# HERMITIAN PERIODICITY AND COHOMOLOGY OF INFINITE ORTHOGONAL GROUPS

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ABSTRACT. As an application of our papers [2], [4], in favourable cases we prove the periodicity of hermitian $K$-groups with a shorter period than previously obtained. We also compute the homology and cohomology with field coefficients of infinite orthogonal and symplectic groups of specific rings of integers in a number field.

## 1. INTRODUCTION

We provide applications of our previous papers [2] and [4] in two directions. Firstly, we prove a refinement of our periodicity theorem proved in [4] which leads to a shorter period for hermitian $K$-groups of specific rings $A$. This is the case if $A$ is an algebra over the real subfield $R$ of a cyclotomic field. The computation of the hermitian $K$-theory of $R$ is detailed in [2] and enables us to define explicit "Bott elements" for $R$. An important subcase is when $A$ is an $F$-algebra, where $F$ is algebraically closed. This leads to the classical 8-periodicity of the hermitian $K$-groups of $A$: see Theorem 2.2.

The second application is the computation of the cohomology and homology with fields coefficients of the infinite orthogonal and symplectic groups associated to specific rings of 2-integers in a number field. This computation is quite explicit and relies partly on computations made in [6] for finite coefficients. For rational coefficients, these results are particular cases of those of Borel [5].

## 2. REFINEMENTS OF THE PERIODICITY THEOREM IN HERMITIAN $K$-THEORY

In this section we establish a more refined periodicity theorem for a class of rings introduced in [4], and there called "hermitian regular". As seen in [3], this class includes many rings (and more generally schemes) of geometric nature. First, we have a lemma that clarifies the definition in the setting of $K$-theoretic Bott periodicity. For this, we use the "positive Bott element" in ${}_1KQ_p(\mathbb{Z}'; \mathbb{Z}/2p)$ constructed in [4, Theorem 1.1] for a 2-power $p \geq 8$.

**Lemma 2.1.** *Let $A$ be a ring with involution such that $1/2 \in A$, and let $m = 2^\nu$ and $p = \sup\big(8, 2^{\nu-1}\big)$. We assume the existence of an integer $d$, such that the cup-product with the Bott element in $K_p(\mathbb{Z}; \mathbb{Z}/m)$ induces an isomorphism*

$$K_n(A; \mathbb{Z}/m) \xrightarrow{\cong} K_{n+p}(A; \mathbb{Z}/m).$$

*for $n \geq d$. Then the following are equivalent.*

(i) *There exists an integer $n$ such that all iterated cup-products with the positive Bott element in ${}_1KQ_p(\mathbb{Z}'; \mathbb{Z}/2p)$ induce isomorphisms*

$$K_n(A; \mathbb{Z}/m) \xrightarrow{\cong} K_{n+ps}(A; \mathbb{Z}/m)$$

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*and*

$$K_{n+1}(A;\mathbb{Z}/m)\xrightarrow{\cong}K_{n+1+ps}(A;\mathbb{Z}/m).$$

(ii) *$A$ is* hermitian regular*, as in Definition 0.5 of* [4]*.*

(iii) *Whenever $n \geq d$, cup-product with the positive Bott element in ${}_1KQ_p(\mathbb{Z}';\mathbb{Z}/2p)$ induces an isomorphism*

$$ {}_\varepsilon KQ_n(A;\mathbb{Z}/m)\xrightarrow{\cong}{}_\varepsilon KQ_{n+p}(A;\mathbb{Z}/m).$$

**Proof.** Since (i) is evidently a special case of (iii), and it is shown in [4, Theorems 0.7 and 4.2, Example 4.3] that (ii) is equivalent to (iii), it remains to show that (i) implies (iii). Without loss of generality, we may assume that $n \geq d$.

This follows by the argument of "downward Karoubi induction" as in [1, Theorem 3.1(b)], applied to the commuting diagrams of exact sequences (notation obviously abbreviated)

$$\begin{array}{ccccccc} {}_{-\varepsilon}KQ_{n+1} & \to & {}_{-\varepsilon}U_n & \to & K_n & & {}_{-\varepsilon}KQ_n \\ \downarrow^{\cup b^+} & & \downarrow^{\cup b^+} & & \downarrow^{\cup b_K=\cup b^+} & & \downarrow^{\cup b^+} \\ {}_{-\varepsilon}KQ_{n+1+p} & \to & {}_{-\varepsilon}U_{n+p} & \to & K_{n+p} & & {}_{-\varepsilon}KQ_{n+p} \end{array}$$

and

$$\begin{array}{ccccccc} K_n & \longrightarrow & {}_\varepsilon V_{n-1} & \longrightarrow & {}_\varepsilon KQ_{n-1} & \longrightarrow & K_{n-1} \\ \downarrow^{\cup b_K} & & \downarrow^{\cup b^+} & & \downarrow^{\cup b^+} & & \downarrow^{\cup b_K} \\ K_{n+p} & \longrightarrow & {}_\varepsilon V_{n-1+p} & \longrightarrow & {}_\varepsilon KQ_{n-1+p} & \longrightarrow & K_{n-1+p} \end{array}$$

that arise from cup-products with Bott elements. □

The most interesting result in this section is the following generalization of the theorem proved in [8]. It is a consequence of more general results which are proved in this paper.

**Theorem 2.2.** *Let $F$ be an algebraically closed field of characteristic $\neq 2$ with the trivial involution, and let $A$ be an involutive $F$-algebra which is hermitian regular as above. Suppose that, with $m$ prime to the characteristic of $F$, for $n \geq d$ the classical periodicity map*

$$K_n(A;\mathbb{Z}/m)\longrightarrow K_{n+2}(A;\mathbb{Z}/m)$$

*is an isomorphism. Then, for $n \geq d$ the hermitian $K$-groups ${}_\varepsilon KQ_n(A;\mathbb{Z}/m)$ are periodic of period 8 with respect to $n$, the isomorphism being given by the cup-product with a "Bott element" in ${}_1KQ_8(F;\mathbb{Z}/m)$.*

In order to prove this theorem and more general ones below, we combine our general periodicity theorems [4, Theorem 0.7] with the more detailed information in [4, Theorem 2.6] for totally real 2-regular number fields, so as to obtain sharper periodicity results for certain algebras $A$ with involution.

Writing $\mathbb{Z}' = \mathbb{Z}[1/2]$, for $\alpha > 2$ define $R = R^+_\alpha$ to be the ring $\mathbb{Z}'[\zeta_{2^\alpha}+\bar{\zeta}_{2^\alpha}]$ of 2-integers in the maximal real subfield $F = \mathbb{Q}(\zeta_{2^\alpha}+\bar{\zeta}_{2^\alpha})$ of the cyclotomic field $\mathbb{Q}(\zeta_{2^\alpha})$, provided with the trivial involution. Then, from e.g. [2], $F$ is a totally real 2-regular number field to which Proposition 2.3 of [4] applies, as follows. Let $\nu \geq 4$, $p = 2^{\nu-1}$ and $M = 2^{\nu+\alpha-2} = 2p\cdot 2^{a-2}$. Then the image of the Bott element in ${}_1KQ_p(R;\mathbb{Z}/2p)$ by the canonical map

$$ {}_1KQ_p(\mathbb{Z}';\mathbb{Z}/2p)\longrightarrow {}_1KQ_p(R;\mathbb{Z}/2p)$$

is the reduction mod $2p$ of an "exotic Bott element" $\bar{b}\in {}_1KQ_p(R;\mathbb{Z}/M)$.

Theorem 2.4 of [4] now takes the following form. The theorem stated afterwards is a direct consequence.

**Lemma 2.3.** [4] *For $n \geq 0$, the cup-product with $\overline{b}$ in ${}_1KQ_p(R; \mathbb{Z}/M)$ induces an isomorphism*

$$\overline{\beta} : {}_\varepsilon KQ_n(R; \mathbb{Z}/M') \xrightarrow{\cong} {}_\varepsilon KQ_{n+p}(R; \mathbb{Z}/M')$$

*whenever $M'$ divides $M$.* □

**Theorem 2.4.** *Let $A$ be an $R$-algebra with an involution that is trivial*[1] *on $R$, and suppose that $A$ is hermitian regular. Let $\nu \geq 4$ and $p = 2^{\nu-1}$, and assume the existence of an integer $d$, such that, for $n \geq d$, the cup-product with the Bott element in $K_p(\mathbb{Z}; \mathbb{Z}/2p)$ induces an isomorphism*

$$K_n(A; \mathbb{Z}/m) \xrightarrow{\cong} K_{n+p}(A; \mathbb{Z}/m)$$

*whenever $m$ divides $2p$. Then for $n \geq d$, the cup-product with the exotic Bott element $\overline{b}$ defined above induces an isomorphism*

$${}_\varepsilon KQ_n(A; \mathbb{Z}/M') \xrightarrow{\cong} {}_\varepsilon KQ_{n+p}(A; \mathbb{Z}/M')$$

*whenever $M'$ divides $2^{\nu+\alpha-2} = 2p \cdot 2^{a-2}$.*

**Proof.** By using cofinality in direct systems as in [4, §4] and applying the lemma above, we observe that the group $\varinjlim {}_\varepsilon KQ_{n+sp}(A; \mathbb{Z}/M')$ may be computed by taking cup-product either with the usual Bott element in ${}_1KQ_p(\mathbb{Z}; \mathbb{Z}/2p)$ or with an exotic Bott element in ${}_1KQ_p(\mathbb{Z}; \mathbb{Z}/(2p \cdot 2^{a-2}))$. Therefore, the theorem is a consequence of Theorem 4.5 in [4, Section 4]. □

When $\ell$ is an odd prime the periodicity statement is quite different as we shall see, in contrast with the situation in algebraic $K$-theory (see also [11] and [3] for very similar arguments). We consider the algebra $R_\alpha = \mathbb{Z}[\zeta_{\ell^\alpha}]$ and the group $K_2(R_\alpha; \mathbb{Z}/\ell^\alpha), \alpha > 2$. From the exact sequence

$$K_2(R_\alpha) \to K_2(R_\alpha) \to K_2(R_\alpha; \mathbb{Z}/\ell^\alpha) \to K_1(R_\alpha) \to K_1(R_\alpha)$$

we can define a "Bott element" $u$ in the group $K_2(R_\alpha; \mathbb{Z}/\ell^\alpha)$ which maps to a generator of the kernel of the map between the $K_1$-groups. Now let $\sigma$ be the involution on the $K$-groups induced by the duality. Then $c = u \cdot \sigma(u)$ in $K_4(R_\alpha; \mathbb{Z}/\ell^\alpha)$ is invariant by $\sigma$, and therefore belongs to the group[2]

$${}_1KQ_4(R_\alpha; \mathbb{Z}/\ell^\alpha)_+ \cong {}_1K_4(R_\alpha; \mathbb{Z}/\ell^\alpha)_+ \cong K_4(R_\alpha^+; \mathbb{Z}/\ell^\alpha)_+ \cong {}_1KQ_4(R_\alpha^+; \mathbb{Z}/\ell^\alpha)_+,$$

where the last isomorphism is a transfer map between $K$-groups and $KQ$-groups. Moreover, if $\nu > 0$, a classical Bockstein argument shows that $c^{\ell^{\nu-1}}$ may also be lifted to an "exotic Bott element" $b_+$ in the group ${}_1KQ_{4\ell^{\nu-1}}(R_\alpha^+; \mathbb{Z}/\ell^{\alpha+\nu-1})_+$. On the other hand, as in Section 5 of [4], we may define a "negative" Bott element $b_-$ in ${}_1KQ_{4\ell^{\nu-1}}(R_a^+; \mathbb{Z}/\ell^{\alpha+\nu-1})_-$ that is the reduction mod $\ell^{\alpha+\nu-1}$ of an integral class in ${}_1KQ_{4\ell^{\nu-1}}(R_a^+)$, as detailed in [7, p. 278]. Thus, following the spirit of [4, §5] we define a "mixed Bott element"

$$b_+ + b_- \in {}_1KQ_{4\ell^{\nu-1}}(R_\alpha^+; \mathbb{Z}/\ell^{\alpha+\nu-1}).$$

The following theorem is a consequence of the analogous one in $K$-theory and the periodicity theorem for the higher Witt groups. More precisely, Theorem 4.3, p. 278 in [7] implies that the higher Witt groups are 4-periodic mod 2-torsion. On the other hand, the symmetric part of hermitian $K$-theory is isomorphic to the symmetric part of $K$-theory modulo 2-torsion: see [4, § 5] for more details.

[1] If $\sigma$ is the involution, this precisely means that $\sigma(\lambda a) = \lambda\sigma(a)$ for $\lambda \in R$ and $a \in A$.

[2] In general, we indicate by $G_+$ the invariant part of an abelian group $G$ (with 2 invertible) by an involution. We also indicate by $G_-$ its anti-invariant part. Finally, we note that the groups $KQ_r(\Lambda; \mathbb{Z}/\ell^\alpha)_-$ are periodic of period 4 according to the periodicity theorem proved in [7].

**Theorem 2.5.** *With the previous notations, let $A$ be an $R^+_\alpha$-algebra, $\alpha > 2$, with an involution that is trivial on $R^+_\alpha$. Writing $p = 2(\ell-1)\ell^{\nu-1}$, $\nu > 0$, we assume the existence of an integer $d$, such that, for $n \geq d$, the cup-product with the Bott element in $K_p(\mathbb{Z};\mathbb{Z}/m)$ induces an isomorphism*

$$K_n(A;\mathbb{Z}/m) \xrightarrow{\cong} K_{n+p}(A;\mathbb{Z}/m)$$

*whenever $m$ divides $\ell^\nu$. Then, for $n \geq d$ and $p' = 4\ell^{\nu-1}$, the cup-product with the mixed Bott element $b_+ + b_-$ defined above induces an isomorphism*

$${}_\varepsilon KQ_n(A;\mathbb{Z}/m') \xrightarrow{\cong} {}_\varepsilon KQ_{n+p'}(A;\mathbb{Z}/m')$$

*whenever $m'$ divides $\ell^{\alpha+\nu-1}$. In particular, if $A$ is an $R^+_\alpha$-algebra for all $\alpha$, we may choose $\nu = 1$ so that the groups ${}_\varepsilon KQ_n(A;\mathbb{Z}/m')$ are periodic of period 4 with respect to $n$ for all powers $m'$ of $\ell$.*

We may now combine this theorem (for $\ell$ odd) with the previous one for 2-primary coefficients to prove Theorem 2.2. It is a generalization of the theorem proved in [8] when $A = F$ (see below).

**Proof of Theorem 2.2.** We remark that $F$ contains all the rings $R^+_\alpha$ considered before. We now decompose the integer $m$ into primary powers. According to the previous theorem, when $m$ is odd the groups ${}_\varepsilon KQ_n(A;\mathbb{Z}/m)$ are periodic of period 4 with respect to $n$. For 2-primary powers, we have 8-periodicity according to Theorem 2.4 (choose $\nu = 4$). Note that the "exotic" Bott element in ${}_1KQ_8(F;\mathbb{Z}/m)$ was already defined in [8]. □

## 3. Cohomology of orthogonal and symplectic groups of rings of 2-integers in 2-regular totally real number fields

Let $A$ be a commutative ring and let $\epsilon = \pm 1$. The group ${}_\varepsilon O_{n,n}(A) \subseteq GL_{2n}(A)$ is the subgroup of automorphisms of $A^n \oplus A^n$ that leave invariant the $\epsilon$-quadratic form

$$\begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix}.$$

Passing to the colimit by componentwise inclusion, we obtain the group ${}_\varepsilon O(A)$ and its classifying space $B_\varepsilon O(A)$. The group cohomology of ${}_\varepsilon O(A)$ coincides with the singular cohomology of $B_\varepsilon O(A)$, and therefore with the singular cohomology of Quillen's $+$-construction $B_\varepsilon O(A)^+$. For a number field $F$ with ring of 2-integers $R_F$, the rational group cohomology of ${}_\varepsilon O(R_F)$ was computed by Borel [5] almost forty years ago. When $F$ is a totally real 2-regular number field, our results in [2] identify the homotopy type of $B_\varepsilon O(R_F)^+$ in terms of classical topological invariants. In the following, we use these results to compute explicitly the group cohomology of ${}_\varepsilon O(R_F)$ with $\mathbb{F}_2$-coefficients.

In this section, all displayed spectra are implicitly 2-completed and connective. For example, we let ${}_\varepsilon\mathcal{KQ}(R_F)$ denote the 2-completion of the connective cover of the hermitian $K$-theory spectrum of $R_F$. Let $r$ denote the number of real embeddings of the number field $F$. We refer to [2] for the choice of the residue field $\mathbb{F}_q$ of $R_F$.

We recall from [2] the homotopy cartesian square:

$$\begin{array}{ccc} {}_\varepsilon\mathcal{KQ}(R_F) & \longrightarrow & {}_\varepsilon\mathcal{KQ}(\mathbb{R})^r \\ \downarrow & & \downarrow \\ {}_\varepsilon\mathcal{KQ}(\mathbb{F}_q) & \longrightarrow & {}_\varepsilon\mathcal{KQ}(\mathbb{C})^r \end{array}$$

When $\varepsilon = 1$, ${}_1\mathcal{KQ}(\mathbb{R}) \simeq \mathcal{K}(\mathbb{R}) \vee \mathcal{K}(\mathbb{R})$ and ${}_1\mathcal{KQ}(\mathbb{C}) \simeq \mathcal{K}(\mathbb{R})$, where we consider the real and complex numbers with their usual topologies. The map from ${}_1\mathcal{KQ}(\mathbb{R})$

to ${}_1\mathcal{KQ}(\mathbb{C})$ is induced by the Whitney sum of real vector bundles. Since it has a splitting [2, Appendix B], we deduce the following theorem.

**Theorem 3.1.** *There is a homotopy equivalence of* 2*-completed connective spectra*

$$ {}_1\mathcal{KQ}(R_F) \simeq {}_1\mathcal{KQ}(\mathbb{F}_q) \vee \mathcal{K}(\mathbb{R})^r. $$

By considering the underlying infinite loop spaces of the spectra in Theorem 3.1 we obtain the following group cohomology computation.

**Corollary 3.2.** *Let $H^*$ denote cohomology with $\mathbb{F}_2$-coefficients. Then there is an isomorphism of Hopf algebras and modules over the mod* 2 *Steenrod algebra*

$$ H^*(O(R_F)) \cong H^*(O(\mathbb{F}_q)) \otimes H^*(BO)^{\otimes r}. $$

*Here $H^*(BO) \cong \mathbb{F}_2[w_1, w_2, \cdots]$ is a polynomial algebra generated by the Stiefel-Whitney classes $w_i$, $i \geq 1$, and $H^*(O(\mathbb{F}_q))$ is a polynomial algebra on generators $x_i$, $\overline{x}_{2i-1}$, $i \geq 1$.*

The cohomology of the classifying space $BO$ is computed in *e.g.* [12, Corollary 16.11], and that of $O(\mathbb{F}_q)$ in [6, IV, Corollary 4.3].

When $\varepsilon = -1$, ${}_{-1}\mathcal{KQ}(\mathbb{R}) \simeq \mathcal{K}(\mathbb{C})$ and ${}_{-1}\mathcal{KQ}(\mathbb{C}) \simeq \mathcal{K}(\mathbb{H})$, where we consider the quaternions with the usual topology. Hence, there is a homotopy cartesian square:

$$ \begin{array}{ccc} {}_{-1}\mathcal{KQ}(R_F) & \longrightarrow & \mathcal{K}(\mathbb{C})^r \\ \downarrow & & \downarrow \\ {}_{-1}\mathcal{KQ}(\mathbb{F}_q) & \longrightarrow & \mathcal{K}(\mathbb{H})^r \end{array} $$

We note that there is a naturally induced isomorphism

$$ {}_{-1}KQ_0(\mathbb{R}) \longrightarrow {}_{-1}KQ_0(\mathbb{C}) \cong \mathbb{Z} $$

corresponding to the (even) rank of the free symplectic $A$-inner product space [10, p.7]. Thus, by considering the underlying infinite loop spaces of these spectra, we find the homotopy cartesian square:

$$ \begin{array}{ccc} B\mathrm{Sp}(R_F)^+ & \longrightarrow & BU^r \\ \downarrow & & \downarrow \\ B\mathrm{Sp}(\mathbb{F}_q)^+ & \longrightarrow & B\mathrm{Sp}^r \end{array} $$

We note that the classifying space $B\mathrm{Sp}$ of the sympletic group is simply-connected. Moreover, see *e.g.* [12, Corollary 16.11], $H^*(BU)$ is a polynomial algebra generated by Chern classes $c_i$, $i \geq 1$, and $H^*(B\mathrm{Sp})$ is the subring generated by $p_i = c_{2i}$. Thus $H^*(BU)$ is a free module over the subalgebra $H^*(B\mathrm{Sp})$. It follows that the Eilenberg-Moore spectral sequence in cohomology [9, §7,8]

$$ Tor_\star^{H^*(B\mathrm{Sp})^r}(H^*(B\mathrm{Sp}(\mathbb{F}_q), H^*(BU^r)) \Longrightarrow H^*(B\mathrm{Sp}(R_F)) $$

collapses to its zeroth column, and we conclude the following result.

**Theorem 3.3.** *Let $H^*$ denote cohomology with $\mathbb{F}_2$-coefficients. Then there is an isomorphism of Hopf algebras and modules over the mod* 2 *Steenrod algebra*

$$ H^*(BU)^{\otimes r} \otimes_{H^*(B\mathrm{Sp})^{\otimes r}} H^*(\mathrm{Sp}(\mathbb{F}_q)) \xrightarrow{\cong} H^*(\mathrm{Sp}(R_F)). $$

The cohomology of $\mathrm{Sp}(\mathbb{F}_q)$ is the tensor product of a polynomial algebra on generators $g_i$, $i \geq 1$, and an exterior algebra on generators $h_j$, $j \geq 1$ [6, IV, §6].

Dually, we may compute the *homology* of $B\mathrm{Sp}(R_F)$ by using the cotensor product $\square$ instead of the usual tensor product over $\mathbb{F}_2$. This gives an isomorphism

$$ H_*(\mathrm{Sp}(R_F)) \xrightarrow{\cong} H_*(BU^r)\square_{H_*(B\mathrm{Sp}^r)} H_*(\mathrm{Sp}(\mathbb{F}_q)). $$

In particular, there is a naturally induced injective map

$$ H_*(\mathrm{Sp}(R_F)) \rightarrowtail H_*(BU)^r \otimes H_*(\mathrm{Sp}(\mathbb{F}_q)). $$

The $\mathbb{F}_2$-homology of $\mathrm{Sp}(\mathbb{F}_q)$ is the tensor product of a polynomial algebra on generators $\sigma_i$, $i \geq 0$, and an exterior algebra on generators $\tau_j$, $j \geq 1$ [6, IV, §5].

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