CLIFFORD MODULES AND INVARIANTS OF QUADRATIC FORMS

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Abstract. We construct new invariants of quadratic forms over commutative rings, using ideas from Topology. More precisely, we define a hermitian analog of the Bott class with target algebraic K-theory, based on the classification of Clifford modules. These invariants of quadratic forms go beyond the classical invariants defined via the Clifford algebra. An appendix by J.-P. Serre, of independent interest, describes the "square root" of the Bott class in the general framework of lambda rings.

0. Introduction

For any integer $k > 0$, the Bott class ρ^k in topological complex K-theory is well-known [8], [14, pg. 259]. If V is a complex vector bundle on a compact space X, $\rho^k(V)$ is defined as the image of 1 by the composition

$$K(X) \xrightarrow{\varphi} K(V) \xrightarrow{\psi^k} K(V) \xrightarrow{\varphi^{-1}} K(X),$$

where φ is Thom's isomorphism in complex K-theory and ψ^k is the Adams operation. This characteristic class is natural and satisfies the following properties which insure its uniqueness (by the splitting principle):

1) $\rho^k(V \oplus W) = \rho^k(V).\rho^k(W)$
2) $\rho^k(L) = 1 \oplus L \oplus ... \oplus L^{k-1}$ if L is a line bundle.

The Bott class may be extended to the full K-theory group if we invert the number k in the group $K(X)$. It induces a morphism from $K(X)$ to the multiplicative group $K(X)\,[1/k]^{\times}$. The Bott class is sometimes called "cannibalistic", since both its source and target are K-groups.

As pointed out by Serre (see the Appendix), the definition of the Bott class and its "square root", may be generalized to λ-rings, for instance in the theory of group representations or in equivariant topological K-theory.

The purpose of this paper is to give a hermitian analog of the Bott class. We shall define it on hermitian K-theory, with target algebraic

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K-theory. For instance, let $X = \mathrm{Spec}(R)$, where R is a commutative ring with $k!$ invertible and let V be an algebraic vector bundle on X provided with a nondegenerate quadratic form (and also a "spin structure"; see below). We shall associate to V a "hermitian Bott class", designated by $\rho_k(V)$, which takes its values in the same type of multiplicative group $K(X)\,[1/k]^{\times}$, where $K(X)$ is algebraic K-theory.

We write ρ_k instead of ρ^k in order to distinguish the new class from the old one, although they are closely related (cf. Theorem 3.5).We also note that the "cannibalistic" character of the new class ρ_k is avoided since the source and target are different groups. We refer to [13] for some basic notions in hermitian K-theory, except that we follow more standard notations, writing this theory $KQ(X)$, instead of $L(X)$ as in [13].

In order to define the new class $\rho_k(V)$, we need a slight enrichment of hermitian K-theory, using "spin modules" and not only quadratic ones. More precisely, a spin module is given by a couple (V, E), where V is a quadratic R-module and E is a finitely generated $\mathbf{Z}/2$-graded projective R-module, such that the Clifford algebra $C(V)$ is isomorphic to $\mathrm{End}(E)$ as $\mathbf{Z}/2$-graded algebras The associated Grothendieck group $K\mathrm{Spin}(X)$ is related to the hermitian[1] K-group $KQ(X)$ by an exact sequence

$$0 \longrightarrow \mathrm{Pic}(X) \xrightarrow{\theta} K\mathrm{Spin}(X) \xrightarrow{\varphi} KQ(X) \xrightarrow{\gamma} \mathsf{BW}(X),$$

where $\mathrm{BW}(X)$ denotes the Brauer-Wall group of X. As a set, $\mathrm{BW}(X)$ is isomorphic to the sum of three étale cohomology groups with $\mathbf{Z}/2$ coefficients [20] [9, Theorem 3.6]. There is a twisted group rule on this direct sum, (compare with [10]). In particular, for the spectrum of fields, the morphism γ is induced by the rank, the discriminant and the Hasse-Witt invariant [20]. From this point of view, the class ρ_k we shall define on $K\mathrm{Spin}(X)$ may be considered as a secondary invariant.

The hyperbolic functor $K(X) \longrightarrow KQ(X)$ admits a natural factorization

$$K(X) \xrightarrow{H} K\mathrm{Spin}(X) \longrightarrow KQ(X).$$

The class ρ_k is more precisely a homomorphism

$$\rho_k : K\mathrm{Spin}(X) \longrightarrow K(X)\,[1/k]^{\times},$$

such that we have the following factorization of the classical Bott class ρ^k :

$$\begin{array}{ccc} K(X) & \xrightarrow{\rho^k} & K(X)\,[1/k]^{\times} \\ H \searrow & & \nearrow \rho_k \\ & K\mathrm{Spin}(X) & \end{array}$$

An important example is when the bundle of Clifford algebras $C(V)$ has a trivial class in $\mathrm{BW}(X)$. In that case, $C(V)$ is the bundle of

[1] We consider here ε-hermitian modules with $\varepsilon = 1$ is the sign of symmetry.

endomorphisms of a $\mathbf{Z}/2$-graded vector bundle E (see Section 8) and we can interpret ρ_k as being defined on a suitable subquotient of $KQ(X)$, thanks to the exact sequence above. If k is odd, using a consequence of a result of Serre quoted in the Appendix (Section 9), we can "correct" the class ρ_k into another one $\overline{\rho}_k$, which is defined on the "spin Witt group"

$$W\mathrm{Spin}(X) = \mathrm{Coker}\,[K(X) \longrightarrow K\mathrm{Spin}(X)]$$

and which takes its values in the 2-torsion of the multiplicative group $K(X)\,[1/k]^{\times}\,/(\mathrm{Pic}(X))^{(k-1)/2}$.

With the same method, for $n > 0$, we define Bott classes in "higher KSpin-theory":

$$\rho_k : K\mathrm{Spin}_n(X) \longrightarrow K_n(X)\,[1/k]$$

There is a canonical homomorphism

$$K\mathrm{Spin}_n(X) \longrightarrow KQ_n(X)$$

which is injective if $n \geqslant 2$ and bijective if $n > 2$ (see Section 3). For all $n \geqslant 0$, the following diagram commutes

$$\begin{array}{ccc} K_n(X) & \xrightarrow{\rho^k} & K_n(X)\,[1/k] \\ H \searrow & & \nearrow \rho_k \\ & K\mathrm{Spin}_n(X) & \end{array} .$$

In Section 4, we make the link with Topology, showing that ρ_k is essentially the Bott class defined for spin bundles (whereas ρ^k is related to complex vector bundles as we have seen before).

Sections 5 and 6 are devoted to characteristic classes for Azumaya algebras, especially generalizations of Adams operations.

In Section 7, we show how to avoid spin structures by defining ρ_k on the full hermitian K-group $KQ(X)$. The target of ρ_k is now an algebraic version of "twisted K-theory" [16]. We recover the previous hermitian Bott class for quadratic modules which are provided with a spin structure, up to a non canonical isomorphism.

In the short section 8, we establish some technical lemmas more or less known on the Brauer group and the graded Brauer group of an Azumaya algebra.

Finally, there is an appendix (in French) written by J.-P. Serre where it is proved that the "modified" Bott class is canonically a square. This technical point is used in the definition of our characteristic classes on the Witt group either in a spin or twisted framework.

Terminology. It will be implicit in this paper that tensor products of $\mathbf{Z}/2$-graded modules or algebras are graded tensor products.

Aknowledgments. As we shall see many times through the paper, our methods are greatly inspired by the papers of Bott [8], Atiyah [1], Atiyah, Bott and Shapiro [2], and Bass [6]. We are indebted to Serre for his letter reproduced in the Appendix, concerning the "square root"

of the modified Bott class. Our Lemma 3.6 is one of its applications. Finally, we are indebted to Deligne, Knus and Tignol for useful remarks about operations on Azumaya algebras which are defined briefly in Sections 5 and 6.

Here is a summary of the paper by sections:

1. Clifford algebras and the spin group. Orientation of a quadratic module
2. Operations on Clifford modules
3. Bott classes in hermitian K-theory
4. Relation with Topology
5. Extension to Azumaya algebras
6. Adams operations revisited
7. Twisted hermitian Bott classes
8. Some lemmas about the Brauer-Wall group
9. Appendix. A letter from Jean-Pierre Serre to Max Karoubi (2 July 2007).

1. Clifford algebras and the spin group. Orientation of a quadratic module

In this section, we closely follow a paper of Bass [6]. The essential prerequisites are recalled here for the reader's convenience and in order to fix the notations.

Let R be a commutative ring and let V be a finitely generated projective R-module provided with a nondegenerate quadratic form q. We denote by $C(V,q)$, or simply $C(V)$, the associated Clifford algebra which is naturally $\mathbf{Z}/2$-graded. The canonical map from V to $C(V)$ is an injection and we shall implicitly identify V with its image.

The Clifford group $\Gamma(V)$ is the subgroup of $C(V)^{\times}$, whose elements u are homogeneous and satisfy the condition

$$uVu^{-1} \subset V.$$

We define a homomorphism from $\Gamma(V)$ to the orthogonal group

$$\phi : \Gamma(V) \longrightarrow \mathbf{O}(V)$$

by the formula

$$\phi(u)(v) = (-1)^{deg(u)} u.v.u^{-1}.$$

The group we are interested in is the 0-degree part of $\Gamma(V)$, i.e.

$$\Gamma^0(V) = \Gamma(V) \cap C^0(V).$$

We then have an exact sequence proved in [6, pg. 172]:

$$1 \to R^* \to \Gamma^0(V) \to \mathbf{SO}(V).$$

The group $\mathbf{SO}(V)$ in this sequence is defined as the kernel of the "determinant map"

$$\det : \mathbf{O}(V) \to \mathbf{Z}/2(R),$$

where $\mathbf{Z}/2(R)$ is the set of locally constant functions from $\mathrm{Spec}(R)$ to $\mathbf{Z}/2$. This set may be identified with the Boolean ring of idempotents in the ring R, according to [6, pg. 159]. The addition of idempotents is defined as follows

$$(e, e') \longmapsto e + e' - ee'.$$

The determinant map is then a group homomorphism. If $\mathrm{Spec}(R)$ is connected and if 2 is invertible in R, we recover the usual notion of determinant which takes its values in the multiplicative group ± 1.

We define an antiautomorphism of order 2 (called an involution through this paper):

$$a \longmapsto \overline{a}$$

of the Clifford algebra by extension of the identity on V (we change here the notation of Bass who writes this involution $a \longmapsto {}^t a$).

If $a \in \Gamma(V)$, its "spinor norm" $N(a)$ is given by the formula

$$N(a) = a\overline{a}.$$

It is easy to see that $N(a) \in R^\times \subset C(V)^\times$. The spin group $\mathbf{Spin}(V)$ is then the subgroup of $\Gamma^0(V)$ whose elements are of spinor norm 1.

We have an exact sequence

$$1 \to \mu_2(R) \to \mathbf{Spin}(V) \to \mathbf{SO}(V) \to \mathrm{Disc}(R).$$

Here $\mu_2(R)$ is the group of 2-roots of the unity in R. It is reduced to ± 1 if R is an integral domain and if 2 is invertible in R. On the other hand, $\mathrm{Disc}(R)$ is an extension

$$1 \to R^*/R^{*2} \to \mathrm{Disc}(R) \to \mathrm{Pic}_2(R) \to 1,$$

where $\mathrm{Pic}_2(R)$ is the 2-torsion of the Picard group [6, pg. 176]. The homomorphism

$$\mathrm{SN} : \mathbf{SO}(V) \to \mathrm{Disc}(R),$$

which is the generalization of the spinor norm if R is a field, is quite subtle and is also detailed in [6].

The map SN stabilizes and defines a homomorphism (where $\mathbf{SO}(R) = \mathop{colim}\limits_{m} \mathbf{SO}(H(R^m))$)

$$\chi : \mathbf{SO}(R) \to \mathrm{Disc}(R).$$

The following theorem is proved in [6, pg. 194].

Theorem 1.1. *The determinant map and the spinor norm define a homomorphism*

$$\widetilde{\chi} : \mathbf{O}(R) \to \mathbf{Z}/2(R) \oplus \mathrm{Disc}(R).$$

which is surjective. It induces a split epimorphism

$$KQ_1(R) \to \mathbf{Z}/2(R) \oplus \mathrm{Disc}(R).$$

The following corollary is immediate.

Corollary 1.2. *We have a central extension*

$$1 \to \mu_2(R) \to \mathrm{Spin}(R) \to \mathbf{SO}^0(R) \to 1,$$

where $\mathbf{SO}^0(R)$ *is the kernel of the epimorphism* $\widetilde{\chi}$ *defined above.*

Let us now assume that 2 is invertible in R and that the quadratic form q is defined by a symmetric bilinear form f, i.e.

$$q(x) = f(x, x).$$

The symmetric bilinear form associated to q is then $(x, y) \mapsto 2f(x, y)$.

Let us also assume that V is an R-module of constant rank which is even, say $n = 2m$. In this case, the n^{th} exterior power $\lambda^n(V)$ is an R-module of rank 1 which may be provided with the quadratic form associated to q. We say that V is orientable (in the quadratic sense) if $\lambda^n(V)$ is isomorphic to R with the standard quadratic form $\theta : x \longmapsto x^2$ (up to a scaling factor which is a square). We say that V is oriented if we fix an isometry between $\lambda^n(V)$ and (R, θ). If V is free with a given basis, this is equivalent to saying that the symmetric matrix associated to f is of determinant 1.

Remark 1.3. One may use the orientation on V to define on $C^0(V)$ a bilinear form

$$\Phi^0 : C^0(V) \times C^0(V) \to C^0(V) \xrightarrow{\sigma} \lambda^n(V) \cong R.$$

The first map is defined by the product in $C(V)$ and the last map σ is defined by the canonical filtration of the Clifford algebra, the associated graded algebra being the exterior algebra. In the same way, we define another bilinear form by taking the composition

$$\Phi^1 : C^1(V) \times C^1(V) \to C^0(V) \xrightarrow{\sigma} \lambda^n(V) \cong R.$$

The following theorem is not really needed for our purposes but is worth recording.

Theorem 1.4. *The previous bilinear forms* Φ^i, $i = 0, 1$, *are non degenerate, i.e. induce isomorphisms between* $C^i(V)$ *and its dual as an* R*-module. Moreover,* Φ^0 *is symmetric while* Φ^1 *is antisymmetric.*

Proof. We can check this theorem by localizing at any maximal ideal (m) of the ring R (see for instance [3, pg. 49]). In this case, there exists an orthogonal basis $(e_1, ..., e_n)$ of $V_{(m)}$. Since V is oriented, we may choose this basis such that the product $q(e_1)...q(e_n)$ is equal to 1. It is also well-known that the various products

$$e_I = e_{i_1 ... } e_{i_r}$$

form a basis of the free $R_{(m)}$-module $C(V_{(m)})$. Here the multiindex $I = (i_1, ..., i_r)$ is chosen such that $i_1 < i_2 < ... < i_r$. By a direct computation we have

$$\Phi(e_I, e_J) = \pm 1$$

if $I \cup J = \{1, ..., n\}$ and 0 otherwise, for $\Phi = \Phi^0$ or Φ^1. Therefore, these bilinear forms are non degenerate with the expected sign of symmetry. Moreover, they are hyperbolic at each localization. □

Remark 1.5. Since V is oriented, the group $\mathbf{SO}(V)$ acts naturally on $C(V)$ and we get two natural representations of this group in the orthogonal and symplectic groups associated to the previous bilinear forms Φ^0 and Φ^1.

Let us now consider the submodule N of $C(V)$ whose elements u satisfy the identity $u.v = -v.u$ for any element v in $V \subset C(V)$. The canonical surjection $V \longrightarrow \lambda^n(V)$ induces a homomorphism

$$\tau : N \longrightarrow \lambda^n(V).$$

Proposition 1.6. *The homomorphism τ is an isomorphism between N and $\lambda^n(V)$. Moreover, N is included in $C^0(V)$.*

Proof. We again localize with respect to all maximal ideals (m) of R and consider an orthogonal basis $\{e_i\}$ of $V_{(m)}$ as above. Then we see that the product $e_1...e_n$ generates N and we get the required isomorphism between $N_{(m)}$ and $\lambda^n(V)_{(m)}$. □

Remark 1.7. If we assume that V is oriented and of even rank as before, the previous proposition provides us with a canonical element u in $C^0(V)$ which anticommutes with all elements v in V, such that $u^2 = 1$. Moreover, $u.\overline{u} = 1$ and therefore u belongs to the spin group $\mathbf{Spin}(V)$.

An important example is the case when the Clifford algebra $C(V)$ has a trivial class in the Brauer-Wall group of R, denoted by BW(R). In other words, $C(V)$ is isomorphic to the algebra End(E) of a graded vector space $E = E_0 \oplus E_1$ where E_0 and E_1 are not reduced to 0 (see Section 8). The only possible choices for u are then one of the two following matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

We always choose E such that u is of the first type and, by a topological analogy, we shall say that V is "spinorial". For instance, let R be the ring of real continuous functions on a compact space X and let V be a real vector bundle provided with a positive definite quadratic form. The triviality of the Clifford bundle $C(V)$ in BW(X) is then equivalent to the following properties: the rank of V is a multiple of 8 and the Stiefel-Whitney classes $w_1(V)$ and $w_2(V)$ are trivial (see [10]).

Remark 1.8. Strictly speaking, in the topological situation, the classical spinor property does not imply that the rank of V is a multiple of 8. We put this extra condition in order to ensure the triviality of $C(V)$ in the Brauer-Wall group of R.

Remark 1.9. In Section 7, we shall remove the spinor condition in order to get suitable classes in twisted K-theory.

2. Operations on Clifford modules

As it is well-known, at least for fields, the standard non trivial invariants of a quadratic form (V, q) are the discriminant and the Hasse-Witt invariant. They are encoded in the class of the Clifford algebra $C(V) = C(V, q)$ in the Brauer-Wall group of R, which we call $\mathrm{BW}(R)$, as in the previous section. For any commutative ring R, this group $\mathrm{BW}(R)$ has been computed by Wall and Caenepeel [20][9]. As a set, it is the sum of the first three étale cohomology groups of $X = \mathrm{Spec}(R)$ with $\mathbf{Z}/2$ coefficients, but with a twisted group rule (compare with [10]). We view this class of $C(V)$ in $\mathrm{BW}(R)$ as a "primary" invariant. In order to define "secondary" invariants, we may proceed as usual by assuming first that this class is trivial. Therefore, we have an isomorphism

$$C(V) \cong \mathrm{End}(E),$$

where E is a $\mathbf{Z}/2$-graded R-module which is projective and finitely generated. We always choose E such that the associated element u defined in the previous section is the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

However, E is not uniquely defined by these conditions. If

$$\mathrm{End}(E) \cong \mathrm{End}(E'),$$

we have $E' \cong E \otimes L$, where L is a module of rank 1, concentrated in degree 0 according to our choice of u (this is a simple consequence of Morita equivalence).

As in the introduction, we may formalize the previous considerations better thanks to the following definition. A "spin module" is a couple (V, E), where E is a finitely generated $\mathbf{Z}/2$-graded projective R-module and $V = (V, q)$ is a quadratic oriented R-module, such that $C(V)$ is isomorphic to $\mathrm{End}(E)$ with the choice of u above. We define the "sum" $(V, E) + (V', E')$ as $(V \oplus V', E \otimes E')$ and the group $K\mathrm{Spin}(R)$ by the usual Grothendieck construction.

Proposition 2.1. *We have an exact sequence*

$$0 \longrightarrow \mathrm{Pic}(R) \xrightarrow{\theta} K\mathrm{Spin}(R) \xrightarrow{\varphi} KQ(R) \xrightarrow{\gamma} \mathrm{BW}(R),$$

where the homomorphisms γ, φ and θ are defined below.

Proof. The map γ was defined previously: it associates to the quadratic module (V, q) the class of the Clifford algebra $C(V) = C(V, q)$ in $\mathrm{BW}(R)$. We note that γ is not necessarily surjective, even on the 2-torsion part: see e.g. [10, pg. 11] for counterexamples. The map φ

sends a couple (V,E) to the class of the quadratic module V. Finally, θ associates to a module L of rank one the difference[2] $(H(R),\Lambda(R)\otimes L)-(H(R),\Lambda(R))$, where H is the hyperbolic functor and Λ the exterior algebra functor, viewed as a module functor. This map θ is a homomorphism since the image of $L\otimes L'$ may be written as follows

$$(H(R),\Lambda(R)\otimes L\otimes L')-(H(R),\Lambda(R)\otimes L)$$
$$+(H(R),\Lambda(R)\otimes L)-(H(R),\Lambda(R)).$$

This image is also

$$(H(R),\Lambda(R)\otimes L')-(H(R),\Lambda(R))+(H(R),\Lambda(R)\otimes L)-(H(R),\Lambda(R))$$

which is $\theta(L)+\theta(L')$. In order to complete the proof, it remains to show that the induced map

$$\sigma:\mathrm{Pic}(R)\longrightarrow \mathrm{Ker}(\varphi)$$

is an isomorphism.

1) The map σ is surjective. Any element of $\mathrm{Ker}(\varphi)$ may be written $(V,E)-(V,E')$. Therefore, we have $E\cong E'\otimes L$, where L is of rank 1. If we add to this element $(H(R),\Lambda(R)\otimes L^{-1})-(H(R),\Lambda(R))$, which belongs to $\mathrm{Im}(\varphi)$, we find 0.

2) The map σ is injective. We define a map backwards

$$\sigma':\mathrm{Ker}(\varphi)\longrightarrow \mathrm{Pic}(R),$$

by sending the difference $(V,E)-(V,E')$ to the unique L such that $E\cong E'\otimes L$. It is clear that $\sigma'\cdot\sigma=Id$, which proves the injectivity of σ. □

Before going any further, we need a convenient definition, due to Atiyah, Bott and Shapiro [2], of the graded Grothendieck group $GrK(A)$ of a $\mathbf{Z}/2$-graded algebra A. It is defined as the cokernel of the restriction map

$$K(A\widehat{\otimes}C^{0,2})\longrightarrow K(A\widehat{\otimes}C^{0,1}),$$

where $C^{0,r}$ is in general the Clifford algebra of R^r with the standard quadratic form $\sum_{i=1}^r {x_i}^2$. We note that if A is concentrated in degree 0, we recover the usual definition of the Grothendieck group $K(A)$, under the assumption that 2 is invertible in A, which we assume from now on. This follows from the fact that a $\mathbf{Z}/2$-graded structure on a module M is equivalent to an involution on M.

Another important example is $A=C(V,q)$, where V is oriented and of even rank. In order to compute the graded Grothendieck group of A, we use the element u introduced in 1.7 to define a natural isomorphism

$$A\widehat{\otimes}C^{0,r}\longrightarrow A\otimes C^{0,r}.$$

It is induced by the map

$$(v,t)\mapsto v\otimes 1+u\otimes t,$$

[2]Note that we can replace R by R^n in this formula.

where $v \in V \subset C(V, q)$ and $t \in R^r \subset C^{0,r}$. If E is a $\mathbf{Z}/2$-graded R-module, the same argument may be applied to $A = \mathrm{End}(E)$. The graded Grothendieck group again coincides with the usual one. Since we consider only these examples in our paper, we simply write $K(A)$ instead of $GrK(A)$ from now on.

The graded algebras we are interested in are the Clifford algebras $\Lambda_k = C(V, kq)$, where $k > 0$ is an invertible integer in R. The interest of this family of algebras is the following. Let M be a $\mathbf{Z}/2$-graded module over Λ_1. Then its k^{th}-power $M^{\widehat{\otimes} k}$ is a graded module over the crossed product algebra $S_k \ltimes C(V)^{\widehat{\otimes} k}) \cong S_k \ltimes C(V^k)$, where S_k is the symmetric group on k letters. One has to remark that the action of the symmetric group S_k on $M^{\widehat{\otimes} k}$ takes into account the grading as in [1, pg. 176]: the transposition (i, j) acts on a decomposable homogeneous tensor

$$m_1 \otimes ... \otimes m_i \otimes ... \otimes m_j \otimes ... \otimes m_k$$

as the permutation of m_i and m_j, up to the sign $(-1)^{\deg(m_i)\deg(m_j)}$.

Let us now consider the diagonal $V \longrightarrow V^k$. It is an isometry if we provide V with the quadratic form kq. Therefore, we have a well defined map

$$\Lambda_k \longrightarrow C(V^k)$$

which is equivariant with respect to the action of the symmetric group S_k. It follows that the correspondence

$$M \longmapsto M^{\widehat{\otimes} k}$$

induces (by restriction of the scalars) a "power map"

$$\mathcal{P} : K(\Lambda_1) \longrightarrow K_{S_k}(\Lambda_k),$$

where K_{S_k} denotes equivariant K-theory, the group S_k acting trivially on Λ_k.

Let us give more details about this definition. First, we notice that V^k splits as the direct sum of (V, kq) and its orthogonal module W. This implies that $C(V)^{\widehat{\otimes} k} \cong C(V^k) \cong C(W)\widehat{\otimes}\Lambda_k$ is a Λ_k-module which is finitely generated and projective. Therefore, the "restriction of scalars" functor from the category of finitely generated projective modules over $C(V^k)$ to the analogous category of modules over Λ_k is well defined. Secondly, we have to show that the map $\mathcal{P}$, which is a priori defined in terms of modules, can be extended to a map between graded Grothendieck groups. This may be shown by using a trick due to Atiyah which is detailed in [1, pg. 175][3]. Finally, we notice that $\mathcal{P}$ is a **set** map, not a group homomorphism.

In order to define K-theory operations in this setting, we may proceed in at least two ways. First, following Grothendieck, we consider a

[3]More precisely, Atiyah is considering complexes in his argument but the same idea may be applied to $\mathbf{Z}/2$-graded modules.

$\mathbf{Z}/2$-graded module M and its k^{th}-exterior power in the graded sense. The specific map $K_{S_k}(\Lambda_k) \longrightarrow K(\Lambda_k)$ which defines the k^{th}-exterior power is the following: we take the quotient of $M^{\widehat{\otimes} k}$ by the relations identifying to 0 all the elements of type

$$m - \varepsilon(\sigma) m^{\sigma}.$$

In this formula, m is an element of $M^{\widehat{\otimes} k}$, m^{σ} its image under the action of the element σ in the symmetric group, with signature $\varepsilon(\sigma)$. The composition

$$K(\Lambda_1) \longrightarrow K_{S_k}(\Lambda_k) \longrightarrow K(\Lambda_k)$$

defines the analog of Grothendieck's λ-operations:

$$\lambda^k : K(\Lambda_1) \longrightarrow K(\Lambda_k),$$

as detailed in [14, pg. 252] for instance.

Remark 2.2. If M is a graded module concentrated in degree 0 (resp. 1), $\lambda^k(M)$ is the usual exterior power (resp. symmetric power) with an extra Λ_k-module structure.

The diagonal map from V into $V \times V$ enables us to define a "cup-product": it is induced by the tensor product of modules with Clifford actions:

$$K(\Lambda_k) \times K(\Lambda_l) \longrightarrow K(\Lambda_{k+l})$$

The following theorem is a consequence of the classical property of the usual exterior (graded) powers, extended to this slightly more general situation.

Theorem 2.3. *Let M and N be two Λ_1-modules. Then one has natural isomorphisms of Λ_r-modules*

$$\lambda^r(M \oplus N) \cong \sum_{k+l=r} \lambda^k(M) \otimes \lambda^l(N).$$

Proof. It is more convenient to consider the direct sum of all the $\lambda^k(M)$, which we view as the $\mathbf{Z}$-graded exterior algebra $\Lambda(M)$. Since the natural algebra isomorphism

$$\Lambda(M) \otimes \Lambda(N) \longrightarrow \Lambda(M \oplus N)$$

is compatible with the Clifford structures, the theorem is proved. □

From these λ-operations, it is classical to associate "Adams operations" Ψ^k. For any element x of $K(\Lambda_1)$, we define $\Psi^k(x) \in K(\Lambda_k)$ by the formula

$$\Psi^k(x) = Q_k(\lambda^1(x), ..., \lambda^k(x)),$$

where Q_k is the Newton polynomial (cf. [14, pg. 253] for instance). The following theorem is a formal consequence of the previous one.

Theorem 2.4. *Let x and y be two elements of $K(\Lambda_1)$. Then one has the identity*

$$\Psi^k(x+y) = \Psi^k(x) + \Psi^k(y)$$

in the group $K(\Lambda_k)$.

Proof. Following Adams [14, pg. 257], we note that the series

$$\Psi_{-t}(x) = \sum_{k=1}^{\infty} (-1)^k t^k \Psi^k(x)$$

is the logarithm differential of $\lambda_t(x)$ multiplied by $-t$, i.e.

$$-t\frac{\lambda_t'(x)}{\lambda_t(x)}.$$

This can be checked by a formal "splitting principle" as in [11, pg. 9] for instance. The additivity of the Adams operation follows from the fact that the logarithm differential of a product is the sum of the logarithm differentials of each factor. □

Another important and less obvious property of the Adams operations is the following.

Theorem 2.5. *Let us assume that $k!$ is invertible in R and let x and y be two elements of $K(\Lambda_1)$. Then one has the following identity in the group $K(\Lambda_{2k})$.*

$$\Psi^k(x \cdot y) = \Psi^k(x) \cdot \Psi^k(y).$$

Proof. In order to prove this theorem, we use the second description of the operations λ^k and Ψ^k due to Atiyah [1, § 2], which we transpose in our situation. In order to define operations in K-theory, Atiyah considers the following composition (where $R(S_k)$ denotes the integral representation group ring of the symmetric group S_k):

$$K(\Lambda_1) \xrightarrow{\mathcal{P}} K_{S_k}(\Lambda_k) \xrightarrow{\cong} K(\Lambda_k) \otimes R(S_k) \xrightarrow{\chi} K(\Lambda_k).$$

In this sequence, $\mathcal{P}$ is the k^{th}-power map introduced before. The second map is defined by using our hypothesis that $k!$ is invertible in R. More precisely, any S_k-module is the direct sum of simple modules: if π runs through all the (integral) irreducible representations of the symmetric group S_k, the natural map

$$\oplus \mathrm{Hom}(\pi, T) \otimes \pi \longrightarrow T$$

is an isomorphism (note that π is of degree 0). Therefore, by linearity, the equivariant K-theory $K_{S_k}(C(V, kq)) = K_{S_k}(\Lambda_k)$ may be written as $K(\Lambda_k) \otimes R(S_k)$. Finally, the map χ is defined once a homomorphism

$$\chi_k : R(S_k) \longrightarrow \mathbf{Z}$$

is given. For instance, the Grothendieck operation $\lambda^k(M)$ is obtained through the specific homomorphism χ_k equal to 0 for all the irreducible representations of S_k, except the sign representation ε, where $\chi_k(\varepsilon) = 1$.

Moreover, we can define the product of two operations associated to χ_k and χ_l using the ring structure on the direct sum $\oplus\mathrm{Hom}(R(S_r), \mathbf{Z})$, as detailed in [1, pg. 169]. This structure is induced by the pairing

$$\begin{aligned}\mathrm{Hom}(R(S_k), \mathbf{Z}) \times \mathrm{Hom}(R(S_l), \mathbf{Z}) &\longrightarrow \mathrm{Hom}(R(S_k \times S_l), \mathbf{Z}) \\ &\longrightarrow \mathrm{Hom}(R(S_{k+l}), \mathbf{Z}).\end{aligned}$$

In particular, as proved formally by Atiyah [1, pg. 179], the Adams operation Ψ^k is induced by the homomorphism

$$\Psi : R(S_k) \longrightarrow \mathbf{Z}$$

associating to a class of representations ρ the trace of $\rho(c_k)$, where c_k is the cycle $(1, 2, ..., k)$. With this interpretation, the multiplicativity of the Adams operation is obvious. □

Remark 2.6. We conjecture[4] that the previous theorem holds without the hypothesis that $k!$ is invertible in R. If we assume that $2k$ is invertible in R, and that R contains the k^{th}-roots of the unity, we propose another closely related operation $\overline{\Psi}^k$ in Section 6. We also conjecture that $\Psi^k = \overline{\Psi}^k$. This is at least true if $k!$ is invertible in R.

3. Bott classes in hermitian K-theory

Let us assume that $k!$ is invertible in R. We consider a spin module (V, E), as in the previous section. The following maps are detailed below:

$$\begin{aligned}\theta_k : K(R) \underset{\cong}{\xrightarrow{\alpha}} K(C(V, q)) \xrightarrow{\mathcal{P}} K_{S_k}(C(V, kq)) \underset{\cong}{\longrightarrow} K(C(V, kq)) \otimes R(S_k) \\ \xrightarrow{\Psi'} K(C(V, kq)) \underset{\cong}{\xrightarrow{(\alpha_q)^{-1}}} K(R).\end{aligned}$$

The morphism α is the Morita isomorphism

$$K(R) \cong K(C(V, q)) \cong K(\mathrm{End}(E))$$

and $\mathcal{P}$ is the k^{th}-power map defined in the previous section. The morphism Ψ' is induced by $\Psi : R(S_k) \longrightarrow \mathbf{Z}$ also defined there. Finally, for the definition of α_q, we remark that the isomorphism between $C(V, q)$ and $\mathrm{End}(E)$ implies the existence of an R-module map

$$f : V \longrightarrow \mathrm{End}(E_0 \oplus E_1)$$

such that

$$f(v) = \begin{bmatrix} 0 & \sigma(v) \\ \tau(v) & 0 \end{bmatrix}$$

[4]For instance, this conjecture holds if M is hyperbolic or if M is of rank one.

with $\sigma(v)\tau(v) = \tau(v)\sigma(v) = q(v).1$. We now define a "$k$-twisted map"

$$f_k : V \longrightarrow \mathrm{End}(E_0 \oplus E_1)$$

by the formula

$$f_k(v) = \begin{bmatrix} 0 & k\sigma(v) \\ \tau(v) & 0 \end{bmatrix}.$$

Since $(f_k(v))^2 = kq(v)$, f_k induces a homomorphim between $C(V, kq)$ and $\mathrm{End}(E_0 \oplus E_1)$ which is clearly an isomorphism, as we can see by localizing at all maximal ideals. The map α_q is then induced by the same type of Morita isomorphism we used to define α.

Remark 3.1. The map α is the algebraic analog of the Thom isomorphism in topological K-theory.

Theorem 3.2. *Let (V, E) be a spin module and let M be an R-module. Then the image of M under θ_k is defined by the following formula*

$$\theta_\kappa(M) = \rho_k(V, E).\Psi^k(M).$$

Therefore, θ_k is determined by $\theta_\kappa(1) = \rho_k(V, E)$, which we shall simply write $\rho_k(V, q)$ or $\rho_k(V)$ if the quadratic form q and the module E are implicit. We call $\rho_k(V)$ the "hermitian Bott class" of V. Moreover, we have the multiplicativity formula

$$\rho_k(V \oplus W) = \rho_k(V) \cdot \rho_k(W)$$

in the Grothendieck group $K(R)$.

Proof. The first formula follows from the multiplicativity of the Adams operation proved in Theorem 2.5. The second one follows from the same multiplicativity property and the well-known isomorphism

$$C(V \oplus W) \cong C(V) \otimes C(W)$$

(graded tensor product as always, according to our conventions). □

Theorem 3.3. *Let $(V, -q)$ be the module V provided with the opposite quadratic form. Then we have the identity*

$$\rho_k(V, q) = \rho_k(V, -q).$$

Proof. According to our hypothesis, the Clifford algebra $C(V)$ is oriented, since it is isomorphic to $\mathrm{End}(E)$. Therefore, we may use the element u defined in 1.7 to show that $C(V, q)$ is isomorphic to $C(V, -q)$ (more generally, $C(V, q)$ is isomorphic to $C(V, kq)$ if k is invertible). Explicitly, we keep the same E as the module of spinors, so that $C(V, q) \cong C(V, -q) \cong \mathrm{End}(E)$. We now write the commutative diagram

$$\begin{array}{ccccccc} K(R) & \to & K(C(V,q)) & \xrightarrow{\Psi^k} & K(C(V,kq)) & \to & K(R) \\ \downarrow Id & & \downarrow\cong & & \downarrow\cong & & \downarrow Id \\ K(R) & \to & K(C(V,-q)) & \xrightarrow{\Psi^k} & K(C(V,-kq)) & \to & K(R) \end{array}.$$

□

Remark 3.4. The isomorphism between the Clifford algebras $C(V,q)$ and $C(V,-q)$ is defined by using the element u of degree 0 and of square 1 in $C(V,q)$ which anticommutes with all the elements of V. It is easy to see that the k-tensor product $u_k = u \otimes ... \otimes u$ satisfies the same properties for the Clifford algebra $C(V^k, q \oplus ... \oplus q)$. Therefore, we have an analogous commutative diagram with the power map $\mathcal{P}$ instead of the Adams operation Ψ^k:

$$\begin{array}{ccc} K(C(V,q)) & \xrightarrow{\mathcal{P}} & K(C(V,kq)) \otimes R(S_k) \\ \downarrow\cong & & \downarrow\cong \\ K(C(V,-q)) & \xrightarrow{\mathcal{P}} & K(C(V,-kq)) \otimes R(S_k) \end{array}$$

Theorem 3.5. *Let (V,q) be the hyperbolic module $H(P)$ and $E = \Lambda(P)$ be the associated module of spinors. Then $\rho_k(V,E)$ is the classical Bott class $\rho^k(P)$ of the R-module P.*

Proof. According to [6, pg. 166], the Clifford algebra $C(V)$ is isomorphic to $\mathrm{End}(\Lambda P)$ as a $\mathbf{Z}/2$-graded algebra, which gives a meaning to our definition. The class $\rho_k(V,q)$ may be identified with the "formal quotient" $\Psi^k(\Lambda P)/\Lambda P$ which satisfies the algebraic splitting principle[5]. Therefore, in order to prove the theorem, it is enough to consider the case when $P = L$ is of rank one. We then have $\Lambda L = 1 - L$, $\Psi^k(\Lambda L) = 1 - L^k$ and therefore, $\Psi^k(\Lambda L)/\Lambda L = 1 + L + ... + L^{k-1}$. □

In order to extend the definition of the hermitian Bott class to "KSpin-theory", we remark that any element x of $K\mathrm{Spin}(R)$ may be written as

$$x = V - H(R^m),$$

where V is a quadratic module (the module of spinors E being implicit). Moreover, $V - H(R^m) = V' - H(R^{m'})$ iff we have an isomorphism

$$V \oplus H(R^{m'}) \oplus H(R^s) \cong V' \oplus H(R^m) \oplus H(R^s)$$

for some s. Therefore, if we invert k in the Grothendieck group $K(R)$, the following definition

$$\rho_k(x) = \rho_k(V - H(R^m)) = \rho_k(V)/k^m$$

does not depend of the choice of V and m.

The previous definitions are not completely satisfactory if we are interested in characteristic classes for the "spin Witt group" of R, denoted by $W\mathrm{Spin}(R)$ and defined as the cokernel of the hyperbolic map

$$K(R) \longrightarrow K\mathrm{Spin}(R).$$

One way to deal with this problem is to consider the underlying module V_0 of (V,E). According to our hypothesis, V_0 is a module of even rank,

[5] More precisely, we write formally P as a sum of rank one modules L_i and its exterior powers as elementary symmetric functions λ^j of the L_i. Therefore, the quotient $\Psi^k(\Lambda P)/\Lambda P$ is an explicit polynomial in the λ^j's.

oriented and isomorphic to its dual. The following lemma is a particular case of a theorem due to Serre proved in the Appendix (Section 9). We summarize Serre's argument in this special case (we need the more general case in Section 7).

Lemma 3.6. *Let us assume that k is odd. With the previous hypothesis, the classical Bott class $\rho^k(V_0)$ is canonically a square in $K(R)$.*

Proof. Let Ω_k be the ring of integers in the k-cyclotomic extension of $\mathbb{Q}$ and let z be a primitive k^{th}-root of the unity. In the computations below, we always embed an abelian group G in $G \otimes_{\mathbf{Z}} \Omega_k$. Let us now write

$$G_{V_0}(t) = 1 + t\lambda^1(V_0) + ... + t^n\lambda^n(V_0).$$

From the algebraic splitting principle, it follows that

$$\rho^k(V_0) = \prod_{r=1}^{k-1} G_{V_0}(-z^r).$$

The identity $\lambda^j(V_0) = \lambda^{n-j}(V_0)$ implies that $G_{V_0}(t) = t^n G_{V_0}(1/t)$. We then deduce from [19] that $\rho^k(V_0)$ has a square root[6] which we may choose to be

$$\sqrt{\rho^k(V_0)} = (-1)^{n(k-1)/4} \prod_{r=1}^{(k-1)/2} G_{V_0}(-z^r) \cdot z^{-nr/2}.$$

This square root is invariant under the action of the Galois group $(\mathbf{Z}/k)^*$ of the cyclotomic extension which is generated by the transformations $z \longmapsto z^j$, where $j \in (\mathbf{Z}/k)^*$. Therefore, it belongs to $K(R)$, as a subgroup of $K(R) \otimes_{\mathbf{Z}} \Omega_k$. $\square$

The previous lemma enables us to "correct" the hermitian Bott class in the following way. We put

$$\overline{\rho}_k(V) = \rho_k(V)(\sqrt{\rho^k(V_0)})^{-1}.$$

If V is a hyperbolic module $H(W) = W \oplus W^*$, we have $\rho_k(V) = \rho^k(W)$. On the other hand, we have $\lambda_t(W \oplus W^*) = J(t) \cdot t^{n/2} \cdot J(1/t) \cdot \sigma$, where

$$J(t) = \lambda_t(W) = 1 + t\lambda^1(W) + ... + t^{n/2}\lambda^{n/2}(W)$$

and $\sigma = \lambda^{n/2}(W^*)$. Therefore,

$$\sqrt{\rho^k((W \oplus W^*)} = (-1)^{n(k-1)/4} \prod_{r=1}^{(k-1)/2} \sigma \cdot J(-z^r) \cdot (-z^r)^{n/2} . J(-1/z^r) \cdot (z^{-nr/2})$$

$$= \sigma^{(k-1)/2} \cdot \prod_{r=1}^{k-1} J(-z^r) = \sigma^{(k-1)/2} \cdot \rho^k(W) = \sigma^{(k-1)/2} \cdot \rho_k(H(W)).$$

[6]We have inserted a normalization sign $(-1)^{n(k-1)/4}$ in Serre's formula for a reason which is explained in the computation below.

From this computation, it follows that $\overline{\rho}_k(V)$ is a $[(k-1)/2]^{th}$-power of an element of the Picard group of R if V is hyperbolic. Moreover,

$$\begin{aligned}\overline{\rho}_k(V)^2 &= (\rho_k(V))^2(\rho^k(V_0))^{-1} = \rho_k(V,q))\rho_k(V,-q))(\rho^k(V_0))^{-1} \\ &= \rho_k(H(V_0))(\rho^k(V_0))^{-1} = \rho^k(V_0)(\rho^k(V_0))^{-1} = 1.\end{aligned}$$

Summarizing this discussion, we have proved the following theorem:

Theorem 3.7. *Let $k > 0$ be an odd number. Then the corrected hermitian Bott class $\overline{\rho}_k(V) = \rho_k(V)(\sqrt{\rho^k(V_0)})^{-1}$ induces a homomorphism, also called $\overline{\rho}_k$:*

$$\overline{\rho}_k : W\mathrm{Spin}(R) \longrightarrow (K(R)\,[1/k])^\times/\mathrm{Pic}(R)^{(k-1)/2},$$

where the right hand side is viewed as a multiplicative group. Moreover, the image of $\overline{\rho}_k$ lies in the 2-torsion of this group.

Remark 3.8. The case k even does not fit with this strategy. However, we shall see in the next section that ρ_2 is not trivial in general on the spin Witt group.

Remark 3.9. We have chosen the Adams operation to define the hermitian Bott class. We could as well consider any operation induced by a homomorphism

$$R(S_k) \longrightarrow \mathbf{Z}.$$

The only reason for our choice is the very pleasant properties of the Adams operations with respect to direct sums and tensor products of Clifford modules.

We would like to extend the previous considerations to higher hermitian K-theory. More precisely, the orthogonal group we are considering to define this K-theory is the group $\mathbf{O}_{m,m}(R)$ which is the group of isometries of $H(R^m)$, together with its direct limit $\mathbf{O}(R) = \underset{m}{colim}\, \mathbf{O}_{m,m}(R)$. For $n \geqslant 1$, the higher hermitian K-groups are defined in a way parallel to higher K-groups, using Quillen's + construction, by the formula

$$KQ_n(R) = \pi_n(B\mathbf{O}(R)^+).$$

However, from classical group considerations, as we already have seen, the spin group behaves better than the orthogonal group for our purposes. Therefore, we shall replace the group $\mathbf{O}_{m,m}(R)$ by the associated spin group $\mathbf{Spin}_{m,m}(R)$ defined in Section 1, which direct limit is denoted by $\mathbf{Spin}(R)$. We have the following two exact sequences (where the first one splits):

$$1 \longrightarrow \mathbf{SO}^0(R) \longrightarrow \mathbf{O}(R) \longrightarrow \mathbf{Z}/2(R) \oplus \mathrm{Disc}(R) \longrightarrow 1,$$

$$1 \longrightarrow \mu_2(R) \longrightarrow \mathbf{Spin}(R) \longrightarrow \mathbf{SO}^0(R) \longrightarrow 1.$$

We may classical tools of Quillen's + construction [12] and [7], applied to the classifying space of central group extensions. We then show that the maps $\mathbf{SO}^0(R) \longrightarrow \mathbf{O}(R)$ and $\mathbf{Spin}(R) \longrightarrow \mathbf{SO}^0(R)$ induce isomorphisms

$$\pi_n(B\mathbf{SO}^0(R)^+) \cong \pi_n(B\mathbf{O}(R)^+) \text{ for } n > 1.$$

$$\pi_n(B\mathbf{Spin}(R)^+) \cong \pi_n(B\mathbf{SO}^0(R)^+) \text{ for } n > 2.$$

Moreover, the maps

$$\pi_1(B\mathbf{SO}^0(R)^+) \longrightarrow \pi_1(B\mathbf{O}(R)^+)$$

and

$$\pi_2(B\mathbf{Spin}(R)^+) \longrightarrow \pi_2(B\mathbf{SO}^0(R)^+)$$

are injective.

Our extension of the Bott class to higher hermitian K-groups will be a map also called ρ_k :

$$\rho_k : \pi_n(B\mathbf{Spin}(R)^+) \longrightarrow \pi_n(B\mathbf{GL}(R)^+)\,[1/k] = K_n(R)\,[1/k]$$

In order to define such a map, we work geometrically, using the description of the various K-theories in terms of flat bundles as detailed in the Appendix 1 to [15]. Any element of $\pi_n(B\mathrm{Spin}(R)^+) = K\mathrm{Spin}_n(R)$ for instance is represented by a formal difference $x = V - T$, where V and T are flat spin bundles of the same rank, say $2m$, over a homology sphere $X = \widetilde{S}^n$ of dimension n. We may also assume that the fibers of V and T are hyperbolic, e.g. $H(R^m)$, and that T is "virtually trivial", which means that T is the pull-back of a flat bundle over an acyclic space. More precisely, we should first consider a flat principal bundle Q of structural group $\mathbf{Spin}_{m,m}(R)$ such that

$$V = Q \times_{\mathbf{Spin}_{m,m}(R)} H(R^m).$$

On the other hand, $\mathbf{Spin}_{m,m}(R)$ acts on $C(H(R^m)) = \mathrm{End}(\Lambda R^m)$ by inner automorphisms. Therefore, the bundle of Clifford algebras $C(V)$ associated to V is the bundle of endomorphisms of the flat bundle

$$Q \times_{\mathbf{Spin}_{m,m}(R)} \Lambda R^m.$$

We now apply our general recipe of Section 2 on each fiber of V and T. In other words, for the bundle V for instance, we consider the following composition

$$K(X) \xrightarrow{\alpha} K^{C(V)}(X) \xrightarrow{\Psi^k} K^{C(V(k))}(X) \xrightarrow{\alpha_k^{-1}} K(X).$$

In this sequence, we write $K(Y)$ for the group of homotopy classes of maps from Y to the classifying space of algebraic K-theory which is homotopically equivalent to $K_0(R) \times B\mathbf{GL}(R)^+$. Its elements are represented by flat bundles over spaces X homologically equivalent to Y. The notation $K^{C(V)}(X)$ means the (graded) K-theory of flat bundles provided with a graded $C(V)$-module structure.

The image of 1 by the composition $\alpha_k^{-1}.\Psi^k.\alpha$ defines an element of $K(X)$, which we call $\rho_k(V)$. On the other hand, since T is virtually trivial, we have $\rho_k(T) = k^m$. We then define $\rho_k(x)$ in the group $K_n(R)\,[1/k]$ by the formula

$$\rho_k(x) = \rho_k(V)/k^m$$

Theorem 3.10. *For $n \geqslant 1$, the correspondance $x \mapsto \rho_k(x)$ induces a group homomorphism*

$$\rho_k : K\mathrm{Spin}_n(R) \longrightarrow K_n(R)\,[1/k]$$

called the n^{th}-hermitian Bott class.

Proof. The map $x \mapsto \rho_k(x)$ is well defined by general homotopy considerations which are given with much details in [15, Appendix 1] in a closely related context. In order to check that we get a group homomorphism, we write the direct sum of the $K_n(R)\,[1/k]$ as the multiplicative group

$$1 + K_{*>0}(R)\,[1/k]$$

where the various products between the K_n-groups are reduced to 0. If we now take two elements x and y in $K\mathrm{Spin}_n(R)$, we write $\rho_k(x)$ as $1 + u$ and $\rho_k(y)$ as $1 + v$. Then

$$\rho_k(x + y) = \rho_k(x) \cdot \rho_k(y) = (1 + u) \cdot (1 + v) = 1 + u + v$$

since $u \cdot v = 0$. □

4. Relation with Topology

Let V be a real vector bundle on a compact space X provided with a positive definite quadratic form. We assume that V has rank $8n$ with a spin structure, so that the bundle of Clifford algebras may be written as $\mathrm{End}(E)$, where E is the $\mathbf{Z}/2$-graded vector bundle of "spinors" (see for instance Section 8 and [10]). Following Bott [8], we define the class $\rho^k_{top}(V)$ as the image of 1 by the composition of the homomorphisms

$$K_{\mathbf{R}}(X) \xrightarrow{\varphi} K_{\mathbf{R}}(V) \xrightarrow{\psi^k} K_{\mathbf{R}}(V) \xrightarrow{\varphi^{-1}} K_{\mathbf{R}}(X),$$

where $K_{\mathbf{R}}$ denotes real K-theory and φ is Thom's isomorphism for this theory. One purpose in this section is to show that this topological class $\rho^k_{top}(V)$ coincides with our hermitian Bott class $\rho_k(V)$ in the group $K_{\mathbf{R}}(X) \cong K(R)$, where R is the ring of real continuous functions on X.

The proof of this statement requires a careful definition of the group $K_{\mathbf{R}}(V)$, since V is not a compact space. A possibility is to define this group as follows (see [14, §2] for instance). One considers couples (G, D), where G is a $\mathbf{Z}/2$-graded real vector bundle on V, provided with a metric, and D an endomorphism of G with the following properties:

1) D is an isomorphism outside a compact subset of V and we identify (G, D) to 0 if this compact set is empty

2) D is self-adjoint and of degree 1.

One can show that the Grothendieck group associated to this semi-group of couples is the reduced K-theory of the one-point compactification of V. With this description, Thom's isomorphism

$$K_{\mathbf{R}}(X) \xrightarrow{\varphi} K_{\mathbf{R}}(V)$$

is easy to describe. It associates to a vector bundle F on X the couple

$$\tau = (G, D) = (F \otimes E, 1 \otimes \rho(v)).$$

In this formula, $C(V) \cong \mathrm{End}(E)$ and $\rho(v)$ denotes the Clifford multiplication by $\rho(v)$ over a point $v \in V \subset C(V)$. We note that $\rho(v)$ is an isomorphism outside the 0-section of V. Therefore, Thom's isomorphism may be interpreted as Morita's equivalence. In fact, φ is the following composition (where $K(C(V))$ is the K-theory of the ring of sections of the algebra bundle $C(V)$)

$$K_{\mathbf{R}}(X) \longrightarrow K(C(V)) \xrightarrow{t} K_{\mathbf{R}}(V),$$

according to [14] for instance.

Remark 4.1. This class $\rho^k_{top}(V)$, which requires a metric and a spin structure on V, is different in general from the algebraic class $\rho^k(V)$, defined for λ-rings. On the other hand, the quotient between $\rho^k_{top}(V)$ and $(\rho^k(V))^2$ is a 2-torsion class which is not trivial in general, as it is shown in an example at the end of this section.

If we apply the Adams operation to the previous couple $\tau = (G, D)$, one finds $(\Psi^k(F)\cdot\Psi^k(E), \Psi^k(1\otimes\rho(v)))$ with obvious definitions. Strictly speaking, $\Psi^k(E)$ should be thought of as a virtual module over $C(V^k)$ and then we use the diagonal $V \longrightarrow V^k$ in order to view $\Psi^k(E)$ as a virtual module over $C(V)$. We use here the functorial definition of the Adams operation detailed in the proof of Theorem 2.5. Moreover, since k has a square root as a positive real number, we always have $C(V, kq) \cong C(V, q)$. To sum up, we have proved the following theorem:

Theorem 4.2. *Let R be the ring of real continuous functions on a compact space X and let V be a real spin bundle of rank $8n$ on X. Then the topological Bott class $\rho^k_{top}(V)$ in $K(R)$ coincides with the hermitian Bott class $\rho_k(V)$ of V, viewed as a finitely generated projective module provided with a positive definite quadratic form and a spin structure.*

For completeness' sake, let us make some explicit computations of this hermitian Bott class when X is a sphere of dimension $8m$. Let V be a real oriented vector bundle of rank $4t$ on S^{8m}, generating the reduced real K-group $\widetilde{K}_{\mathbf{R}}(S^{8m})$ and let $W = V \oplus V$ be its complexification which generates $\widetilde{K}_{\mathbf{C}}(S^{8m})$, where $\widetilde{K}_{\mathbf{C}}$ is reduced complex K-theory. Let us denote by $W_{\mathbf{R}}$ the underlying real vector bundle with the associated spin structure. According to [14, Proposition 7.27], we have the formula

$$c(\rho^k_{top}(W_{\mathbf{R}})) = \rho^k_{\mathbf{C}}(W) = \rho^k(W),$$

where $c : K_{\mathbf{R}}(S^{8m}) \xrightarrow{\cong} K_{\mathbf{C}}(S^{8m})$ denotes the complexification. Therefore, we are reduced to computing the class ρ^k for complex vector bundles on even dimensional spheres X.

If $X = S^2$, $K_{\mathbf{C}}(S^2)$ is free of rank 2, generated by 1 and the Hopf line bundle L. The classical Bott class is then computed from the formula

$$\rho^k(L) = 1 + L + ... + L^{k-1}.$$

Since $(L-1)^2 = 0$, another way to write this sum is to consider Taylor's expansion of the polynomial

$$1 + X + ... + X^{k-1}$$

at $X = 1$. We get the formula

$$\rho^k(L) = k + [1 + 2 + ... + (k-1)]\,(L-1) = k + k(k-1)(L-1)/2.$$

If x_2 denotes the class $L - 1$, we also can write

$$\rho^k(x_2) = 1 + [1 + 2 + ... + (k-1)]\,/k \cdot x_2.$$

We compute in the same way the Bott class on $\widetilde{K}_{\mathbf{C}}(S^4) \cong \mathbf{Z}$, generated by the product

$$x_4 = (L_1 - 1) \cdot (L_2 - 1)$$

where $L_1$and L_2 are two copies of the Hopf line bundle on S^2. Since we again have $(L_i - 1)^2 = 0$, it is sufficient to compute the first terms of Taylor's expansion of the polynomial

$$f(X, Y) = 1 + XY + X^2Y^2 + ... + X^{k-1}Y^{k-1}$$

at the point $(1, 1)$. We get the second derivative $(\delta^2 f/\delta x \delta y)/k^2$ at the point $(1, 1)$ multiplied by x_4. In other words, we have

$$\rho^k(x_4) = 1 + \left[1 + 2^2 + ... + (k-1)^2)\right]/k^2 \cdot x_4.$$

More generally, on $\widetilde{K}_{\mathrm{C}}(S^{2r}) \cong \mathbf{Z}$, generated by

$$x_{2r} = (L_1 - 1) \cdots (L_r - 1),$$

we find the formula

$$\rho^k(x_{2r}) = 1 + [1 + 2^r + ... + (k-1)^r]\,/k^r \cdot x_{2r}.$$

Since $c(\rho^k(x_{8m})) = \rho^k_{top}(y_{8m})$, where y_{8m}(resp x_{8m}) generates $\widetilde{K}_{\mathbf{R}}(S^{8m})$ (resp. $\widetilde{K}_{\mathbf{C}}(S^{8m})$), we deduce from the last formula the following proposition.

Proposition 4.3. *Let V be a real vector bundle of rank $8t$ generating the real reduced K-theory of the sphere S^{8m} and let $y_{8m} = V - 8t$. We then have the formula*

$$\rho^k_{top}(y_{8m}) = 1 + \left[1 + 2^{4m} + ... + (k-1)^{4m}\right] \cdot y_{8m}.$$

Remark 4.4. If we assume that k is odd, the sum $1+2^r+...+(k-1)^r$ has the same parity as $1+2+...+(k-1)$ or equivalently $(k-1)/2$ which is also odd for an infinite number of odd $k's$.

A more delicate example is the case of the sphere $X = S^{8m+2}$ with $m > 0$. It is well-known that the realification map

$$\mathbf{Z} \cong \widetilde{K}_{\mathbf{C}}(S^{8m+2}) \longrightarrow \widetilde{K}_{\mathbf{R}}(S^{8m+2}) \cong \mathbf{Z}/2$$

is surjective. Let V be a complex vector bundle over S^{8m+2} which generates $\widetilde{K}_{\mathbf{C}}(S^{8m+2})$. We consider the following diagram

$$\begin{array}{ccc} K_{\mathbf{C}}(V) & \xrightarrow{\Psi^k} & K_{\mathbf{C}}(V) \\ \uparrow \phi_{\mathbf{C}} & & \downarrow \phi_{\mathbf{C}}^{-1} \\ K_{\mathbf{C}}(S^{8m+2}) & & K_{\mathbf{C}}(S^{8m+2}) \end{array},$$

where $\phi_{\mathbf{C}}$ is Thom's isomorphism in complex K-theory. By definition, we have

$$\rho^k(V) = \phi_{\mathbf{C}}^{-1}(\Psi^k(\phi_{\mathbf{C}}(1))).$$

Since $m > 0, V$ is also a spin bundle and we therefore have a commutative diagram up to isomorphism

$$\begin{array}{ccc} K_{\mathbf{C}}(V) & \xrightarrow{r} & K_{\mathbf{R}}(V) \\ \uparrow \phi_{\mathbf{C}} & & \uparrow \phi_{\mathbf{R}} \\ K_{\mathbf{C}}(S^{8m+2}) & \xrightarrow{r} & K_{\mathbf{R}}(S^{8m+2}) \end{array},$$

where $\phi_{\mathbf{R}}$ is Thom's isomorphism in real K-theory and r is the realification. Since the Adams operation Ψ^k commutes with r, we have the identity

$$\rho^k_{top}(y_{8m+2}) = 1 + \left[1 + 2^{4m+1} + ... + (k-1)^{4m+1}\right] \cdot y_{8m+2} = 1 + y_{8m+2}$$

if k and $(k-1)/2$ are odd.

Let V_0 be the underlying real vector bundle of V. The last identity implies that $\rho^k_{top}(V_0) = (1 + y_{8m+2}) \cdot k^{4t}$ if V_0 is of rank $8t$. Therefore $\rho^k_{top}(V_0)$ cannot be a square, even modulo the Picard group (which is trivial in this case). This implies that the corrected hermitian Bott class defined in 3.7:

$$\overline{\rho}_k : W\mathrm{Spin}(R) \longrightarrow K(R)^\times/(\mathrm{Pic}(R))^{(k-1)/2} = K(R)^\times$$

is not trivial either (we recall that R is the ring of real continuous functions on the sphere S^{8m+2}).

Remark 4.5. We should add a few words if k is even. If $k = 2$ for instance and if X is the sphere S^{8n}, we find that

$$\rho^2(y_{8n}) = 1/2^{4n} \cdot y_{8n}$$

We get the same result for bundles with negative definite quadratic forms. Since the hyperbolic map

$$K(C_{\mathbf{R}}(S^{8n})) \cong \mathbf{Z} \to \ KQ(C_{\mathbf{R}}(S^{8n})) \cong \mathbf{Z}\oplus\mathbf{Z}$$

is the diagonal, we see that the class ρ_2 of an hyperbolic module belongs to $1/2^{4n-1}\mathbf{Z}$. Therefore, at least for this example, the class ρ_2 also detects non trivial Witt classes.

5. Extension to Azumaya algebras

Another purpose of this paper is the extension of our definitions to Azumaya algebras [4][6], beyond the example of Clifford algebras.

Definition 5.1. Let A be an Azumaya algebra. We say that A is "oriented" if the permutation of the two copies of A in $A^{\otimes 2}$ is given by an inner automorphism associated to an element $\tau \in (A^{\otimes 2})^\times$ of order 2. If A is $\mathbf{Z}/2$-graded, we moreover assume that τ may be choosen on degree 0;

As a matter of fact, as it was pointed out to us by Knus and Tignol, any ungraded Azumaya algebra is oriented[7]. This is a theorem quoted by Knus and Ojanguren[17, Proposition 4.1, p. 112.] and attributed to O. Goldman. We shall illustrate it by a few typical examples.

The first easy but fundamental example is $A = \mathrm{End}(P)$, where P is a faithful finitely generated projective module. We identity $B = A \otimes A$ with $\mathrm{End}(P \otimes P)$ and $B^\times$ with $\mathrm{Aut}(P \otimes P)$. The element τ required is simply the permutation of the two copies of P, viewed as an element of $(A \otimes A)^\times = \mathrm{Aut}(P \otimes P)$, as it can be shown by a direct computation.

Let now D be a division algebra over a field F. We claim that D is also oriented. In order to show this, we consider the tensor product $A = D \otimes_F F_1$, where F_1 is a finite Galois extension of F, such that A is F-isomorphic to a matrix algebra $\mathrm{M}_n(F_1) = \mathrm{End}(F_1^n)$ and is therefore oriented according to our first example. Let G be the Galois group of F_1 over F, so that D is the fixed algebra of G acting on A. If we compose this action by the usual action of the Galois group on $\mathrm{M}_n(F_1)$, we get automorphisms of $\mathrm{M}_n(F_1)$ as a F_1-algebra which are inner by Skolem-Noether's theorem. If $g \in G$, we let α_g be an element of $\mathrm{Aut}(F_1^n)$ so that the action $\rho(g)$ of g on A is given by the composition of the inner automorphism associated to a_g with the usual Galois action on $\mathrm{M}_n(F_1)$.

Let now τ' be the permutation of the two copies of A in the tensor product $A \otimes_{F_1} A$. It is induced by the inner automorphism associated to a specific element τ in $\mathrm{Aut}(F_1^n \otimes_{F_1} F_1^n)$ of order 2 which commutes with $\rho(g) \otimes \rho(g)$. Therefore, τ is invariant by the action of G and belongs to $(D \otimes_F D)^\times$, considered as a subgroup of $(A \otimes_{F_1} A)^\times$.

From a different point of view, let us consider the algebra R of complex continuous functions on a connected compact space X. According to a well-known dictionary of Serre and Swan, one may consider an Azumaya algebra A over R as a bundle $\widetilde{A}$ of algebras over X with fiber

[7]However, the situation is more subtle in the $\mathbf{Z}/2$-graded case as we shall show below.

End(P), where $P = \mathbf{C}^n$. The structural group of this bundle is the projective linear group Aut$(P)/\mathbf{C}^\times$. In the same way, the structural group of $A^{\otimes 2}$ is Aut$(P \otimes P)/\mathbf{C}^\times$. Therefore, the inner automorphism of $A^{\otimes 2}$, permuting the two copies of A, is induced by the permutation of the two copies of P. This is well defined globally since this permutation commutes with the transition functions of $\widetilde{A}$.

Let A be any Azumaya algebra. We would like to lift the action σ_k of the symmetric group S_k on $A^{\otimes k}$ to $(A^{\otimes k})^\times$, such that we have a commutative diagram

$$\begin{array}{ccc} & & (A^{\otimes k})^\times \\ & \overset{\widetilde{\sigma}_k}{\nearrow} & \downarrow \gamma \\ S_k & \xrightarrow{\sigma_k} & \mathrm{Aut}(A^{\otimes k}) \end{array},$$

where $\widetilde{\sigma}_k$ is a group homomorphism and γ induces inner automorphisms. This task is achieved in the "Book of Involutions" [18, Proposition 10.1, pg. 115.], using again the "Goldman element" quoted above. For completeness' sake, we shall sketch a proof below, since we shall need it in the graded case too.

In order to define $\widetilde{\sigma}_k$, we use the classical description of the symmetric group in terms of generators $\tau_i = (i, i+1), i = 1, ..., k-1$, with the relations $(\tau_i)^2 = 1, \tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1}$ and $\tau_i\tau_j = \tau_j\tau_i$ if $[i - j| > 1$. Since A is oriented, we may view the τ_i in $(A^{\otimes k})^\times$ as the tensor product of τ by the appropriate number of copies of $1 = Id_A$. We easily check the previous relations, except the typical one

$$\tau_1\tau_2\tau_1 = \tau_2\tau_1\tau_2.$$

(one may replace the couple $(1, 2)$ by $(i, i + 1)$). However, we already have $\tau_1\tau_2\tau_1 = \lambda\tau_2\tau_1\tau_2$, where $\lambda \in R^\times$. The identity

$$(\tau_1\tau_2\tau_1)^2 = (\tau_2\tau_1\tau_2)^2 = 1$$

also implies that $(\lambda)^2 = 1$. The solution to our lifting problem is then to keep the τ_i for i odd and replace the τ_i for i even by $\lambda\tau_i$, in order to get the required relations among the $\tau's$.

The previous considerations may be translated in the framework of $\mathbf{Z}/2$-graded Azumaya algebras [6, pg. 160]. In this case, we must require the element τ in the definition to be of degree 0. Unfortunately, in general, a Clifford algebra is not oriented in the graded sense. As a counterexample, we may choose $A = C^{0,1}$. Then the permutation of the two copies of A in $A \otimes A = C^{0,2}$ is given by the inner automorphism associated to $e_1 + e_2$ which is of degree 1 and not of degree 0, as required in our definition. However, if V is a module which is oriented and of even rank, the associated graded Clifford algebra $C(V)$ is oriented, as we shall see below.

To start with, let V and V' be two quadratic modules such that V is of even rank and oriented. The argument used in Section 1 shows the existence of an element v in $C^0(V) \otimes C^0(V') \subset C(V)\widehat{\otimes}C(V') \cong C(V \oplus V')$ which anticommutes with the elements of V and commutes with the elements of V' : one puts $v = u \otimes 1$ with the notations of Section 1 (see Remark 1.7). Moreover, $(v)^2 = 1$ and $v \in \mathrm{Spin}(V \oplus V')$.

Let us choose $V' = V$ and put $T = V \oplus V$. Since 2 is invertible in R, T is isomorphic to the orthogonal sum $T_1 \oplus T_2$, where $T_1 = \{v, -v\}$ and $T_2 = \{v, v\}$. Thanks to this isomorphism, the permutation of the two summands of $V \oplus V$ is translated into the involution $(t_1, t_2) \mapsto (-t_1, t_2)$ on $T_1 \oplus T_2$. Therefore, the previous argument shows the existence of a canonical element $u_{12} \in \mathrm{Spin}(V \oplus V)$ of square 1 such that the transformation

$$x \mapsto u_{12}^{-1}.x.u_{12}$$

permutes the two summands of $V \oplus V$. It follows immediately that the Clifford algebra $C(V)$ is oriented (in the graded sense) if V is oriented and of even rank.

From the previous general considerations, we deduce a natural representation of the symmetric group S_k in the group $\mathbf{Spin}(V^k)$ which lifts the canonical representation of S_k in $\mathbf{SO}(V^k)$. To sum up, we have proved the following theorem:

Theorem 5.1. *Let V be a quadratic module of even rank which is oriented and let*

$$\sigma_k : S_k \longrightarrow \mathbf{SO}(V^k)$$

be the standard representation. Then there is a canonical lifting

$$\widetilde{\sigma}_k : S_k \longrightarrow \mathbf{Spin}(V^k),$$

such that the following diagram commutes

$$\begin{array}{ccc} & & \mathbf{Spin}(V^k) \\ & \overset{\widetilde{\sigma}_k,}{\nearrow} & \downarrow \pi \\ S_k & \xrightarrow{\sigma_k} & \mathbf{SO}(V^k) \end{array} .$$

In other words, the Clifford algebra $C(V)$ is a $\mathbf{Z}/2$-graded oriented Azumaya algebra.

Remark 5.2. One can also make an explicit computation in the Clifford algebra $C(V^k)$ with the obvious elements $\tau_i = u_{i,i+1}.$; one checks they satisfy the required relations for the generators of the symmetric group S_k. Moreover, these liftings for various $k's$ are of course compatible with each other. If R is an integral domain, we note that $\widetilde{\sigma}_k$ is unique, once $\widetilde{\sigma}_2$ is given.

6. Adams operations revisited

In this section we assume that V is a quadratic R-module which is oriented and of even rank, so that the Clifford algebra is a $\mathbf{Z}/2$-graded oriented Azumaya algebra.

If $k!$ is invertible in R we have defined Adams operations in a functorial way:

$$\Psi^k : K(C(V)) \longrightarrow K(C(V(k))).$$

The purpose of this section is to define similar operation $\overline{\Psi}^k$ under another type of hypothesis: $2k$ is invertible in R and R contains the ring of integers in the k-cyclotomic extension of $\mathbb{Q}$ which is

$$\Omega_k = \mathbf{Z}(\omega) = \mathbf{Z}\left[x\right]/(\Phi_k(x)).$$

Here $\Phi_k(x)$ is the cyclotomic polynomial and ω is the class of x. We conjecture that $\overline{\Psi}^k = \Psi^k$ (defined more generally via Newton polynomials from the λ-operations) but we are not able to prove it, unless $k!$ is invertible in R as we have assumed. We also want to extend these operations Ψ^k and $\overline{\Psi}^k$ to $\mathbf{Z}/2$-graded oriented Azumaya algebras (not only Clifford algebras) which were defined in the previous section.

The idea to define $\overline{\Psi}^k$ is a remark by Atiyah [1, Formula 2.7] (used already in Section 3) that Adams operations may be defined using the cyclic group $\mathbf{Z}/k$ instead of the symmetric group S_k (if $k!$ is invertible in R). More precisely, the Adams operation Ψ^k is induced by the homomorphism $R(S_k) \longrightarrow \mathbf{Z}$ which associates to a representation σ its character on the cycle $(1, 2..., k)$. Therefore, if we put $F = E^{\otimes k}$, we see that

$$\Psi^k(E) = \sum_{j=0}^{k-1} F_{\omega^j} \cdot \omega^j,$$

where ω is a primitive k^{th}-root of unity and where F_{ω^j} is the eigenmodule corresponding to the eigenvalue ω^j. The previous sum belongs in fact to the subgroup $K(C(V(k))$ of $K(C(V(k)) \otimes_{\mathbf{Z}} \Omega_k$.

If we only assume that k is invertible in R, we can consider the previous sum as a new operation. More precisely, we define

$$\overline{\Psi}^k(E) = \sum_{j=0}^{k-1} F_{\omega^j} \cdot \omega^j.$$

In this new setting, this sum belongs to the group $K(C(V(k))) \otimes_{\mathbf{Z}} \Omega_k$ and not necessarily to the subgroup $K(C(V(k)))$. This definition makes sense since we have assumed k invertible in R, so that F splits as the direct sum of the eigenmodules associated to the eigenvalues ω^j, where $0 \leqslant j \leqslant k-1$. Since we work in the $\mathbf{Z}/2$-graded case, we also have to assume that 2 is invertible in R.

If k is prime, because of the underlying action of the symmetric group on F, the eigenmodules F_{ω^j} are isomorphic to each other when $1 \leqslant j \leqslant k-1$, so that this definition of $\overline{\Psi}^k(E)$ reduces to $F_0 - F_\omega$.We may be more precise and choose as a model of the symmetric group S_k the group of permutations of the set $\mathbf{Z}/k$. One generator T of the cyclic group $\mathbf{Z}/k$ is the permutation $x \mapsto x+1$. If α is a generator of the multiplicative cyclic group $(\mathbf{Z}/k)^\times$, the permutation $x \longmapsto \alpha^s x$, where s runs from 1 to $k-2$, enables us to identify all the eigenmodules $F_{\omega^j}, j = 2, .., k-1$ with F_ω. We therefore get the following theorem:

Theorem 6.1. *Let E be a graded $C(V)$-module and let us assume that $2k$ is invertible in R and that the k^{th}-roots of unity belong to R. We define $\overline{\Psi}^k(E)$ in the group $K(C(V(k))) \otimes_{\mathbf{Z}} \Omega_k$ by the following formula*

$$\overline{\Psi}^k(E) = \sum_{j=0}^{k-1} F_{\omega^j} \cdot \omega^j.$$

If E_0 and E_1 are two such modules, we have

$$\overline{\Psi}^k(E_0 \otimes E_1) = \overline{\Psi}^k(E_0) \cdot \overline{\Psi}^k(E_1)$$

in the Grothendieck groups $K(C(V(2k))) \otimes_{\mathbf{Z}} \Omega_k$. Moreover, if k is prime, $\overline{\Psi}^k(E)$ belongs to $K(C(V(k))) \subset K(C(V(k))) \otimes_{\mathbf{Z}} \Omega_k$ and we have the following formula in $K(C(V(k)$:

$$\overline{\Psi}^k(E_0 \oplus E_1) = \overline{\Psi}^k(E_0) + \overline{\Psi}^k(E_1).$$

Finally, the operation $\overline{\Psi}^k$ coincides with the usual Adams operation Ψ^k if $k!$ is invertible in R.

Proof. When k is prime, we have the isomorphism

$$(E_0 \oplus E_1)^{\otimes k} \cong (E_0)^{\otimes k} \oplus (E_1)^{\otimes k} \oplus \Gamma,$$

where Γ is a module of type $(H)^k$ with an action of S_k permuting the factors of $(H)^k$. From elementary algebra, we see that Γ is not contributing to the computation of $\overline{\Psi}^k(E_0 \oplus E_1)$, hence the second formula.

For the first formula, we compute $\overline{\Psi}^k(E_0 \otimes E_1)$ by looking formally at the eigenmodules of $T \otimes T$ acting on $(E_0)^{\otimes k} \otimes (E_1)^{\otimes k}$, considered as a module over $C(V(k)) \otimes C(V(k))$. They are of course associated to the eigenvalues $\omega^i \otimes \omega^j = \omega^{i+j}$. Using the remark above, we can write

$$\begin{aligned}\overline{\Psi}^k(E_0 \otimes E_1) &= \sum_{r=0}^{k-1} \left[(E_0 \otimes E_1)^{\otimes k}\right]_r \cdot \omega^r \\ &= \sum_{r=0}^{k-1} \sum_{i+j=r} \left[(E_0)^{\otimes k}\right]_i \cdot \omega^i \cdot \left[(E_1)^{\otimes k}\right]_j \cdot \omega^j = \overline{\Psi}^k(E_0) \cdot \overline{\Psi}^k(E_1).\end{aligned}$$

Finally, for $k!$ invertible in R, the fact that $\overline{\Psi}^k = \Psi^k$ is just the remark made by Atiyah [1] quoted above. □

Let now A be any $\mathbf{Z}/2$-graded Azumaya algebra which is oriented. We would like to define operations on the K-theory of A of the following type

$$K(A) \longrightarrow K(A^{\otimes k}).$$

For this, we again follow the scheme defined by Atiyah [1, Formula 2.7] (if $k!$ is invertible in R). The only point which requires some care is the definition of the "power map"

$$\mathcal{P} : K(A) \longrightarrow K_{S_k}(A^{\otimes k}) = K(A^{\otimes k}) \otimes R(S_k).$$

A priori, the target of this map is the K-group of the cross-product algebra $S_k \ltimes A^{\otimes k}$. However, as we have seen in the previous section, the representation of S_k in $\mathrm{Aut}(A^{\otimes k})$ lifts as a homomorphism from S_k to $(A^{\otimes k})^{\times}$. Therefore, this cross product algebra is the tensor product of the group algebra $\mathbf{Z}\,[S_k]$ with $A^{\otimes k}$.

Therefore, any homomorphism

$$\lambda : R(S_k) \longrightarrow \mathbf{Z}$$

gives rise to an operation

$$\lambda_* : K(A) \longrightarrow K(A^{\otimes k}),$$

as we showed in Section 3. However, one has to be careful that this operation depends on the orientation chosen on A, i.e. on the lifting of the representation $\sigma_k : S_k \longrightarrow \mathrm{Aut}(A^{\otimes k})$ to a representation $\widetilde{\sigma}_k : S_k \longrightarrow (A^{\otimes k})^{\times}$, in such a way that the diagram

$$\begin{array}{ccc} & & (A^{\otimes k})^{\times} \\ & \overset{\widetilde{\sigma}_k}{\nearrow} & \downarrow \\ S_k & \xrightarrow{\sigma_k} & \mathrm{Aut}(A^{\otimes k}) \end{array}$$

commutes. If R is an integral domain, this lifting is defined up to the sign representation. However, in this case, we get a canonical choice of $\widetilde{\sigma}_k$ as follows. Let F be the quotient ring of R and $\overline{F}$ its algebraic closure. If we extend the scalar to $\overline{F}$, A becomes a matrix algebra $\mathrm{End}(E)$ over $\overline{F}$, in which case $(A^{\otimes k})^{\times}$ is identified with $\mathrm{Aut}(E^k)$. We then choose the sign of the lifting $\widetilde{\sigma}_k$ in such a way that it corresponds to the canonical lifting $S_k \longrightarrow \mathrm{Aut}(E^k)$ by extension of the scalars.

Let us be more explicit and define the k^{th}-exterior power $\lambda^k(M)$ of M as an $A^{\otimes k}$-module in our setting. We take the quotient of $M^{\otimes k}$ by the usual relations (where the m_i are homogeneous elements):

$$m_{s(1)} \otimes m_{s(2)} \otimes ... \otimes m_{s(k)} = \varepsilon(s)\widetilde{\sigma}_k(s)\deg(m_s) m_1 \otimes m_2 \otimes ... \otimes m_k.$$

Here $\varepsilon(s)$ is the signature of the permutation s, $\widetilde{\sigma}_k$ the lifting defined above and $\deg(m_s)$ the signature of the representation s restricted to elements of odd degree. We note that $\lambda^k(M)$ is a graded module over $A^{\otimes k}$.

Example. Let $A = \mathrm{End}(E)$ with the trivial grading and let $\widetilde{\sigma}_k$ be the canonical lifting. Then, by Morita equivalence, all left A-modules M may be written as $E \otimes N$, where N is an R-module. It is then easy to see that $\lambda^k(M) \cong E^{\otimes k} \otimes \lambda^k(N)$, where $\lambda^k(N)$ is the usual k^{th}-exterior power over the commutative ring $R, E^{\otimes k}$ being viewed as a module over $A^{\otimes k} \cong \mathrm{End}(E^{\otimes k})$. We note that if we change the sign of the orientation, we get the symmetric power $E^{\otimes k} \otimes S^k(N)$ instead of the exterior power.

It is convenient to consider the full exterior algebra $\Lambda(M)$ of M which is the direct sum of all the $\lambda^k(M)$. As usual, $\Lambda(M)$ is the solution of a universal problem. If $g : M \longrightarrow C$ is an R-module map where C is an R-algebra and if

$$g(m_{s(1)})g(m_{s(2)}), ...g(m_{s(k)}) = \varepsilon(s)\widetilde{\sigma}_k(s)\deg(m_s)s(m_1)s(m_2)...s(m_k),$$

there is an algebra map $\Lambda(M) \longrightarrow C$ which makes the obvious diagram commutative. If M is a finitely generated projective A-module, $\lambda^k(M)$ as a finitely generated projective $A^{\otimes k}$-module: this is a consequence of the following theorem.

Theorem 6.2. *Let A be an oriented $\mathbf{Z}/2$-graded Azumaya algebra and let M and N be two finitely generated projective A-modules Then the exterior algebra of $M \oplus N$ is canonically isomorphic to $\Lambda(M) \otimes_R \Lambda(N)$. Moreover, in each degree k, we get an isomorphism of $A^{\otimes k}$-modules.*

Proof. The canonical map from $M \oplus N$ to $\Lambda(M) \otimes_R \Lambda(N)$ induces the usual isomorphism

$$\Lambda(M \oplus N) \longrightarrow \Lambda(M) \otimes_R \Lambda(N).$$

In each degree k, this map induces an isomorphism between $\lambda^k(M \oplus N)$ and the sum of the

$$\lambda^i(M) \otimes_R \lambda^{k-i}(N),$$

$0 \leqslant i \leqslant k$, viewed as $A^{\otimes k}$-modules. □

Following Grothendieck and Atiyah again, we define λ-operations on K-groups:

$$\lambda^k : K(A) \longrightarrow K(A^{\otimes k})$$

satisfying the usual identity

$$\lambda^r(M \oplus N) = \sum_{k+l=r} \lambda^k(M) \cdot \lambda^l(N)$$

as $A^{\otimes(k+l)}$-modules. We can also define the Adams operations by the usual formalism.

We may view operations in this type of K-theory as compositions

$$K(A) \xrightarrow{\mathcal{P}} K(A^{\otimes k}) \otimes_{\mathbf{Z}} R(S_k) \xrightarrow{\theta} K(A^{\otimes k}).$$

Here $\mathcal{P}$ is the power map defined through the lifting $\widetilde{\sigma}_k$ above. The second map θ is induced by an homomorphism $R(S_k) \longrightarrow \mathbf{Z}$. In particular, the Adams operation Ψ^k is given by the homomorphism

$$R(S_k) \longrightarrow \mathbf{Z}$$

which associates to a representation ρ the trace of the cycle action $(1, 2, ..., k)$.

Remark 6.3. A careful analysis of these considerations shows that we don't need $k!$ to be invertible in order to define the λ-operations in the non graded case. However, we need 2 to be invertible in the graded case and, moreover, $k!$ invertible in order to define the Adams operations with good formal properties.

Another approach to the Adams operations, as we showed at the beginning of this section, only assumes that $2k$ is invertible in R and that R contains the k^{th}-roots of unity. If E is a finitely generated projective A-module, the tensor power $E^{\otimes k}$ is an $S_k \ltimes A^{\otimes k}$ -module. We can "untwist" the two actions of S_k and $A^{\otimes k}$, thanks to the orientation of A and we end up with an $A^{\otimes k}$-module F, with an independant action of S_k. We put formally

$$\overline{\Psi}^k(E) = \sum_{j=0}^{k-1} F_j \cdot \omega^j$$

where F_j is the eigenmodule associated to the eigenvalue ω^j. The previous sum lies in $K(A^{\otimes k}) \otimes_{\mathbf{Z}} \Omega_k$ and even in the subgroup $K(A^{\otimes k})$ if k is prime. This second definition is very pleasant, since the formal properties of the Adams operations can be checked easily with this formula (at least for k prime). We conjecture that $\Psi^k = \overline{\Psi}^k$ in this case too.

7. Twisted hermitian Bott classes

We are going to define more subtle operations, associated not only to the K-theory of A but also to the K-theory of $A \otimes B$, where $A = C(V)$ and $B = C(W)$ are two Clifford algebras. We no longer assume that V and W are of even rank or oriented. However, we assume k odd, $2k$ invertible in R and that the k^{th}-roots of unity belong to R. We also replace the symmetric group S_k by the cyclic group $\mathbf{Z}/k$ in our previous arguments. The reason for this change is the following remark. The natural representation $\sigma_k : \mathbf{Z}/k \longrightarrow \mathbf{O}(V^k)$ has its image in the subgroup $\mathbf{SO}^0(V^k)$ defined in Section 1 and lifts uniquely to a representation of $\mathbf{Z}/k$ in $\mathbf{Spin}(V^k)$, so that the following diagram commutes:

$$\begin{array}{ccc} & & \mathbf{Spin}(V^k) \\ & \nearrow & \downarrow \\ \mathbf{Z}/k & \longrightarrow & \mathbf{SO}^0(V^k) \end{array} .$$

This lifting does not exist in general for the symmetric group S_k, except if V is even dimensional and oriented, as we have seen in Section 5.

Let now M be a finitely generated projective module over $A \otimes B = C(V) \otimes C(W) = C(V \oplus W)$. We can compose the power map

$$K(A \otimes B) \longrightarrow K(\mathbf{Z}/k \ltimes (A \otimes B)^{\otimes k}) \cong K(\mathbf{Z}/k \ltimes C(V^k \oplus W^k))$$

with the "half-diagonal"

$$K(\mathbf{Z}/k \ltimes C(V^k \oplus W^k)) \longrightarrow K(\mathbf{Z}/k \ltimes C(V(k) \oplus W^k))$$
$$\cong K(C(V(k)) \otimes (\mathbf{Z}/k \ltimes C(W^k))),$$

as we did in Section 2 for $W = 0$. From the considerations in Section 6, we can "untwist" the action of $\mathbf{Z}/k$ on the $\mathbf{Z}/2$-graded Azumaya algebra $C(W^k)$, so that $\mathbf{Z}/k \ltimes C(W^k)$ is isomorphic to the usual group algebra $\mathbf{Z}\,[\mathbf{Z}/k] \otimes C(W^k)$. Using the methods of Section 2 and of the previous section, we get a more precise power map:

$$K(C(V) \otimes C(W)) \longrightarrow K(C(V(k)) \otimes C(W^k)) \otimes R(\mathbf{Z}/k).$$

Therefore, according to Atiyah again [1], any homomorphism

$$\lambda : R(\mathbf{Z}/k) \longrightarrow \Omega_k$$

gives rise to a "twisted operation"

$$\lambda_* : K(C(V) \otimes C(W)) \longrightarrow K(C(V(k)) \otimes C(W^k)) \otimes \Omega_k.$$

We apply this formalism to $W = V(-1)$, in which case $C(W)$ is the (graded) opposite algebra of $C(V)$. Therefore, $K(C(V) \otimes C(W)) \cong K(R)$ by Morita equivalence. If we choose for λ the map above, we define the "twisted hermitian Bott class" as the image of 1 by the composition

$$K(R) \cong K(C(V) \otimes C(W)) \longrightarrow K(C(V(k)) \otimes C(W^k)) \otimes R(\mathbf{Z}/k)$$
$$\longrightarrow K(C(V(k)) \otimes C(W^k) \otimes \Omega_k.$$

We have proved the following theorem:

Theorem 7.1. *Let V be an arbitrary quadratic module and $W = V(-1)$. The "twisted hermitian Bott class" $\rho_k(V)$ belongs to the following group*

$$\rho_k(V) \in K(C(V(k)) \otimes C(W^k)) \otimes_{\mathbf{Z}} \Omega_k.$$

It satisfies the multiplicative property

$$\rho_k(V_1 \oplus V_2) = \rho_k(V_1) \cdot \rho_k(V_2),$$

taking into account the following algebra identifications:

$$C((V_1 \oplus V_2)(k)) \otimes C(W_1^k \oplus W_2^k)$$
$$\cong \left[C(V_1(k)) \otimes C(W_1^k))\right] \otimes \left[C(V_2(k)) \otimes C(W_2^k)\right].$$

.

Remark 7.2. Let (V, E) be a spin module and let us identify the four $\mathbf{Z}/2$-graded algebras $C(V)$, $\mathrm{End}(E)$, $C(V(k))$ and $C(W)$. Then, by Morita equivalence, we see that the twisted hermitian Bott class coincides (non canonically) with the untwisted one.

Remark 7.3. It is easy to show that V^4 is an orientable quadratic module which implies by 1.7 that the Clifford algebra $C(V^4)$ is isomorphic to its opposite. Let now k be an odd square which implies that $k \equiv 1 \bmod 8$. Since $C(V(k)) \cong C(V)$ and $C(W^k)$ are Morita equivalent to $C(W)$, the target group of the twisted hermitian Bott class is isomorphic to

$$K(C(V) \otimes C(W)) \otimes_{\mathbf{Z}} \Omega_k \cong K(R) \otimes_{\mathbf{Z}} \Omega_k.$$

This shows that we have a commutative diagram up to isomorphism

$$\begin{array}{ccc} K\mathrm{Spin}(R) & \longrightarrow & KQ(R) \\ \downarrow & & \downarrow \\ K(R)\,[1/k]^{\times} & \longrightarrow & [K(R) \otimes_{\mathbf{Z}} \Omega_k]\,[1/k]^{\times} \end{array},$$

where the vertical maps are defined by hermitian Bott classes, twisted and untwisted.

Finally, as we did in Section 3, we can "correct" the twisted hermitian Bott class by using the result of Serre about the square root of the "modified" Bott class (Section 9). More precisely, if V is a self-dual module of dimension n, there is an explicit class $\sigma_k(V)$ in $K(R) \otimes_{\mathbf{Z}} \Omega_k$ which only depends on the exterior powers of V, such that

$$\sigma_k(V)^2 = \delta^{(k-1)/2} \rho^k(V),$$

with $\delta = (-1)^n \lambda^n(V)$. The corrected twisted hermitian Bott class is then defined by the formula

$$\overline{\rho}_k(V) = \rho_k(V)(\sigma_k(V))^{-1},$$

taking into account the fact that $K(C(V(k)) \otimes C(W^k))$ is a module over the ring $K(R)$. We have $\overline{\rho}_k(V) \in \pm(\mathrm{Pic}(R))^{(k-1)/2}$ if V is hyperbolic[8], as we showed in Section 3. Therefore, the previous formula for $\overline{\rho}_k$ defines a morphism also called $\overline{\rho}_k$, between the classical Witt group $W(R)$ and twisted K-theory modulo $\pm(\mathrm{Pic}(R))^{(k-1)/2}$(as a multiplicative group), more precisely

$$\overline{\rho}_k : W(R) \longrightarrow \left[K(C(V(k)) \otimes C(W^k)) \otimes_{\mathbf{Z}} \Omega_k\right] [1/k] /^{\times} \pm (\mathrm{Pic}(R))^{(k-1)/2}.$$

Remark 7.4. If $k \equiv 1 \bmod 4$, we can multiply $\sigma_k(V)$ by the sign $(-1)^{n(k-1)4}$, as we did in Section 3. If we apply this sign change, the new corrected twisted hermitian Bott class takes its values in the group

$$\left[K(C(V(k)) \otimes C(W^k)) \otimes_{\mathbf{Z}} \Omega_k\right] [1/k] /^{\times} / (\mathrm{Pic}(R))^{(k-1)/2},$$

without any sign ambiguity.

[8]The sign ambiguity is unavoidable, since V is of arbitrary dimension.

8. Some lemmas about the Brauer-Wall group

The purpose of this section is to prove the following lemma and its applications which are also found in [4, Proposition 5.3 and Corollary 5.4] for the non graded case. They are added to this paper for completeness' sake with Z/2-graded variants.

Lemma 8.1. *Let R be a commutative ring. Let A be an R-algebra which is projective, finitely generated and faithful as an R-module. Let P and Q be faithful projective finitely generated R-modules such that*

$$A \otimes \mathrm{End}(P) \cong \mathrm{End}(Q).$$

Then A is isomorphic to some End(E), where E is also faithful, projective and finitely generated. The same statement is true for $\mathbf{Z}/2$-graded algebras and modules if 2 is invertible in R.

In order to prove the lemma, we need another one:

Lemma 8.2. *Let P be a faithful finitely generated projective R-module. Then there exists an R-module Q such that $F = P \otimes Q$ is free. Moreover, if P is $\mathbf{Z}/2$-graded and if 2 is invertible in R, we may choose Q such that $F = R^{2m} = R^m \oplus R^m$, with the obvious grading.*

Proof. (compare with [10, pg. 14] and [5, Corollary 16.2]). Since any module P of this type is locally the image of a projection operator p of rank $r > 0$, we can look at the "universal example". This universal ring R is generated by variables p_i^j where $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant n$ such that the matrix $p = (p_i^j)$ is idempotent of trace r. According to [5, p. 39], since R is of finite stable range, the element $y = [P] - [r]$ is nilpotent in the Grothendieck group $K(R)$, say $y^N = 0$ for some N. Let us now consider the element

$$x = r^{N-1} - r^{N-2}y + ... + (-1)^{N-1}y^{N-1}.$$

We have the identity $(r + y)Mx = M(r^N - (-1)^{N-1}y^N) = Mr^N$. Since the rank of x is $r^{N-1} > 0$ and since the stable range of R is finite, the element Mx in $K(R)$ is the class of a module Q for sufficiently large M. If follows that $P \otimes Q$ is stably free and therefore free if M is again large enough. Finally, the case of $\mathbf{Z}/2$-graded modules follows by the same argument, considering graded R-modules as $R\,[\mathbf{Z}/2]$-modules. □

Proof. (of the first lemma). Let us first consider the non graded case. Without restriction of generality, we may assume that A is of constant rank and that P and Q are also of constant rank such that $A \otimes \mathrm{End}(P)$ is isomorphic to End(Q). According to the previous lemma, we may also assume that P is free of constant rank, say n. Therefore, we have an algebra isomorphism

$$A \otimes M_n(R) \cong \mathrm{End}(Q).$$

Let us now consider the fundamental idempotents in the matrix algebra $M_n(R)$ defined by the diagonal matrices with all elements $= 0$ except one which is 1. Thanks to the previous isomorphism, we may use these idempotents to split Q as the direct sum of n copies of E. Since the commutant of $M_n(R)$ in $A \otimes M_n(R)$ is A, it follows that the representation of A in $\mathrm{End}(Q)$ is the orthogonal sum of n copies of a representation ρ from A to $\mathrm{End}(E)$. From the previous algebra isomorphism, we therefore deduce the required identity

$$A \cong \mathrm{End}(E).$$

Finally, we make the obvious modifications of the previous argument in the $\mathbf{Z}/2$-graded case by writing the previous algebra isomorphism in the form

$$A \widehat{\otimes} M_{2n}(R) \cong \mathrm{End}(Q),$$

with the obvious grading on $M_{2n}(R)$. We use again the fundamental idempotents in $M_{2n}(R)$ in order to split Q as a direct sum E^n, where E is $\mathbf{Z}/2$-graded. □

9. Appendix. Letter from Jean-Pierre Serre to Max Karoubi (2 July 2007)

Paris, le 2 juillet 2007

Cher Karoubi,

Voici quelques détails sur la “racine carrée de Bott”. Comme je te l'ai dit, je trouve commode de travailler avec les λ-anneaux universels associés à la situation. Cela a plusieurs avantages :

1) On n'a pas à se demander si on s'intéresse (comme toi[9]) à la K-théorie topologique[10], ou (comme moi) à la catégorie des représentations d'un groupe (ou d'un groupe algébrique).

2) On obtient un résultat plus fort : existence d'une racine carrée ne faisant intervenir que les puissances extérieures.

3) Les anneaux considérés ont une structure simple; en particulier, ils sont libres sur $\mathbf{Z}$. Cela servira par la suite.

Je commence donc par parler de ces anneaux.

[9]The English footnotes are written by Max Karoubi.

[10]In 2007, Karoubi proved that the modified Bott class (§ 9.4) in equivariant topological K-theory has a square root in K-theory. The goal of Serre's letter is to prove this kind of result in a more general context which happens to be useful for our purpose.

9.1. **L'anneau R_n.** C'est celui que tout le monde connaît. On se donne un entier $n > 0$ et l'on s'intéresse à un élément v d'un λ-anneau qui est "moralement" de degré n, i.e. tel que $\lambda^j v = 0$ pour $j > n$ et l'élément $d = \lambda^n v$ est inversible. L'anneau universel pour cette situation est l'anneau R_n des polynômes en n indéterminées $s_1, ..., s_n$, localisé par rapport à s_n. Une façon un peu incorrecte[11] de le noter est

$$R_n = \mathbf{Z}[s_1, ..., s_n, s_n^{-1}].$$

On a $v = s_1$, $d = s_n$, $\lambda^j v = s_j$ pour $0 < j \leqslant n$ et $\lambda^j v = 0$ pour $j > n$.

Interprétation concrète de R_n : c'est l'anneau $R(\mathbf{GL}_n)$ des représentations linéaires (algébriques) du groupe $\mathbf{GL}_n$ sur un corps k (ou même sur $\mathbf{Z}$, cf. Publ. Math. IHES, 34 (1968), p. 52). Variante topologique : l'anneau des représentations complexes continues du groupe unitaire $\mathbf{U}_n$.

Le "splitting principle" consiste à voir R_n comme sous-anneau de l'anneau

$$C_n = \mathbf{Z}[x_1, ..., x_n, (x_1...x_n)^{-1}]$$

des polynômes de Laurent en $x_1, ..., x_n$; en particulier, on a $v = x_1 + ... + x_n$: on a "splitté" v. Le groupe symétrique S_n opère sur C_n et R_n est le sous-anneau de C_n fixé par S_n. Interprétation concrète de C_n : c'est l'anneau $R(\mathbf{T}^n)$ des représentations du tore $\mathbf{T}^n$, produit de n copies du groupe multiplicatif (ou du cercle, si tu préfères l'aspect "groupes compacts"). Bien sûr, le fait que $\mathbf{T}^n$ soit un tore maximal de $\mathbf{GL}_n$ explique le plongement de R_n dans C_n.

Il y a une involution $x \mapsto x^*$ sur R_n et C_n qui correspond à la *dualité* des fibrés vectoriels (ou des représentations linéaires). Sur C_n, elle est caractérisée par $x_i^* = x_i^{-1}$; sur R_n, elle est caractérisée par $d^* = d^{-1}$ et $s_i^* = d^{-1}s_{n-i}$. On peut voir cette involution comme l'opération d'Adams Ψ^{-1}.

9.2. **L'anneau R'_n.** Je note R'_n le quotient de R_n par l'idéal engendré par les éléments de la forme $x - x^*$, avec $x \in R_n$. On peut le voir comme engendré sur $\mathbf{Z}$ par les s_i avec les relations

$$d^2 = 1, \quad s_i = ds_{n-i} \ ,$$

où $d = s_n$ comme plus haut.

La structure de cet anneau est moins évidente que celle de R_n et je ne suis pas sûr qu'on la trouve explicitement dans la littérature (c'est pourtant R'_n qui intervient vraiment dans ton théorème !).

Les cas $n = 1$ et $n = 2$ sont faciles à décrire directement :

1) Pour $n = 1$, R'_n est $\mathbf{Z}$-libre de base $(1, d)$ avec la relation $d^2 = 1$ (et bien sûr $v = d$). On peut le voir comme le sous-anneau de $\mathbf{Z} \times \mathbf{Z}$ formé

[11] More pedantically, one should introduce an extra variable d_n and the extra relation $s_n d_n = 1$. Serre's way to write the ring R_n is of course more suggestive.

des couples (a, b) tels que $a \equiv b \pmod 2$. Dans cette description, on a $v = d = (1, -1)$. On peut aussi voir cet anneau comme l'anneau $R(G)$ des représentations du groupe orthogonal à une variable, le groupe $\mathbf{O}_1 = \mu_2$.

2) Pour $n = 2$, on a $v = s_1, d = s_2$ avec les relations $d^2 = 1, dv = v$. Une $\mathbf{Z}$-base de R'_2 est $\{1, d, v, v^2, v^3, ...\}$. Cet anneau est l'anneau $R(G)$ où $G = \mathbf{O}_2$. C'est le sous-anneau d'indice 2 de $\mathbf{Z}[t] \times \mathbf{Z}$ formé des couples (f, b) tels que $f(0) \equiv b \pmod 2$.

Pour n quelconque, R'_n s'identifie à $R(\mathbf{O}_n)$. On peut l'expliciter comme ceci[12]:

1) Pour n impair, écrivons n sous la forme $2m + 1$. Alors

$$R'_n = \mathbf{Z}[s_1, ..., s_m] \otimes \mathbf{Z}[d]/(d^2 - 1),$$

$v = s_1$, et les λ-opérations sont données par

$$\lambda^j v = s_j \quad (0 < j \leqslant m),$$

$$\lambda^j v = ds_{n-j} \quad (m < j < n),$$

$$\lambda^n v = d.$$

2) Pour n pair, écrivons n sous la forme $2m + 2$. Alors

$$R'_n = \mathbf{Z}[s_1, ..., s_m] \otimes \mathbf{Z}[s_{m+1}, d]/(d^2 - 1, ds_{m+1} - s_{m+1}),$$

où le deuxième facteur est soumis aux mêmes relations que plus haut pour $n = 2$, à savoir $d^2 = 1$ et $ds_{m+1} = s_{m+1}$.

Note que les anneaux R'_n ne sont pas intègres, mais sont des sous-anneaux d'un produit de *deux* anneaux intègres. C'est raisonnable: le groupe orthogonal $\mathbf{O}_n$ a *deux* composantes connexes.

9.3. **Quelques notations.** Comme toi, je note k un entier impair > 0, mais je ne suppose pas qu'il soit sans facteur carré ; en fait le cas où k est un carré est intéressant.

Je note z une racine primitive k-ième de l'unité (dans $\mathbf{C}$ pour fixer les idées). Soit $A = \mathbf{Z}[z]$; on a $[A : \mathbf{Z}] = \varphi(k)$. Soit G le groupe de Galois de $\mathbf{Q}(z)$ sur $\mathbf{Q}$. Il s'identifie au groupe multiplicatif $(\mathbf{Z}/k\mathbf{Z})^\times$; si j est un élément de $(\mathbf{Z}/k\mathbf{Z})^\times$, je note σ_j l'élément correspondant de G. Il est caractérisé par $\sigma_j(z) = z^j$.

Soit $k' = \pm k$, le signe étant choisi pour que $k' \equiv 1 \pmod 4$; autrement dit $k' = (-1)^{(k-1)/2}k$. Il est bien connu que $\mathbf{Q}(\sqrt{k'})$ est contenu dans $\mathbf{Q}(z)$: je noterai A' l'anneau des entiers de $\mathbf{Q}(\sqrt{k'})$; on a $A' = \mathbf{Z}$ si k' est un carré. Le groupe $H = \mathrm{Gal}(\mathbf{Q}(z)/\mathbf{Q}(\sqrt{k'}))$ est le noyau d'un caractère

$$\chi : G \longrightarrow \{1, -1\}$$

[12]Référence pour l'isomorphisme $R'_n \simeq R(\mathbf{O}_n)$: T. Bröcker & T. tom Dieck, *Representations of Compact Lie Groups*, Springer-Verlag, New York, 1985, Chap. VI.7, 7.2 (n impair) et 7.7 (n pair). [Note ajoutée le 15 septembre 2010.]

qui est d'ordre 2 si k n'est pas un carré. On peut voir χ comme un caractère de Dirichlet

$$(\mathbf{Z}/k\mathbf{Z})^\times \longrightarrow \{1, -1\}\,.$$

En voici deux descriptions concrètes :

(a) Soit P l'ensemble des nombres premiers p tels que $v_p(k) \equiv 1 \pmod 2$. Alors χ est le produit des caractères de Legendre relatifs aux p appartenant à P : on a

$$\chi(j) = \prod_{p\in P} (\frac{j}{p}).$$

(b) Soit $S = \{1, ..., (k-1)/2\}$. Si j est premier à k et si $s \in S$, on a soit $js \in S \pmod k$, soit $js \in -S \pmod k$. Notons $r(j)$ le nombre de s du second type. Alors $\chi(j) = (-1)^{r(j)}$.

Le fait que ces deux descriptions coïncident n'est pas tout à fait évident mais n'est pas difficile[13].

9.4. **La racine carrée de l'élément de Bott.** Je note[14] $\mathrm{Bott}(v)$ l'élément de R_n' obtenu à partir de v et du polynôme[15]

$$F(t) = 1 + t + ... + t^{k-1}.$$

En utilisant la factorisation

$$F(t) = \prod_{\substack{i \bmod k \\ i\neq 0}} (1 - z^i t),$$

on voit que l'on peut factoriser $\mathrm{Bott}(v)$ en

$$\mathrm{Bott}(v) = \prod_{\substack{i \bmod k \\ i\neq 0}} P(-z^i),$$

où $P(t) = \lambda_t(v) = 1 + s_1 t + ... + s_n t^n$.

L'élément de Bott modifié comme tu l'as indiqué[16] est

$$\mathrm{Bott}'(v) = \delta^{(k-1)/2} \cdot \mathrm{Bott}(v),$$

avec $\delta = (-1)^n d$ (note que $\delta^2 = 1$).

Soit $u = z^{(1-k)/2}$; on a $u^2 = z$. Si i est un entier non divisible par k, posons

$$b(i) = u^{-ni} \cdot P(-z^i).$$

[13]If k is prime, this is found for instance in [19, pg. 19].

[14]The element $\mathrm{Bott}(v)$ may as well be defined in the ring R_n.

[15]Recall from [11, §1] that given a polynomial $F(t)$, there is a standard construction which associates to an element x of degree n of a λ-ring, written formally as $\sum_1^n x_i$, the element $F(x) = \prod F(x_i)$, expressed as a polynomial in the elementary symmetric function of the x_i, as summarized in 9.1.

[16]See footnote Nr. 10.

Le fait que v soit self-dual[17] entraîne que $t^n \cdot P(1/t) = d \cdot P(t)$. En appliquant ceci à $t = -z^i$, on en déduit par un petit calcul que $b(-i) = \delta \cdot b(i)$.

Soit $S = \{1, 2, ..., (k-1)/2\}$. Posons

$$c = \prod_{i \in S} b(i).$$

Le produit analogue, pris sur $-S$, est égal à $c.\delta^{(k-1)/2}$. En regroupant tout ceci, on obtient

$$\text{Bott}'(v) = \delta^{(k-1)/2} \cdot \prod_{i \in S} b(i) \cdot \prod_{i \in -S} b(i) = \delta^{k-1} \cdot c^2 = c^2.$$

Ainsi, l'élément c est une racine carrée de $\text{Bott}'(v)$ dans $A \otimes R'_n$ Pour voir où se trouve cette racine carrée, il suffit de regarder comment elle est transformée par le groupe de Galois G de $\mathbf{Q}(z)/\mathbf{Q}$. Le résultat est très simple :

Théorème. *Si $\sigma \in G$, écrivons $\chi(\sigma)$ comme $(-1)^{r(\sigma)}$ avec $r(\sigma) \in \mathbf{Z}/2\mathbf{Z}$. On a alors $\sigma(c) = \delta^{r(\sigma)} \cdot c$.*

Corollaire. *L'élément c appartient à $A' \otimes R'_n$ (et même à R'_n si k est un carré).*

(C'est ce que tu voulais. Note que le théorème est plus précis que son corollaire : il dit comment la racine carrée c se comporte lorsqu'on change $\sqrt{k'}$ en $-\sqrt{k'}$.)

Démonstration du théorème. On écrit σ sous la forme σ_j, avec $j \in (\mathbf{Z}/k\mathbf{Z})^\times$. On écrit l'ensemble $\mathbf{Z}/k\mathbf{Z} - \{0\}$ comme $S \sqcup -S$ avec S comme ci-dessus. L'élément $\sigma(c)$ est égal au produit des $b(i)$ pour $i \in jS$. Lorsque $i \in S$ est tel que ji appartienne à $-S$, cela introduit un facteur δ. On en conclut que $\sigma(c)$ est égal au produit de c par $\delta^{n(j)}$, où $n(j)$ est le nombre des $i \in S$ tels que $ij \in -S$. D'où le résultat.

Bien à toi,

J.-P. Serre

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[17] which means that v belongs to R'_n.

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