CLIFFORD MODULES AND TWISTED *K*-THEORY

by Max KAROUBI

The purpose of this short paper is to make the link between the fundamental work of Atiyah, Bott and Shapiro [1] and twisted K-theory as defined by P. Donovan, J. Rosenberg and the author [2] [8] [7]. This link was implicit in the literature (for bundles over spheres as an example) but was not been explicitly defined before.

The setting is the following: V is a real vector bundle on a compact space X, provided with a non degenerate quadratic form to which we associate a bundle of (real or complex) Clifford algebras denoted by $C(V)$; the quadratic form is implicit in this notation. We denote by $M(V)$ the Grothendieck group associated to the category of (real or complex) vector bundles provided with a structure of (twisted) $\mathbb{Z}/2$-graded $C(V)$-module. Another way to describe $M(V)$ is to consider the bundle $V \oplus 1$, where the symbol "1" denotes the trivial vector bundle of rank one with a positive quadratic form. Then $M(V)$ is just the Grothendieck group $K(\Lambda_1)$ of the category $\mathcal{P}(\Lambda_1)$ whicho objects are finitely generated projective modules over Λ_1. The notation Λ_n means in general $\Lambda\widehat{\otimes}C^{0,n}$, where Λ is the ring of continuous sections of the $\mathbb{Z}/2$-graded bundle $C(V)$ and $C^{0,n}$ is the Clifford algebra of $\mathbb{R}^n$ with a positive quadratic form.

Following [1], we define $A(V)$ as the cokernel of the homomorphism induced by restriction of the scalars :

$$A(V) = Coker[M(V \oplus 1) \to M(V)] = Coker[K(\Lambda_2) \to K(\Lambda_1)].$$

Remark. Let us denote by V^- the vector bundle V with the opposite quadratic form. It is quite easy to see[1] that the category of $C(V \oplus 1)$-modules is isomorphic to the category of $C(V^- \oplus 1)$-modules. From now on, we assume that the quadratic form on V is positive (in which case $A(V^-)$ was the original definition of [1]).

With these definitions, we have the following theorem, where $K(V)$ denotes the real or complex reduced K-theory of the Thom space of V.

Theorem . *There is an exact sequence between K-groups*[2]

$$K(\Lambda_2) \to K(\Lambda_1) \to K(V) \to K^1(\Lambda_2) \to K^1(\Lambda_1)$$

In particular, $A(V)$ is a subgroup of $K(V)$ which coincides with it in the following important cases :

[1] If (v,t) is a symbol for the action of $V \oplus 1$, with $t^2 = 1$, we change it in (vt, t) which represents the action of $V^- \oplus 1$,

[2] One might also write K_{-1} instead of K^1.

a) $K^1(\Lambda_2) = 0$*, for instance when* X *is reduced to a point.*

b) V *is oriented of rank divisible by 4.*

c) V *is oriented of even rank in the framework of complex* K*-theory.*

Proof. According to the general theory developed in [3], $K(V) \equiv K^1(V \oplus 1)$ is canonically isomorphic to the K^1-group of the Banach functor

$$\phi : \mathcal{E}^{V\oplus 2}(X) \to \mathcal{E}^{V\oplus 1}(X),$$

where $\mathcal{E}^W(X)$ denotes the category of vector bundles provided with a $C(W)$-module structure. According to the Serre-Swan theorem, the categories involved are equivalent to categories $\mathcal{P}(R)$ for suitable rings R, in this case Λ_2 or Λ_1. The first part of the theorem follows from these general considerations.
If X is a point, the category $\mathcal{P}(R)$ is finite dimensional and therefore its K^1-group is trivial. On the other hand, if V is oriented of rank n divisible by 4, let us choose an orthonormal *oriented* basis $e_1, \ldots, e_n$ on each fiber V_x, $x \in X$. Then the product $\varepsilon = e_1 \ldots e_n$ in the Clifford algebra $C(V_x)$ is independant of the choice of the basis since ε commutes with the action of $SO(n)$ and defines therefore a continuous section of $C(V)$. On the other hand, for any W, there is an isomorphism between the graded tensor product $C(V)\widehat{\otimes}C(W)$ and the nongraded one $C(V) \otimes C(W)$. In order to see it, we send $V \oplus W$ to $C(V) \otimes C(W)$ by the formula

$$(v, w) \mapsto v \otimes 1 + \varepsilon \otimes w,$$

The fact that n is even shows that ε anticommutes with v. Moreover, if 4 divides n, the square of ε is 1. Therefore, by the universal property of Clifford algebras, the previous map induces the required isomorphism $C(V)\widehat{\otimes}C(W) \equiv C(V) \otimes C(W)$. If $n = 4k + 2$ and in the framework of complex K-theory, one may replace ε by $\varepsilon\sqrt{-1}$ in order to get the same result.
In our situation, W is of dimension one or two and the Banach functor

$$\mathcal{P}(\Lambda) \sim \mathcal{P}(\Lambda\widehat{\otimes}C^{0,2}) \to \mathcal{P}(\Lambda\widehat{\otimes}C^{0,1}) \sim \mathcal{P}(\Lambda) \times \mathcal{P}(\Lambda)$$

may be identified to the diagonal functor through the previous category isomorphisms. This shows that the map

$$K^1(\Lambda\widehat{\otimes}C^{0,2}) \to K^1(\Lambda\widehat{\otimes}C^{0,1})$$

is injective and concludes the proof of the theorem. □

Example. When X is reduced to a point, the theorem implies that the reduced K-theory of the sphere S^n is the cokernel of the map

$$K(C^{0,n+2}) \to K(C^{0,n+1})$$

which is the same as the cokernel of the map

$$K(C^{n+1,1}) \to K(C^{n,1})$$

as we noticed earlier. This is the starting remark in [1] which was the inspiration of [3], where the notation $C^{p,q}$ is used.

Generalizations. Since the main tool used here is the real Thom isomorphism proved in [3] and [5], the previous theorem might be generalized to the equivariant case. For instance, if G is a finite group acting linearly on $\mathbb{R}^n$, the group $K_G(\mathbb{R}^n)$ is isomorphic to the cokernel of the following map

$$K(G \ltimes C^{0,n+2}) \to K(G \ltimes C^{0,n+1})$$

where the involved rings are crossed products of G by Clifford algebras. More precise results may be found in [6].

Another generalization is to consider modules over bundles of $\mathbb{Z}/2$-graded Azumaya algebras $\mathscr{A}$ as in [2], instead of bundles of Clifford algebras. The analog of the group $A(V)$ is now what we might call the "algebraic twisted K-theory" of $\mathscr{A}$, denoted by $K_{alg}^{\mathscr{A}}(X)$ which is the cokernel of the map $K(\Lambda_2) \to K(\Lambda_1)$, where Λ denotes the ring of continuous sections of the $\mathbb{Z}/2$-graded algebra bundle $\mathscr{A}$. We can prove, as in the previous theorem, that this new group is a subgroup[3] of the usual twisted K-theory of X denoted by $K^{\mathscr{A}}(X)$. It coincides with it in some important cases, for instance if $\mathscr{A}$ is oriented (in the graded sense) with fibers modelled on matrix algebras over the real or complex numbers.

The multiplicative structure. It is well known that the twisted K-groups $K^{\mathscr{A}}(X)$ can be provided with a cup-product structure (see [2] §7 and also [4]). This cup-product makes a heavy use of Fredholm operators in Hilbert spaces. This machinery is unavoidable, especially for the odd K-groups.

More precisely, as shown in [2], the elements which define the group $K^{\mathscr{A}}(X)$ are pairs (E, D) where E is a $\mathbb{Z}/2$-graded Hilbert bundle provided by an $\mathscr{A}$-module structure and $D : E \to E$ is a family of Fredholm operators which are self-adjoint, of degree one and commute (in the graded sense) with the action of $\mathscr{A}$. According to the Thom isomorphism in twisted K-theory [7], $K^{\mathscr{A}}(X)$ is also the K^1-group of the Banach functor

$$\phi : \mathcal{E}^{\mathscr{A}\widehat{\otimes}C^{0,2}}(X) \to \mathcal{E}^{\mathscr{A}\widehat{\otimes}C^{0,1}}(X).$$

We have therefore the following exact sequence (as for Clifford modules)

$$K(\mathscr{A}\widehat{\otimes}C^{0,2}) \to K(\mathscr{A}\widehat{\otimes}C^{0,1}) \to K^{\mathscr{A}}(X) \to K^1(\mathscr{A}\widehat{\otimes}C^{0,2}) \to K^1(\mathscr{A}\widehat{\otimes}C^{0,1}).$$

A closer look at the connecting homomorphism

$$K(\mathscr{A}\widehat{\otimes}C^{0,1}) \to K^{\mathscr{A}}(X)$$

shows that it associates the pair $(E, 0)$ to a (finite dimensional) vector bundle E which is a module over $A\widehat{\otimes}C^{0,1}$. In other words, the elements in $K^{\mathscr{A}}(X)$ corresponding to finite dimensional bundles E are just elements of the cokernel of the

[3]One has to use again the Thom isomorphism in twisted K-theory as stated in [7].

map $K(\mathscr{A}\widehat{\otimes}C^{0,2}) \to K(\mathscr{A}\widehat{\otimes}C^{0,1})$ which we might call $A(\mathscr{A})$, if we follow the conventions of [1] or simply $K^{\mathscr{A}}_{alg}(X)$ as we did before.

Since the elements of $K^{\mathscr{A}}_{alg}(X)$ are associated to finite dimensional bundles (with the Fredholm operator reduced to 0), the usual cup-product

$$K^{\mathscr{A}}(X) \times K^{\mathscr{A}'}(X) \to K^{\mathscr{A}\widehat{\otimes}\mathscr{A}'}(X)$$

induces a pairing between the algebraic parts

$$K^{\mathscr{A}}(X)_{alg} \times K^{\mathscr{A}'}_{alg}(X) \to K^{\mathscr{A}\widehat{\otimes}\mathscr{A}'}_{alg}(X).$$

On the other hand, it might be interesting to characterize the elements of $K^{\mathscr{A}}(X)$ which are "algebraic". They belong to the kernel of the map

$$\phi : K^{\mathscr{A}}(X) \to K^1(\mathscr{A}\widehat{\otimes}C^{0,2}) = K(\mathscr{B}/\mathscr{A}\widehat{\otimes}C^{0,2}).$$

The notation $\mathscr{B}/\mathscr{A}$ represents here the bundle of Calkin algebras associated to $\mathscr{A}$ (the structural group of $\mathscr{A}$ is $PU(H)$ where H is an infinite dimensional Hilbert space, as mentioned in [7] and [8]). This map ϕ is easy to describe: it associates to a couple (E, D) as before the space of sections of the $\mathscr{B}/\mathscr{A}-$bundle associated to E. It is provided with the action of $C^{0,2}$ described by the grading and the involution on the Calkin bundle induced by the polar decomposition of D. This description of the "algebraic" elements also holds in the generalization of twisted K-theory considered by J. Rosenberg [8][7].

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Max Karoubi
Université Paris 7 - Denis Diderot
2 place Jussieu
75251 PARIS (FRANCE)
E-mail: max.karoubi@gmail.com