMARK 1.2 These definitions extend immediately to the real case. Hence, there is a real index map

$$\mu_{\mathbb{R}}(X, \Gamma) : KO_j^{\Gamma}(X) \longrightarrow K_j(C_r^*(\Gamma; \mathbb{R})).$$

The *real* Baum–Connes conjecture for the group states that $\mu_{\mathbb{R}}(\underline{E}\Gamma, \Gamma) = \mu_{\mathbb{R}}(\Gamma)$ is an isomorphism for all $j \in \mathbb{Z} \mod 8$.

2. Index maps and connecting homomorphisms in K-theory

2.1 The strategy for proving our theorem is as follows. We will describe (in this section) a C^*-algebra whose K-theory (real or complex) vanishes precisely when the corresponding version of the Baum–Connes conjecture is true.† In the next section we will apply to this C^*-algebra the result of [**4**], according to which the K-theory of a real C^*-algebra vanishes if and only if the K-theory of its complexification vanishes.

† In the analogous context of surgery theory, the K-theory of this C^*-algebra would be the 'fibre of assembly' or 'structure set' term in the surgery exact sequence.

2.2 To construct the required C^*-algebra we have chosen to use the method of [**3**, **7**, **8**]. Let X be a locally compact space[‡] X and let Γ be a countable discrete group Γ acting properly on X. Choose a separable Hilbert space H with a representation of C^*-algebras $\psi : C_0(X) \longrightarrow \mathcal{B}(H)$ and a unitary group representation $\tau : \Gamma \longrightarrow \mathcal{U}(H)$ which are compatible in the sense that $\psi(\gamma.f) = \tau(\gamma).\psi(f).\tau(\gamma)^*$, where $\gamma.f$ is the function $x \mapsto f(\gamma^{-1}x)$. (Note that these conditions imply that we have in fact a representation of the crossed product C^*-algebra $C_0(X) \rtimes \Gamma$ in $\mathcal{B}(H)$.) It is also required that H be a 'large' representation in a certain technical sense; it is sufficient to take $H = L^2(X;\mu) \otimes \ell^2(\Gamma) \otimes H'$, where H' is an auxiliary infinite-dimensional Hilbert space and μ is a Borel measure on X whose support is all of X.

Within this setting, we define the support in $X \times X$ of an operator T, denoted by $\mathrm{Supp}(T)$, as the complement of the points (x, y) such that there exists a neighborhood $U \times V$ of (x, y) such that $\psi(f)T\psi(g) = 0$, for f supported in U and g supported in V.

2.3 Following [**3**, **7**], we now define the C^*-algebra $D^*_\Gamma(X)$ and a closed ideal $C^*_\Gamma(X)$. Thus there is an exact sequence of C^*-algebras

$$0 \longrightarrow C^*_\Gamma(X) \longrightarrow D^*_\Gamma(X) \longrightarrow D^*_\Gamma(X)/C^*_\Gamma(X) \longrightarrow 0. \qquad (\mathcal{E})$$

By definition, $D^*_\Gamma(X)$ is the closure of the algebra in $\mathcal{B}(H)$ consisting of all the (bounded) operators T such that

(1) T is Γ-invariant, that is, $T.\tau(\gamma) = \tau(\gamma)T$ for all γ in Γ.

(2) $\mathrm{Supp}(T)$ is Γ-compact, that is, its quotient[§] by Γ is compact in $(X \times X)/\Gamma$.

(3) For all f in $C_0(X)$, $T\psi(f) - \psi(f)T$ is a compact operator on H.

The ideal $C^*_\Gamma(X)$ is the closure of the algebra in $\mathcal{B}(H)$ consisting of the (bounded) operators T which satisfy (1), (2), and a stronger condition.

(3′) For all f in $C_0(X)$, $T\psi(f)$ and $\psi(f)T$ are compact operators on H.

EXAMPLE 2.4 If Γ is a finite group and X is compact, it is well known that the K-theory of the C^*-algebra $D^*_\Gamma(X)/C^*_\Gamma(X)$ is the K-homology, with a shift of dimension, of the cross-product algebra $C(X) \rtimes \Gamma$ (this is 'Paschke duality' [**6**]). In the simplest case when X is a point, the exact sequence above is essentially equivalent to a direct sum of exact sequences of the form

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{B}(H) \longrightarrow \mathcal{B}(H)/\mathcal{K} \longrightarrow 0$$

as many as the number of conjugacy classes in Γ.

THEOREM 2.5 [**8**] *For any proper cocompact Γ-space X, there is a canonical Morita equivalence between the C^*-algebra $C^*_\Gamma(X)$ and $C^*_r(\Gamma)$, the reduced C^*-algebra of the group Γ.*

THEOREM 2.6 [**3**, **6**] *For any proper Γ-space X, there is a natural isomorphism*

$$K^\Gamma_j(X) := \operatorname*{colim}_{\Delta} KK^j_\Gamma(C_0(\Delta), \mathbb{C}) \xrightarrow{\cong} K_{j+1}(D^*_\Gamma(X)/C^*_\Gamma(X)),$$

where Δ runs over all the Γ-compact subspaces of X.

[‡] We assume X to be also second countable in order to get separable Hilbert spaces.

[§] Here Γ is acting on $X \times X$ by the diagonal action.

THEOREM 2.7 [**8**] *For any proper* Γ*-space* X*, we have a commutative diagram*

$$\begin{array}{ccc} K_j^{\Gamma}(X) & \stackrel{\mu}{\longrightarrow} & K_j(C_r^*(\Gamma)) \\ \cong\downarrow & & \cong\downarrow \\ K_{j+1}(D_{\Gamma}^*(X)/C_{\Gamma}^*(X)) & \stackrel{\delta}{\longrightarrow} & K_j(C_{\Gamma}^*(X)) \end{array}$$

where μ *is the Baum–Connes map and where* δ *is the* K*-theory connecting homomorphism associated to the exact sequence* $(\mathcal{E})$ *above.*

REMARK 2.8 It is important to notice that the three theorems above are also true in the real case (see [**9**] for a detailed account of this 'real Paschke duality'). In this case, $C_r^*(\Gamma)$ has to be replaced by $C_r^*(\Gamma;\mathbb{R})$. The real analogues of the C^*-algebras $D_{\Gamma}^*(X)$ and $C_{\Gamma}^*(X)$ will be denoted $D_{\Gamma}^*(X;\mathbb{R})$ and $C_{\Gamma}^*(X;\mathbb{R})$.

COROLLARY 2.9 *The Baum–Connes map* $\mu : K_j^{\Gamma}(X) \longrightarrow K_j(C_r^*(\Gamma))$ *is an isomorphism for all* $j \in \mathbb{Z} \mod 2$ *if and only if the* K*-groups* $K_j(D_{\Gamma}^*(X)) = 0$ *for all* j*. In the same way, the real Baum–Connes map* $\mu_{\mathbb{R}} : KO_j^{\Gamma}(X) \longrightarrow K_j(C_{\Gamma}^*(X;\mathbb{R}))$ *is an isomorphism for all* $j \in \mathbb{Z} \mod 8$ *if and only if the* K*-groups* $K_j(D_{\Gamma}^*(X;\mathbb{R})) = 0$ *for all* j*.*

3. Proof of the Baum–Connes conjecture in the real case for a given group Γ (assuming its validity for Γ in the complex case)

3.1 As we have shown in section 2, the complex (resp. real) Baum–Connes conjecture is equivalent to the vanishing of the K-groups $K_j(D_{\Gamma}^*(X))$ (resp. $K_j(D_{\Gamma}^*(X;\mathbb{R}))$) for $X = \underline{E}\Gamma$. If we put $A = D_{\Gamma}^*(X;\mathbb{R})$, its complexification $A' = A \otimes_{\mathbb{R}} \mathbb{C}$ is isomorphic to $D_{\Gamma}^*(X)$. The scheme of the argument is then the following, where $BC(\Gamma)$ (resp. $BC_{\mathbb{R}}(\Gamma)$) stands for the Baum–Connes conjecture (resp. the real Baum–Connes conjecture) for a given discrete group Γ:

$$BC(\Gamma) \Longleftrightarrow K_*(A') = 0 \Longrightarrow K_*(A) = 0 \Longleftrightarrow BC_{\mathbb{R}}(\Gamma).$$

3.2 The only point to show is the implication $K_*(A') = 0 \Longrightarrow K_*(A) = 0$, which follows from the descent theorem stated in [**4**] in the general framework of Banach algebras. More precisely, let A be any Banach algebra over the real numbers and A' denote its complexification $A \otimes_{\mathbb{R}} \mathbb{C}$. There is then a cohomology spectral sequence with $E_2^{pq} = H^p(\mathbb{RP}_2; K_{-q}(A'))$ converging to $K_{-q-p}(A) \oplus K_{-q-p+4}(A)$, where $\mathbb{RP}_2$ is the real projective plane and H^p means the usual singular cohomology with local coefficients.[¶] The hypothesis $K_*(A') = 0$ implies that the E_2 term of the spectral sequence is 0. Therefore the E_∞ term is also 0. Since moreover the filtration is finite (because $\mathbb{RP}_2$ is finite dimensional), $K_*(A)$ must be also 0.

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