ALGEBRAIC AND HERMITIAN K-THEORY OF $\mathcal{K}$-RINGS

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ABSTRACT. The main purpose of the present article is to establish the real case of "Karoubi's Conjecture" [20] in algebraic K-theory. The complex case was proved in 1990-91 (cf. [32] for the C^*-algebraic form of the conjecture, and [38] for both the Banach and C^*-algebraic forms). Compared to the case of complex algebras, the real case poses additional difficulties. This is due to the fact that topological K-theory of real Banach algebras has period 8 instead of 2. The method we employ to overcome these difficulties can also be used for complex algebras, and provides some simplifications to the original proofs. We also establish a natural analog of Karoubi's Conjecture for Hermitian K-theory.

INTRODUCTION

Let A be a complex C^*-algebra and $\mathcal{K} = \mathcal{K}(H)$ be the ideal of compact operators on a separable complex Hilbert space H. One of us conjectured around 1977 that the natural comparison map between the algebraic and topological K-groups

$$\epsilon_n\colon K_n(\mathcal{K}\bar{\otimes}A) \longrightarrow K_n^{\mathrm{top}}(\mathcal{K}\bar{\otimes}A) \simeq K_n^{\mathrm{top}}(A)$$

is an isomorphism for a suitably completed tensor product $\bar{\otimes}$. The conjecture was announced in [20] where accidentally only $\mathcal{K}\widehat{\otimes}_\pi A$ was mentioned, and it was proved there for $n \leq 0$. In [23] the conjecture was established for $n \leq 2$ and $\mathcal{K}\bar{\otimes}A$ having the meaning of the C^*-algebra completion of the algebraic tensor product (in view of *nuclearity* of $\mathcal{K}$, there is only one such completion).

The C^*-algebraic form of the conjecture was established for all $n \in \mathbf{Z}$ in 1990 [31], [32]. A year later the conjecture was proved also for all $n \in \mathbf{Z}$ and $\mathcal{K}\widehat{\otimes}_\pi A$ where A was only assumed to be a Banach $\mathbf{C}$-algebra with one-sided bounded approximate identity [38].

Here we prove analogous theorems for K-theory of real Banach and C^*-algebras, and then deduce similar results in Hermitian K-theory.

The article is organized as follows. In Chapter 1 we set the stage by introducing the concept of a $\mathcal{K}$-*ring*, which is a slight generalization and a modification to what was called a "stable" algebra in [23]. We also introduce a novel notion of a *stable retract*. Then we proceed to demonstrate that the comparison map between algebraic and topological K-groups in degrees $n \leq 0$ is an isomorphism for Banach algebras that are stable retracts of $\mathcal{K}\otimes_{\max}A$ for some C^*-algebra A, or of $\mathcal{K}\widehat{\otimes}_\pi A$, for some Banach algebra A (Theorem 1.1).

We recall how to endow $K_*(\mathcal{K})$ with a canonical structure of a $\mathbf{Z}$-graded, associative, graded commutative, and unital ring in Chapter 2. If a $\mathcal{K}$-ring is H-unital as a $\mathbf{Q}$-algebra, we equip its $\mathbf{Z}$-graded algebraic K-groups with a structure of a $\mathbf{Z}$-graded unitary $K_*(\mathcal{K})$-module. We distinguish two cases: $\mathcal{K}_{\mathbf{R}}$-rings and

1991 *Mathematics Subject Classification.* 19K99.
Key words and phrases. operator ideals, algebraic K-theory, Bott periodicity.
Mariusz Wodzicki was partially supported by the NSF grant 1001846.

$\mathcal{K}_{\mathbf{C}}$-rings, where $\mathcal{K}_{\mathbf{R}}$ stands for the ring of compact operators on a real Hilbert space, and $\mathcal{K}_{\mathbf{C}}$ — on the complex Hilbert space.

These structures are used to prove several results, two of them: 2-periodicity of algebraic K-groups for $\mathcal{K}_{\mathbf{C}}$-rings (Theorem 2.2) and a comparison theorem for Banach $\mathcal{K}_{\mathbf{C}}$-rings (Theorem 2.3), are derived in Chapter 2. Both results were known before [38], [32].

Relying on a rather delicate argument employing K-theory with coefficients mod 16 and certain classical results of stable homotopy theory, we detect in Chapter 3 the existence of an element $v_8 \in K_8(\mathcal{K}_{\mathbf{R}})$ which maps onto a generator of $K_8^{\mathrm{top}}(\mathcal{K}_{\mathbf{R}})$. We showed that $K_{-8}(\mathcal{K}_{\mathbf{R}}) \simeq \mathbf{Z}$ in Chapter 1. The element v_8 is the multiplicative inverse of a generator of that group. This allows us to establish 8-periodicity of algebraic K-groups for arbitrary $\mathcal{K}_{\mathbf{R}}$-rings (Theorem 3.4), and to prove that the comparison map is an isomorphism for Banach $\mathcal{K}_{\mathbf{R}}$-rings ("the real case of Karoubi's Conjecture"). Both results are new.

The next chapter is devoted to Hermitian K-theory. The goal is to deduce comparison theorems in Hermitain K-theory from the corresponding results in algebraic K-theory (for complex C^*-algebras this was partially done in [6]). The primary tool in this task is a pair of *Comparison Induction Theorems* which are among the consequences of the "Fundamental Theorem of Hermitian K-theory" [22]. In the same chapter we also provide a rather thorough discussion of Hermitian K-groups for nonunital rings, we compare two approaches to defining relative Hermitian K-groups and, in the end, we deduce excision properties in Hermitian K-theory from the corresponding properties in algebraic K-theory, cf. [5], [6]. This is achieved with a different kind of Induction Theorems.

In two appendices that follow we collect material frequently used throughout the article. We review some facts about multiplicative structures in algebraic K-theory in Appendix B, indicating difficulties of the nonunital case. Appendix B provides a well known characterization of pure-exact extensions of Banach spaces, and a much less known criterion of pure-exactness for extensions of Banach modules associated with ideals in Banach algebras. Complete proofs are given for the reader's convenience.

The present article was written so that the presentation would be accessible to experts in theory of Banach and operator algebras. We hope to attract their attention to the possibilities offered by the interplay between real structures, involutive algebras, and K-theory.

We would like to thank Thierry Fack who kindly agreed to write an appendix where he supplies a short proof of exactness of the maximal tensor product on the category of C^*-algebras—the result we could not find a satisfactory reference to in the existing literature.

Aknowledgments. We thank C. Anantharaman, T. Fack, A. Guichardet and S. Wassermann for discussions about C^*-algebra tensor products, and J. A. Neisendorfer for providing an up to date information on multiplicative structures in homotopy groups with finite coefficients.

1. Natural transformations $\epsilon_n \colon K_n \to K_n^{\mathrm{top}}$ for $n \leq 0$

For $n < 0$, the algebraic and the topological K-groups both can be defined in terms of K_0 using the notions of the cone and suspension functor suitable for rings and, respectively, for Banach algebras. The compatibility of the two constructions leads to a sequence of natural transformations of functors $\epsilon_n \colon K_n \to K_n^{\mathrm{top}}$ and the associated connecting homomorphisms, which means that any extension in the category of real or complex Banach algebras,

$$A \overset{\pi}{\twoheadleftarrow} B \overset{\iota}{\hookleftarrow} C \quad ,$$

induces a morphism of half-infinite exact sequences

(1.1)
$$\begin{tikzcd}
\cdots & K_n(B) \arrow[l, "\pi_n"'] \arrow[d, "\epsilon_n"'] & K_n(C) \arrow[l, "\iota_n"'] \arrow[d, "\epsilon_n"'] & K_{n+1}(A) \arrow[l, "\partial_{n+1}"'] \arrow[d, "\epsilon_{n+1}"'] & K_{n+1}(B) \arrow[l, "\pi_{n+1}"'] \arrow[d, "\epsilon_{n+1}"'] & \cdots \arrow[l, "\pi_{n+1}"'] \\
\cdots & K_n^{\mathrm{top}}(B) \arrow[l, "\pi_n^{\mathrm{top}}"'] & K_n^{\mathrm{top}}(C) \arrow[l, "\iota_n^{\mathrm{top}}"'] & K_{n+1}^{\mathrm{top}}(A) \arrow[l, "\partial_{n+1}^{\mathrm{top}}"'] & K_{n+1}^{\mathrm{top}}(B) \arrow[l, "\pi_{n+1}^{\mathrm{top}}"'] & \cdots \arrow[l, "\pi_{n+1}^{\mathrm{top}}"']
\end{tikzcd} ,$$

where $n < 0$. Below, we will refer to ϵ_n as the *comparison maps*.

When the comparison map

$$\epsilon_n \colon K_n(A) \longrightarrow K_n^{\mathrm{top}}(A)$$

is an isomorphism, we say that the Banach algebra A is *K_n-stable*.

We say that a Banach algebra R is a *stable retract* of a Banach algebra A if, for some $n \geq 1$, the *stabilization homomorphism*

$$\iota_n \colon R \hookrightarrow M_n(R), \qquad r \longmapsto \left(\begin{array}{c:c} r & 0_{1,n-1} \\ \hdashline 0_{n-1,1} & 0_{n-1,n-1} \end{array}\right) \tag{1.2}$$

factorizes through A,

$$\begin{tikzcd}
R \arrow[rr, rightarrowtail, "\iota_n"] \arrow[dr, dashed, "\kappa"'] & & M_n(R) \\
& A \arrow[ur, dashed, "\nu"'] &
\end{tikzcd} ,$$

with κ and ν being Banach algebra homomorphisms. Note that *to be a stable retract of* is a transitive relation: if A is a stable retract of B, then R is also a stable retract of B. A stable rectract of a K_n-stable algebra is automatically K_n-stable.

Let us consider the C^*-algebra $\mathcal{B}(H)$ of bounded linear operators on a Hilbert space H. The space itself can be *real* or *complex*. We will denote by F the corresponding field of coefficients, which is $\mathbf{R}$ in the former case, and $\mathbf{C}$ in the latter.

In the case when H is infinite dimensional $\mathcal{B}(H)$ is an example of what Farrell and Wagoner call an *infinite sum ring* [13, p. 477]. For an infinite sum ring R, its general linear group $GL(R)$ is acyclic [33, Corollary 2.5] which means that $BGL(R)^+$ is contractible. Since the property of being an infinite sum ring is preserved under suspensions, $K_n(R \otimes_k A) = 0$ for all $n \in \mathbf{Z}$. If R is an infinite sum k-algebra, which means that the endomorphism

$$r \longmapsto r^\infty \qquad (r \in R), \tag{1.3}$$

the key ingredient of the structure of an infinite sum ring, is k-linear, then $R \otimes_k A$ is again an infinite sum k-algebra and, accordingly, $K_n(R \otimes_k A) = 0$ for all unital k-algebras and arbitrary $n \in \mathbf{Z}$.

If R is an infinite sum Banach algebra, which means that endomorphism (1.3) is continuous, then so is the completed projective tensor product $R \widehat{\otimes}_\pi B$ with any unital Banach algebra B, and both algebraic and topological K-groups of $K_n(R \widehat{\otimes}_\pi B)$ vanish in all degrees. Finally, when R is an infinite sum C^*-algebra, which means that (1.3) is a C^*-endomorphism, then so is the tensor product $R\otimes_{\max}C$ with any unital C^*-algebra C (the maximal tensor product can be replaced by any functorial C^*-product).

The algebra of bounded linear operators on an infinite dimensional Hilbert space is an infinite sum C^*-algebra. In the course of this work we shall repeatedly use the fact that

$$K_*(\mathcal{B}(H)\otimes A) = K_*(\mathcal{B}(H)\widehat{\otimes}_\pi B) = K_*(\mathcal{B}(H)\otimes_{\max}C) = 0 \tag{1.4}$$

and

$$K_*^{\mathrm{top}}(\mathcal{B}(H)\widehat{\otimes}_\pi B) = K_*^{\mathrm{top}}(\mathcal{B}(H)\otimes_{\max}C) = 0 \tag{1.5}$$

for any unital F-algebra A, Banach algebra B, and C^*-algebra C.

Vanishing of algebraic and topological K-groups of $\mathcal{B}(H)$ and various tensor products with unital algebras can also be established using the language of *flabby categories* [16], and has been a part of the K-theoretical lore for more than four decades.

When H is infinite dimensional and separable, then $\mathcal{K} = \mathcal{K}(H)$ is a unique proper, nonzero and closed ideal in $\mathcal{B} = \mathcal{B}(H)$; it contains every proper nonzero ideal in $\mathcal{B}$ as a dense subset and the quotient $\mathcal{B}/K$ is called the *Calkin algebra*. Below, H will always denote an infinite-dimensional separable Hilbert space.

For any closed subspace $H' \subseteq H$, there is an associated nonunital embedding of C^*-algebras

$$\iota^{H'H}\colon \mathcal{B}(H') \hookrightarrow \mathcal{B}(H), \qquad a \longmapsto i^{H'H}\circ a\circ p^{HH'},$$

where $p^{HH'}\colon H \to H'$ denotes the projection onto H' with kernel $(H')^\perp$ and $i^{H'H}$ denotes the inclusion of H' into H. It sends $\mathcal{K}(H')$ to $\mathcal{K}(H)$. When H' is finite dimensional, then

$$\mathcal{K}(H') = \mathcal{B}(H') = \mathrm{End}_F H'.$$

In the one dimensional case we obtain a nonunital algebra homomorphism

$$\iota\colon F = \mathrm{End}_F H' \hookrightarrow \mathcal{K}(H), \qquad 1 \longmapsto i^{H'H}\circ p^{HH'}. \tag{1.6}$$

Let us consider the following 2-parameter family of C^*-algebras

$$F_{lm} := \mathcal{K}^{\otimes_{\max} l}\otimes_{\max}(\mathcal{B}/\mathcal{K})^{\otimes_{\max} m} \qquad (l,m\in\mathbf{N}),$$

and, given a C^*-algebra A, the associated 2-parameter family

$$A_{lm} := F_{lm}\otimes_{\max}A,$$

where the *maximal* tensor product is formed in the category of C^*-algebras over F. By tensoring the extension of C^*-algebras

$$\mathcal{B}/\mathcal{K} \twoheadleftarrow \mathcal{B} \longleftarrow\!\prec \mathcal{K} \quad , \tag{1.7}$$

with A_{lm}, we obtain the extension

$$A_{l,m+1} \twoheadleftarrow \mathcal{B}\otimes_{\max}A_{lm} \longleftarrow\!\prec A_{l+1,m} \quad . \tag{1.8}$$

Exactness of (1.8) is, at least in the complex case, a well known feature of the maximal tensor product of C^*-algebras, see [12].

In view of (1.4)–(1.5), commutative diagram (1.1), whose rows are formed by the long exact sequences associated with extension (1.8), splits into a sequence of commutative squares with connecting homomorphisms being isomorphisms

$$\begin{tikzcd} K_n(A_{l+1,m}) \arrow[d, "\epsilon_n^{l+1,m}"'] & K_{n+1}(A_{l,m+1}) \arrow[l, "\partial_{n+1}"', "\sim"] \arrow[d, "\epsilon_{n+1}^{l,m+1}"'] \\ K_n^{\mathrm{top}}(A_{l+1,m}) & K_{n+1}^{\mathrm{top}}(A_{l,m+1}) \arrow[l, "\partial_{n+1}^{\mathrm{top}}"', "\sim"] \end{tikzcd}.$$

This demonstrates that $A_{l+1,m}$ is K_n-stable if and only if $A_{l,m+1}$ is K_{n+1}-stable. Since every Banach algebra is K_0-stable, we deduce that A_{lm} is K_n-stable if $-l \leq n \leq 0$.

Note that

$$F_{l0} = \mathcal{K}(H)^{\otimes_{\max} l} = \mathcal{K}\big(H^{\tilde{\otimes} l}\big) \simeq F_{10}, \qquad (l \in \mathbf{N}),$$

where $H^{\tilde{\otimes} l}$ is the completion of the pre-Hilbert space $H^{\otimes_F l}$. In particular, for any C^*-algebra, one has a C^*-algebra isomorphism

$$A_{10} \simeq A_{l0} \qquad (l > 0).$$

In the theory of C^*-algebras, an algebra A is said to be *stable* if it is isomorphic to A_{10}. The terminology reflects the fact $\mathcal{K}\otimes_{\max}A$ is the (unique) C^*-algebra completion of $M_\infty(A)$, as well as the fact that $\mathcal{K}\otimes_{\max}A \simeq A$ if A is a stable C^*-algebra. The above argument demonstrates that stable C^*-algebras are K_n-stable for $n \leq 0$.

We shall adapt the above argument to general Banach algebras by replacing $\otimes_{\max}$ with $\widehat{\otimes}_\pi$ in the definition of the family F_{lm},

$$\hat{F}_{lm} := \mathcal{K}^{\widehat{\otimes}_\pi l}\widehat{\otimes}_\pi(\mathcal{B}/\mathcal{K})^{\widehat{\otimes}_\pi m} \qquad (l, m \in \mathbf{N}),$$

and, given a Banach algebra A, in the definition of the associated 2-parameter family A_{lm},

$$\hat{A}_{lm} := \hat{F}_{lm}\widehat{\otimes}_\pi A,$$

remembering that the projective tensor product is formed in the category of Banach spaces over F. As an extension of Banach spaces, (1.7) is *pure* (see Appendix B), therefore projective tensor product with $\hat{A}_{lm}$ preserves its exactness and we obtain the extension

$$\hat{A}_{l,m+1} \twoheadleftarrow \mathcal{B}\widehat{\otimes}_\pi\hat{A}_{lm} \leftarrowtail \hat{A}_{l+1,m} \quad .$$

By invoking (1.4)–(1.5) again, we conclude precisely as we did for C^*-algebras that $\hat{A}_{lm}$ is K_n-stable if $-l \leq n \leq 0$.

For a given orthogonal projection $p \in \mathcal{K}$ of rank 1, the correspondence

$$\mathcal{K}(H) \longrightarrow \mathcal{K}(H)^{\widehat{\otimes}_\pi l}, \qquad c \longmapsto p^{\widehat{\otimes}_\pi (l-1)}\widehat{\otimes}_\pi c,$$

induces a Banach algebra homomorphism $\kappa_l \colon \hat{F}_{10} \to \hat{F}_{l0}$. For $L = p(H)^{\otimes_F(l-1)}$ let $L^\perp$ be the orthogonal complement of L in $H^{\tilde{\otimes}(l-1)}$. The obvious identification of $L \otimes_F H$ with H combined with an arbitrary Hilbert space isomorphism $L^\perp\tilde{\otimes}H \simeq H$, induces an isomorphism of C^*-algebras $\mathcal{K}\big(H^{\tilde{\otimes} l}\big) \simeq M_2(\mathcal{K}(H))$. When composed with the tensor-product-of-operators map

$$\mathcal{K}(H)^{\widehat{\otimes}_\pi l} \hookrightarrow \mathcal{K}\big(H^{\tilde{\otimes} l}\big),$$

it produces a homomorphism of Banach algebras

$$\nu_l\colon \mathcal{K}^{\widehat{\otimes}_\pi l} \longrightarrow M_2(\mathcal{K}(H))$$

and the composite $\nu_l \circ \kappa_l$ coincides with $\iota_2\colon \mathcal{K} \hookrightarrow M_2(\mathcal{K})$. Setting $\kappa = \kappa_l \otimes_F \mathrm{id}_A$ and $\nu = \nu_l \otimes_F \mathrm{id}_A$ provides then a similar factorization of $\iota_2\colon \hat{A}_{10} \hookrightarrow M_2(\hat{A}_{10})$ through $\hat{A}_{l0}$. This demonstrates that, for any Banach algebra A and any $l > 0$, the algebra $\hat{A}_{10}$ is a stable retract of $\hat{A}_{l0}$. In particular, $\hat{A}_{10}$ is K_n-stable for any $n \leq 0$.

We arrive at the following theorem which is a generalization of the main result of [20]. We emphasize that it holds for real and complex Banach algebras alike.

Theorem 1.1. *Any Banach algebra which is a stable retract of $\mathcal{K}\otimes_{\max}A$, for some C^*-algebra A, or of $\mathcal{K}\widehat{\otimes}_\pi A$, for some Banach algebra A, is K_n-stable for $n \leq 0$.*

Remark 1.2. The bilinear pairing

$$\mathcal{K} \times (\mathcal{K}\otimes_{\max}A) \longrightarrow \mathcal{K}\otimes_{\max}(\mathcal{K}\otimes_{\max}A), \qquad (c, \alpha) \longmapsto c \otimes \alpha,$$

gives rise to a homomorphism of Banach algebras

$$\mathcal{K}\widehat{\otimes}_\pi(\mathcal{K}\otimes_{\max}A) \longrightarrow \mathcal{K}\otimes_{\max}(\mathcal{K}\otimes_{\max}A).$$

Composition with $\iota\widehat{\otimes}_\pi \mathrm{id}_{\mathcal{K}\otimes_{\max}A}$, where $\iota\colon F \hookrightarrow \mathcal{K}$ is the inclusion defined in (1.6), is a homomorphism

$$\mathcal{K}\otimes_{\max}A \longrightarrow \mathcal{K}\otimes_{\max}(\mathcal{K}\otimes_{\max}A) \simeq (\mathcal{K}\otimes_{\max}\mathcal{K})\otimes_{\max}A$$

which is isomorphic to the inclusion $\mathcal{K}\otimes_{\max}A \hookrightarrow M_2(\mathcal{K}\otimes_{\max}A)$. In particular, $\mathcal{K}\widehat{\otimes}_\pi A$ is a stable retract of $\mathcal{K}\widehat{\otimes}_\pi(\mathcal{K}\otimes_{\max}A)$, thus a separate argument used by us to demonstrate K_n-stability of $\mathcal{K}\otimes_{\max}A$ is, in fact, redundant.

2. K-THEORY OF $\mathcal{K}_{\mathbf{C}}$-RINGS

In 1970-ties one of us [20] discovered that algebraic K-theory of unital rings and topological K-theory of Banach algebras both possess $\mathbf{Z}$-graded multiplicative structures that are associative, graded-commutative, and compatible with the comparison map

$$\epsilon_*\colon K_*(A) \longrightarrow K_*^{\mathrm{top}}(A),$$

where

$$K_*(A) = \bigoplus_{n\in\mathbf{Z}} K_n(A) \qquad \text{and} \qquad K_*^{\mathrm{top}}(A) := \bigoplus_{n\in\mathbf{Z}} K_n^{\mathrm{top}}(A),$$

and A denotes a Banach algebra. That early observation provides a convenient starting point for this chapter. For any unital ring R, the tensor product map

$$\otimes_{\mathbf{Z}}\colon R \times R \to R\otimes_{\mathbf{Z}}R, \qquad (r, r') \mapsto r \otimes r',$$

is a *bimultiplicative* pairing in the sense of Appendix A and therefore induces a homomorphism of $\mathbf{Z}$-graded abelian groups

$$K_*(R)\otimes_{\mathbf{Z}}K_*(R) \to K_*(R\otimes_{\mathbf{Z}}R).$$

It follows that any unital ring homomorphism

$$\mu : R\otimes_{\mathbf{Z}}R \to R \tag{2.1}$$

induces a $\mathbf{Z}$-graded (nonassociative) ring structure on $K_*(R)$.

If the multiplication map (2.1) is α-associative for a certain ring automorphism $\alpha \in \mathrm{Aut}(R)$, i.e., if

$$\mu \circ (\mathrm{id}_R \otimes \mu) = \alpha \circ \mu(\mu \otimes \mathrm{id}_R),$$

then the induced multiplication on $K_*(R)$ is α_*-associative, where α_* denotes the induced automorphism of $K_*(R)$.

Similarly, if μ is β-commutative for a certain ring automorphism $\beta \in \operatorname{Aut}(R)$, i.e., if

$$\mu \circ \tau = \beta \circ \mu,$$

where τ transposes factors of $R \otimes_{\mathbf{Z}} R$, then the induced multiplication on $K_*(R)$ is graded β_*-commutative,

$$vu = (-1)^{\tilde{u}\tilde{v}} \beta_*(uv),$$

where $\tilde{w}$ denotes the *parity* of the degree of an element $w \in K_n(R)$.

In a similar vein, any unital ring homomorphism

$$\lambda : R \otimes_{\mathbf{Z}} S \to S$$

induces a graded map

$$K_*(R) \otimes_{\mathbf{Z}} K_*(S) \longrightarrow K_*(S).$$

If R itself is equipped with a multiplication (2.1), and if there exists an automorphism $\gamma \in \operatorname{Aut}(S)$ such that

$$\lambda \circ (\mathrm{id}_R \otimes \lambda) = \gamma \circ \lambda \circ (\mu \otimes \mathrm{id}_S),$$

then the product maps induced on the corresponding $\mathbf{Z}$-graded K-groups by λ and μ satisfy a similar associativity identity twisted by γ_*.

Some of the most interesting "multiplications" μ and "actions" λ involve nonunital rings. Extension of the above multiplicative structures to K-theory of nonunital rings is straightforward provided each of the following rings,

$$R \otimes_{\mathbf{Z}} R, \qquad R \otimes_{\mathbf{Z}} S, \qquad R \otimes_{\mathbf{Z}} R \otimes_{\mathbf{Z}} R, \qquad \text{and} \qquad R \otimes_{\mathbf{Z}} R \otimes_{\mathbf{Z}} S,$$

satisfies excision in algebraic K-theory.

The problem how to characterize rings that satisfy excision in rational algebraic K-theory was solved in a pair of articles [36] and [32]:

(2.2) *a ring R satisfies excision in rational algebraic K-theory if and only if $R \otimes_{\mathbf{Z}} \mathbf{Q}$ is an H-unital $\mathbf{Q}$-algebra.*

H-unitality was introduced in [35] where it was shown to characterize algebras satisfying excision in Hochschild and cyclic homology. For $\mathbf{Q}$-algebras, excision in K-theory and in rational K-theory are equivalent, hence (2.2) provides also a complete characterization of $\mathbf{Q}$-algebras that satisfy excision in algebraic K-theory [32, Theorem B and the remark preceding it]. Validity of the above theorem is closely related to the following surprisingly delicate fact [32, Theorem 7.10]

(2.3) *the category of H-unital k-algebras over an arbitrary ground ring k is closed under $\otimes_k$.*

This result is particularly useful when extending the multiplicative structures of K-theory to H-unital $\mathbf{Q}$-algebras. It guarantees, in particular, that the "multiplication" and "action" induced in K-theory by μ and λ satisfy twisted associativity and commutativity properties, mirroring the ones satisfied by μ and λ themselves.

Remarkably, H-unitality is a relatively frequent phenomenon among rings originating in Operator Theory and Functional, as well as Global Analysis. For example, any Banach algebra with bounded left approximate identity satisfies the Triple Factorization Property (Φ) introduced in [37] (cf. [32, Proposition 10.2]). Any ring R with this property satisfies excision in algebraic K-theory [32, Theorem C]. Such a ring is also a flat right module over an arbitrary unital ring A that contains R as a right ideal (cf. [37, Proposition 4] where this property of R is called *right* *universal flatness*). Similarly for Banach algebras with bounded right approximate identity: they are left universally flat.

A universally flat ring is H-unital in the category of k-algebras for any ground ring k and any *flat* k-algebra structure on R, cf. [37, Theorem 1]. In particular, Banach algebras with bounded one-sided approximate identities satisfy excision in algebraic K-theory and are H-unital as k-algebras for every flat k-algebra structure on them. Projective tensor product of Banach algebras with bounded left approximate identity has a bounded left approximate identity itself. Finally, tensor products over $\mathbf{Z}$ of Banach algebras with bounded approximate identity are H-unital as $\mathbf{Z}$-algebras, and therefore also as $\mathbf{Q}$-algebras, according to Theorem (2.3) quoted above.

This litany of remarkable algebraic and homological properties means that the mutiplicative structures discussed at the beginning of this chapter extend to any Banach algebra with one-sided bounded approximate identity. This includes every C^*-algebra as well as every closed one-sided ideal in a C^*-algebra. It also includes $\mathcal{K}\widehat{\otimes}_\pi A$ where A is an arbitrary Banach algebra with one-sided bounded approximate identity (note that $\mathcal{K}$ itself possesses a two-sided approximate identity bounded by 1).

Let us record one more favorable property of H-unital algebras. If a $\mathbf{Q}$ algebra R is H-unital, so is the matrix algebra $M_n(R)$ [36, Corollary 9.8] (this is, of course, a special case of Theorem (2.3)), and therefore satisfies excision in algebraic K-theory. This implies that the stabilization map of (1.2) induces an isomorphism in algebraic K-theory. The argument utilised in the proof of Corollary 9.10 in [36] then shows that conjugation by any element $a \in \mathrm{GL}_1(A)$, for any unital ring A containing R as a two-sided ideal, acts trivially on $K_*(R)$.

If R and S are Banach algebras, if λ, μ, and the constraint maps, α, β, and γ, are continuous, then $K_*^{\mathrm{top}}(R)$ becomes a $\mathbf{Z}$-graded, α_*-associative, graded β_*-commutative ring, and $K_*^{\mathrm{top}}(S)$ becomes a $\mathbf{Z}$-graded, γ_*-associative $K_*^{\mathrm{top}}(R)$-module. Since excision holds in topological K-theory for all Banach algebras without exception, neither R nor S are subjected to further restrictions similar to H-unitality. For Banach algebras which are H-unital as $\mathbf{Q}$-algebras, the multiplicative structures in algebraic and topological K-theory are compatible with the comparison map, i.e., the following diagrams,

$$\begin{array}{ccc} K_*(R)\times K_*(R) & \xrightarrow{\mu_*} & K_*(R) \\ {\scriptstyle \epsilon_*\times\epsilon_*}\downarrow & & \downarrow{\scriptstyle \epsilon_*} \\ K_*^{\mathrm{top}}(R)\times K_*^{\mathrm{top}}(R) & \xrightarrow{\mu_*} & K_*^{\mathrm{top}}(R) \end{array} \quad \text{and} \quad \begin{array}{ccc} K_*(R)\times K_*(S) & \xrightarrow{\lambda_*} & K_*(S) \\ {\scriptstyle \epsilon_*\times\epsilon_*}\downarrow & & \downarrow{\scriptstyle \epsilon_*} \\ K_*^{\mathrm{top}}(R)\times K_*^{\mathrm{top}}(S) & \xrightarrow{\lambda_*} & K_*^{\mathrm{top}}(S) \end{array},$$

commute.

An important example of a noncommutative ring equipped with a multiplication (2.1) is provided by the ring $\mathcal{K} = \mathcal{K}(H)$ of compact operators on a separable, real or complex, infinite-dimensional Hilbert space. Let us consider the map induced by tensor product of operators

$$\mathcal{K}(H)\otimes_{\mathbf{Z}}\mathcal{K}(H) \longrightarrow \mathcal{K}(H)\otimes_F \mathcal{K}(H) \hookrightarrow \mathcal{K}(H\tilde{\otimes}H) \tag{2.4}$$

where $H\tilde{\otimes}H$ denotes the Hilbert-space completion of the pre-Hilbert space $H\otimes_F H$ and F denotes the field of coefficients of H.

Composition of (2.4) with the isomorphism

$$\mathcal{K}(H\tilde{\otimes}H) \simeq \mathcal{K}(H) \tag{2.5}$$

induced by a fixed but otherwise arbitrary identification $H\tilde{\otimes}H \simeq H$ of Hilbert spaces, defines a multiplication μ on $\mathcal{K}$ which depends on the choice of that identification. This multiplication is ad_U-associative and ad_V-commutative for suitable bounded invertible operators $U, V \in \mathcal{B}(H)$ where $\mathrm{ad}_E(X) := EXE^{-1}$.

Any orthogonal projection $p \in \mathcal{K}$ of rank 1 is a "stable quasiidentity" for this multiplication, which means that the following diagram commutes

$$\begin{tikzcd}
& \mathcal{K} \arrow[dr, "\mathrm{ad}_{W'}"] & \\
\mathcal{K} \arrow[ur, "\mu(p\otimes\ )"] \arrow[rr, tail, "\iota_2"] \arrow[dr, "\mu(\ \otimes p)"'] & & M_2(\mathcal{K}) \\
& \mathcal{K} \arrow[ur, "\mathrm{ad}_{W''}"'] &
\end{tikzcd}$$

for suitable isomorphisms W' and W'' between H and $H \oplus H$, where ι_2, as usual, denotes the stabilization map of (1.2).

We are ready to introduce a certain class of rings equipped with a quasiassociative action of $\mathcal{K}$, i.e., with a ring homomorphism

$$\lambda\colon \mathcal{K} \otimes_{\mathbf{Z}} R \to R \tag{2.6}$$

which is γ-associative for some $\gamma \in \operatorname{Aut} R$. Note that quasiassociativity involves both λ and a fixed multiplication μ on $\mathcal{K}$. If (2.6) is "stably quasiunitary" which means that the following diagram commutes

$$\begin{tikzcd}
R \arrow[r, tail, "\iota_2"] & M_2(R) \arrow[d, "\theta"] \\
R \arrow[u, "\lambda(p\otimes\ )"] \arrow[r, tail, "\iota_2"] & M_2(R)
\end{tikzcd}$$

for a suitable automorphism θ of $M_2(R)$, we will call R a *$\mathcal{K}$-ring*. More precisely, we will call it a *real $\mathcal{K}$-ring*, if $\mathcal{K} = \mathcal{K}_{\mathbf{R}}$ is the algebra of compact operators on a real Hilbert space, and a *complex $\mathcal{K}$-ring*, if $\mathcal{K} = \mathcal{K}_{\mathbf{C}}$ is the algebra of compact operators on a complex Hilbert space. Alternatively, we will be talking of $\mathcal{K}_{\mathbf{R}}$- and $\mathcal{K}_{\mathbf{C}}$-rings.

A Banach algebra will be called a *Banach $\mathcal{K}$-ring* if (2.6) induces a homomorphism of Banach algebras $\mathcal{K}\widehat{\otimes}_\pi R \to R$ while γ and θ are continuous.

The ring of the form $R = \mathcal{K} \otimes_k A$, where A is an algebra over a subring $k \subseteq F$, is an example of a $\mathcal{K}$-ring provided R is H-unital over $\mathbf{Q}$. The projective tensor product $R = \mathcal{K}\widehat{\otimes}_\pi A$ by a Banach algebra A with a one-sided bounded approximate identity supplies an example of a Banach $\mathcal{K}$-ring. Finally, *stable* C^*-algebras $R = \mathcal{K}\otimes_{\max} A$ are examples of Banach $\mathcal{K}$-rings among C^*-algebras.

These are the obvious examples, so to speak. Less obvious is the Calkin algebra $\mathcal{B}/\mathcal{K}$ and its tensor products in any of the above three senses. Another example of a unital Banach $\mathcal{K}$-ring is suppplied by the algebra defined as the fibered square of $\mathcal{B}$ over $\mathcal{B}/\mathcal{K}$

$$\mathcal{B}^{(2)} := \{(a, b) \in \mathcal{B} \times \mathcal{B} \mid a - b \in \mathcal{K}\}. \tag{2.7}$$

The C^*-algebra defined in (2.7) fits into an extension

$$\mathcal{B} \overset{p}{\twoheadleftarrow} \mathcal{B}^{(2)} \overset{i}{\leftarrowtail} \mathcal{K} \quad ,$$

where

$$i\colon \mathcal{K} \longrightarrow \mathcal{B}^{(2)}, \qquad c \longmapsto (c,0), \qquad (c \in \mathcal{K}). \tag{2.8}$$

The long exact sequence in either algebraic or topological K-theory demonstrates independently of Theorem 2.2 that (2.8) induces isomorphisms in K-theory

$$K_*(\mathcal{K}) \simeq K_*\big(\mathcal{B}^{(2)}\big) \qquad \text{and} \qquad K_*^{\text{top}}(\mathcal{K}) \simeq K_*^{\text{top}}\big(\mathcal{B}^{(2)}\big).$$

Noting that the group $\mathrm{GL}_1(\mathcal{B}) = \mathrm{GL}_1(\mathcal{B}(H))$ acts trivially on $K_*(\mathcal{K})$, we infer that the multiplication induced by μ on $K_*(\mathcal{K})$ is associative, graded commutative, and does not depend on the isomorphism in (2.5) induced by the chosen identification $H\tilde{\otimes}H \simeq H$.

For a one dimensional subspace $H' \subset H$, the map induced by tensor product of operators

$$\mathcal{K}(H') \times \mathcal{K}(H') \longrightarrow \mathcal{K}(H' \otimes_F H')$$

coincides with multiplication in the ground field. Moreover, the embedding of F into $\mathcal{K}(H)$ which corresponds to the inclusion $H' \hookrightarrow H$, cf. (1.6), is a homomorphism of the ground field into $(\mathcal{K}, +, \mu)$ provided we choose an isomorphism $H\tilde{\otimes}H \simeq H$ which identifies the one-dimensional subspace $H' \otimes H' \subset H\tilde{\otimes}H$ with $H' \subset H$. The induced map $\iota_*\colon K_*(F) \to K_*(\mathcal{K})$ in this case is a homomorphism of graded rings and simultaneously a homomorphism of graded $K_*(F)$-modules.

In conclusion, the $\mathbf{Z}$-graded algebraic K-group of any $\mathcal{K}$-ring R which is H-unital as a $\mathbf{Q}$-algebra, becomes a graded γ_*-associative module over $K_*(\mathcal{K})$, and the class $[p]$ of any rank 1 projection, which is a generator of $K_0(\mathcal{K}) \simeq \mathbf{Z}$, acts on $K_*(R)$ via automorphism θ_*

$$[p]w = \theta_*(w), \qquad (w \in K_*(R)).$$

For $R = \mathcal{K} \otimes_k A$, as well as for $\mathcal{K}\widehat{\otimes}_\pi A$ and $\mathcal{K}\otimes_{\max}A$, constraints γ and θ can be realized as inner automorphisms of unital rings that contain R and, respectively, $M_2(R)$, as two-sided ideals. Accordingly, $\gamma_* = \mathrm{id}$ and $\theta_* = \mathrm{id}$, and $K_*(R)$ is in those cases a strictly associative and unitary $K_*(\mathcal{K})$-module.

The $\mathbf{Z}$-graded *topological* K-theory ring of $\mathbf{C}$ is canonically isomorphic to the ring of Laurent polynomials $\mathbf{Z}[u, u^{-1}]$ where $u \in K_2^{\text{top}}(\mathbf{C})$ corresponds to the generator of $\tilde{K}_{\mathbf{C}}(S^2)$, usually chosen to be the class of the line bundle $\mathcal{O}(1)$ on $P^1(\mathbf{C}) \simeq S^2$ [2, Theorem 2.4.9]. Accordingly, the $\mathbf{Z}$-graded topological K-theory ring of any complex Banach algebra A becomes a graded unitary module over $K_*^{\text{top}}(\mathbf{C}) \simeq \mathbf{Z}[u, u^{-1}]$. Multiplication by u defines canonical isomorphisms $K_n^{\text{top}}(A) \simeq K_{n+2}^{\text{top}}(A)$, $n \in \mathbf{Z}$, multiplication by u^{-1} provides the inverse isomorphisms. This is, of course, Bott periodicity in topological K-theory of complex Banach algebras, seen through the multiplicative structures of topological K-theory.

Any Banach algebra homomorphism $f\colon A \to B$ induces a homomorphism $f_*\colon K_*^{\text{top}}(A) \to K_*^{\text{top}}(B)$ of graded modules over $K_*^{\text{top}}(\mathbf{C}) \simeq \mathbf{Z}[u, u^{-1}]$. In particular, f_* is an isomorphism if and only if f_n is an isomorphism for at least a single even and a single odd degree n. This is so for the inclusion $\iota\colon F \hookrightarrow \mathcal{K}$ defined in (1.6) since

$$\iota_0\colon K_0(\mathbf{C}) \simeq K_0(\mathcal{K}_{\mathbf{C}}) \qquad \text{and} \qquad K_1^{\text{top}}(\mathbf{C}) = K_1^{\text{top}}(\mathcal{K}_{\mathbf{C}}) = 0.$$

Hence ι_* is an isomorphism of graded $K_*^{\text{top}}(\mathbf{C})$-modules and, being also a ring homomorphism, is an isomorphism of graded rings.

In [20] and [23] we demonstrated that the comparison maps

$$\epsilon_{\pm 2}\colon K_{\pm 2}(\mathcal{K}_{\mathbf{C}}) \longrightarrow K_{\pm 2}^{\text{top}}(\mathcal{K}_{\mathbf{C}})$$

are surjective. Note that existence of u_{-2} was already established in Chapter 1 where we proved that $K_n(\mathcal{K}) \simeq K_n^{\mathrm{top}}(\mathcal{K})$ for all $n \leq 0$, in real and complex cases alike.

Since ϵ_0 is the identity map and the comparison map is a homomorphism of graded rings, there must exist elements $u_{\pm 2} \in K_{\pm 2}(\mathcal{K}_{\mathbf{C}})$ such that $u_{-2}u_2 = 1 \in K_0(\mathcal{K}_{\mathbf{C}}) \simeq \mathbf{Z}$. The graded ring $K_*(\mathcal{K}_{\mathbf{C}})$ thus possesses an invertible element of degree 2. By invoking Theorem 1.1, which says that ϵ_* is an isomorphism in degrees ≤ 0 we arrive at the following result.

Theorem 2.1 ([32], [38])**.** *The comparison map $\epsilon_*\colon K_*(\mathcal{K}(H)) \to K_*^{\mathrm{top}}(\mathcal{K}(H))$, where H denotes a complex infinite dimensional separable Hilbert space, is an isomorphism of $\mathbf{Z}$-graded unital rings.*

The following result in essence was predicted by the first author, and was proved by the second [38]. It is an immediate corollary of Theorem 2.1 and the fact that u_2 acts on $K_*(R)$, for any $\mathcal{K}_{\mathbf{C}}$-ring which is H-unital as a $\mathbf{Q}$-algebra, via an isomorphism of degree 2.

Theorem 2.2 ([38])**.** *Let R be any $\mathcal{K}_{\mathbf{C}}$-ring which is H-unital as a $\mathbf{Q}$-algebra. Then the algebraic K-groups are periodic with period 2, the periodicity isomorphism realized as multiplication by $u_2 \in K_2(\mathcal{K}_{\mathbf{C}})$.*

Let us reiterate that besides the rings of the form $R = \mathcal{K}_{\mathbf{C}} \otimes_k A$ which are H-unital over $\mathbf{Q}$, the above periodicity theorem applies to any Banach $\mathcal{K}_{\mathbf{C}}$-ring, e.g., to $R = \mathcal{K}_{\mathbf{C}} \widehat{\otimes}_\pi A$, where A is any real or complex Banach algebra with bounded one-sided approximate identity, and to $\mathcal{K}_{\mathbf{C}} \otimes_{\max} A$ where A is any C^*-algebra.

For a Banach $\mathcal{K}_{\mathbf{C}}$-ring, the comparison map $\epsilon_*\colon K_*(R) \to K_*^{\mathrm{top}}(R)$ is a homomorphism of graded $\mathbf{Z}[u, u^{-1}]$-modules which, in view of Theorem 1.1, is an isomorphism in degrees ≤ 0. This leads to the following result.

Theorem 2.3 ([38])**.** *Any Banach $\mathcal{K}_{\mathbf{C}}$-ring R is K_n-stable for every $n \in \mathbf{Z}$. In other words, the comparison map $\epsilon_*\colon K_*(R) \to K_*^{\mathrm{top}}(R)$ is an isomorphism of $\mathbf{Z}$-graded rings.*

Remark 2.4. Theorem 2.3 contains as a special case Theorem 10.9 of [32] which states that every complex stable C^*-algebra is K_n-stable for every $n \in \mathbf{Z}$ (so called *Karoubi's Conjecture*). The proof of that theorem in [32] does not exploit multiplicative structures on $\mathbf{Z}$-graded K-groups. Instead, it utilises the characterization of homotopy invariant functors from the category of C^*-algebras to the category of abelian groups due to Higson [14] whose work extended earlier results and techniques of Kasparov and Cuntz. It also invokes the fact that any complex stable C^*-algebra is K_1-stable (the proof of that statement is elementary).

3. K-theory of $\mathcal{K}_{\mathbf{R}}$-rings

Let H denote an arbitrary *real* Hilbert space. The diagonal embedding

$$\Delta_2\colon \mathcal{K}(H) \hookrightarrow \mathcal{K}(H \oplus H), \qquad c \longmapsto \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}, \tag{3.1}$$

factorizes

$$\begin{tikzcd} \mathcal{K}(H) \arrow[rr, tail, "\Delta_2"] \arrow[dr, tail, "\gamma"'] & & \mathcal{K}(H \oplus H) \\ & \mathcal{K}(H \otimes_{\mathbf{R}} \mathbf{C}) \arrow[ur, tail, "\rho"'] & \end{tikzcd} , \tag{3.2}$$

where γ is the complexification map, $c \mapsto c \otimes_{\mathbf{R}} \mathrm{id}_{\mathbf{C}}$, while ρ is the inclusion $\mathcal{K}_{\mathbf{C}}(H \otimes_{\mathbf{R}} \mathbf{C}) \hookrightarrow \mathcal{K}_{\mathbf{R}}(H \otimes_{\mathbf{R}} \mathbf{C})$. Out of these three ring homomorphisms only γ is also a homomorphism of the corresponding μ-multiplications, hence it induces a homomorphism of graded K-theory rings, the other two induce homomorphisms of $K_*(\mathcal{K}(H))$-modules in algebraic K-theory, and of $K_*^{\mathrm{top}}(\mathcal{K}(H))$-modules—in topological K-theory.

The homomorphism defined in (3.1) induces multiplication by 2 in algebraic and in topological K-theory alike. Since $\epsilon_*\colon K_*(\mathcal{K}(H \otimes_{\mathbf{R}} \mathbf{C})) \to K_*^{\mathrm{top}}(\mathcal{K}(H \otimes_{\mathbf{R}} \mathbf{C}))$ is an isomorphism in view of Theorem 2.1, the kernel and the cokernel of the comparison map

$$\epsilon_*\colon K_*(\mathcal{K}(H)) \to K_*^{\mathrm{top}}(\mathcal{K}(H)) \tag{3.3}$$

are abelian groups of exponent 2.

Theorem 3.1. *The comparison map $\epsilon_*\colon K_*(\mathcal{K}(H)) \to K_*^{\mathrm{top}}(\mathcal{K}(H))$, where H denotes a real infinite dimensional separable Hilbert space, is an isomorphism of $\mathbf{Z}$-graded unital rings.*

By Theorem 1.1, the comparison map is an isomorphism in degrees less or equal 0. It remains to find an invertible element of degree not equal 0 in the ring $K_*(\mathcal{K}(H))$. Such an element is sent by (3.3) to an invertible element in $K_*^{\mathrm{top}}(\mathcal{K}(H))$. Recalling that inclusion of a one-dimensional subspace into H induces an inclusion of $\mathbf{R}$ into $\mathcal{K}(H)$, cf. (1.6), we turn our attention to the ring structure of $K_*^{\mathrm{top}}(\mathbf{R})$,

$$K_*^{\mathrm{top}}(\mathbf{R}) \;\simeq\; \mathbf{Z}[\eta, w, v, v^{-1}]/(\eta^3, 2\eta, \eta w, w^2 - 4v)\,.$$

Here η, w and v have degrees 1, 4 and, respectively, 8 [19, p. 157]. In particular, $\{\pm v^n \mid n \in \mathbf{Z}\}$ is its group of invertible elements; the degrees of such elements are multiples of 8.

For a one-dimensional Hilbert space, diagram (3.2) becomes

$$\begin{tikzcd} \mathbf{R} \arrow[rr, tail, "\Delta_2"] \arrow[dr, tail, "\gamma"'] & & M_2(\mathbf{R}) \\ & \mathbf{C} \arrow[ur, tail, "\rho"'] & \end{tikzcd},$$

where γ is the inclusion map, and ρ sends $a + ib$ to $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

The induced ring homomorphism $\gamma_*^{\mathrm{top}}\colon K_*^{\mathrm{top}}(\mathbf{R}) \to K_*^{\mathrm{top}}(\mathbf{C})$ sends w to $2u^2$ and v to u^4, while

$$\rho_*^{\mathrm{top}}(u^{4l+m}) = \begin{cases} 2v^l & \text{if} \quad m = 0 \\ \eta^2 v^l & \text{if} \quad m = 1 \\ w v^l & \text{if} \quad m = 2 \\ 0 & \text{if} \quad m = 3 \end{cases}$$

cf. [19, pp. 157–158].

Let us consider the algebraic and topological K-groups of $\mathcal{K}_{\mathbf{R}} = \mathcal{K}(H)$ with coefficients $\mathbf{Z}/2^l\mathbf{Z}$, cf. [1], [9], [26]. They fit into the commutative diagram

$$\begin{tikzcd}
K_8(\mathcal{K}_{\mathbf{R}}; \mathbf{Z}/2^l\mathbf{Z}) \arrow[dd, "\bar{\epsilon}"'] & K_8(\mathcal{K}_{\mathbf{R}}) \arrow[l, "r"'] \arrow[dd, "\epsilon_{\mathbf{R}}"'] & & K_8(\mathcal{K}_{\mathbf{R}}) \arrow[ll, "\times 2^l"'] \arrow[dl, "\gamma"] \arrow[dd, "\epsilon_{\mathbf{R}}"'] \\
& & K_8(\mathcal{K}_{\mathbf{C}}) \arrow[ul, "2^{l-1}\rho"] \arrow[dd, "\epsilon_{\mathbf{C}}"] & \\
K_8^{\mathrm{top}}(\mathcal{K}_{\mathbf{R}}; \mathbf{Z}/2^l\mathbf{Z}) & K_8^{\mathrm{top}}(\mathcal{K}_{\mathbf{R}}) \arrow[l, "r^{\mathrm{top}}"'] & & K_8^{\mathrm{top}}(\mathcal{K}_{\mathbf{R}}) \arrow[ll, "\times 2^l"'] \arrow[dl, "\gamma^{\mathrm{top}}"] \\
& & K_8^{\mathrm{top}}(\mathcal{K}_{\mathbf{C}}) \arrow[ul, "2^{l-1}\rho^{\mathrm{top}}"] &
\end{tikzcd}$$

whose rows are portions of the long exact sequence relating the homotopy groups of a space to the homotopy groups with coefficients $\mathbf{Z}/2^l\mathbf{Z}$. The *reduction* mod 2^l maps, denoted r, are induced by the *pinching* maps $P^n(2^l) \to S^n$ from *Peterson* spaces

$$P^n(d) = S^{n-2} \wedge P^2(d), \qquad (n \geq 2),$$

which are the cofibers of maps between spheres $S^{n-1} \to S^{n-1}$ of degree d. The vertical arrows are the appropriate comparison maps.

The comparison map for $\mathcal{K}_{\mathbf{C}} = \mathcal{K}(H \otimes_R \mathbf{C})$, denoted $\epsilon_{\mathbf{C}}$, is an isomorphism by Theorem 2.3 and it sends element $u_2^4 \in K_8(\mathcal{K}_{\mathbf{C}})$ to $u^4 \in K_8^{\mathrm{top}}(\mathcal{K}_{\mathbf{C}})$. In view of

$$\mathrm{GL}_\infty(\mathbf{R}) \hookrightarrow (1 + \mathcal{K}(H))^* \qquad \text{and} \qquad \mathrm{GL}_\infty(\mathbf{C}) \hookrightarrow (1 + \mathcal{K}(H \otimes_{\mathbf{R}} \mathbf{C}))^*$$

being homotopy equivalences [27], the vertical arrows in the commutative diagram

$$\begin{tikzcd}
K_*^{\mathrm{top}}(\mathbf{C}) \arrow[r, "\rho^{\mathrm{top}}"] \arrow[d, "\iota"', "\simeq"] & K_*^{\mathrm{top}}(\mathbf{R}) \arrow[d, "\iota"', "\simeq"] \\
K_*^{\mathrm{top}}(\mathcal{K}_{\mathbf{C}})) \arrow[r, "\rho^{\mathrm{top}}"] & K_*^{\mathrm{top}}(\mathcal{K}_{\mathbf{R}})
\end{tikzcd}$$

are isomorphisms and we infer that

$$\left(\bar{\epsilon} \circ r \circ 2^{l-1}\rho\right)\left(u_2^4\right) = \left(r^{\mathrm{top}} \circ 2^{l-1}\rho^{\mathrm{top}}\right)\left(u^4\right) = r^{\mathrm{top}}\left(2^l v\right) = 0.$$

If $\bar{\epsilon}$ is injective, then exactness of the top row implies that

$$2^{l-1}\rho\left(u_2^4\right) = 2^l v_8$$

for some element $v_8 \in K_8(\mathcal{K}_{\mathbf{R}})$. Accordingly,

$$2^l v = 2^{l-1}\rho^{\mathrm{top}}\left(u^4\right) = \epsilon_{\mathbf{C}}\left(u_2^4\right) = 2^l \epsilon_{\mathbf{R}}(v_8)$$

in $K_8^{\mathrm{top}}(\mathcal{K}_R) \simeq \mathbf{Z}$. As the latter group is torsion free, it follows that $v = \epsilon_{\mathbf{R}}(v_8)$, i.e., the comparison map $K_8(\mathcal{K}_{\mathbf{R}}) \to K_8^{\mathrm{top}}(\mathcal{K}_{\mathbf{R}})$ is surjective. The $\mathbf{Z}$-graded comparison map (3.3), being a unital ring homomorphism and an isomorphism in degrees less or equal 0, is then seen to be an isomorphism in all degrees, with multiplication by v_8 providing 8-periodicity isomorphisms $K_n(\mathcal{K}_{\mathbf{R}}) \simeq K_{n+8}(\mathcal{K}_{\mathbf{R}})$.

The proof of Theorem 3.1 will be complete if we show that $\bar{\epsilon}$ is injective for at least one $l > 0$. This follows, for example, from a celebrated result of Suslin about K-theory of $\mathbf{R}$ and $\mathbf{C}$ with finite coefficients. According to [30, Corollary 4.8], one has

$$\bar{\epsilon}_n \colon K_n(\mathbf{R}; \mathbf{Z}/d\mathbf{Z}) \xrightarrow{\ \sim\ } K_n^{\mathrm{top}}(\mathbf{R}; \mathbf{Z}/d\mathbf{Z}) \qquad (n > 0,\ d > 1).$$

Below we offer a proof that does not rely on Suslin's Theorem.

Lemma 3.2. *The* **Z***-graded comparison map*

$$\bar{e}_*\colon K_*(\mathcal{K}_{\mathbf{R}};\mathbf{Z}/16\mathbf{Z}) \longrightarrow K_*^{\mathrm{top}}(\mathcal{K}_{\mathbf{R}};\mathbf{Z}/16\mathbf{Z}) \tag{3.4}$$

is an isomorphism.

Proof. There exists an element of order 16 in $\pi_8\left(\Omega^\infty S^\infty;\mathbf{Z}/16\mathbf{Z}\right)$ which under the sequence of obvious maps

$$\Omega^\infty S^\infty \sim \mathrm{B}\Sigma_\infty^+ \longrightarrow \mathrm{BGL}_\infty(\mathbf{Z})^+ \longrightarrow \mathrm{BGL}_\infty(\mathbf{R})^+ \longrightarrow \mathrm{BGL}_\infty^{\mathrm{top}}(\mathbf{R})$$

is sent to the generator of $K_8^{\mathrm{top}}(\mathbf{R};\mathbf{Z}/16\mathbf{Z}) \simeq \mathbf{Z}/16\mathbf{Z}$, cf. e.g., [9, Theorem 4.1]. It follows that $\bar{e}_{\mathbf{R}}$, the left comparison map in the commutative diagram

$$\begin{tikzcd} K_8(\mathbf{R};\mathbf{Z}/16\mathbf{Z}) \arrow[r, "\iota"] \arrow[d, two heads, "\bar{e}_{\mathbf{R}}"] & K_8(\mathcal{K}_{\mathbf{R}};\mathbf{Z}/16\mathbf{Z}) \arrow[d, "\bar{e}_{\mathcal{K}_{\mathbf{R}}}"] \\ K_8^{\mathrm{top}}(\mathbf{R};\mathbf{Z}/16\mathbf{Z}) \arrow[r, "\simeq", "\iota"'] & K_8^{\mathrm{top}}(\mathcal{K}_{\mathbf{R}};\mathbf{Z}/16\mathbf{Z}) \end{tikzcd},$$

is surjective. Since the bottom arrow is an isomorphism by the result of Palais mentioned above [27], also $\bar{e}_{\mathcal{K}_{\mathbf{R}}}$ is surjective. Let us denote by $\bar{v}_8$ any element of $K_8(\mathcal{K}_{\mathbf{R}};\mathbf{Z}/16\mathbf{Z})$ which is mapped by $\bar{e}_{\mathcal{K}_{\mathbf{R}}}$ to the generator of $K_8^{\mathrm{top}}(\mathcal{K}_{\mathbf{R}};\mathbf{Z}/16\mathbf{Z})$.

For any $d \neq 2 \mod 4$, there exists a coproduct map $P^4(d) \to P^2(d) \wedge P^2(d)$ which, by using suspension, generates a sequence of compatible coproduct maps on Peterson spaces

$$P^{m+n}(d) \longrightarrow P^m(d) \wedge P^n(d) \qquad (m, n \geq 2). \tag{3.5}$$

The coproduct maps are known to be coassociative for d *odd* and $d \neq 3 \mod 9$. This fact is relatively easy to see when d is not divisible by 2 or 3. The case when $9|d$ is significantly more delicate, and it has been settled recently by Neisendorfer [26, Theorem 25.1]. No positive coassociativity results about (3.5) seem to have been established for any d divisible by powers of 2^l, $l > 1$. The most recent as well as seemingly the most reliable source of information on the subject of multiplicative structures in homotopy groups with finite coefficients is [26].

While the exact extent of coassociativity still remains unclear, the coproducts in (3.5) exhibit what we would like to call *limited coassociativity*. In the context of the corresponding product structures on $K_*(\ ;\mathbf{Z}/d\mathbf{Z})$, this means that

$$(\alpha\beta)\gamma = \alpha(\beta\gamma)$$

if at least one of the three elements is the reduction mod d of an element in the *integral* K-group, i.e., belongs to the image of the reduction map

$$r\colon K_*(\ ) \to K_*(\ ;\mathbf{Z}/d\mathbf{Z}).$$

This has been known already to Araki and Toda [1] who refer to it as *quasiassociativity*.

The comparison mod 16 map (3.4) is a homomorphism of **Z**-graded binary rings, as are the reduction mod 16 maps. If we denote by $\bar{v}_{-8}$ the reduction of $v_{-8} \in K_{-8}(\mathcal{K}_{\mathbf{R}})$, and by $\bar{v}$ the reduction of $v \in K_8^{\mathrm{top}}(\mathcal{K}_{\mathbf{R}})$, then

$$\bar{e}_0(\bar{v}_8\bar{v}_{-8}) = \bar{e}_8(\bar{v}_8)\bar{e}_{-8}(\bar{v}_{-8}) = \bar{v}\bar{v}^{-1} = r(vv^{-1}) = r(1) = 1$$

in $K_0^{\mathrm{top}}(\mathcal{K}_{\mathbf{R}};\mathbf{Z}/16\mathbf{Z}) \simeq \mathbf{Z}/16\mathbf{Z}$. Since $\bar{e}_n$ is an isomorphism for $n \leq 0$, one has $\bar{v}_{-8} = \bar{v}_8^{-1}$. Limited associativity of the product structure in $K_*(\mathcal{K}_{\mathbf{R}};\mathbf{Z}/16\mathbf{Z})$

combined with the fact that $\bar{v}_{-8} = r(v_{-8})$, implies that multiplication by $\bar{v}_{-8}$, viewed as an operation on the $\mathbf{Z}$-graded abelian group $K_*(\mathcal{K}_{\mathbf{R}}; \mathbf{Z}/16\mathbf{Z})$, is the inverse to the operation of multiplication by $\bar{v}_8$. In other words, (3.4) is an isomorphism of graded modules over the ring of Laurent polynomials in one variable of degree 8. □

Remark 3.3. Note that Theorem 3.1 implies that $\bar{v}_8$ is the reduction mod 16 of a generator of $K_8(\mathcal{K}_{\mathbf{R}}) \simeq K_8^{\mathrm{top}}(\mathcal{K}_{\mathbf{R}}) \simeq \mathbf{Z}$, unlike the element in $K_8(\mathbf{R}; \mathbf{Z}/16\mathbf{Z})$ which allowed us to prove the existence of $\bar{v}_8$: according to [30, Theorem 4.9], $K_8(\mathbf{R})$ is a uniquely divisible group.

The following two results follow now from Theorems 3.1 and 1.1 precisely in the same way as Theorems 2.2 and 2.3 follow from Theorem 2.1 and Theorem 1.1.

Theorem 3.4. *Let R be any $\mathcal{K}_{\mathbf{R}}$-ring which is H-unital as a $\mathbf{Q}$-algebra. Then the algebraic K-groups are periodic with period 8, the periodicity isomorphism realized as multiplication by $v_8 \in K_8(\mathcal{K}_{\mathbf{R}})$.*

Theorem 3.5. *Any Banach $\mathcal{K}_{\mathbf{R}}$-ring R is K_n-stable for every $n \in \mathbf{Z}$. In other words, the comparison map $\epsilon_* \colon K_*(R) \to K_*^{\mathrm{top}}(R)$ is an isomorphism of $\mathbf{Z}$-graded rings.*

4. Hermitian K-theory of $\mathcal{K}$-rings

Let A be a unital ring with anti-involution α. We shall assume throughout that (A, α) is *split*, i.e., that there exists an element λ in the center of A such that $\lambda + \alpha(\lambda) = 1$. This condition is automatically satisfied when A is a $\mathbf{Z}[\frac{1}{2}]$-algebra. One can associate with (A, α) the Hermitian K-theory spectrum ${}_\varepsilon\mathbf{KQ}(A)$ where $\varepsilon = \pm 1$ (see [22] and [29] for basic references).[1] This parallels the construction of the algebraic K-theory spectrum $\mathbf{K}(A)$ for a unital ring.

The Hermitian K-groups ${}_\varepsilon KQ_n(A)$ are the homotopy groups of ${}_\varepsilon\mathbf{KQ}(A)$ in the same way as Quillen's K-groups $K_n(A)$ are the homotopy groups of $\mathbf{K}(A)$. For instance, if A is a commutative ring and the anti-involution is trivial, then

$$ {}_{-1}KQ_n(A) = \pi_n\left(\mathrm{BSp}(A)^+\right) \qquad (n > 0) $$

where $\mathrm{Sp}(A)$ denotes the infinite symplectic group $\bigcup_{d\geq 1} \mathrm{Sp}_{2d}(A)$.

For $n > 0$, one has ${}_\varepsilon KQ_n(A) = \pi_n\left(\mathrm{B}_\varepsilon\mathrm{O}(A)^+\right)$ where

$$ {}_\varepsilon\mathrm{O}(A) = \bigcup_{l>0} {}_\varepsilon\mathrm{O}_{l,l}(A) $$

is the group of ε-orthogonal matrices with coefficients in A [17, p. 64], [18, p. 308]. The classical Whitehead's Theorem $[\mathrm{GL}(A), \mathrm{GL}(A)] = \mathrm{E}(A)$ is replaced by

$$ [{}_\varepsilon\mathrm{O}(A), {}_\varepsilon\mathrm{O}(A)] = {}_\varepsilon\mathrm{EO}(A) $$

where ${}_\varepsilon\mathrm{EO}(A)$ is a suitable replacement for the group of elementary matrices [34, теоремы 1.4-1.4$_0$].

The cone and suspension functors familiar from algebraic K-theory are naturally equipped with the induced anti-involutions and the extension

$$ SA \twoheadleftarrow CA \hookleftarrow MA \tag{4.1} $$

leads to the homotopy fibration

$$ \mathrm{B}_\varepsilon\mathrm{EO}(SA)^+ \longleftarrow \mathrm{B}_\varepsilon\mathrm{O}(CA)^+ \longleftarrow \mathrm{B}_\varepsilon\mathrm{O}(A)^+ \quad . $$

[1] We use notation $\mathbf{KQ}$ instead of $\mathbf{L}$, the notation employed in [22] and [7], in order to avoid confusion with the surgery spectrum and the surgery groups.

This is a consequence of the fact that ${}_\varepsilon\mathrm{O}(CA)$ acts trivially on the homology of

$$ {}_\varepsilon\mathrm{O}(MA) = \bigcup_{r>0} {}_\varepsilon\mathrm{O}(M_r(A)). $$

Indeed, for any unital ring with anti-involution, the group ${}_\varepsilon\mathrm{EO}(A)$ possesses a system of generators $E_\beta(a)$, labeled by elements $a \in A$ and by elements β of a certain set of "roots" Φ , such that

$$ E_\beta(a+a') = E_\beta(a)E_\beta(a'), \qquad (a, a' \in A), $$

for every $\beta \in \Phi$ [34, p. 329], cf. also [4, 3.5(a), p. 29]. In view of this, for any $c \in CA$ and $r > 0$, one can represent c as $m + c'$ so that $m \in M_r(A)$ and $c'M_r(A) = M_r(A)c' = 0$. In particular, conjugation of elements of ${}_\varepsilon\mathrm{O}(M_r(A)$ by $E_\beta(c)$ coincides with conjugation by the matrix $E_\beta(m)$ which, as an element of ${}_\varepsilon\mathrm{O}(M_r(A)$, acts trivially on the homology of the group to which it belongs.

Acyclicity of $\mathrm{B}_\varepsilon\mathrm{O}(CA)^+$, which is proved very much like the acyclicity of $\mathrm{BGL}(CA)^+$, implies contractibility of $\mathrm{B}_\varepsilon\mathrm{O}(CA)^+$ and yields functorial isomorphisms

$$ {}_\varepsilon KQ_n(A) \simeq {}_\varepsilon KQ_{n+1}(SA) \qquad (n > 0), \tag{4.2} $$

exactly like in algebraic K-theory. In degrees $n \leq 0$, isomorphism (4.2) is a consequence of the long exact sequence of KQ-groups, associated with extension (4.1), used in conjunction with ${}_\varepsilon KQ_n(CA) = 0$ and isomorphisms ${}_\varepsilon KQ_n(A) = {}_\varepsilon KQ_n(MA) = 0$ holding for all $n \in \mathbf{Z}$.

If A is a real or complex Banach algebra with anti-involution, one can also define the topological Hermitian K-theory spectrum ${}_\varepsilon\mathbf{KQ}^{\mathrm{top}}(A)$ (see [18]). This can be achieved in a manner similar to how the usual topological K-theory spectrum $\mathbf{K}^{\mathrm{top}}(A)$ is built. The definition easily extends to nonunital Banach algebras.

In [18] it was proved that the Hermitian topological K-theory spectrum is 8-periodic:

$$ {}_\varepsilon\mathbf{KQ}^{\mathrm{top}}(A) \sim \Omega^8({}_\varepsilon\mathbf{KQ}^{\mathrm{top}}(A)). $$

This homotopy equivalence is, in fact, induced by the product with a "Bott element" in ${}_1KQ_8^{\mathrm{top}}(\mathbf{R}) \simeq \mathbf{Z} \oplus \mathbf{Z}$.

The functorial *forgetful* and *hyperbolic* morphisms ${}_\varepsilon F$ and ${}_\varepsilon H$ connect the K-theory and ${}_\varepsilon KQ$-theory spectra

$$ {}_\varepsilon F\colon {}_\varepsilon\mathbf{KQ}(A) \longrightarrow \mathbf{K}(A), \qquad {}_\varepsilon H\colon \mathbf{K}(A) \longrightarrow {}_\varepsilon\mathbf{KQ}(A), $$

and similarly for topological K-theory

$$ {}_\varepsilon F^{\mathrm{top}}\colon {}_\varepsilon\mathbf{KQ}^{\mathrm{top}}(A) \longrightarrow \mathbf{K}^{\mathrm{top}}(A), \qquad {}_\varepsilon H^{\mathrm{top}}\colon \mathbf{K}^{\mathrm{top}}(A) \longrightarrow {}_\varepsilon\mathbf{KQ}^{\mathrm{top}}(A). $$

Following [22] we denote the homotopy fibers of ${}_\varepsilon F$ and ${}_\varepsilon H$ by, respectively, ${}_\varepsilon\mathbf{V}(A)$ and ${}_\varepsilon\mathbf{U}(A)$, and their homotopy groups by, respectively,

$$ {}_\varepsilon V_n(A) = \pi_n({}_\varepsilon\mathbf{V}(A)) \qquad \text{and} \qquad {}_\varepsilon U_n(A) = \pi_n({}_\varepsilon\mathbf{U}(A)) \qquad (n \in \mathbf{Z}). $$

We shall call the cokernel of the hyperbolic map,

$$ {}_\varepsilon W_n(A) := \mathrm{Coker}\left(K_n(A) \xrightarrow{{}_\varepsilon H_n} {}_\varepsilon KQ_n(A)\right), $$

the n-th *Witt* group, and the kernel of the forgetful map,

$$ {}_\varepsilon W'_n(A) := \mathrm{Ker}\left({}_\varepsilon KQ_n(A) \xrightarrow{{}_\varepsilon F_n} K_n(A)\right), $$

the n-th *co-Witt* group of a ring with anti-involution A.

In the same way we define, for a Banach algebra with anti-involution, the spectra ${}_{\varepsilon}\mathbf{V}^{\mathrm{top}}(A)$ and ${}_{\varepsilon}\mathbf{U}^{\mathrm{top}}(A)$, their homotopy groups ${}_{\varepsilon}V_n^{\mathrm{top}}(A)$ and ${}_{\varepsilon}U_n^{\mathrm{top}}(A)$, and the topological Witt and co-Witt groups. One can show that

$$ {}_1W_n^{\mathrm{top}}(A) \simeq K_n^{\mathrm{top}}(A) $$

for any real or complex C^*-algebra A with α being the $*$-operation on A [21, Definition 2.2 and Théorème 2.3].

The following theorem was initially established for Banach algebras [18] and later for general unital rings with anti-involution [22]. Recently, it was extended to Grothendieck-Witt rings of pointed pretriangulated DG-categories with weak equivalences and duality [29].

Theorem 4.1. (The Fundamental Theorem of Hermitian K-theory) *For any unital ring with split anti-involution, there exists a natural homotopy equivalence of spectra*

$$ {}_{\varepsilon}\mathbf{V}(A) \sim \Omega({}_{-\varepsilon}\mathbf{U}(A)). $$

If A is a Banach algebra, then there exists a similar homotopy equivalence of spectra

$$ {}_{\varepsilon}\mathbf{V}^{\mathrm{top}}(A) \sim \Omega({}_{-\varepsilon}\mathbf{U}^{\mathrm{top}}(A)), \tag{4.3} $$

and the corresponding functorial comparison morphisms of spectra making the diagram

$$
\begin{tikzcd}
{}_{\varepsilon}\mathbf{V}(A) \arrow[r] \arrow[d] & \Omega({}_{-\varepsilon}\mathbf{U}(A)) \arrow[d] \\
{}_{\varepsilon}\mathbf{V}^{\mathrm{top}}(A) \arrow[r] & \Omega({}_{-\varepsilon}\mathbf{U}^{\mathrm{top}}(A))
\end{tikzcd}
$$

commute up to homotopy. In particular,

$$ {}_{\varepsilon}V_n(A) \simeq {}_{-\varepsilon}U_{n+1}(A) \qquad \text{and} \qquad {}_{\varepsilon}V_n^{\mathrm{top}}(A) \simeq {}_{-\varepsilon}U_{n+1}^{\mathrm{top}}(A) \qquad (n \in \mathbf{Z}). $$

We illustrate the Fundamental Theorem in topological Hermitian K-theory by considering Banach algebras $\mathbf{R}$ and $\mathbf{C}$, equipped with trivial involution, and the algebra of quaternions $\mathbf{H}$, equipped with conjugation

$$ w = a + bi + cj + dk \longmapsto \overline{w} = a - bi - cj - dk. $$

The relevant information is collected in Table 1. The groups in columns 2–4 are filtered unions of the compact forms of the corresponding Lie groups and they have the same homotopy type as $\mathrm{GL}(A)$, ${}_1\mathrm{O}$ and ${}_{-1}\mathrm{O}$. The spaces in the four right columns represent, up to homotopy, the connected components of the spaces occurring as degree 0 terms of the corresponding Ω-spectra. The spaces in columns 5 and 7 are, up to connected components, the loop spaces of their neighbors located in columns 6 and 8, respectively.

Table 1

algebra	GL	${}_1\mathrm{O}$	${}_{-1}\mathrm{O}$	${}_1\mathrm{V}$	${}_{-1}\mathrm{U}$	${}_{-1}\mathrm{V}$	${}_1\mathrm{U}$
R	O	O×O	U	BO	U/O	O/U	O
C	U	O	Sp	U/O	Sp/U	U/Sp	O/U
H	Sp	Sp×Sp	U	BSp	U/Sp	Sp/U	Sp

If we equip $\mathbf{C}$ with the complex conjugation automorphism σ, or if we consider $A = B \times B^{\mathrm{op}}$, where B is one of $\mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$, with the anti-involution transposing the factors, then in all four cases one has ${}_{-1}\mathrm{O}(\mathrm{A}) = {}_{1}\mathrm{O}(\mathrm{A})$. Accordingly, we use the collective notation ${}_{\pm}\mathrm{O}$, ${}_{\pm}\mathrm{V}$ and ${}_{\pm}\mathrm{U}$ when presenting the relevant information in Table 2.

The classical homotopy equivalences of the complex Bott Periodicity correspond to the homotopy equivalences of (4.3) when $(A, \alpha) = (\mathbf{C}, \sigma)$ or $A = B \times B^{\mathrm{op}}$. Six homotopy equivalences of Table 1 and the remaining two homotopy equivalences of Table 2 together describe eight homotopy equivalences of the real Bott Periodicity.

The three equivalences for $A = B \times B^{\mathrm{op}}$ are instances of the standard homotopy equivalence $\mathrm{G} \sim \Omega\mathrm{BG}$, and thus can be considered "trivial". The remaining seven are not.

Table 2

algebra	GL	${}_{\pm}$O	${}_{\pm}$V	${}_{\pm}$U
$\mathbf{R} \times \mathbf{R}^{\mathrm{op}}$	O×O	O	O	BO
$\mathbf{C} \times \mathbf{C}^{\mathrm{op}}$	U×U	U	U	BU
$\mathbf{H} \times \mathbf{H}^{\mathrm{op}}$	Sp×Sp	Sp	Sp	BSp
$(\mathbf{C}, \sigma)$	U	U×U	BU	U

Nonunital algebras with anti-involution

A standard way to extend an additive functor G from unital to nonunital k-algebras is to consider

$$G(A)_k := \mathrm{Ker}\big(G(\tilde{A}_k) \longrightarrow G(k)\big) \tag{4.4}$$

where $\tilde{A}_k := k \ltimes A$ is the *unitalization* of A in the category of k-algebras. The subscript indicates that (4.4) in general depends on the choice of k-algebra structure even though the original functor on the category of unital k-algebras may not. Algebraic K-functor is an example. Additivity of G implies that $G(A)_k$ is canonically isomorphic to $G(A)$ if A is unital.

Let us say that (A, α) is a (k, φ)*-algebra with anti-involution* if A is an algebra over a commutative unital ring k equipped with an involution φ such that anti-involution α is φ-linear

$$\alpha(ca) = \varphi(c)\alpha(a) \qquad (c \in k,\, a \in A).$$

The corresponding unitalization $\tilde{A}_k := k \ltimes A$ is then naturally equipped with an anti-involution that extends α and

$$k \overset{\pi}{\twoheadleftarrow} \tilde{A}_k \longleftarrow\!\!\!\prec A$$

is a split extension in the category of (k, φ)-algebras with anti-involution. In what follows we will limit ourselves to the case when (k, φ) is a split ring with anti-involution. If (A, α) admits such a (k, φ)-algebra structure, we shall say that (A, α) is a *split* nonunital ring with anti-involution.

If G is a functor on the category of (k, φ)-algebras with anti-involution, we shall be denoting the object defined in (4.4) by $G(A)_{k,\varphi}$. If the result *does not* depend on

φ, we shall drop subscript φ. If it does not depend on (k,φ), we shall drop both subscripts and simply write $G(A)$. Thus, we obtain

$K_n(A)_k$, $\quad {}_\varepsilon KQ_n(A)_{k,\varphi}$, $\quad {}_\varepsilon V_n(A)_{k,\varphi}$, $\quad {}_\varepsilon U_n(A)_{k,\varphi}$, $\quad {}_\varepsilon W_n(A)_{k,\varphi}$, $\quad$ and $\quad {}_\varepsilon W'_n(A)_{k,\varphi}$.

Note that any homomorphism $(k,\varphi) \to (k',\varphi')$ of rings with anti-involution induces the corresponding homomorphisms

$$ {}_\varepsilon KQ_n(A)_{k,\varphi} \longrightarrow {}_\varepsilon KQ_n(A)_{k',\varphi'} \tag{4.5}$$

and likewise for ${}_\varepsilon V_n$ and ${}_\varepsilon U_n$ as well as the Witt and co-Witt groups. The maps in (4.5) are instances of the morphisms to be discussed in a section devoted to relative Hermitian K-theory and excision, cf. (4.22).

Analogous considerations apply to functors on the category of Banach algebras with anti-involution. In this case $k = F$ is the ground field, either $\mathbf{R}$ or $\mathbf{C}$, and $\varphi = \mathrm{id}$ unless A is a $\mathbf{C}$-algebra with a sesquilinear anti-involution which can happen only when $\varphi = \sigma$, the complex-conjugation automorphism of $\mathbf{C}$. The topological groups

$$K_n^{\mathrm{top}}(A)_{F,\varphi} \qquad \text{and} \qquad {}_\varepsilon KQ_n^{\mathrm{top}}(A)_{F,\varphi}\ ,$$

indeed do not depend on (F,φ) and therefore will be denoted simply $K_n^{\mathrm{top}}(A)$ and ${}_\varepsilon KQ_n^{\mathrm{top}}(A)$.

A natural transformation of additive functors

$$\tau\colon G \longrightarrow G'$$

on the category of (k,φ)-algebras induces a morphism of functorially split short exact sequences

$$\begin{tikzcd} G(k) \arrow[d, "\tau_k"] & G(\tilde{A}_k) \arrow[l, two heads, "G\pi"'] \arrow[d, "\tau_{\tilde{A}_k}"] & G(A)_{k,\varphi} \arrow[l, tail] \arrow[d, "(\tau_A)_{k,\varphi}"] \\ G'(k) & G'(\tilde{A}_k) \arrow[l, two heads, "G'\pi"'] & G'(A)_{k,\varphi} \arrow[l, tail] \end{tikzcd} \tag{4.6}$$

which implies that $\tau_{\tilde{A}_k}$ is isomorphic to $\tau_k \oplus (\tau_A)_{k,\varphi}$ with respect to the induced decompositions $G(\tilde{A}_k) \simeq G(k) \oplus G(A)_{k,\varphi}$ and $G'(\tilde{A}_k) \simeq G'(k) \oplus G'(A)_{k,\varphi}$. As a consequence, if the target category has kernels and cokernels, and we set

$$\Gamma(A) := \operatorname{Ker} \tau_A \qquad \text{and} \qquad \Gamma'(A) := \operatorname{Coker} \tau_A\ ,$$

we obtain from (4.6) functorially split short exact sequences, one for Γ,

$$\Gamma(k) \xleftarrow{\ \Gamma\pi\ } \Gamma(\tilde{A}_k) \longleftarrow \operatorname{Ker}(\tau_A)_{k,\varphi}\quad ,$$

and one for Γ',

$$\Gamma'(k) \xleftarrow{\ \Gamma'\pi\ } \Gamma'(\tilde{A}_k) \longleftarrow \operatorname{Coker}(\tau_A)_{k,\varphi}\quad .$$

In particular, the functorial morphisms

$$\operatorname{Ker}(\tau_A)_{k,\varphi} \xrightarrow{\ \sim\ } \Gamma(A)_{k,\varphi} \qquad \text{and} \qquad \operatorname{Coker}(\tau_A)_{k,\varphi} \xrightarrow{\ \sim\ } \Gamma'(A)_{k,\varphi} \tag{4.7}$$

are isomorphisms.

Another consequence is that $(\tau_A)_{k,\varphi}$ is an isomorphism if both τ_k and $\tau_{\tilde{A}_{k,\varphi}}$ are isomorphisms.

After making these preliminary remarks we are ready to establish the following useful fact.

Proposition 4.2. *For any Banach algebra A, the canonical comparison maps*

$$ {}_{\varepsilon}W_1(A)_{F,\varphi} \longrightarrow {}_{\varepsilon}W_1^{\mathrm{top}}(A) \tag{4.8}$$

and

$$ {}_{\varepsilon}W'_{-1}(A)_{F,\varphi} \longrightarrow {}_{\varepsilon}W'^{\,\mathrm{top}}_{-1}(A) \tag{4.9}$$

are isomorphisms.

Proof. The map in (4.8) is an isomorphism for a unital Banach algebra as was observed in [18, pp. 404–405]. The following commutative diagram

$$\begin{tikzcd}
{}_{\varepsilon}W'_0(A) \arrow[d, equal] & k'_0(A) \arrow[l] \arrow[d, equal] & {}_{-\varepsilon}W'_{-1}(A) \arrow[l] \arrow[d] & {}_{\varepsilon}W_1(A) \arrow[l] \arrow[d, "\simeq"] & k_0(A) \arrow[l] \arrow[d, equal] \\
{}_{\varepsilon}W'_0(A) & k'_0(A) \arrow[l] & {}_{-\varepsilon}W'^{\,\mathrm{top}}_{-1}(A) \arrow[l] & {}_{\varepsilon}W_1^{\mathrm{top}}(A) \arrow[l] & k_0(A) \arrow[l]
\end{tikzcd}$$

whose rows are portions of a 12-term exact sequence, cf. [22, Théorème 4.3, p. 278], demonstrates the same for the map in (4.9). Here $k_0(A)$ and $k'_0(A)$ denote the even and, respectively, the odd Tate cohomology groups of $\mathbf{Z}/2\mathbf{Z}$ acting on $K_0(A)$ by

$$[P] \longmapsto [P^{\dagger}]$$

where $P^{\dagger}$ denotes the module of α-linear maps $P \to A$ from a finitely generated projective right A-module P to A. Recall that the Tate groups of the cyclic group of order 2 acting on an abelian group V are 2-periodic. The *even* group equals

$$H^{\mathrm{ev}}(\mathbf{Z}/2\mathbf{Z}; V) = \{v \in V \mid v - \bar{v} = 0\}/\{w + \overline{w} \mid w \in V\},$$

where $w \mapsto \overline{w}$ denotes the action of the generator, while the *odd* one equals

$$H^{\mathrm{odd}}(\mathbf{Z}/2\mathbf{Z}; V) = \{v \in V \mid v + \bar{v} = 0\}/\{w - \overline{w} \mid w \in V\}.$$

The nonunital case follows from a remark that preceded Proposition 4.2. □

As was already mentioned in the context of general additive functors, if A is a ring with identity, groups ${}_{\varepsilon}KQ_n(A)_{k,\varphi}$ are canonically isomorphic to ${}_{\varepsilon}KQ_n(A)$. In particular, the former do not depend on the choice of a (k,φ)-algebra structure on (A,α). The same holds also for the ${}_{\varepsilon}V_n$ and ${}_{\varepsilon}U_n$ groups, as well as the Witt and co-Witt groups.

The long exact sequences associated with fibrations

$$\mathbf{K} \longleftarrow {}_{\varepsilon}\mathbf{KQ} \longleftarrow {}_{\varepsilon}\mathbf{V} \qquad \text{and} \qquad {}_{\varepsilon}\mathbf{KQ} \longleftarrow \mathbf{K} \longleftarrow {}_{\varepsilon}\mathbf{U} \tag{4.10}$$

for unital rings, induce functorial long exact sequences

$$\cdots \longleftarrow {}_{\varepsilon}V_{n-1}(A)_{k,\varphi} \longleftarrow K_n(A)_k \longleftarrow {}_{\varepsilon}KQ_n(A)_{k,\varphi} \longleftarrow {}_{\varepsilon}V_n(A)_{k,\varphi} \longleftarrow \cdots, \tag{4.11}$$

and, respectively,

$$\cdots \longleftarrow {}_{\varepsilon}U_{n-1}(A)_{k,\varphi} \longleftarrow {}_{\varepsilon}KQ_n(A)_k \longleftarrow K_n(A)_k \longleftarrow {}_{\varepsilon}U_n(A)_{k,\varphi} \longleftarrow \cdots. \tag{4.12}$$

While working with the long exact sequences of (4.11)–(4.12), it is important to bear in mind existence of functorial isomorphisms

$$\operatorname{Coker}\left(K_n(A)_k \xrightarrow{\ {}_{\varepsilon}H_n\ } {}_{\varepsilon}KQ_n(A)_{k,\varphi}\right) \simeq {}_{\varepsilon}W_n(A)_{k,\varphi}$$

and

$$\operatorname{Ker}\left({}_{\varepsilon}KQ_n(A)_{k,\varphi} \xrightarrow{\ {}_{\varepsilon}F_n\ } K_n(A)_k\right) \simeq {}_{\varepsilon}W'_n(A)_{k,\varphi}$$

which are special instances of the general isomorphisms of (4.7).

There are similar long exact sequences involving *topological* groups for nonunital Banach algebras.

Induction Theorems

The two results we are going to discuss now, collectively called *Induction Theorems*, reflect a key feature of Hermitian K-theory and admit a number of variants. Formally speaking, all those variants sound exactly the same, the only difference being in the meaning of the symbols

$$K_n, \quad {}_\varepsilon KQ_n, \quad {}_\varepsilon V_n, \quad {}_\varepsilon U_n, \quad {}_\varepsilon W_n, \quad {}_\varepsilon W'_n, \tag{4.13}$$

and

$$\overline{K}_n, \quad {}_\varepsilon \overline{KQ}_n, \quad {}_\varepsilon \overline{V}_n, \quad {}_\varepsilon \overline{U}_n, \quad {}_\varepsilon \overline{W}_n, \quad {}_\varepsilon \overline{W}'_n, \tag{4.14}$$

present in their formulations.

We begin by considering one situation served by Induction Theorems: the case of a homomorphism $f\colon A \to \bar{A}$ between unital rings with anti-involution. In the two theorems that follow, the lists of symbols in (4.13)–(4.14) have the following meaning

$$K_n = K_n(A), \qquad {}_\varepsilon KQ_n = {}_\varepsilon KQ_n(A), \qquad {}_\varepsilon V_n = {}_\varepsilon V_n(A), \qquad \dots,$$

and

$$\overline{K}_n = K_n(\bar{A}), \qquad {}_\varepsilon \overline{KQ}_n = {}_\varepsilon \overline{KQ}_n(\bar{A}), \qquad {}_\varepsilon \overline{V}_n = {}_\varepsilon V_n(\bar{A}), \qquad \dots,$$

and the maps between those K-groups, ${}_\varepsilon KQ$-groups, ${}_\varepsilon V$-groups, etc., are all assumed to be induced by $f\colon A \to \bar{A}$.

Theorem 4.3 (Upwards Induction)**.** *Let us assume that*

$$K_n \simeq \overline{K}_n, \quad K_{n+1} \simeq \overline{K}_{n+1}, \quad {}_\varepsilon KQ_n \simeq {}_\varepsilon \overline{KQ}_n, \quad \textit{and} \quad {}_\varepsilon W_{n+1} \simeq {}_\varepsilon \overline{W}_{n+1}$$

for both $\varepsilon = 1$ *and* -1. *Then*

$$ {}_\varepsilon U_n \simeq {}_\varepsilon \overline{U}_n, \qquad {}_\varepsilon V_n \simeq {}_\varepsilon \overline{V}_n, \qquad {}_\varepsilon U_{n+1} \simeq {}_\varepsilon \overline{U}_{n+1}, \qquad {}_\varepsilon V_{n-1} \simeq {}_\varepsilon \overline{V}_{n-1}, \tag{4.15}$$

as well as

$$ {}_\varepsilon KQ_{n+1} \simeq {}_\varepsilon \overline{KQ}_{n+1} \qquad \textit{and} \qquad {}_\varepsilon W_{n+2} \simeq {}_\varepsilon \overline{W}_{n+2} \tag{4.16}$$

for both $\varepsilon = 1$ *and* -1.

Proof. We shall analyze a sequence of commutative diagrams whose rows are portions of the long exact sequences of homotopy groups of two fibrations in (4.10) and whose vertical arrows are the maps induced by f.

In view of the hypothesis, the diagram

$$\begin{tikzcd}
{}_\varepsilon KQ_n \arrow[d, "\simeq"] & K_n \arrow[l] \arrow[d, "\simeq"] & {}_\varepsilon U_n \arrow[l] \arrow[d] & {}_\varepsilon KQ_{n+1} \arrow[l] \arrow[d] & K_{n+1} \arrow[l] \arrow[d, "\simeq"] \\
{}_\varepsilon \overline{KQ}_n & \overline{K}_n \arrow[l] & {}_\varepsilon \overline{U}_n \arrow[l] & {}_\varepsilon \overline{KQ}_{n+1} \arrow[l] & \overline{K}_{n+1} \arrow[l]
\end{tikzcd}$$

yields the diagrams

$$\begin{tikzcd}
{}_\varepsilon KQ_n \arrow[d, "\simeq"] & K_n \arrow[l] \arrow[d, "\simeq"] & {}_\varepsilon U_n \arrow[l] \arrow[d] & {}_\varepsilon W_{n+1} \arrow[l] \arrow[d, "\simeq"] & 0 \arrow[l] \\
{}_\varepsilon \overline{KQ}_n & \overline{K}_n \arrow[l] & {}_\varepsilon \overline{U}_n \arrow[l] & {}_\varepsilon \overline{W}_{n+1} \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

and

$$\begin{tikzcd} 0 & {}_\varepsilon W_{n+1} \arrow[l] \arrow[d, "\simeq"] & {}_\varepsilon KQ_{n+1} \arrow[l] \arrow[d] & K_{n+1} \arrow[l] \arrow[d] \\ 0 & {}_\varepsilon \overline{W}_{n+1} \arrow[l] & {}_\varepsilon \overline{KQ}_{n+1} \arrow[l] & \overline{K}_{n+1} \arrow[l] \end{tikzcd}.$$

The Five Lemma implies that ${}_\varepsilon U_n \simeq {}_\varepsilon \overline{U}_n$ and ${}_\varepsilon KQ_{n+1} \twoheadrightarrow {}_\varepsilon \overline{KQ}_{n+1}$. By applying it again to the diagram

$$\begin{tikzcd} K_n \arrow[d, "\simeq"] & {}_\varepsilon KQ_n \arrow[l] \arrow[d, "\simeq"] & {}_\varepsilon V_n \arrow[l] \arrow[d] & K_{n+1} \arrow[l] \arrow[d, "\simeq"] & {}_\varepsilon KQ_{n+1} \arrow[l] \arrow[d, two heads] \\ \overline{K}_n & {}_\varepsilon \overline{KQ}_n \arrow[l] & {}_\varepsilon \overline{V}_n \arrow[l] & \overline{K}_{n+1} \arrow[l] & {}_\varepsilon \overline{KQ}_{n+1} \arrow[l] \end{tikzcd}.$$

we obtain ${}_\varepsilon V_n \simeq {}_\varepsilon \overline{V}_n$.

The remaining two isomorphisms in (4.15) follow from the two proven ones in view of the Fundamental Theorem of Hermitian K-theory.

The Five Lemma applied to the diagrams

$$\begin{tikzcd} K_n \arrow[d, "\simeq"] & {}_\varepsilon U_n \arrow[l] \arrow[d, "\simeq"] & {}_\varepsilon KQ_{n+1} \arrow[l] \arrow[d] & K_{n+1} \arrow[l] \arrow[d, "\simeq"] & {}_\varepsilon U_{n+1} \arrow[l] \arrow[d, "\simeq"] \\ \overline{K}_n & {}_\varepsilon \overline{U}_n \arrow[l] & {}_\varepsilon \overline{KQ}_{n+1} \arrow[l] & \overline{K}_{n+1} \arrow[l] & {}_\varepsilon \overline{U}_{n+1} \arrow[l] \end{tikzcd}$$

and

$$\begin{tikzcd} {}_\varepsilon KQ_{n+1} \arrow[d, "\simeq"] & K_{n+1} \arrow[l] \arrow[d, "\simeq"] & {}_\varepsilon U_{n+1} \arrow[l] \arrow[d, "\simeq"] & {}_\varepsilon W_{n+2} \arrow[l] \arrow[d] & 0 \arrow[l] \\ {}_\varepsilon \overline{KQ}_{n+1} & \overline{K}_{n+1} \arrow[l] & {}_\varepsilon \overline{U}_{n+1} \arrow[l] & {}_\varepsilon \overline{W}_{n+2} \arrow[l] & 0 \arrow[l] \end{tikzcd}$$

yields the two isomorphisms of (4.16). □

Corollary 4.4. *If*

$${}_\varepsilon KQ_l \simeq {}_\varepsilon \overline{KQ}_l, \qquad {}_\varepsilon KQ_{l+1} \simeq {}_\varepsilon \overline{KQ}_{l+1}, \qquad \text{and} \qquad K_n \simeq \overline{K}_n,$$

for $n \geq l$ and both $\varepsilon = 1$ and -1, then

$${}_\varepsilon KQ_n \simeq {}_\varepsilon \overline{KQ}_n, \qquad {}_\varepsilon V_n \simeq {}_\varepsilon \overline{V}_n, \qquad \text{and} \qquad {}_\varepsilon U_n \simeq {}_\varepsilon \overline{U}_n,$$

for $n \geq l$ and both $\varepsilon = 1$ and -1.

□

Theorem 4.5 (Downwards Induction). *Let us assume that*

$$K_n \simeq \overline{K}_n, \quad K_{n-1} \simeq \overline{K}_{n-1}, \quad {}_\varepsilon KQ_n \simeq {}_\varepsilon \overline{KQ}_n, \quad \text{and} \quad {}_\varepsilon W'_{n-1} \simeq {}_\varepsilon \overline{W}'_{n-1}$$

for both $\varepsilon = 1$ and -1. Then

$$(4.17) \quad {}_\varepsilon V_{n-1} \simeq {}_\varepsilon \overline{V}_{n-1}, \qquad {}_\varepsilon U_{n-1} \simeq {}_\varepsilon \overline{U}_{n-1}, \qquad {}_\varepsilon U_n \simeq {}_\varepsilon \overline{U}_n, \qquad {}_\varepsilon V_{n-2} \simeq {}_\varepsilon \overline{V}_{n-2},$$

as well as

$$({4.18}) \qquad {}_{\varepsilon}KQ_{n-1} \simeq {}_{\varepsilon}\overline{KQ}_{n-1} \qquad \textit{and} \qquad {}_{\varepsilon}W'_{n-2} \simeq {}_{\varepsilon}\overline{W}'_{n-2}$$

for both $\varepsilon = 1$ *and* -1.

Proof. The proof is analogous. We provide the details for the reader's convenience. In view of the hypothesis, the diagram

$$\begin{tikzcd}
K_{n-1} \arrow[d, "\simeq"] & {}_{\varepsilon}KQ_{n-1} \arrow[l] \arrow[d] & {}_{\varepsilon}V_{n-1} \arrow[l] \arrow[d] & K_n \arrow[l] \arrow[d, "\simeq"] & {}_{\varepsilon}KQ_n \arrow[l] \arrow[d, "\simeq"] \\
\overline{K}_{n-1} & {}_{\varepsilon}\overline{KQ}_{n-1} \arrow[l] & {}_{\varepsilon}\overline{V}_{n-1} \arrow[l] & \overline{K}_n \arrow[l] & {}_{\varepsilon}\overline{KQ}_n \arrow[l]
\end{tikzcd}$$

yields the diagrams

$$\begin{tikzcd}
0 & {}_{\varepsilon}W'_{n-1} \arrow[l] \arrow[d, "\simeq"] & {}_{\varepsilon}V_{n-1} \arrow[l] \arrow[d] & K_n \arrow[l] \arrow[d, "\simeq"] & {}_{\varepsilon}KQ_n \arrow[l] \arrow[d, "\simeq"] \\
0 & {}_{\varepsilon}\overline{W}'_{n-1} \arrow[l] & {}_{\varepsilon}\overline{V}_{n-1} \arrow[l] & \overline{K}_n \arrow[l] & {}_{\varepsilon}\overline{KQ}_n \arrow[l]
\end{tikzcd}$$

and

$$\begin{tikzcd}
K_{n-1} \arrow[d, "\simeq"] & {}_{\varepsilon}KQ_{n-1} \arrow[l] \arrow[d] & {}_{\varepsilon}W'_{n-1} \arrow[l] \arrow[d, "\simeq"] & 0 \arrow[l] \\
\overline{K}_{n-1} & {}_{\varepsilon}\overline{KQ}_{n-1} \arrow[l] & {}_{\varepsilon}\overline{W}'_{n-1} \arrow[l] & 0 \arrow[l]
\end{tikzcd}.$$

The Five Lemma implies that ${}_{\varepsilon}V_{n-1} \simeq {}_{\varepsilon}\overline{V}_{n-1}$ and ${}_{\varepsilon}KQ_{n+1} \rightarrowtail {}_{\varepsilon}\overline{KQ}_{n-1}$. By applying it again to the diagram

$$\begin{tikzcd}
{}_{\varepsilon}KQ_{n-1} \arrow[d, tail] & K_{n-1} \arrow[l] \arrow[d, "\simeq"] & {}_{\varepsilon}U_{n-1} \arrow[l] \arrow[d] & {}_{\varepsilon}KQ_n \arrow[l] \arrow[d, "\simeq"] & K_n \arrow[l] \arrow[d, "\simeq"] \\
{}_{\varepsilon}\overline{KQ}_{n-1} & \overline{K}_{n-1} \arrow[l] & {}_{\varepsilon}\overline{U}_{n-1} \arrow[l] & {}_{\varepsilon}\overline{KQ}_n \arrow[l] & \overline{K}_n \arrow[l]
\end{tikzcd}$$

we obtain ${}_{\varepsilon}U_{n-1} \simeq {}_{\varepsilon}\overline{U}_{n-1}$.

The remaining two isomorphisms in (4.17) follow from the two proven ones in view of the Fundamental Theorem of Hermitian K-theory.

The Five Lemma applied to the diagrams

$$\begin{tikzcd}
K_n \arrow[d, "\simeq"] & {}_{\varepsilon}U_n \arrow[l] \arrow[d, "\simeq"] & {}_{\varepsilon}KQ_{n-1} \arrow[l] \arrow[d] & K_{n-1} \arrow[l] \arrow[d, "\simeq"] & {}_{\varepsilon}U_{n-1} \arrow[l] \arrow[d, "\simeq"] \\
\overline{K}_n & {}_{\varepsilon}\overline{U}_n \arrow[l] & {}_{\varepsilon}\overline{KQ}_{n-1} \arrow[l] & \overline{K}_{n-1} \arrow[l] & {}_{\varepsilon}\overline{U}_{n-1} \arrow[l]
\end{tikzcd}$$

and

$$
\begin{array}{ccccccccc}
0 & \longleftarrow & {}_\varepsilon W'_{n-2} & \longleftarrow & {}_\varepsilon V_{n-2} & \longleftarrow & K_{n-1} & \longleftarrow & {}_\varepsilon KQ_{n-1} \\
 & & \downarrow & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\
0 & \longleftarrow & {}_\varepsilon \overline{W}'_{n-2} & \longleftarrow & {}_\varepsilon \overline{V}_{n-2} & \longleftarrow & \overline{K}_{n-1} & \longleftarrow & {}_\varepsilon \overline{KQ}_{n-1}
\end{array}
$$

yields the two isomorphisms of (4.18). □

Corollary 4.6. *If*

$$ {}_\varepsilon KQ_l \simeq {}_\varepsilon \overline{KQ}_l, \qquad {}_\varepsilon KQ_{l-1} \simeq {}_\varepsilon \overline{KQ}_{l-1}, \qquad \textit{and} \qquad K_n \simeq \bar{K}_n, $$

for $n \leq l$ and both $\varepsilon = 1$ and -1, then

$$ {}_\varepsilon KQ_n \simeq {}_\varepsilon \overline{KQ}_n, \qquad {}_\varepsilon V_n \simeq {}_\varepsilon \overline{V}_n, \qquad \textit{and} \qquad {}_\varepsilon U_n \simeq {}_\varepsilon \overline{U}_n $$

for $n \leq l$ and both $\varepsilon = 1$ and -1.

□

Nonunital and topological variants. Suppose (A, α) is a nonunital (k, φ)-algebra with anti-involution and $(\bar{A}, \bar{\alpha})$ is a nonunital $(\bar{k}, \bar{\varphi})$-algebra with anti-involution, and (f, f_k) is a corresponding homomorphism, i.e., a pair of homomorphisms of rings with anti-involution $f\colon A \to \bar{A}$ and $f_k\colon k \to \bar{k}$ such that

$$ f(ca) = f_k(c)f(a) \qquad (c \in k;\ a \in A). $$

Theorems 4.3 and 4.5 hold if

$$ K_n = K_n(A)_{k,\varphi}, \qquad {}_\varepsilon KQ_n = {}_\varepsilon KQ_n(A)_{k,\varphi}, \qquad {}_\varepsilon V_n = {}_\varepsilon V_n(A)_{k,\varphi}, \qquad \dots, $$

and

$$ \bar{K}_n = K_n(\bar{A})_{k,\varphi}, \qquad {}_\varepsilon \overline{KQ}_n = {}_\varepsilon \overline{KQ}_n(\bar{A})_{k,\varphi}, \qquad {}_\varepsilon \overline{V}_n = {}_\varepsilon V_n(\bar{A})_{k,\varphi}, \qquad \dots, $$

and the maps between K-groups, ${}_\varepsilon KQ$-groups, ${}_\varepsilon V$-groups, etc., are all assumed to be induced by (f, f_k). We shall refer to these theorems as the *Nonunital Homomorphism Induction Theorems* in order to distinguish them from the original formulations of Theorems 4.3 and 4.5 which we shall refer to as the *Unital Homomorphism Induction Theorems*. They occur as special cases of more general theorems involving *relative* K-groups, ${}_\varepsilon KQ$-groups, ${}_\varepsilon V$-groups, etc. Properly introducing the corresponding relative versions of all the objects would significantly increase the size of this chapter while it is not needed for the results in the present article.

Then there are obvious versions of both pairs of induction theorems for Banach algebras and topological K-theory. We shall treat them as a single pair of theorems acknowledging the fact that $K_*^{\mathrm{top}}(\ )_F$ and ${}_\varepsilon KQ *^{\mathrm{top}} (\ )_{F,\varphi}$, and the functors derived from them, do not depend on the (F, φ)-algebra structure, as was already pointed out on p. 19. We shall refer to this pair as the *Topological* (or, *Banach*) *Induction Theorems*.

This produces six theorems (plus six corollaries) in total. Now, one can replace the ordinary homotopy groups by the homotopy groups with coefficients, e.g., finite or rational. This will produce the corresponding versions "with coefficients" of all those theorems and their corollaries. Several such results have been previously employed, cf. e.g., [7] and [6]. In the next section we will encounter yet another variety: the *Comparison Induction Theorems* for Banach algebras.

The comparison map in Hermitian K-theory

There are functorial comparison maps for nonunital Banach algebras,

$$ {}_{\varepsilon}KQ_*(A)_{k,\varphi} \longrightarrow {}_{\varepsilon}KQ_*^{\mathrm{top}}(A), \tag{4.19}$$

and similar comparison maps for ${}_{\varepsilon}V_*$, ${}_{\varepsilon}U_*$, ${}_{\varepsilon}W_*$ and ${}_{\varepsilon}W'_*$, where $k \subseteq F$ denotes any subring of the ground field and $\varphi = \mathrm{id}$ unless $F = \mathbf{C}$ and the anti-involution on A is sesquilinear. In the last case k is supposed to be a subring of $\mathbf{C}$ invariant under complex conjugation and φ is complex conjugation restricted to k.

We shall study those maps in relation to the comparison maps in algebraic K-theory,

$$ K_n(A)_k \longrightarrow K_n^{\mathrm{top}}(A), \tag{4.20}$$

using yet another variety of Induction Theorems. In the *Comparison Induction Theorems* only a single Banach algebra A is present, and the groups in (4.14) are the topological counterparts of the groups in (4.13). The comparison maps of (4.20) play the role of the maps $K_n \to \overline{K}_n$, the comparison maps of (4.19) play the role of the maps ${}_{\varepsilon}KQ_n \to {}_{\varepsilon}\overline{KQ}_n$, etc.

The following theorem is a direct consequence of Proposition 4.2 combined with the Comparison Induction Theorems.

Theorem 4.7. *Let A be a Banach algebra with anti-involution. If the comparison maps of* (4.20) *are isomorphisms in the range* $0 < n \leq n'$ *(respectively, in the range* $n'' \leq n < 0$*), then the comparison maps of* (4.19) *are isomorphisms in precisely the same range.*

Equipped with Theorem 4.7 we deduce the following result from Theorem 1.1.

Theorem 4.8. *If a Banach algebra A is a stable retract of $\mathcal{K}\otimes_{\max}A$ for some C^*-algebra A, or of $\mathcal{K}\widehat{\otimes}_{\pi}A$, for some Banach algebra A, then, for any anti-involution α on A, the comparison maps in Hermitian K-theory,* (4.19), *are isomorphisms for $n \leq 0$.* ☐

The next result is similarly deduced from Theorems 2.3 and 3.5

Theorem 4.9. *If A is a Banach $\mathcal{K}_{\mathbf{R}}$ or $\mathcal{K}_{\mathbf{C}}$-ring, then, for any anti-involution α on A, the comparison maps in Hermitian K-theory,* (4.19), *are isomorphisms for all $n \in \mathbf{Z}$.* ☐

The case of a complex stable C^*-algebra was deduced from the results of [32] by Battikh [6].

All rings satisfy excision in K-theory in degrees less or equal 0, while rings satisfying the hypothesis of Theorem 4.9 satisfy excision in all degrees. In the next section we prove that such rings satisfy excision also in Hermitian K-theory which explains why the groups ${}_{\varepsilon}KQ_*(A)_{k,\varphi}$ are according to Theorem 4.9 all isomorphic to each other.

Relative ${}_{\varepsilon}KQ$-groups and Excision

If $A \subseteq R$ is a two-sided ideal in a unital ring with anti-involution α and A is α-invariant, then the relative ${}_{\varepsilon}KQ$-groups are defined in terms of the homotopy fiber $\mathbf{F}(R, A)$ of the morphism of Ω-spectra,

$$ {}_{\varepsilon}\mathbf{KQ}(R) \longrightarrow {}_{\varepsilon}\mathbf{KQ}(R/A)), $$

induced by the quotient homomorphism $R \to R/A$. The relative ${}_{\varepsilon}KQ$-groups,

$$ {}_{\varepsilon}KQ_n(R, A) := \pi_n(\mathbf{F}(R, A)) \qquad (n \in \mathbf{Z}), $$

become functors on the category of triples (R, A, α). One has

$$ {}_\varepsilon KQ_n(R, A) = \pi_n(F(R, A)) \qquad (n > 0) $$

where $F(R, A)$ denotes the homotopy fiber of the map

$$ \mathrm{B}_\varepsilon \mathrm{O}(R)^+ \longrightarrow \mathrm{B}_\varepsilon \mathrm{O}(R/A)^+. $$

The connected component of $F(R, A)$ is naturally identified with $\bar{F}(R, A)$, the homotopy fiber of

$$ \mathrm{B}_\varepsilon \mathrm{O}(R)^+ \longrightarrow \mathrm{B}_\varepsilon \bar{\mathrm{O}}(R/A)^+, $$

where ${}_\varepsilon\bar{\mathrm{O}}(R/A)$ denotes the image of ${}_\varepsilon\mathrm{O}(R)$ in ${}_\varepsilon\mathrm{O}(R/A)$ and

$$ \pi_0(F(R, A)) = \mathrm{Coker}\big({}_\varepsilon KQ_1(R) \longrightarrow {}_\varepsilon KQ_1(R/A)\big) \simeq \mathrm{Ker}\big({}_\varepsilon KQ_0(R, A) \longrightarrow {}_\varepsilon KQ_0(R)\big), $$

while the homotopy equivalence

$$ F(R, A) \ \sim \ \Omega FE(SR, SA), $$

where $FE(R, A)$ is simultaneously the homotopy fiber of the map

$$ \mathrm{B}_\varepsilon \mathrm{EO}(R)^+ \longrightarrow \mathrm{B}_\varepsilon \mathrm{EO}(R/A)^+, $$

and a covering of $\bar{F}(R, A)$, induces suspension isomorphisms

$$ \pi_n(R, A) \ \simeq \ \pi_{n+1}(SR, SA) \qquad (n > 0). $$

For any homomorphism of unital rings with anti-involution $f\colon R \to R'$, let us consider, following [33], the ring $\Gamma(f)$ defined by the pull-back diagram

$$ \begin{tikzcd} CR' \arrow[d, two heads] & \Gamma(f) \arrow[l, dashed] \arrow[d, dashed, two heads] \\ SR' & SR \arrow[l, two heads] \end{tikzcd} . \tag{4.21} $$

The diagram in (4.21) induces the following commutative diagram of group extensions

$$ \begin{tikzcd} {}_\varepsilon\mathrm{O}(MR') \arrow[r, equal] \arrow[d, tail] & {}_\varepsilon\mathrm{O}(MR') \arrow[d, tail] \\ {}_\varepsilon\mathrm{O}(CR') \arrow[d, two heads] & {}_\varepsilon\mathrm{O}(\Gamma(f)) \arrow[l] \arrow[d, two heads] \\ {}_\varepsilon\mathrm{EO}(SR') & {}_\varepsilon\bar{\mathrm{O}}(SR) \arrow[l] \end{tikzcd} $$

where ${}_\varepsilon\bar{\mathrm{O}}(SR)$ is the image of ${}_\varepsilon\mathrm{O}(\Gamma(f))$ in ${}_\varepsilon\mathrm{O}(SR)$. The group ${}_\varepsilon\mathrm{O}(\Gamma(f))$ acts trivially on the homology of ${}_\varepsilon\mathrm{O}(MR')$ because the action factorizes through the action of ${}_\varepsilon\mathrm{O}(CR')$ which is trivial, as we pointed above. It follows that

$$ \mathrm{B}_\varepsilon\mathrm{EO}(SR')^+ \longleftarrow \mathrm{B}_\varepsilon\bar{\mathrm{O}}(SR)^+ \longleftarrow \mathrm{B}_\varepsilon\mathrm{O}(\Gamma(f))^+ \longleftarrow \mathrm{B}_\varepsilon\mathrm{O}(R')^+ $$

is an initial segment of the Puppe sequence of homotopy fibrations.

In the special case when $f\colon R \to R/A$ is the quotient homomorphism, we obtain the natural map

$$ F(R, A) \xrightarrow{\ \sim\ } \Omega FE(SR, SA) \longrightarrow \Omega \mathrm{B}_\varepsilon\bar{\mathrm{O}}(\Gamma(R, A))^+ $$

where $\Gamma(R, A) := \Gamma(f)$. It induces isomorphism on π_n, for $n > 0$, and is injective on π_0. In particular, a morphism $\phi\colon (R, A) \to (R', A')$ induces an isomorphism of relative ${}_\varepsilon KQ$-groups for a particular $n > 0$,

$$ {}_\varepsilon KQ_n(R, A) \simeq {}_\varepsilon KQ_n(R', A'), \tag{4.22} $$

if and only if

$$ {}_\varepsilon KQ_{n+1}(\Gamma(R, A)) \simeq {}_\varepsilon KQ_{n+1}(\Gamma(R', A')). \tag{4.23} $$

All the arguments in this section have exact counterparts for the general linear group GL, the group of elementary matrices E, and algebraic K-theory. In particular,

$$ K_n(R, A) \simeq K_n(R', A'), \tag{4.24} $$

for a given $n > 0$, if and only if

$$ K_{n+1}(\Gamma(R, A)) \simeq K_{n+1}(\Gamma(R', A')). \tag{4.25} $$

Recall that a nonunital ring A is said to satisfy excision for K_n if any morphism

$$ \phi\colon (R, A) \longrightarrow (R', A) \tag{4.26} $$

which on A restricts to the identity map, induces an isomorphism on K_n

$$ K_n(R, A) \xrightarrow[\sim]{K_n\phi} K_n(R', A) \quad . $$

In the same way, one can define excision for any other functor on the category of pairs (R, A). Analogously, one defines excision for nonunital rings with anti-involution and ${}_\varepsilon KQ$.

According to Bass' Excision Theorem [3, Sections VII.6 and XII.8], every ring satisfies excision for K_n and $n \leq 0$. A similar result holds for nonunital rings with anti-involution [17, Section 3, Théorème 4.1]. The following result demonstrates usefulness of the Induction Theorems.

Theorem 4.10 (cf. [5], [6]). *If a ring A satisfies excision for K_n, $n \leq n'$, then, for any anti-involution on A which admits a split unitalization, (A, α) satisfies excision in ${}_\varepsilon KQ_n$ in the same range $n \leq n'$.*

Proof. Consider the commutative diagram of ring extensions

$$ \begin{tikzcd} C(R/A) \arrow[d] & \Gamma(R, A) \arrow[l, two heads] \arrow[d, "\Gamma\phi"] & SA \arrow[l, tail] \arrow[d, equal] \\ C(R'/A) & \Gamma(R', A) \arrow[l, two heads] & SA \arrow[l, tail] \end{tikzcd} $$

where $\Gamma\phi$ is induced by (4.26). The associated long exact sequences of K-groups and ${}_\varepsilon KQ$-groups produce two sequences of commutative squares, $n \in \mathbf{Z}$,

$$ \begin{tikzcd} K_n(\Gamma(R, A)) \arrow[d, "\Gamma\phi"] & K_n(\Gamma(R, A), SA) \arrow[l, "\sim"'] \arrow[d] \\ K_n(\Gamma(R', A)) & K_n(\Gamma(R', A), SA) \arrow[l, "\sim"'] \end{tikzcd} \tag{4.27} $$

and

$$
(4.28)\qquad
\begin{tikzcd}
{}_{\varepsilon}KQ_n(\Gamma(R,A)) \arrow[d, "\Gamma\phi"] & {}_{\varepsilon}KQ_n(\Gamma(R,A),SA) \arrow[l, "\sim"'] \arrow[d] \\
{}_{\varepsilon}KQ_n(\Gamma(R',A)) & {}_{\varepsilon}KQ_n(\Gamma(R',A),SA) \arrow[l, "\sim"']
\end{tikzcd}
.
$$

In view of Bass' Excision Theorem, right vertical arrows in (4.27) are isomorphisms for $n \leq 0$. Similarly for (4.28), in view of [17, Section 3, Théorème 4.1].

Equivalence of (4.24) and (4.25) used in conjunction with the Upwards Induction Theorem proves that $\Gamma\phi$ induces isomorphisms ${}_{\varepsilon}KQ_{n+1}(\Gamma(R,A)) \simeq {}_{\varepsilon}KQ_{n+1}(\Gamma(R',A))$ for $n \leq n'$. Invoking equivalence of (4.23) and (4.22) completes the proof. □

Appendix A. Multiplicative structures in K-theory

A.1. **Product in K-theory.** We say that a pairing between rings

$$(A.1)\qquad \varphi : A \times B \longrightarrow C$$

is *bimultiplicative* if

$$\varphi(aa', bb') = \varphi(a,b)\varphi(a',b') \qquad (a, a' \in A;\, b, b' \in B).$$

If rings are unital, we say that the pairing is *unital* if

$$\varphi(1_A, 1_B) = 1_C.$$

A unital bimultiplicative pairing induces pairings between algebraic K-groups

$$(A.2)\qquad K_m(A) \times K_n(B) \longrightarrow K_{m+n}(C) \qquad (m, n \in \mathbf{Z}).$$

A detailed treatment can be found, for instance, in [21, pp. 219-227]. The pairings in (A.2) are associative, graded-commutative and functorial. One calls them collectively the *product structure* in algebraic K-theory of unital rings,

Similarly, a *continuous* unital bimultiplicative pairing between Banach algebras induces pairings between topological K-groups

$$(A.3)\qquad K_m^{\text{top}}(A) \times K_n^{\text{top}}(B) \longrightarrow K_{m+n}^{\text{top}}(C) \qquad (m, n \in \mathbf{Z}),$$

and the comparison map between algebraic and topological K-groups carries the pairings in (A.2) to the pairings in (A.3), cf. [21, Sections 1.25 and 2.24].

The algebraic K-groups of a non unital ring A are usually defined as

$$K_n(A) := \operatorname{Ker}\left(K_n(\tilde{A}) \longrightarrow K_n(\mathbf{Z})\right)$$

where $\tilde{A} := \mathbf{Z} \ltimes A$ denotes the *unitalization* of ring A. If A is a k-algebra over a unital ring k, then one can use instead the unitalization $\tilde{A}_k := k \ltimes A$ in the category of k-algebras:

$$K_n(A)_k := \operatorname{Ker}\left(K_n(\tilde{A}_k) \longrightarrow K_n(k)\right).$$

The corresponding K-groups, however, depend on the choice of k if $n > 0$.

A bimultiplicative pairing (A.1) between nonunital rings does not extend to a pairing

$$\tilde{A}_{\mathbf{Z}} \times \tilde{B}_{\mathbf{Z}} \longrightarrow \widetilde{C}_{\mathbf{Z}}.$$

This is underscored by the fact that the universal pairing

$$A \times B \longrightarrow A \otimes_{\mathbf{Z}} B, \qquad (a, b) \longmapsto a \otimes b \qquad (a \in A \;\; b \in B),$$

induces the unital pairing
$$\tilde{A}_{\mathbf{Z}} \times \tilde{B}_{\mathbf{Z}} \longrightarrow \tilde{A}_{\mathbf{Z}} \otimes_{\mathbf{Z}} \tilde{B}_{\mathbf{Z}}$$
rather than
$$\tilde{A}_{\mathbf{Z}} \times \tilde{B}_{\mathbf{Z}} \longrightarrow (A \otimes_{\mathbf{Z}} B)\tilde{}_{\mathbf{Z}}$$
and it, accordingly, induces pairings between the relative K-groups
$$K_m(A) \times K_n(B) \longrightarrow K_{m+n}\big(\tilde{A}_{\mathbf{Z}} \otimes_{\mathbf{Z}} \tilde{B}_{\mathbf{Z}}, A \otimes_{\mathbf{Z}} B\big)$$
instead of
$$K_m(A) \times K_n(B) \longrightarrow K_{m+n}\big(A \otimes_{\mathbf{Z}} B\big). \tag{A.4}$$
The canonical map
$$K_*\big(A \otimes_{\mathbf{Z}} B\big) \longrightarrow K_*\big(\tilde{A}_{\mathbf{Z}} \otimes \tilde{B}_k, A \otimes_{\mathbf{Z}} B\big) \tag{A.5}$$
is an isomorphism in degrees less or equal 0, and rarely so in positive degrees. If the ring $A \otimes_{\mathbf{Z}} B$ satisfies excision in K-theory, then the map in (A.5) is an isomorphism in all degrees, and any bimultiplicative pairing (A.1) induces binary pairings
$$K_m(A) \times K_n(B) \longrightarrow K_{m+n}(C).$$
Questions of associativity for such pairings would, however, depend on whether the triple tensor products $A \otimes_{\mathbf{Z}} B \otimes_{\mathbf{Z}} C$ satisfy excision as well.

Rings satisfying excision in rational algebraic K-theory were completely characterized in a pair of articles [36] and [32]. It was also proved that a $\mathbf{Q}$-algebra satisfies excision in K-theory if and only if it is H-unital. The category of H-unital $\mathbf{Q}$-algebras is closed under $\otimes_{\mathbf{Z}}$, cf. [32, Theorem 7.10]. This in turn implies that the universal pairings of (A.4) are for such rings associative.

Appendix B. Bounded approximate identities and $\widehat{\otimes}_\pi$

We collect here some important facts that relate presence of a bounded approximate identity in a Banach algebra to certain exactness properties of the projective tensor product of Banach spaces, introduced by Schatten [28]. These results are quite well known, and apply equally to complex and real algebras. In particular, the terms *normed algebra*, *Banach algebra* and *Banach space* will be used below collectively, to cover the complex and the real cases alike.

We shall say that an extension in the category of Banach spaces (with bounded linear maps as morphisms),
$$D \overset{p}{\twoheadleftarrow} E \overset{i}{\hookleftarrow} F\ , \tag{B.1}$$
is *pure* or, more precisely, $\widehat{\otimes}_\pi$-pure, if the functor $C\widehat{\otimes}_\pi$ preserves exactness of (B.1) for any Banach space C.

Lemma B.1. *The following conditions are equivalent:*

(a) *the extension in* (B.1) *is pure;*
(b) *the sequence*
$$F^*\widehat{\otimes}_\pi D \xleftarrow{\mathrm{id}_{F^*}\widehat{\otimes}_\pi p} F^*\widehat{\otimes}_\pi E \xleftarrow{\mathrm{id}_{F^*}\widehat{\otimes}_\pi i} F^*\widehat{\otimes}_\pi F \tag{B.2}$$
is exact;
(c) *the dual extension*
$$D^* \overset{p^*}{\rightarrowtail} E^* \overset{i^*}{\twoheadrightarrow} F^* \tag{B.3}$$
is split.[2]

[2]In the existing literature on Banach spaces such extensions are often called *weakly split*.

Proof. Since a one-dimensional Banach space is an injective cogenerator in the category of Banach spaces (the Hahn-Banach theorem), the sequence

$$C\widehat{\otimes}_\pi D \xleftarrow{\mathrm{id}_C\widehat{\otimes}_\pi p} C\widehat{\otimes}_\pi E \xleftarrow{\mathrm{id}_C\widehat{\otimes}_\pi i} C\widehat{\otimes}_\pi F \tag{B.4}$$

is exact if and only if

$$(C\widehat{\otimes}_\pi D)^* \xrightarrow{(\mathrm{id}_C\widehat{\otimes}_\pi p)^*} (C\widehat{\otimes}_\pi E)^* \xrightarrow{(\mathrm{id}_C\widehat{\otimes}_\pi i)^*} (C\widehat{\otimes}_\pi F)^* \tag{B.5}$$

is exact. Since $(X\widehat{\otimes}_\pi Y)^*$ is isometric to the space $\mathcal{L}(X, Y^*)$ of bounded linear operators from X to Y^*, the sequence in (B.5) coincides with

$$\mathcal{L}(C, D^*) \xrightarrow{p^*\circ} \mathcal{L}(C, E^*) \xrightarrow{i^*\circ} \mathcal{L}(C, F^*) \quad . \tag{B.6}$$

If (B.3) is split, the sequence in (B.6), and thus also in (B.5), are split-exact. In particular, the sequence in (B.4) is exact. In reverse, if the sequence in (B.2) is exact, then its dual

$$\mathcal{L}(F, D^*) \xrightarrow{p^*\circ} \mathcal{L}(F, E^*) \xrightarrow{i^*\circ} \mathcal{L}(F, F^*)$$

is exact which implies that (B.3) is split. □

An example of a pure-exact extension is provided by

$$F^{**}/F \xleftarrow{\pi} F^{**} \xleftarrow{\kappa} F \tag{B.7}$$

where κ denotes the canonical isometric embedding of an arbitrary Banach space F into its second dual. Pure-exactness of (B.7) follows from the fact that $\kappa^*: X^{***} \to X^*$ has a canonical splitting of norm 1. The extension in (B.7) is split precisely when F is isomorphic to a complemented subspace of E^* for some Banach space E.

Recall that a net $(e_i)_{i\in I}$ in a normed algebra A is a *left approximate identity* if

$$\lim_{i\in I} \|e_i a - a\| = 0 \qquad \text{for any } a \in A.$$

A right approximate identity is defined similarly. We say that the approximate identity is *bounded* if $\sup_{i\in I} \|e_i\| < \infty$.

Among numerous examples of normed algebras with bounded approximate identity one should mention that every right (respectively, left) ideal in a C^*-algebra possesses a bounded left (respectively, right) approximate identity [10, 1.7.3].

Proposition B.2. *For any Banach algebra A, the following conditions are equivalent:*

(a) *for any unital Banach algebra B which contains A as a closed right ideal, the extension*

$$(B/A)^* \rightarrowtail B^* \twoheadrightarrow A^*$$

admits a bounded B-linear splitting;

(b) *there exists a unital Banach algebra B_0 which contains A as a closed right ideal, such that the extension*

$$(B_0/A)^{**} \twoheadleftarrow B_0^{**} \leftarrowtail A^{**}$$

admits a bounded B_0-linear splitting;

(c) *A possesses a bounded left approximate identity.*

Proof. Let us consider the commutative diagram of extensions of Banach right B_0-modules

$$\begin{tikzcd}
B_0/A \arrow[d, tail, "\kappa"] & B_0 \arrow[l, two heads, "p"'] \arrow[d, tail, "\kappa"] & A \arrow[l, tail, "i"'] \arrow[d, tail, "\kappa"] \\
(B_0/A)^{**} & B_0^{**} \arrow[l, two heads, "p^{**}"'] & A^{**} \arrow[l, tail, "i^{**}"']
\end{tikzcd}$$

where the vertical arrows correspond to the canonical B_0-linear embeddings into the second dual. Suppose $s\colon B_0^{**} \to A^{**}$ is a bounded B_0-linear map such that $s \circ i^{**} = \mathrm{id}_{A^{**}}$. Let $\epsilon = (s \circ \kappa)(1_{B_0}) \in A^{**}$. Since $s \circ \kappa\colon B_0 \to A^{**}$ is B_0-linear, it is necessarily of the form

$$b \mapsto (s \circ \kappa)(b) = \epsilon b \qquad (b \in B_0)$$

and

$$\epsilon a = (s \circ \kappa \circ i)(a) = (s \circ i^{**} \circ \kappa)(a) = \kappa(a) \qquad (a \in A).$$

Every Banach space X is dense in its second dual in the weak* topology. Indeed, for any $\theta \in X^{**}$, and any linearly independent n-tuple $\xi_1, \ldots, \xi_n \in X^*$, there exists $x \in X$ such that

$$\theta(\xi_1) = \xi_1(x), \qquad \ldots \qquad , \theta(\xi_n) = \xi_n(x).$$

This follows from the fact that the linear mapping

$$\begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} : X \longrightarrow F^n$$

is surjective (F denotes the ground field, either $\mathbf{R}$ or $\mathbf{C}$). Any net $(e_i)_{i \in I}$ in A which converges to ϵ in the weak* topology is a left approximate identity in A. Since the unit ball of A is weak*-dense in the unit ball of A^{**} (Goldstine's theorem, cf., e.g. [11, V.4.5]) one can find $(e_i)_{i \in I}$ with $\sup_{i \in I} \|e_i\| = \|\epsilon\|$. This completes the proof that (b) implies (c).

Given any bounded left approximate identity on A and a unital Banach algebra B which contains A as a closed right ideal, let $(l_i)_{i \in I}$ be the corresponding net of bounded B-linear maps

$$l_i\colon B \longrightarrow A, \qquad b \longmapsto l_i(b) = e_i b \qquad (b \in B).$$

The net of adjoint maps $l_i^*\colon A^* \to B^*$ is bounded, hence, by the Banach-Alaoğlu theorem, there exists a subnet $(l_i^*)_{i \in I'}$ which converges in the weak* topology of $\mathcal{L}(A^*, B^*) \simeq (A^* \widehat{\otimes}_\pi B)^*$ to a map $\lambda \in \mathcal{L}(A^*, B^*)$. That map, being the limit of B-linear maps is B-linear, and

$$(\lambda(\alpha))(a) = \lim_{i \in I'} (l_i(\alpha))(a) = \lim_{i \in I'} \alpha(e_i a) = \alpha(a) \qquad (\alpha \in A^*, \, a \in A),$$

shows that $i^* \circ l = \mathrm{id}_{A^*}$. □

Corollary B.3. *Let A be a Banach algebra with bounded left approximate identity. Then, for any Banach algebra B which contains A as a closed right ideal,*

$$B/A \twoheadleftarrow B \leftarrowtail A \tag{B.8}$$

is a pure-exact sequence of Banach spaces.

The sequence in (B.8) is pure-exact if it is pure exact when B is replaced by its unitalization and for unital B the assertion in B.3 is an immediate corollary of Proposition B.2.

Note that Proposition B.2 is for Banach algebras what Proposition 2 in [37] is for rings.

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